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A Jacobi Spectral Collocation Scheme Based on Operational Matrix for Time-fractional Modified Korteweg-de Vries Equations

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Abstract: In this paper, a high accurate numerical approach is investigated for solving the time-fractional linear and nonlinear Korteweg-de Vries (KdV) equations. These equations are the most appropriate and desirable definition for physical modeling. The spectral collocation method and the operational matrix of fractional derivatives are used together with the help of the Gauss-quadrature formula in order to reduce such problem into a problem consists of solving a system of algebraic equations which greatly simplifying the problem. Our approach is based on the shifted Jacobi polynomials and the fractional derivative is described in the sense of Caputo. In addition, the presented approach is applied also to solve the time-fractional modified KdV equation. For testing the accuracy, validity and applicability of the developed numerical approach, we apply it to provide high accurate approximate solutions for four test problems.

Keywords: KdV equation, Jacobi polynomials, Operational matrix, Gauss quadrature, Collocation spectral method, Caputo derivative

1 Introduction

In recent years, many engineering and physical scientists have interested in studying the fractional calculus (theories of derivatives and integrals with any non-integer arbitrary order) for its ability to describe many engineering, physical and mathematical phenomena, see [Alcoutlabi and Martinez-Vega (1998); Chen, Han and Liu (2014); Biswas, Bhrawy, Abdelkawy, Alshaery and Hilal (2014); Pang, Chen and Sze (2014); Kumar, Singh and Kumar (2015); Lin, Wang and Wei (2015);

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Garrappa (2015); Li (2014); Wang, Liu, Chen, Liu and Liu (2014); Chen, Liu, Li and Sun (2014); Sadati, Ghaderi and Ranjbar (2013); Dehghan, Abbaszadeh and Mohebbi (2015a); Kumar, Singh and Sushila (2013); Abdelkawy, Zaky, Bhrawy and Baleanu, (2015)]. Furthermore, studying the properties of the fractional differential equations (differential equations with non-integer arbitrary order) and finding effective analytical and numerical solutions for them have become very important to be studied, for instance, the fractional sub-equation method [Wang and Xu (2014a) and Wang and Xu (2014b)]; the kernel-based approximation technique [Dou and Hon (2014)], the predictor-corrector method [Yu, Liu, Turner and Burrage (2014)], the sumudu decomposition method [Al-Khaled (2015)]; the waveform relaxation methods [Jiang and Ding (2013)], the Haar wavelet operational matrix [Ray (2012)], the fast alternating-direction finite difference method [Wang and Du (2014)], the Taylor matrix method [Gulsu, Ozturk and Anapal (2013)] and others [Wang, Du, Tan, Li and Nie (2013); Wei, Chen and Sun (2014); Shukla, Tamsir, Srivastava and Kumar (2014); Hwang and Geubelle (2000); El-Danaf and Hadhoud (2012); Wei and Zeng (2012); Valipour, Yaghoobi and Mashinchi (2014); Li, Chen and Ye (2011); Garrappa and Popolizio (2011)]. Recently, the spectral methods have been used based on some orthogonal polynomials to solve high-order differential and fractional differential equations, see [Abd-Elhameed (2014); Avila, Ramos and Atluri (2009); Bhrawy and Abdelkawy (2015); Bhrawy and Zaky (2015b); Dehghan, Abbaszadeh and Mohebbi (2015b)].

The operational matrices of fractional derivatives have been derived for some types of orthogonal polynomials such as, Legendre polynomials [Saadatmandi and Dehghan (2010)], Chebyshev polynomials [Doha, Bhrawy and Ezz-Eldien (2011)] and Jacobi polynomials [Doha, Bhrawy and Ezz-Eldien (2012)] that used together with the tau- and collocation spectral methods to solve types of ordinary fractional differential equations. Also, the operational matrices of fractional integrals have been derived for some types of orthogonal polynomials such as Chebyshev polynomials [Bhrawy and Alofi (2013)], Jacobi polynomials [Doha, Bhrawy and Ezz-Eldien (2012)] and Laguerre polynomials [Bhrawy, Alghamdi and Taha (2012)] that used together with the tau- and collocation spectral methods to solve fractional differential equations. Recently, the operational matrices of fractional derivatives and those of fractional integrals have been used with the help of the tau-spectral method to solve types of partial fractional differential equations, see [Saadatmandi and Dehghan (2011); Bhrawy and Zaky (2015a); Doha, Bhrawy and Ezz-Eldien (2015)]. More recently, the operational matrices have been used for obtaining the numerical solution of types of optimal control problems [Bhrawy, Doha, Baleanu, Ezz-Eldien and Abdelkawy (2015)] and the Lane-Emden type equations [Doha, Abd-Elhameed and Bassuony (2015)].

The KdV equation has been used to describe a large number of engineering and physical phenomena, see [Karpman (1998); Leblond and Sanchez (2003); Liu, Zhou, Liu and Luo (2003); Gao and Tian (2001)]. The time-fractional KdV (modified KdV) equation is a generalization of the classical KdV (modified KdV) equation that obtained by replacing the first-order time derivative term by a fractional derivative one of order ν, $0 < \nu \leq 1$. Wang [Wang (2007)] applied the homotopy perturbation method for an analytical solution of the fractional KdV equation, while in [El-Wakil, Abulwafa, Zahran and Mahmoud (2011)], the authors applied the He's variational-iteration method to solve the time-fractional KdV equation. Recently, Guo at al. [Guo, Mei, Fang and Qiu (2012)] used the fractional variational iteration method based on the He's polynomials to introduce compacton and solitary pattern solutions for the nonlinear time-fractional dispersive KdV-type equations involving Jumarie's fractional derivative. On the other hand, Abdulaziz et al. [Abdulaziz, Hashim and Ismail (2009)] introduced the fractional modified KdV equation and applied the homotopy-perturbation method for its approximate solution, while in [Kurulay and Bayram (2010)], the authors applied the two-dimensional differential transform method for an approximate analytical solution of the fractional modified KdV equation.

The main goal of the current paper is to introduce some efficient numerical techniques to solve spectrally the time-fractional linear, nonlinear and modified KdV equations. Our numerical techniques are based on the shifted Jacobi collocation spectral method and the operational matrix of fractional derivative together with the help of the Gauss quadrature formula to reduce thus problems into a problem consists of solving a system of algebraic equations that can be solved by any iterative method.

The current paper is organized as follows. In Section 2, we introduce some definitions and notations of fractional calculus with some properties of Jacobi polynomials. In Sections 3, 4 and 5, the operational matrix of fractional derivative is used together with the Jacobi tau-spectral method to solve the time-fractional linear, nonlinear and modified KdV equations, respectively. In Section 6, some numerical examples are introduced for ensuring the validity and accuracy of the presented technique. Also a conclusion is given in Section 7.

2 Preliminaries and notation

2.1 Fractional calculus definitions

Riemann-Liouville and Caputo fractional definitions are the two most used from other definitions of fractional derivatives which have been introduced recently.

Definition 1.1. The integral of order $\gamma \geq 0$ (fractional) according to Riemann-

Liouville is given by

$$I^{\gamma} f(x) = \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} f(t) dt, \quad \gamma > 0, \quad x > 0,$$
$$I^0 f(x) = f(x), \tag{1}$$

where

$$\Gamma(\gamma) = \int_0^{\infty} x^{\gamma-1} e^{-x} dx$$

is gamma function.
The operator I^{γ} satisfies the following properties

$$\begin{aligned} & I^{\gamma} I^{\delta} f(x) = I^{\gamma+\delta} f(x), \\ & I^{\gamma} I^{\delta} f(x) = I^{\delta} I^{\gamma} f(x), \\ & I^{\gamma} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\gamma)}\, x^{\beta+\gamma}. \end{aligned} \tag{2}$$

Definition 1.2. The Caputo fractional derivative of order γ is defined by

$$D^{\gamma} f(x) = \frac{1}{\Gamma(m-\gamma)} \int_0^x (x-t)^{m-\gamma-1} \frac{d^m}{dt^m} f(t) dt, \quad m-1 < \gamma \leq m,\ x > 0, \tag{3}$$

where m is the ceiling function of γ.
The operator D^{γ} satisfies the following properties

$$\begin{aligned} & D^{\gamma} C = 0, \qquad \text{(C is constant)}, \\ & I^{\gamma} D^{\gamma} f(x) = f(x) - \sum_{i=0}^{m-1} f^{(i)}(0^+) \frac{x^i}{i!}, \\ & D^{\gamma} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\gamma)}\, x^{\beta-\gamma}, \\ & D^{\gamma}(\lambda f(x) + \mu g(x)) = \lambda D^{\gamma} f(x) + \mu D^{\gamma} g(x). \end{aligned} \tag{4}$$

2.2 Shifted Jacobi polynomials

The Jacobi polynomial of degree j, denoted by $P_j^{(\alpha,\beta)}(z)$; $\alpha \geq -1$, $\beta \geq -1$ and defined on the interval $[-1,1]$, constitute an orthogonal system with respect to the weight function $\omega^{(\alpha,\beta)}(z) = (1-z)^{\alpha}(1+z)^{\beta}$, i.e.,

$$\int_{-1}^{1} P_j^{(\alpha,\beta)}(z) P_k^{(\alpha,\beta)}(z) \omega^{(\alpha,\beta)}(z) dz = \delta_{jk} \gamma_k^{(\alpha,\beta)},$$

where δ_{jk} is the Kronecker function and

$$\gamma_k^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)k!\Gamma(k+\alpha+\beta+1)}.$$

The shifted Jacobi polynomial of degree j, denoted by $P_{L,j}^{(\alpha,\beta)}(x);\ \alpha \geq -1,\ \beta \geq -1$ and defined on the interval $[0,L]$, is generated by introducing the change of variable $z = \frac{2x}{L} - 1$, i.e., $P_j^{(\alpha,\beta)}(\frac{2x}{L}-1) \equiv P_{L,j}^{(\alpha,\beta)}(x)$. Then the shifted Jacobi polynomials are constituting an orthogonal system with respect to the weight function $\omega_L^{(\alpha,\beta)}(x) = x^\beta (L-x)^\alpha$ with the orthogonality property

$$\int_0^L P_{L,j}^{(\alpha,\beta)}(x) P_{L,k}^{(\alpha,\beta)}(x)\omega_L^{(\alpha,\beta)}(x)dx = h_{L,k}^{(\alpha,\beta)}, \tag{5}$$

where

$$h_{L,k}^{(\alpha,\beta)} = (\frac{L}{2})^{\alpha+\beta+1}\delta_{jk}\gamma_j^{(\alpha,\beta)} = \frac{L^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)k!\Gamma(k+\alpha+\beta+1)}\delta_{jk}.$$

The shifted Jacobi polynomials are generated from the three-term recurrence relations

$$P_{L,j+1}^{(\alpha,\beta)}(x) = (\mu_j x - \xi_j)P_{L,j}^{(\alpha,\beta)}(x) - \zeta_j P_{L,j-1}^{(\alpha,\beta)}(x), \quad j \geq 1,$$

with

$$P_{L,0}^{(\alpha,\beta)}(x) = 1, \quad P_{L,1}^{(\alpha,\beta)}(x) = \frac{1}{L}(\alpha+\beta+2)x - (\beta+1),$$

where

$$\begin{aligned}
\mu_j &= \frac{(2j+\alpha+\beta+1)(2j+\alpha+\beta+2)}{L(j+1)(j+\alpha+\beta+1)},\\
\xi_j &= \frac{(2j+\alpha+\beta+1)(2j^2+(1+\beta)(\alpha+\beta)+2j(\alpha+\beta+1))}{(j+1)(j+\alpha+\beta+1)(2j+\alpha+\beta)},\\
\zeta_j &= \frac{(2j+\alpha+\beta+2)(j+\alpha)(j+\beta)}{(j+1)(j+\alpha+\beta+1)(2j+\alpha+\beta)}.
\end{aligned}$$

The explicit analytic form of the shifted Jacobi polynomials $P_{L,j}^{(\alpha,\beta)}(x)$ of degree j is given by

$$P_{L,j}^{(\alpha,\beta)}(x) = \sum_{k=0}^{j} (-1)^{j-k} \frac{\Gamma(j+\beta+1)\Gamma(j+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\Gamma(j+\alpha+\beta+1)(j-k)!k!L^k} x^k, \tag{6}$$

and this in turn, enables one to get

$$P_{L,i}^{(\alpha,\beta)}(0) = (-1)^i \frac{\Gamma(i+\beta+1)}{\Gamma(\beta+1)\ i!},$$
$$D^q P_{L,i}^{(\alpha,\beta)}(0) = (-1)^{i-q} \frac{\Gamma(i+\beta+1)(i+\alpha+\beta+1)_q}{L^q \Gamma(i-q+1)\ \Gamma(q+\beta+1)}, \quad q \leq i,$$
$$P_{L,i}^{(\alpha,\beta)}(L) = \frac{\Gamma(i+\alpha+1)}{\Gamma(\alpha+1)\ i!},$$
$$D^q P_{L,i}^{(\alpha,\beta)}(L) = \frac{\Gamma(i+\alpha+1)(i+\alpha+\beta+1)_q}{L^q \Gamma(i-q+1)\ \Gamma(q+\alpha+1)}, \qquad q \leq i,$$

which will be of important use later.

Assume $y(x)$ is a square integrable function with respect to the Jacobi weight function $\omega_L^{(\alpha,\beta)}(x)$ in $(0,L)$, then it can be expressed in terms of shifted Jacobi polynomials as

$$y(x) = \sum_{j=0}^{\infty} a_j P_{L,j}^{(\alpha,\beta)}(x),$$

from which the coefficients a_j are given by

$$a_j = \frac{1}{h_{L,j}^{(\alpha,\beta)}} \int_0^L \omega_L^{(\alpha,\beta)}(x) y(x) P_{L,j}^{(\alpha,\beta)}(x) dx, \qquad j = 0, 1, \cdots . \tag{7}$$

If we approximate $y(x)$ by the first $(N+1)$-terms, then we can write

$$y_N(x) \simeq \sum_{j=0}^{N} a_j P_{L,j}^{(\alpha,\beta)}(x), \tag{8}$$

which alternatively may be written in the matrix form:

$$y_N(x) \simeq \mathbf{A}^T \Delta_{L,N}(x), \tag{9}$$

with

$$\mathbf{A} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_j \\ \vdots \\ a_N \end{pmatrix}, \qquad \Delta_{L,N}(t) = \begin{pmatrix} P_{L,0}^{(\alpha,\beta)}(x) \\ P_{L,1}^{(\alpha,\beta)}(x) \\ \vdots \\ P_{L,j}^{(\alpha,\beta)}(x) \\ \vdots \\ P_{L,N}^{(\alpha,\beta)}(x) \end{pmatrix}. \tag{10}$$

Similarly, let $y(x,t)$ be an infinitely differentiable function defined on $0 < x \leq L$ and $0 < t \leq \tau$. Then it is possible to express as

$$y_{M,N}(x,t) = \sum_{i=0}^{M}\sum_{j=0}^{N} y_{ij} P_{\tau,i}^{(\alpha,\beta)}(t) P_{L,j}^{(\alpha,\beta)}(x) = \Delta_{\tau,M}^{T}(t)\mathbf{Y}\Delta_{L,N}(x), \tag{11}$$

with

$$\mathbf{Y} = \begin{pmatrix} y_{00} & y_{01} & \cdots & y_{0j} & \cdots & y_{0N} \\ y_{10} & y_{11} & \cdots & y_{1j} & \cdots & y_{1N} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ y_{i0} & y_{i1} & \cdots & y_{ij} & \cdots & y_{iN} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ y_{M0} & y_{M1} & \cdots & y_{Mj} & \cdots & y_{MN} \end{pmatrix},$$

and

$$\begin{gathered} y_{ij} = \frac{1}{h_{\tau,i}^{(\alpha,\beta)} h_{L,j}^{(\alpha,\beta)}} \int_0^{\tau}\int_0^{L} y(x,t) P_{\tau,i}^{(\alpha,\beta)}(t)\, P_{L,j}^{(\alpha,\beta)}(x)\, \omega_{\tau}^{(\alpha,\beta)}(t)\, \omega_{L}^{(\alpha,\beta)}(x)\, dxdt, \\ i = 0,1,\cdots,M, \qquad j = 0,1,\cdots,N. \end{gathered} \tag{12}$$

The fractional differentiation of order ν of $\Delta_{L,N}(x)$ can be expressed as

$$D^{\nu}\Delta_{L,N}(x) \simeq \mathbf{D}_{(\nu)}\Delta_{L,N}(x), \tag{13}$$

where $\mathbf{D}_{(\nu)}$ is the $(N+1)\times(N+1)$ Jacobi operational matrix of differentiation of order ν in the Caputo sense and is defined as follows:

$$\mathbf{D}_{(\nu)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \Psi_{\nu}^{(\alpha,\beta)}(\lceil\nu\rceil,0) & \Psi_{\nu}^{(\alpha,\beta)}(\lceil\nu\rceil,1) & \cdots & \Psi_{\nu}(\lceil\nu\rceil,j) & \cdots & \Psi_{\nu}^{(\alpha,\beta)}(\lceil\nu\rceil,N) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \Psi_{\nu}^{(\alpha,\beta)}(i,0) & \Psi_{\nu}^{(\alpha,\beta)}(i,1) & \cdots & \Psi_{\nu}^{(\alpha,\beta)}(i,j) & \cdots & \Psi_{\nu}^{(\alpha,\beta)}(i,N) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \Psi_{\nu}^{(\alpha,\beta)}(N,0) & \Psi_{\nu}^{(\alpha,\beta)}(N,1) & \cdots & \Psi_{\nu}^{(\alpha,\beta)}(N,j) & \cdots & \Psi_{\nu}^{(\alpha,\beta)}(N,N) \end{pmatrix},$$

where

$$\Psi_{\nu}^{(\alpha,\beta)}(i,j)$$
$$= \sum_{k=\lceil\nu\rceil}^{i} \frac{(-1)^{i-k}\, L^{\alpha+\beta-\nu+1}\Gamma(j+\beta+1)\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)}{h_{L,j}^{(\alpha,\beta)}\Gamma(j+\alpha+\beta+1)\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)\Gamma(k-\nu+1)(i-k)!}$$
$$\times \sum_{l=0}^{j} \frac{(-1)^{j-l}\,\Gamma(j+l+\alpha+\beta+1)\,\Gamma(\alpha+1)\,\Gamma(l+k+\beta-\nu+1)}{\Gamma(l+\beta+1)\,\Gamma(l+k+\alpha+\beta-\nu+2)\,(j-l)!\,l!}.$$

Note that in $\mathbf{D}_{(\nu)}$, the first $\lceil\nu\rceil$ rows, are all zero, (see [Doha, Bhrawy and Ezz-Eldien (2011)] for proof).

3 Time-fractional linear KdV equation

In this section, we use the operational matrix of fractional derivatives, the collocation spectral method and the Gauss quadrature formula with the shifted Jacobi polynomials as the basis functions to solve the time-fractional linear KdV equation:

$$\frac{\partial^{\nu} u(x,t)}{\partial t^{\nu}} + A\frac{\partial u(x,t)}{\partial x} + B\frac{\partial^{3}\phi(x,t)}{\partial x^{3}} = s(x,t), \tag{14}$$

with the initial condition

$$u(x,0) = q(x), \qquad 0 < x \leq L, \tag{15}$$

and boundary conditions

$$u(0,t) = f(t), \quad \frac{\partial u(L,t)}{\partial x} = g(t), \quad \frac{\partial^{2} u(L,t)}{\partial x^{2}} = h(t), \qquad 0 < t < \tau, \tag{16}$$

where ν, $(0 < \nu \leq 1)$, A, B are real constants and $s(x,t)$ is the source function. The function $u(x,t)$ is assumed to be the causal function of time and space, i.e., vanishing for $t < 0$ and $x < 0$.

First, the initial-boundary value problem (14)-(16) is equivalent to the boundary value problem

$$\frac{\partial^{\nu} u(x,t)}{\partial t^{\nu}} + u(x,0) - q(x) + A\frac{\partial u(x,t)}{\partial x} + B\frac{\partial^{3}\phi(x,t)}{\partial x^{3}} = s(x,t), \tag{17}$$

with the boundary conditions (16).

Now, we approximate $u(x,t)$ and $s(x,t)$ by the shifted Jacobi polynomials as

$$\begin{aligned} u_{M,N}(x,t) &\simeq \Delta_{\tau,M}^{T}(t)\mathbf{U}\Delta_{L,N}(x), \\ s_{M,N}(x,t) &\simeq \Delta_{\tau,M}^{T}(t)\mathbf{S}\Delta_{L,N}(x), \end{aligned} \tag{18}$$

where $\mathbf{U}$ is an unknown coefficients $(M+1)\times(N+1)$ matrix, while $\mathbf{S}$ is a known matrix that can be written as

$$\mathbf{S}=\begin{pmatrix} s_{00} & s_{01} & \cdots & s_{0j} & \cdots & s_{0N}\\ s_{10} & s_{11} & \cdots & s_{1j} & \cdots & s_{1N}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ s_{i0} & s_{i1} & \cdots & s_{ij} & \cdots & s_{iN}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ s_{M0} & s_{M1} & \cdots & s_{Mj} & \cdots & s_{MN} \end{pmatrix},$$

where the coefficients $s_{ij};\ i=0,1,\cdots,M,\ j=0,1,\cdots,N$ can be evaluated by

$$s_{ij}=\frac{1}{h_{\tau,i}^{(\alpha,\beta)}h_{L,j}^{(\alpha,\beta)}}\int_0^\tau\int_0^L s(x,t)P_{\tau,i}^{(\alpha,\beta)}(t)\ P_{L,j}^{(\alpha,\beta)}(x)\ \omega_\tau^{(\alpha,\beta)}(t)\ \omega_L^{(\alpha,\beta)}(x)\ dxdt,$$
$$i=0,1,\cdots,M,\qquad j=0,1,\cdots,N.$$

For general function $s(x,t)$, it is more difficult to compute the previous integral exactly. Using the Jacobi-Gauss quadrature formula, we can approximate the coefficients s_{ij} as

$$s_{ij}=\frac{1}{h_{\tau,i}^{(\alpha,\beta)}h_{L,j}^{(\alpha,\beta)}}\sum_{\delta=0}^{M}\sum_{\varepsilon=0}^{N}s(x_{L,N,\varepsilon}^{(\alpha,\beta)},t_{\tau,M,\delta}^{(\alpha,\beta)})P_{\tau,i}^{(\alpha,\beta)}(t_{\tau,M,\delta}^{(\alpha,\beta)})P_{L,j}^{(\alpha,\beta)}(x_{L,N,\varepsilon}^{(\alpha,\beta)})\varpi_{\tau,M,\delta}^{(\alpha,\beta)}\varpi_{L,N,\varepsilon}^{(\alpha,\beta)},$$
$$i=0,1,\cdots,M,\qquad j=0,1,\cdots,N,$$

where $x_{L,N,\varepsilon}^{(\alpha,\beta)},\ 0\le\varepsilon\le N$ are the zeros of Jacobi-Gauss quadrature in the interval $(0,L)$, with $\varpi_{L,N,\varepsilon}^{(\alpha,\beta)},\ 0\le\varepsilon\le N$ are corresponding Christoffel numbers and $t_{\tau,M,\delta}^{(\alpha,\beta)},\ 0\le\delta\le M$ are the zeros of Jacobi-Gauss quadrature in the interval $(0,\tau)$, with $\varpi_{\tau,M,\delta}^{(\alpha,\beta)},\ 0\le\delta\le M$ are corresponding Christoffel numbers.
Using Eqs. (13) and (18), we can write

$$\begin{aligned}
&\frac{\partial^\nu u(x,t)}{\partial t^\nu}\simeq\Delta_{\tau,M}^T(t)\mathbf{D}_{(\nu)}^T\mathbf{U}\Delta_{L,N}(x),\\
&\frac{\partial u(x,t)}{\partial x}\simeq\Delta_{\tau,M}^T(t)\mathbf{U}\mathbf{D}_{(1)}\Delta_{L,N}(x),\\
&\frac{\partial^3 u(x,t)}{\partial x^3}\simeq\Delta_{\tau,M}^T(t)\mathbf{U}\mathbf{D}_{(3)}\Delta_{L,N}(x),\\
&u(x,0)\simeq\Delta_{\tau,M}^T(0)\mathbf{U}\Delta_{L,N}(x).
\end{aligned}\tag{19}$$

By substituting (18) and (19) in (17), we get

$$\begin{aligned}&\Delta_{\tau,M}^{T}(t)\mathbf{D}_{(\nu)}^{T}\mathbf{U}\Delta_{L,N}(x)+A\Delta_{\tau,M}^{T}(t)\mathbf{U}\mathbf{D}_{(1)}\Delta_{L,N}(x)+B\Delta_{\tau,M}^{T}(t)\mathbf{U}\mathbf{D}_{(3)}\Delta_{L,N}(x)\\&+\Delta_{\tau,M}^{T}(0)\mathbf{U}\Delta_{L,N}(x)-q(x)=\Delta_{\tau,M}^{T}(t)\mathbf{S}\Delta_{L,N}(x).\end{aligned}\tag{20}$$

We collocate (20) at $(M+1)(N-2)$ points, as

$$\begin{aligned}&\Delta_{\tau,M}^{T}(t_i)\mathbf{D}_{(\nu)}^{T}\mathbf{U}\Delta_{L,N}(x_j)+A\Delta_{\tau,M}^{T}(t_i)\mathbf{U}\mathbf{D}_{(1)}\Delta_{L,N}(x_j)+B\Delta_{\tau,M}^{T}(t_i)\mathbf{U}\mathbf{D}_{(3)}\Delta_{L,N}(x_j)\\&+\Delta_{\tau,M}^{T}(0)\mathbf{U}\Delta_{L,N}(x_j)-q(x_j)=\Delta_{\tau,M}^{T}(t_i)\mathbf{S}\Delta_{L,N}(x_j),\end{aligned}\tag{21}$$

where $t_i,\ i=0,1,\cdots,M$ are the roots of $P_{\tau,M+1}^{(\alpha,\beta)}(t)$, while $x_j,\ j=0,1,\cdots,N-3$ are the roots of $P_{L,N-2}^{(\alpha,\beta)}(x)$, this generates a system of $(M+1)(N-2)$ nonlinear algebraic equations in the unknown expansion coefficients, $u_{ij},\ i=0,1,\cdots,M;\ j=0,1,\cdots,N-2$, and the rest of this system is obtained from the boundary conditions (16), as

$$\begin{aligned}&\Delta_{\tau,M}^{T}(t_i)\mathbf{U}\Delta_{L,N}(0)=f(t_i),\\&\Delta_{\tau,M}^{T}(t_i)\mathbf{U}\mathbf{D}_{(1)}\Delta_{L,N}(L)=g(t_i), \qquad\qquad i=0,1,\cdots,M.\\&\Delta_{\tau,M}^{T}(t_i)\mathbf{U}\mathbf{D}_{(2)}\Delta_{L,N}(L)=h(t_i),\end{aligned}\tag{22}$$

The $(M+1)(N-2)$ system (21) may be combined with the $3(M+1)$ system (22) to be written as a $(M+1)(N+1)$ system of nonlinear algebraic equations in the unknown expansion coefficients u_{ij}, that can solved using Newton's iterative method. Consequently $u_{M,N}(x,t)$ given in (18) can be calculated.

4 Time-fractional nonlinear KdV equation

In this section, we apply the numerical technique obtained in the previous section to solve the time-fractional nonlinear KdV equation:

$$\frac{\partial^{\nu}u(x,t)}{\partial t^{\nu}}+Au(x,t)\frac{\partial u(x,t)}{\partial x}+B\frac{\partial^{3}\phi(x,t)}{\partial x^{3}}=s(x,t),\tag{23}$$

with the initial condition (15) and the boundary conditions (16).

As in the previous section, we can rewrite (23) as in the following form:

$$\frac{\partial^{\nu}u(x,t)}{\partial t^{\nu}}+u(x,0)-q(x)+Au(x,t)\frac{\partial u(x,t)}{\partial x}+B\frac{\partial^{3}\phi(x,t)}{\partial x^{3}}=s(x,t),\tag{24}$$

with the boundary conditions (16).

After approximating $u(x,t)$ and $s(x,t)$ by the shifted Jacobi polynomials as in Eq. (18), and writing $\dfrac{\partial^{\nu} u(x,t)}{\partial t^{\nu}}$, $\dfrac{\partial u(x,t)}{\partial x}$, $\dfrac{\partial^3 u(x,t)}{\partial x^3}$ and $u(x,0)$ as in Eq. (19), we get

$$\begin{aligned}
&\Delta^T_{\tau,M}(t)\mathbf{D}^T_{(\nu)}\mathbf{U}\Delta_{L,N}(x)+A\left(\Delta^T_{\tau,M}(t)\mathbf{U}\Delta_{L,N}(x)\right)\left(\Delta^T_{\tau,M}(t)\mathbf{U}\mathbf{D}_{(1)}\Delta_{L,N}(x)\right)\\
&+B\Delta^T_{\tau,M}(t)\mathbf{U}\mathbf{D}_{(3)}\Delta_{L,N}(x)+\Delta^T_{\tau,M}(0)\mathbf{U}\Delta_{L,N}(x)-q(x)\\
&=\Delta^T_{\tau,M}(t)\mathbf{S}\Delta_{L,N}(x).
\end{aligned}\tag{25}$$

Now, we collocate Eq. (25) at $(M+1)(N-2)$ points, as

$$\begin{aligned}
&\Delta^T_{\tau,M}(t_i)\mathbf{D}^T_{(\nu)}\mathbf{U}\Delta_{L,N}(x_j)+A\left(\Delta^T_{\tau,M}(t_i)\mathbf{U}\Delta_{L,N}(x_j)\right)\left(\Delta^T_{\tau,M}(t_i)\mathbf{U}\mathbf{D}_{(1)}\Delta_{L,N}(x_j)\right)\\
&+B\Delta^T_{\tau,M}(t_i)\mathbf{U}\mathbf{D}_{(3)}\Delta_{L,N}(x_j)+\Delta^T_{\tau,M}(0)\mathbf{U}\Delta_{L,N}(x_j)-q(x_j)\\
&=\Delta^T_{\tau,M}(t_i)\mathbf{S}\Delta_{L,N}(x_j),
\end{aligned}\tag{26}$$

this generates a system of $(M+1)(N-2)$ nonlinear algebraic equations in the unknown expansion coefficients, $u_{ij},\ i=0,1,\cdots,M;\ j=0,1,\cdots,N-2$, and the rest of this system is obtained from the boundary conditions (16), as in Eq. (22).

The $(M+1)(N-2)$ system (26) may be combined with the $3(M+1)$ system (22) to be written as a $(M+1)(N+1)$ system of nonlinear algebraic equations in the unknown expansion coefficients u_{ij}, that can solved using Newton's iterative method. Consequently $u_{M,N}(x,t)$ given in (18) can be calculated.

5 Time-fractional modified KdV equation

In this section, we consider the following time-fractional modified KdV equation

$$\frac{\partial^{\nu} u(x,t)}{\partial t^{\nu}}+Au^2(x,t)\frac{\partial u(x,t)}{\partial x}+B\frac{\partial^3 \phi(x,t)}{\partial x^3}=s(x,t),\quad 0<x\le L,\ 0<t<\tau,\tag{27}$$

with the initial condition (15) and the boundary conditions (16).

As in the previous section, Eq. (27) can be written as

$$\frac{\partial^{\nu} u(x,t)}{\partial t^{\nu}}+u(x,0)-q(x)+Au^2(x,t)\frac{\partial u(x,t)}{\partial x}+B\frac{\partial^3 \phi(x,t)}{\partial x^3}=s(x,t),\tag{28}$$

with the boundary conditions (16).

After approximating $u(x,t)$ and $s(x,t)$ by the shifted Jacobi polynomials as in Eq. (18), and writing $\dfrac{\partial^{\nu} u(x,t)}{\partial t^{\nu}}$, $\dfrac{\partial u(x,t)}{\partial x}$, $\dfrac{\partial^3 u(x,t)}{\partial x^3}$ and $u(x,0)$ as in Eq. (19), we get

$$\begin{aligned}
&\Delta^T_{\tau,M}(t)\mathbf{D}^T_{(\nu)}\mathbf{U}\Delta_{L,N}(x)+A\left(\Delta^T_{\tau,M}(t)\mathbf{U}\Delta_{L,N}(x)\right)^2\left(\Delta^T_{\tau,M}(t)\mathbf{U}\mathbf{D}_{(1)}\Delta_{L,N}(x)\right)\\
&+B\Delta^T_{\tau,M}(t)\mathbf{U}\mathbf{D}_{(3)}\Delta_{L,N}(x)+\Delta^T_{\tau,M}(0)\mathbf{U}\Delta_{L,N}(x)-q(x)\\
&=\Delta^T_{\tau,M}(t)\mathbf{S}\Delta_{L,N}(x).
\end{aligned}\tag{29}$$

Now, we collocate Eq. (29) at $(M+1)(N-2)$ points, as

$$\begin{aligned}&\Delta^T_{\tau,M}(t_i)\mathbf{D}^T_{(\nu)}\mathbf{U}\Delta_{L,N}(x_j)+A\left(\Delta^T_{\tau,M}(t_i)\mathbf{U}\Delta_{L,N}(x_j)\right)^2\left(\Delta^T_{\tau,M}(t_i)\mathbf{U}\mathbf{D}_{(1)}\Delta_{L,N}(x_j)\right)\\&+B\Delta^T_{\tau,M}(t_i)\mathbf{U}\mathbf{D}_{(3)}\Delta_{L,N}(x_j)+\Delta^T_{\tau,M}(0)\mathbf{U}\Delta_{L,N}(x_j)-q(x_j)\\&=\Delta^T_{\tau,M}(t_i)\mathbf{S}\Delta_{L,N}(x_j),\end{aligned}\tag{30}$$

this generates a system of $(M+1)(N-2)$ nonlinear algebraic equations in the unknown expansion coefficients, $u_{ij},\ i=0,1,\cdots,M;\ j=0,1,\cdots,N-2$, and the rest of this system is obtained from the boundary conditions (16), as in Eq. (22).

The $(M+1)(N-2)$ system (30) may be combined with the $3(M+1)$ system (22) to be written as a $(M+1)(N+1)$ system of nonlinear algebraic equations in the unknown expansion coefficients u_{ij}, that can solved using Newton's iterative method. Consequently $u_{M,N}(x,t)$ given in (18) can be calculated.

6 Numerical results

For ensuring the efficiency of the proposed numerical techniques, the numerical results of some numerical examples of the time-fractional linear, nonlinear and modified KdV equation have been introduced in this section. Also, comparisons between our results and the exact solutions of such problems are introduced.

6.1 Linear fractional KdV equation

As the first example, we consider the linear time-fractional KdV equation studied in [Momani, Odibat and Alawanh (2008)]:

$$\frac{\partial^\nu u(x,t)}{\partial t^\nu}+\frac{\partial u(x,t)}{\partial x}+\frac{\partial^3\phi(x,t)}{\partial x^3}=\frac{2t^{2-\nu}}{\Gamma(3-\nu)}\cos(x),\tag{31}$$

with the initial condition

$$u(x,0)=0,$$

$$u(0,t)=t^2,\quad \frac{\partial u(\frac{\pi}{2},t)}{\partial x}=-t^2,\quad \frac{\partial^2 u(\frac{\pi}{2},t)}{\partial x^2}=0,$$

and the exact solution is $u(x,t)=t^2\cos(x)$.

Momani et al. [Momani, Odibat and Alawanh (2008)] introduced this problem and applied the variational iteration method for introducing an approximate solution for it. In order to show the high accuracy of the numerical technique presented in Section 3, we have applied it to solve problem (31). In Table 1, we list the absolute errors at $\alpha=\beta=1$, $x=\frac{\pi}{2}$ with $\nu=0.50,\ \nu=0.90$ and different values of M, $(M=N)$. Also, Figs. 1-2 show the absolute error functions at $\alpha=\beta=0,\ N=M=12$ with $\nu=0.50$ and $\nu=0.90$, respectively.

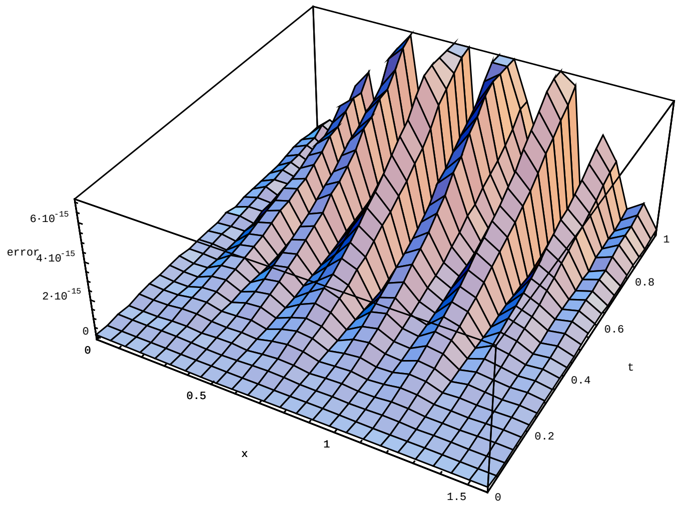

Figure 1: Absolute error function at $\alpha = \beta = 0$ with $N = M = 12$ and $\nu = 0.5$ for problem (31).

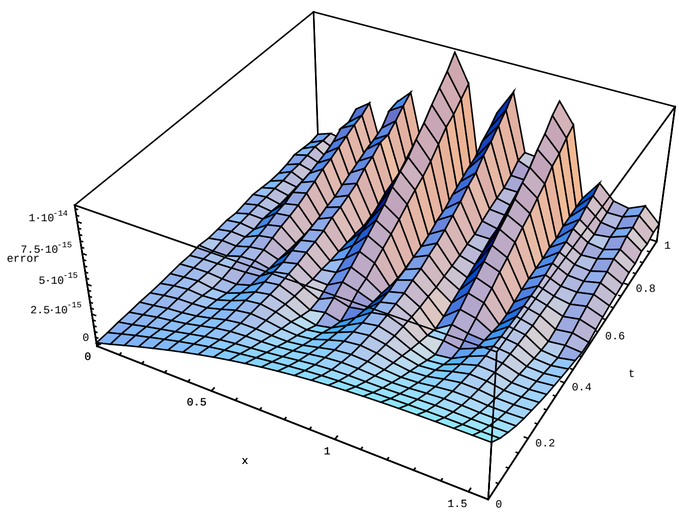

Figure 2: Absolute error function at $\alpha = \beta = 0$ with $N = M = 12$ and $\nu = 0.9$ for problem (31).

Table 1: Absolute errors at $\alpha=\beta=1,\ x=\frac{\pi}{2}$ with two different choices of ν for problem (31).

x	$\nu=0.50$			$\nu=0.90$		
	$M=4$	$M=8$	$M=12$	$M=4$	$M=8$	$M=12$
0.0	$7.80.10^{-5}$	$3.20.10^{-10}$	$2.13.10^{-15}$	$7.84.10^{-4}$	$1.16.10^{-9}$	$8.92.10^{-15}$
0.1	$1.04.10^{-4}$	$3.36.10^{-9}$	$1.10.10^{-14}$	$5.72.10^{-4}$	$2.10.10^{-9}$	$1.05.10^{-14}$
0.2	$3.90.10^{-4}$	$1.60.10^{-8}$	$5.77.10^{-14}$	$5.48.10^{-4}$	$8.64.10^{-9}$	$3.34.10^{-14}$
0.3	$9.63.10^{-4}$	$4.03.10^{-8}$	$1.46.10^{-13}$	$7.61.10^{-4}$	$2.36.10^{-8}$	$8.72.10^{-14}$
0.4	$1.84.10^{-3}$	$7.72.10^{-8}$	$2.80.10^{-13}$	$1.24.10^{-3}$	$4.91.10^{-8}$	$1.79.10^{-13}$
0.5	$3.05.10^{-3}$	$1.27.10^{-7}$	$4.64.10^{-13}$	$2.05.10^{-3}$	$8.69.10^{-8}$	$3.16.10^{-13}$
0.6	$4.61.10^{-3}$	$1.92.10^{-7}$	$6.98.10^{-13}$	$3.19.10^{-3}$	$1.38.10^{-7}$	$5.03.10^{-13}$
0.7	$6.53.10^{-3}$	$2.70.10^{-7}$	$9.85.10^{-13}$	$4.71.10^{-3}$	$2.04.10^{-7}$	$7.43.10^{-13}$
0.8	$8.82.10^{-3}$	$3.64.10^{-7}$	$1.32.10^{-12}$	$6.62.10^{-3}$	$2.86.10^{-7}$	$1.04.10^{-12}$
0.9	$1.14.10^{-2}$	$4.73.10^{-7}$	$1.72.10^{-12}$	$8.95.10^{-3}$	$3.84.10^{-7}$	$1.39.10^{-12}$
1.0	$1.45.10^{-2}$	$5.97.10^{-7}$	$2.17.10^{-12}$	$1.16.10^{-2}$	$4.99.10^{-7}$	$1.81.10^{-12}$

6.2 Homogeneous fractional KdV equation

As the second example, we consider the nonlinear homogeneous time-fractional KdV equation [Momani (2005); Odibat and Momani (2009)]

$$\frac{\partial^{\nu}u(x,t)}{\partial t^{\nu}}+6u(x,t)\frac{\partial u(x,t)}{\partial x}+\frac{\partial^{3}\phi(x,t)}{\partial x^{3}}=0, \tag{32}$$

with the initial-boundary conditions

$$u(x,0)=\frac{1}{1+\cosh(x)},\qquad u(0,t)=\frac{1}{1+\cosh(t)},$$
$$\frac{\partial u(1,t)}{\partial x}=\frac{2e^{t+1}(e^{t}-e)}{(e^{t}+e)^{3}},\quad \frac{\partial^{2}u(1,t)}{\partial x^{2}}=\frac{1}{4}(-2+\cosh(t-1))\mathrm{sech}^{4}(\frac{t-1}{2}),$$

and the exact solution $u(x,t)=\frac{1}{2}\mathrm{sech}^{2}(\frac{1}{2}(x-t))$.

In [Momani (2005)] and [Odibat and Momani (2009)], the Adomian decomposition method and the variational iteration method have been applied respectively to approximate the solution of this problem. In Fig. 3, we plot the absolute error function at $\alpha=\beta=0$ with $N=M=12$ and $\nu=1$ for problem (32), while Fig. 4 present the approximate values of $u(x,1)$ as function of space at $N=10,\ \alpha=\beta=1$ and various choices of $\nu,\ \nu=1,\ 0.90,\ 0.70,\ 0.50$ and 0.30.

From Fig. 3, it is clear that adding few terms of shifted Jacobi polynomials, good approximations of the exact solution were achieved. On the other hand, Fig. 4 obtain that as ν approaches to 1, the solution for the integer order system is recovered.

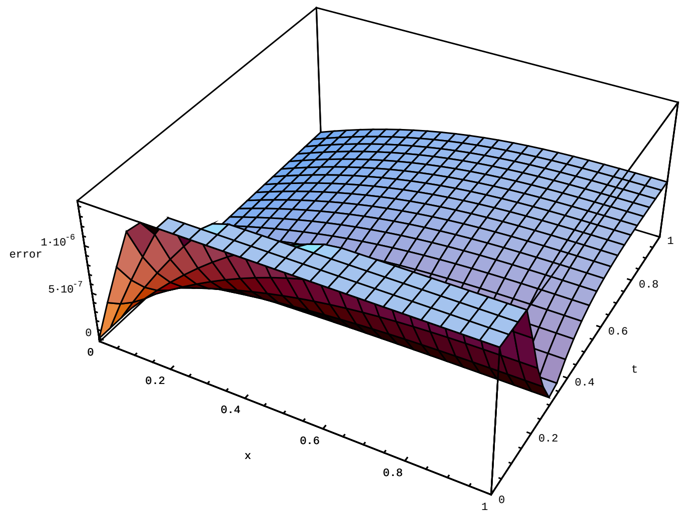

Figure 3: Absolute error function at $\alpha = \beta = 0$ with $N = M = 12$ and $\nu = 1$ for problem (32).

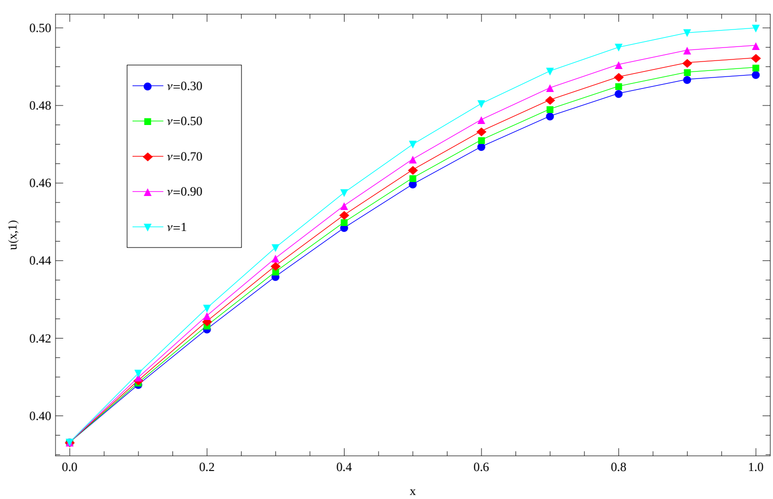

Figure 4: Approximate solution of $u(x,1)$ at $\alpha = \beta = 1$, $N = 10$ and $\nu = 1,\ 0.90,\ 0.70,\ 0.50$ and 0.30 for Example (32).

6.3 Inhomogeneous fractional KdV equation

Consider the following inhomogeneous time-fractional KdV equation

$$\frac{\partial^{\nu} u(x,t)}{\partial t^{\nu}} + 6u(x,t)\frac{\partial u(x,t)}{\partial x} + \frac{\partial^3 \phi(x,t)}{\partial x^3} = e^x\left(\frac{6t^{3-\nu}}{\Gamma(4-\nu)} + t^3 + 6t^6\right), \tag{33}$$

with the initial-boundary conditions

$$u(x,0) = 0,$$
$$u(0,t) = t^3, \quad \frac{\partial u(1,t)}{\partial x} = \frac{\partial^2 u(1,t)}{\partial x^2} = et^3,$$

and the exact solution $u(x,t) = t^3 e^x$.

Table 2 lists the L_∞ and L_2 errors at $\alpha = \beta = 0$ and $\nu = 0.40,\ \nu = 0.80$ with different values of M, $(M = N)$, while in Table 3, we obtain the absolute errors at $\alpha = \beta = 1$, $t = 1$ at $\nu = 0.40,\ \nu = 0.80$ with different values of M, $(M = N)$. Also, in Figs. 5-6, we plot the absolute error functions at $\alpha = \beta = 1$, $N = M = 12$ with $\nu = 0.20$ and $\nu = 0.60$, respectively.

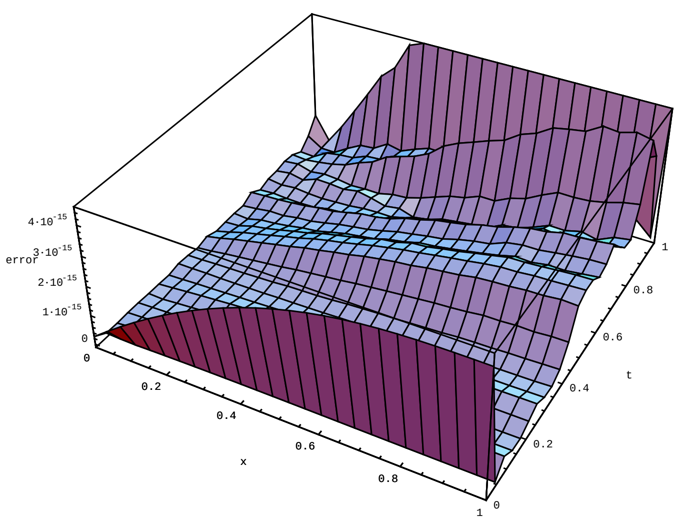

Figure 5: Absolute error function at $\alpha = \beta = 1$ with $N = M = 12$ and $\nu = 0.2$ for problem (33).

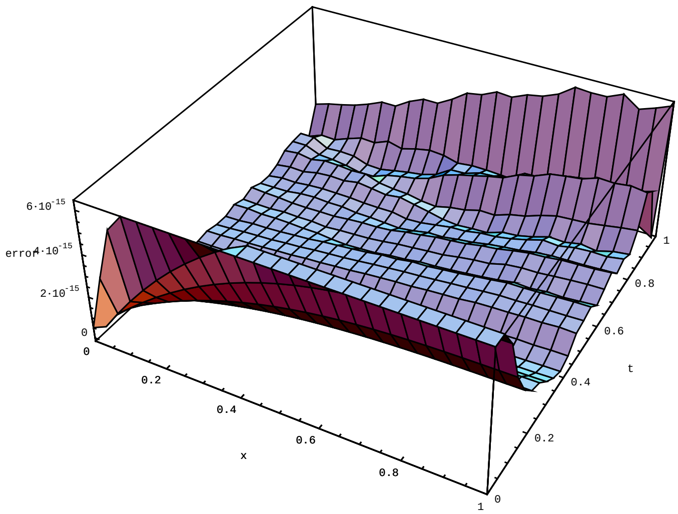

Figure 6: Absolute error function at $\alpha = \beta = 1$ with $N = M = 12$ and $\nu = 0.6$ for problem (33).

Table 2: L_∞ and L_2 errors at $\alpha = \beta = 0$ with two different choices of ν for problem (33).

M	$\nu = 0.40$		$\nu = 0.80$	
	L_∞	L_2	L_∞	L_2
4	$1.95959.10^{-3}$	$1.23297.10^{-3}$	$1.25798.10^{-3}$	$3.26505.10^{-3}$
6	$1.62731.10^{-6}$	$5.53182.10^{-7}$	$1.59100.10^{-6}$	$5.29697.10^{-7}$
8	$1.46047.10^{-9}$	$2.95973.10^{-10}$	$1.39004.10^{-9}$	$2.74151.10^{-10}$
10	$4.86055.10^{-13}$	$8.59688.10^{-14}$	$4.77173.10^{-13}$	$8.53537.10^{-14}$
12	$2.66453.10^{-15}$	$2.73496.10^{-16}$	$1.33226.10^{-15}$	$2.94995.10^{-16}$

6.4 Fractional modified KdV equation

Here, we consider the fractional modified KdV equation in the form:

$$\begin{aligned} &\frac{\partial^\nu u(x,t)}{\partial t^\nu} + 6u^2(x,t)\frac{\partial u(x,t)}{\partial x} + \frac{\partial^3 \phi(x,t)}{\partial x^3} \\ &= \frac{2t^{2-\nu}}{\Gamma(3-\nu)}\cos(x) + t^2\sin(x) - 6t^6\cos^2(x)\sin(x), \end{aligned} \tag{34}$$

Table 3: Absolute errors at $\alpha = \beta = 1,\ t = 1$ with two different choices of ν for problem (33).

x	$\nu = 0.20$			$\nu = 0.60$		
	$M=4$	$M=8$	$M=12$	$M=4$	$M=8$	$M=12$
0.1	$8.19.10^{-4}$	$6.38.10^{-10}$	$1.33.10^{-15}$	$6.27.10^{-4}$	$5.81.10^{-10}$	$1.77.10^{-15}$
0.2	$1.49.10^{-3}$	$6.45.10^{-10}$	$3.10.10^{-15}$	$1.13.10^{-3}$	$5.41.10^{-10}$	$2.66.10^{-15}$
0.3	$1.96.10^{-3}$	$3.11.10^{-10}$	$5.10.10^{-15}$	$1.45.10^{-3}$	$1.70.10^{-10}$	$2.88.10^{-15}$
0.4	$2.21.10^{-3}$	$5.08.10^{-10}$	$5.99.10^{-15}$	$1.58.10^{-3}$	$6.76.10^{-10}$	$3.33.10^{-15}$
0.5	$2.26.10^{-3}$	$1.89.10^{-9}$	$7.77.10^{-15}$	$1.54.10^{-3}$	$2.07.10^{-9}$	$4.44.10^{-15}$
0.6	$2.18.10^{-3}$	$3.46.10^{-9}$	$8.88.10^{-15}$	$1.40.10^{-3}$	$3.65.10^{-9}$	$4.88.10^{-15}$
0.7	$2.04.10^{-3}$	$4.82.10^{-9}$	$8.88.10^{-15}$	$1.21.10^{-3}$	$5.02.10^{-9}$	$4.88.10^{-15}$
0.8	$1.91.10^{-3}$	$6.07.10^{-9}$	$9.76.10^{-15}$	$1.06.10^{-3}$	$6.27.10^{-9}$	$5.32.10^{-15}$
0.9	$1.84.10^{-3}$	$7.23.10^{-9}$	$9.32.10^{-15}$	$9.86.10^{-4}$	$7.43.10^{-9}$	$5.32.10^{-15}$
1.0	$1.83.10^{-3}$	$7.64.10^{-9}$	$9.76.10^{-15}$	$9.70.10^{-4}$	$7.85.10^{-9}$	$5.32.10^{-15}$

with the initial-boundary conditions

$$u(x,0) = 0,$$
$$u(0,t) = t^2, \quad \frac{\partial u(\frac{\pi}{2},t)}{\partial x} = -t^2, \quad \frac{\partial^2 u(\frac{\pi}{2},t)}{\partial x^2} = 0,$$

and the exact solution $u(x,t) = t^2 \cos(x)$.

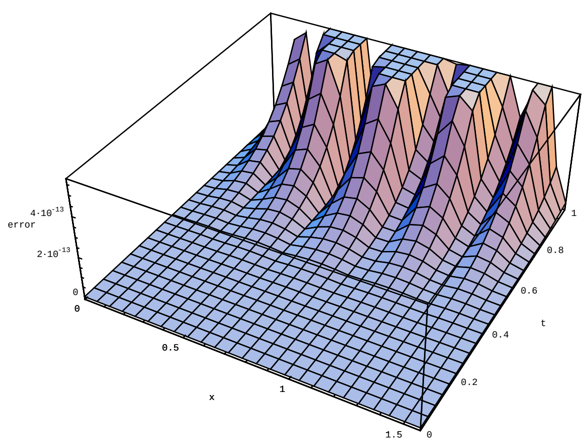

Figure 7: Absolute error function at $\alpha = \beta = 0$ with $N = M = 12$ and $\nu = 0.8$ for problem (34).

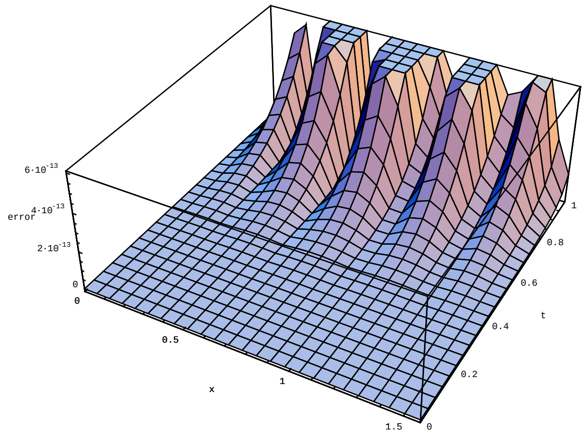

Figure 8: Absolute error function at $\alpha = \beta = 0$ with $N = M = 12$ and $\nu = 0.2$ for problem (34).

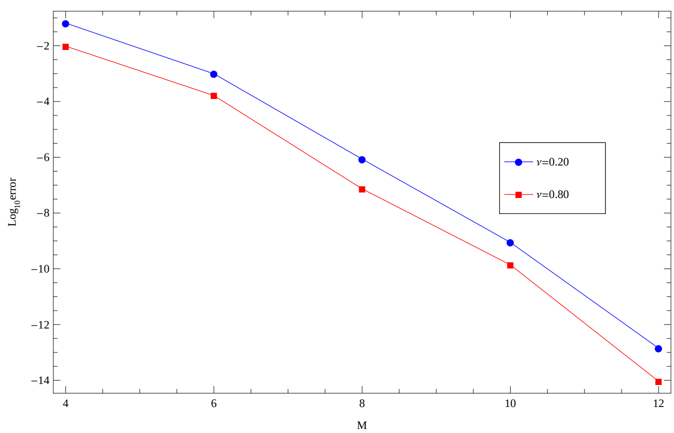

Figure 9: $Log_{10}MAE$ of $u(x,t)$ at $\alpha = \beta = 0$ for problem (34).

This problem has been solved by using the technique discussed in Section 5. Table 4 list the maximum absolute errors (MAEs) at $\alpha = \beta = 0$ with different choices of M, $(M = N)$ and ν. Also, Figs. 7-8 plot the absolute error functions at $\alpha = \beta = 0$, $N = M = 12$ with $\nu = 0.20$ and 0.80, respectively. Finally, in Fig. 9, we plot the logarithmic graphs of the MAEs ($log_{10}Error$) at two different choices of ν and various choices of M, $(N = M)$; by using the presented algorithm. From this figures, it is shown that the numerical errors decay rapidly as M increase.

Table 4: MAEs at $\alpha = \beta = 0$ with different choices of M, $(M = N)$ and ν for problem (34).

M	$\nu = 0.2$	$\nu = 0.6$	$\nu = 0.8$
4	$6.51996.10^{-2}$	$1.47553.10^{-2}$	$9.70825.10^{-3}$
6	$9.89460.10^{-4}$	$2.67555.10^{-4}$	$1.63932.10^{-4}$
8	$1.28407.10^{-6}$	$4.95732.10^{-7}$	$3.97162.10^{-7}$
10	$1.65863.10^{-9}$	$1.01662.10^{-9}$	$9.37753.10^{-10}$
12	$2.67075.10^{-12}$	$2.56228.10^{-12}$	$2.54873.10^{-12}$

7 Conclusions

In the current paper, an accurate numerical technique is constructed and applied to solve the linear and nonlinear time-fractional KdV equations. The operational matrix of fractional derivatives is used together with the collocation spectral method based on the shifted Jacobi polynomials for reducing such problems into a problem consists of solving a system of algebraic equations which simplifying the problem. The fractional derivative is described in the sense of Caputo. In addition, the presented technique is applied also to solve the time-fractional modified KdV equation. The numerical results have been achieved demonstrated the high efficiency and accuracy of our techniques. Moreover, only a small number of shifted Jacobi polynomials is needed to obtain a satisfactory solution.

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