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A New Coupled Fractional Reduced Differential Transform Method for the Numerical Solution of Fractional Predator-Prey System

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Abstract: In the present article, a relatively very new technique viz. Coupled Fractional Reduced Differential Transform, has been executed to attain the approximate numerical solution of the predator-prey dynamical system. The fractional derivatives are defined in the Caputo sense. Utilizing the present method we can solve many linear and nonlinear coupled fractional differential equations. The results thus obtained are compared with those of other available methods. Numerical solutions are presented graphically to show the simplicity and authenticity of the method.

Keywords: Coupled Fractional Reduced Differential Transform, Predator-Prey System, Caputo fractional derivative, Riemann-Liouville fractional derivative.

1 Introduction

In the field of engineering, physics, chemistry, and other sciences, many phenomena can be modelled very successfully by using mathematical tools in the form of fractional calculus, e.g. anomalous transport in disordered systems, some percolations in porous media and the diffusion of biological populations [Podlubny (1999); Hilfer (2000); Saha Ray and Bera (2006); Saha Ray (2007, 2008)]. Fractional calculus has been used to model physical and engineering systems that are found to be more accurately described by fractional differential equations. Thus, we need a reliable and competent technique for the solution of fractional differential equations. In this paper, the predator-prey system [Petrovskii et al. (2005)] has been discussed in the form of fractional coupled reaction-diffusion equation. In the present analysis, a new approximate numerical technique, Coupled Fractional Reduced Differential transform method (CFRDTM), has been presented which is appropriate for coupled fractional differential equations. The proposed method is

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an impressive solver for linear and non-linear coupled fractional differential equations. It is comparatively a new approach to provide the solution very effectively and competently.

The significant advantage of the proposed method is the fact that it provides its user with an analytical approximation, in many instances an exact solution, in a rapidly convergent sequence with elegantly computed terms. This technique does not involve any linearization, discretization or small perturbations and therefore it reduces significantly the numerical computation. This method provides extraordinary accuracy for the approximate solutions when compared to the exact solutions, particularly in large scale domain. It is not affected by computation round off errors and hence one does not face the need of large computer memory and time. The results reveal that the CFRDTM is very effective, convenient and quite accurate to the system of nonlinear equations.

In the present analysis, we consider a system of two species competitive model with prey population A and predator population B. For prey population $A \rightarrow 2A$, at the rate $a(a>0)$ express the natural birth rate. Similarly, for predator population $B \rightarrow 2B$, at the rate $c(c>0)$ represents the natural death rate. The interactive term between predator and prey population is $A+B \rightarrow 2B$, at rate $b(b>0)$ where b denotes the competitive rate. According to the knowledge of fractional calculus and biological population, the time fractional dynamics of a predator-prey system can be described as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + au - buv, \quad u(x,y,0) = \varphi(x,y) \tag{1}$$

$$\frac{\partial^\beta v}{\partial t^\beta} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + buv - cv, \quad v(x,y,0) = \phi(x,y) \tag{2}$$

where $t>0$, $x,y \in R$, $a,b,c>0$, $u(x,y,t)$ denotes the prey population density and $v(x,y,t)$ represents the predator population density. Here $\varphi(x,y)$ and $\phi(x,y)$ represent the initial conditions of population system. The fractional derivatives are considered in Caputo sense. Caputo fractional derivative is used because of its advantage that it permits the initial and boundary conditions included in the formulation of the problem. Here, $u(x,y,t)$ and $v(x,y,t)$ are analytic functions. The physical interpretations of eqns. (1) and (2) indicate that the prey-predator population system is analogous to the behaviour of fractional order model of anomalous biological diffusion.

Several analytical as well as numerical methods have been implemented by various authors to solve fractional differential equations. Wei et al. (2014) applied homotopy method to determine the unknown parameters of solute transport with spatial fractional derivative advection-dispersion equation. Saha Ray and Gupta proposed numerical schemes based on the Haar wavelet method for finding numerical solutions of Burger-Huxley, Huxley, modified Burgers and mKdV equations [Saha Ray and Gupta (2013, 2014)]. An approximate analytical solution of the time-fractional Cauchy-reaction diffusion equation by using fractional-order reduced differential transform method (FRDTM) has been proposed by Shukla et al. (2014).

In this paper, the fractional nonlinear predator-prey population model has been considered. The paper is systematized as follows: in Section 2, a brief review of the theory of fractional calculus has been presented for the specific purpose of this paper. In Section 3, the Coupled Fractional Reduced Differential Transform method has been analyzed in details. In Section 4, CFRDTM has been applied to determine the approximate solution for the nonlinear coupled fractional predator-prey equation. In Section 5, three examples have been examined to demonstrate the simplicity and competence of the proposed method. Finally, conclusions are presented in Section 6.

2 Mathematical Preliminaries of Fractional Calculus

The fractional calculus was first anticipated by Leibnitz, was one of the founders of standard calculus, in a letter written in 1695. This calculus comprises different definitions of the fractional operators along with the Riemann–Liouville fractional derivative, Caputo derivative, Riesz derivative and Grunwald–Letnikov fractional derivative [Podlubny (1999)]. The fractional calculus has gained substantial importance during the past decades mostly due to its applications in various fields of science and engineering. For the purpose of this paper the Caputo's definition of fractional derivative will be used, taking the advantage of Caputo's approach that the initial conditions for fractional differential equations with Caputo's derivatives take on the traditional form as for integer-order differential equations.

2.1 Definition-Riemann-Liouville integral

The most frequently encountered definition of an integral of fractional order is the Riemann-Liouville integral [Podlubny (1999)], in which the fractional integral of order α (>0) is defined as

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad t > 0, \quad \alpha \in \boldsymbol{R}^{+} \tag{3}$$

where R^{+} is the set of positive real numbers.

2.2 Definition-Caputo Fractional Derivative

The fractional derivative, introduced by Caputo [Caputo (1967, 1969)] in the late sixties, is called Caputo Fractional Derivative. The fractional derivative of $f(t)$ in the Caputo sense is defined by

$$D_t^\alpha f(t) = J^{m-\alpha} D^m f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int\limits_0^t (t-\tau)^{(m-\alpha-1)} \frac{d^m f(\tau)}{d\tau^m} d\tau, \\ \quad \text{if } m-1 < \alpha < m, \quad m \in \boldsymbol{N} \\ \frac{d^m f(t)}{dt^m}, \\ \quad \text{if } \alpha = m, \ m \in \boldsymbol{N} \end{cases} \tag{4}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex. In this paper only real and positive α will be considered.

For the Caputo's derivative we have

$$D^\alpha C = 0, (C \text{ is a constant}) \tag{5}$$

$$D^\alpha t^\beta = \begin{cases} 0, & \beta \le \alpha - 1 \\ \frac{\Gamma(\beta+1) t^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}, & \beta > \alpha - 1 \end{cases} \tag{6}$$

Similar to integer order differentiation Caputo's derivative is linear.

$$D^\alpha(\gamma f(t) + \delta g(t)) = \gamma D^\alpha f(t) + \delta D^\alpha g(t) \tag{7}$$

where γ and δ are constants, and satisfies so called Leibnitz's rule.

$$D^\alpha(g(t) f(t)) = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} g^{(k)}(t) D^{\alpha-k} f(t) \tag{8}$$

If $f(\tau)$ is continuous in $[0,t]$ and $g(\tau)$ has n+1 continuous derivatives in $[0,t]$.

2.3 Lemma

If $m-1 < \alpha < m, m \in \boldsymbol{N}$, then

$$D^\alpha J^\alpha f(t) = f(t) \tag{9}$$

and

$$J^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(0+), \quad t > 0 \tag{10}$$

2.4 Theorem

(Generalized Taylor's formula) [Odibat and Shawagfeh (2007)] Suppose that $D_a^{k\alpha} f(t) \in C(a,b]$ for $k = 0,1,...,n+1$, where $0 < \alpha \le 1$, we have

$$f(t) = \sum_{i=0}^{n} \frac{(t-a)^{i\alpha}}{\Gamma(i\alpha+1)} \left[D_a^{k\alpha} f(t)\right]_{t=a} + \Re_n^{\alpha}(t;a) \tag{11}$$

with $\Re_n^{\alpha}(t;a) = \frac{(t-a)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} \left[D_a^{(n+1)\alpha} f(t)\right]_{t=\xi}$, $a \le \xi \le t, \forall t \in (a,b]$,

where $D_a^{k\alpha} = D_a^{\alpha}.D_a^{\alpha}.D_a^{\alpha}...D_a^{\alpha}$ (k times)

2.5 Coupled Fractional Reduced Differential Transform Method (CFRDTM)

In order to introduce coupled fractional reduced differential transform, $U(h,k-h)$ is considered as the coupled fractional reduced differential transform of $u(x,y,t)$. If function $u(x,y,t)$ is analytic and differentiated continuously with respect to time t, then we define the fractional coupled reduced differential transform of $u(x,y,t)$ as

$$U(h,k-h) = \frac{1}{\Gamma(h\alpha+(k-h)\beta+1)} \left[D_t^{(h\alpha+(k-h)\beta)} u(x,y,t)\right]_{t=0} \tag{12}$$

whereas the inverse transform of $U(h,k-h)$is

$$u(x,y,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{k} U(h,k-h) t^{h\alpha+(k-h)\beta} \tag{13}$$

which is one of the solution of coupled fractional differential equations.

Theorem 1 Suppose that $U(h,k-h)$, $V(h,k-h)$ and $W(h,k-h)$ are the Coupled Fractional Reduced Differential Transform of the functions $u(x,y,t)$, $v(x,y,t)$ and $w(x,y,t)$ respectively.

1. If $u(x,y,t) = f(x,y,t) \pm g(x,y,t)$ then $U(h,k-h) = F(h,k-h) \pm G(h,k-h)$.
2. If $u(x,y,t) = af(x,y,t)$, where $a \in R$, then $U(h,k-h) = aF(h,k-h)$.
3. If $f(x,y,t) = u(x,y,t)v(x,y,t)$, then
$$F(h,k-h) = \sum_{l=0}^{h} \sum_{s=0}^{k-h} U(h-l,s)V(l,k-h-s).$$
4. If $f(x,y,t) = D_t^{\alpha} u(x,y,t)$, then
$$F(h,k-h) = \frac{\Gamma((h+1)\alpha+(k-h)\beta+1)}{\Gamma(h\alpha+(k-h)\beta+1)} U(h+1,k-h).$$

5. If $f(x,y,t) = D_t^{\beta} v(x,y,t)$, then

$$F(h,k-h) = \frac{\Gamma(h\alpha + (k-h+1)\beta + 1)}{\Gamma(h\alpha + (k-h)\beta + 1)} V(h,k-h+1).$$

3 Approximate Solution for Fractional Predator-Prey Equation

In order to assess the advantages and the accuracy of the CFRDTM, we consider three cases with different initial conditions of predator-prey system [Liu and Xin (2011)]. Firstly, we derive the recursive formula obtained from predator-prey system of equations (1)-(2). Now, $U(h,k-h)$ and $V(h,k-h)$ are considered as the coupled fractional reduced differential transform of $u(x,y,t)$ and $v(x,y,t)$ respectively, where $u(x,y,t)$ and $v(x,y,t)$ are the solutions of coupled fractional differential equations. Here, $U(0,0) = u(x,y,0)$, $V(0,0) = v(x,y,0)$ are the given initial conditions. Without loss of generality, the following assumptions have taken

$U(0,j) = 0, \quad j = 1,2,3,\cdots$ and $V(i,0) = 0, \quad i = 1,2,3,\cdots$

Applying CFRDTM to eq. (1), we obtain the following recursive formula

$$\frac{\Gamma((h+1)\alpha + (k-h)\beta + 1)}{\Gamma(h\alpha + (k-h)\beta + 1)} U(h+1,k-h) = \frac{\partial^2}{\partial x^2} U(h,k-h) + \frac{\partial^2}{\partial y^2} U(h,k-h) + aU(h,k-h) - b\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h} U(h-l,s)V(l,k-h-s)\right) \tag{14}$$

From the initial condition of eq. (1), we have

$$U(0,0) = u(x,y,0) \tag{15}$$

In the same manner, we can obtain the following recursive formula from eq. (2)

$$\frac{\Gamma(h\alpha + (k-h+1)\beta + 1)}{\Gamma(h\alpha + (k-h)\beta + 1)} V(h,k-h+1) = \frac{\partial^2}{\partial x^2} V(h,k-h) + \frac{\partial^2}{\partial y^2} V(h,k-h) + b\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h} U(l,k-h-s)V(h-l,s)\right) - cV(h,k-h) \tag{16}$$

From the initial condition of eq. (2), we have

$$V(0,0) = v(x,y,0) \tag{17}$$

4 Applications and Results

Now, let us consider the three cases of predator-prey system

Case 1:

Here we consider the fractional predator-prey equation with constant initial condition

$$u(x,y,0) = u_0, \quad v(x,y,0) = v_0 \tag{18}$$

According to CFRDTM, using recursive scheme eq. (14) with initial condition eq. (15) and also using recursive scheme eq. (16) with initial condition eq. (17) simultaneously, we obtain

$$U[0,0] = u(x,y,0) = u_0, \quad V[0,0] = v(x,y,0) = v_0$$

$$U[1,0] = \frac{u_0(a-bv_0)}{\Gamma(1+\alpha)} \quad V[0,1] = \frac{(bu_0v_0 - cv_0)}{\Gamma(1+\beta)},$$

$$U[2,0] = \frac{u_0(a-bv_0)^2}{\Gamma(1+2\alpha)},$$

$$V[0,2] = \frac{v_0(c-bu_0)^2}{\Gamma(1+2\beta)}$$

$$U[1,1] = -\frac{bu_0(-cv_0+bu_0v_0)}{\Gamma(1+\alpha+\beta)}$$

$$V[1,1] = \frac{bu_0v_0(a-bv_0)}{\Gamma(1+\alpha+\beta)}$$

$$U[1,2] = -\frac{b(c-bu_0)^2u_0v_0}{\Gamma(1+\alpha+2\beta)}$$

$$V[1,2] = \frac{bu_0(c-bu_0)v_0(-(a-2bv_0)\Gamma(1+\alpha)\Gamma(1+\beta)+(-a+bv_0)\Gamma(1+\alpha+\beta))}{\Gamma(1+\alpha+2\beta)\Gamma(1+\alpha)\Gamma(1+\beta)}$$

$$U[2,1] = \frac{bu_0v_0(a-bv_0)((c-2bu_0)\Gamma(1+\alpha)\Gamma(1+\beta)+(c-bu_0)\Gamma(1+\alpha+\beta))}{\Gamma(1+2\alpha+\beta)\Gamma(1+\alpha)\Gamma(1+\beta)}$$

$$V[2,1] = \frac{bu_0v_0(a-bv_0)^2}{\Gamma(1+2\alpha+\beta)}$$

$$U[3,0] = \frac{u_0(a-bv_0)^3}{\Gamma(1+3\alpha)}$$

$$V[0,3] = -\frac{v_0(c-bu_0)^3}{\Gamma(1+3\beta)}$$

The approximate solutions, obtained in the series form, are given by

$$\begin{aligned} u(x,y,t) &= U(0,0) + \sum_{k=1}^{\infty}\sum_{h=1}^{k} U(h,k-h)t^{(h\alpha+(k-h)\beta)} \\ &= u_0 + \frac{u_0(a-bv_0)t^{\alpha}}{\Gamma(1+\alpha)} + \frac{u_0(a-bv_0)^2 t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{u_0(a-bv_0)^3 t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &\quad - \frac{bu_0(-cv_0+bu_0v_0)t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \cdots \end{aligned} \tag{19}$$

$$\begin{aligned} v(x,y,t) &= V(0,0) + \sum_{k=1}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{(h\alpha+(k-h)\beta)} \\ &= v_0 + \frac{(bu_0v_0-cv_0)t^{\beta}}{\Gamma(1+\beta)} + \frac{bu_0v_0(a-bv_0)t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \frac{bu_0v_0(a-bv_0)^2 t^{2\alpha+\beta}}{\Gamma(1+2\alpha+\beta)} \cdots \end{aligned} \tag{20}$$

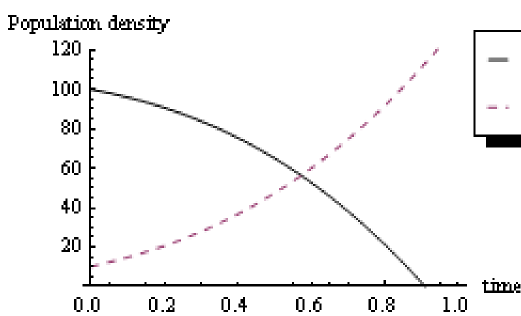

Figure 1: Time Evolution of population of $u(x,y,t)$ and $v(x,y,t)$ obtained from eqs. (19) and (20), when $\alpha = 1,\ \beta = 1$.

Figure 1 cites the numerical solutions for eq. (1)-(2) obtained by the proposed FCRDTM method for the constant initial conditions $u_0 = 100$, $v_0 = 10$, a= 0.05, b= 0.03 and c= 0.01. Figure 2 shows the time evolution of population of $u(x,y,t)$ and $v(x,y,t)$obtained from eqs. (19) and (20) for different values of $\alpha$and β. In the present numerical analysis, Table-1 shows the comparison of the numerical solutions with the proposed method with Homotopy perturbation method and Variational iteration method, when a= 0.05, b= 0.03 and c= 0.01. From the Table 1, it is evidently clear that CFRDTM used in this paper has high accuracy. The numerical results obtained in this proposed method coincide precisely with values obtained in Homotopy perturbation method.

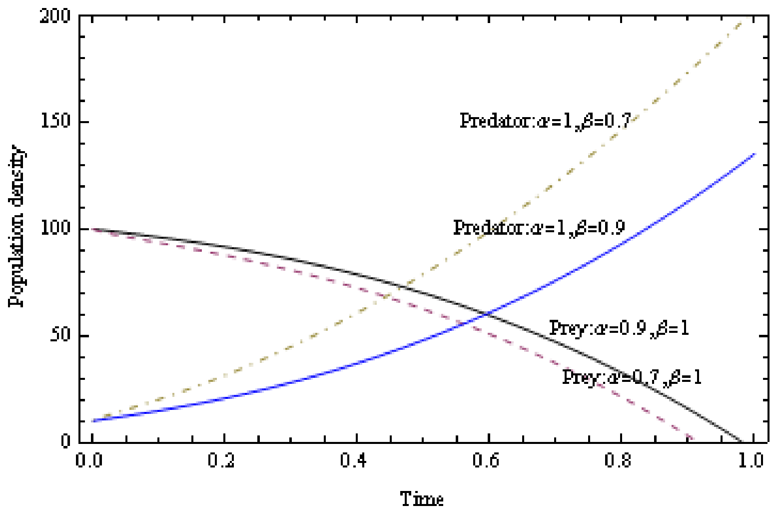

Figure 2: Time Evolution of population of $u(x,y,t)$ and $v(x,y,t)$obtained from eqs. (19) and (20) for different values of $\alpha$and β.

Table 1: Comparison of the numerical solutions of the proposed method with Homotopy perturbation method and Variational iteration method.

t	$\alpha = \beta$	**Numerical value (u, v) by HPM**	**Numerical value (u, v) by VIM**	**Numerical value (u, v) by CFRDTM**
0.02	1	(99.4831, 10.6146)	(99. 4834, 10.6323)	(99.4831, 10.6146)
	0.9	(99.1865, 10.9633)	(99.3065, 10. 8375)	(99.1865, 10.9633)
0.2	1	(93.0910, 17.8514)	(93. 3908, 17.7382)	(93.0910, 17.8514)
	0.9	(90.5735, 20.5567)	(92.4584, 18.8198)	(90.5735, 20.5567)
0.3	1	(87.9348, 23.4430)	(88. 9466, 22. 7237)	(87.9348, 23.4430)
	0.9	(83.7993, 27.7785)	(87. 8005, 24.0532)	(83.7993, 27.7785)

Case 2:

In this case, the initial conditions of eq. (1)-(2) are given by

$$u(x,y,0) = e^{x+y}, \quad v(x,y,0) = e^{x+y} \tag{21}$$

By using eqs. (14) to (17), we can successively obtain

$$U[0,0] = u(x,y,0) = e^{x+y}, \quad V[0,0] = v(x,y,0) = e^{x+y}$$

$$U[1,0] = \frac{2e^{x+y} + ae^{x+y} - be^{2x+2y}}{\Gamma(1+\alpha)}$$

$$V[0,1] = \frac{2e^{x+y} - ce^{x+y} + be^{2x+2y}}{\Gamma(1+\beta)}$$

$$U[1,1] = \frac{be^{2(x+y)}(2 - c + be^{x+y})}{\Gamma(1+\alpha+\beta)}$$

$$V[1,1] = -\frac{be^{2(x+y)}(-2 - a + be^{x+y})}{\Gamma(1+\alpha+\beta)}$$

$$U[2,0] = \frac{e^{x+y}(4 + a^2 - 10be^{x+y} + b^2e^{2(x+y)} + a(4 - 2be^{x+y}))}{\Gamma(1+2\alpha)}$$

$$V[0,2] = \frac{e^{x+y}(4 + c^2 + 10be^{x+y} + b^2e^{2(x+y)} - 2c(2 + be^{x+y}))}{\Gamma(1+2\beta)}$$

$$U[1,2] = -\frac{be^{2(x+y)}(4 + c^2 + 10be^{x+y} + b^2e^{2(x+y)} - 2c(2 + be^{x+y}))}{\Gamma(1+\alpha+\beta)}$$

$$\begin{aligned} V[1,2] = {} & (be^{2(x+y)}(-(a(-8 + c - be^{x+y}) \\ & + 2(-8 + c + 9be^{x+y} - bce^{x+y} + b^2e^{2(x+y)}))\Gamma(1+\alpha)\Gamma(1+\beta) \\ & + (2 + a - be^{x+y})(2 - c + be^{x+y})\Gamma(1+\alpha+\beta))/(\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+\alpha+2\beta)) \end{aligned}$$

$$\begin{aligned} U[3,0] = {} & e^{x+y}(8 + a^3 - 84be^{x+y} + 28b^2e^{2(x+y)} - b^3e^{3(x+y)} + a^2(6 - 3be^{x+y}) \\ & + 3a(4 - 10be^{x+y} + b^2e^{2(x+y)})))/\Gamma(1+3\alpha) \end{aligned}$$

$$\begin{aligned} V[0,3] = {} & e^{x+y}(8 - c^3 + 84be^{x+y} + 28b^2e^{2(x+y)} + b^3e^{3(x+y)} + 3c^2(2 + be^{x+y}) \\ & - 3c(4 + 10be^{x+y} + b^2e^{2(x+y)})))/\Gamma(1+3\beta) \end{aligned}$$

The explicit approximate solution is

$$\begin{aligned} u(x,y,t) = {} & e^{x+y} + \frac{(2e^{x+y} + ae^{x+y} - be^{2x+2y})t^\alpha}{\Gamma(1+\alpha)} \\ & + \frac{e^{x+y}(4 + a^2 - 10be^{x+y} + b^2e^{2(x+y)} + a(4 - 2be^{x+y}))t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots \end{aligned} \tag{22}$$

and

$$v(x,y,t) = e^{x+y} + \frac{(2e^{x+y} - ce^{x+y} + be^{2x+2y})t^{\beta}}{\Gamma(1+\beta)} - \frac{be^{2(x+y)}(-2-a+be^{x+y})t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \cdots \tag{23}$$

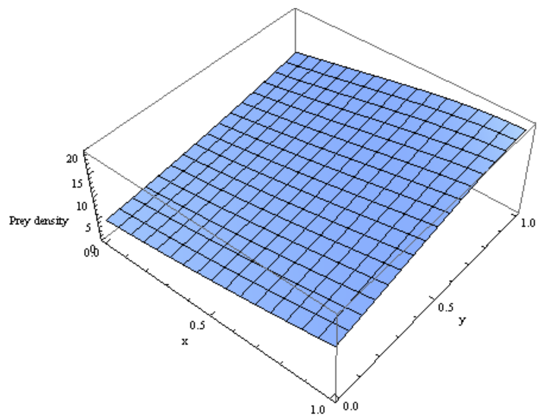

Figure 3: The surface shows the numerical approximate solution of $u(x,y,t)$ when $\alpha = 0.88 \quad \beta = 0.54$, a= 0.7, b= 0.03, c= 0.3, and t=0.53.

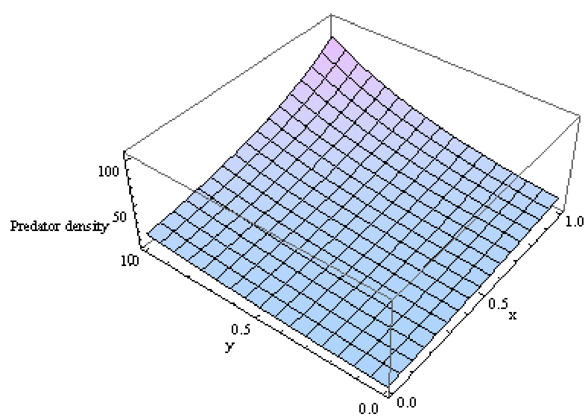

Figure 4: The surface shows the numerical approximate solution of $v(x,y,t)$ when $\alpha = 0.88$, $\beta = 0.54$, a= 0.7, b= 0.03, c= 0.9, and t=0.6.

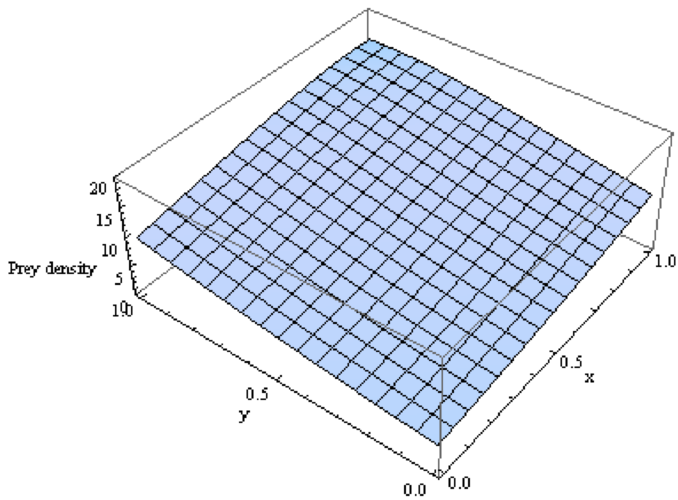

Figure 5: The surface shows the numerical approximate solution of $u(x,y,t)$ when $\alpha = 0.88$, $\beta = 0.54$, a= 0.5, b= 0.03, c= 0.3, and t=0.53.

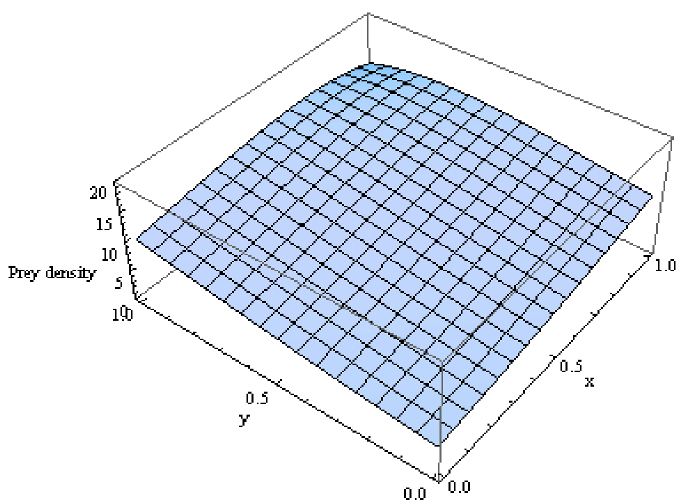

Figure 6: The surface shows the numerical approximate solution of $u(x,y,t)$ when $\alpha = 0.88$, $\beta = 0.54$, a= 0.7, b= 0.04, c= 0.3, and t=0.53.

Figures 3 and 4 cite the numerical approximate solutions for predator-prey system with appropriate parameter. The obtained results of predator-prey population system indicate that this model exhibits the same behaviour observed in the anomalous biological diffusion fractional model.

Figures 5 and 6 show the numerical solutions for prey population density for different values of parameters a, b, i.e. natural birth rate of prey population and competitive rate between predator and prey population. The results depicted in graphs agree with the realistic data.

Case 3:

In this case, we consider the initial condition of fractional predator-prey equation (1)-(2)

$$U[0,0] = u(x,y,0) = \sqrt{xy}, \quad V[0,0] = v(x,y,0) = e^{x+y} \tag{24}$$

$$U[1,0] = \frac{-\frac{x^2}{4(xy)^{3/2}} - \frac{y^2}{4(xy)^{3/2}} + a\sqrt{xy} - be^{x+y}\sqrt{xy}}{\Gamma(1+\alpha)}$$

$$V[0,1] = \frac{2e^{x+y} - ce^{x+y} + be^{x+y}\sqrt{xy}}{\Gamma(1+\beta)}$$

$$U[1,1] = \frac{be^{x+y}\sqrt{xy}(2-c+b\sqrt{xy})}{\Gamma(1+\alpha+\beta)}$$

$$V[1,1] = \frac{-be^{x+y}(y^2+x^2(1-4ay^2+4be^{x+y}y^2))}{4(xy)^{3/2}\Gamma(1+\alpha+\beta)}$$

$$U[2,0] = \frac{1}{16x^4y^4\Gamma(1+2\alpha)}\sqrt{xy}\left(-15y^4 - 16be^{x+y}x^3y^4 + x^2(2y^2 - 8(a-be^{x+y})y^4\right) +$$
$$x^4(-15+16a^2y^4+16b^2e^{2(x+y)}y^4 - 8be^{x+y}y^2(-1+2y+4y^2) - 8ay^2(1+4be^{x+y}y^2)))$$

$$V[0,2] = \frac{e^{x+y}(4(-2+c)^2(xy)^{3/2}+4b^2(xy)^{5/2}-b(y^2-4xy^2+x^2(1-4y+8(-2+c)y^2)))}{4(xy)^{3/2}\Gamma(1+2\beta)}$$

The solution becomes

$$u(x,y,t) = \sqrt{xy} + \frac{(-\frac{x^2}{4(xy)^{3/2}} - \frac{y^2}{4(xy)^{3/2}} + a\sqrt{xy} - be^{x+y}\sqrt{xy})t^{\alpha}}{\Gamma(1+\alpha)} + \cdots \tag{25}$$

and

$$\begin{aligned} v(x,y,t) =& e^{x+y} + \frac{(2e^{x+y} - ce^{x+y} + be^{x+y}\sqrt{xy})t^{\beta}}{\Gamma(1+\beta)} \\ &+ \frac{(-be^{x+y}(y^2+x^2(1-4ay^2+4be^{x+y}y^2))t^{\alpha+\beta}}{4(xy)^{3/2}\Gamma(1+\alpha+\beta)} + \ldots \end{aligned} \tag{26}$$

5 Convergence Analysis and error estimate

Theorem 5.1:

Let, $D_t^{\alpha} u = \mathrm{F}(u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx}, \ldots)$ and $D_t^{\beta} v = \mathrm{H}(u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx}, \ldots)$ be the general coupled fractional differential equations and let, the Caputo derivatives $D_t^{k\alpha} u(x,t)$ and $D_t^{k\beta} v(x,t)$ are continuous functions on $[0,L]\times[0,T]$ i.e. $D_t^{k\alpha} u(x,t) \in C([0,L]\times[0,T])$ and $D_t^{k\beta} v(x,t) \in C([0,L]\times[0,T])$ for $k = 0,1,2,\cdots,n+1$, where $0 < \alpha,\beta < 1$, then the approximate solutions $\tilde{u}(x,t)$ and $\tilde{v}(x,t)$ of the above general coupled fractional differential equations are

$$\tilde{u}(x,t) \cong \sum_{k=0}^{n} \sum_{h=0}^{k} U(h,k-h) t^{h\alpha+(k-h)\beta}.$$

and

$$\tilde{v}(x,t) \cong \sum_{k=0}^{n} \sum_{h=0}^{k} V(h,k-h) t^{h\alpha+(k-h)\beta}$$

where $U(h,k-h)$ and $V(h,k-h)$ are Coupled Fractional Reduced Differential transforms of $u(x,t)$ and $v(x,t)$ respectively.

Moreover, there exists values $\xi_1,\ \xi_2$ where $0 \le \xi_1,\ \xi_2 \le t$ so that the error $E_n(x,t)$ for the approximate solution $\tilde{u}(x,t)$ has the form

$$\|E_n(x,t)\| = \underset{\substack{0\le x\le L\\ 0\le t\le T}}{Sup} \left| \frac{D^{(n+1)\beta} u(x,0+)}{\Gamma((n+1)\beta+1)}\, t^{(n+1)\beta} \right|, \quad \text{if } \xi_1,\ \xi_2 \to 0+$$

Proof:

From lemma 2.3, we have

$$J^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} \frac{t^k}{\Gamma(k+1)} f^{(k)}(0+), \quad m-1 < \alpha < m$$

The error term

$$E_n(x,t) = u(x,t) - \tilde{u}(x,t)$$

where

$$u(x,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{k} \frac{D^{h\alpha+\beta(k-h)} u(x,0)}{\Gamma(h\alpha+\beta(k-h)+1)}\, t^{h\alpha+\beta(k-h)}$$

and

$$\tilde{u}(x,t)=\sum_{k=0}^{n}\sum_{h=0}^{k}\frac{D^{h\alpha+\beta(k-h)}u(x,0)}{\Gamma(h\alpha+\beta(k-h)+1)}\,t^{h\alpha+\beta(k-h)}$$

Now, for $0<\alpha<1$,

$$\begin{aligned}
&J^{h\alpha+\beta(k-h)}D^{h\alpha+\beta(k-h)}u(x,t)-J^{(h+1)\alpha+\beta(k-h)}D^{(h+1)\alpha+\beta(k-h)}u(x,t)\\
&=J^{h\alpha+\beta(k-h)}\left(D^{h\alpha+\beta(k-h)}u(x,t)-J^{\alpha}D^{\alpha}\left(D^{h\alpha+\beta(k-h)}u(x,t)\right)\right)\\
&=J^{h\alpha+\beta(k-h)}D^{h\alpha+\beta(k-h)}u(x,0),\quad \text{since }\ 0<\alpha<1,\ \text{ using eq. (10)}\\
&=\frac{D^{h\alpha+\beta(k-h)}u(x,0)}{\Gamma(h\alpha+\beta(k-h)+1)}t^{h\alpha+\beta(k-h)}
\end{aligned}\tag{27}$$

The n-th order approximation for $u(x,t)$ is

$$\begin{aligned}
\tilde{u}(x,t)&=\sum_{k=0}^{n}\sum_{h=0}^{k}\frac{D^{h\alpha+\beta(k-h)}u(x,0)}{\Gamma(h\alpha+\beta(k-h)+1)}t^{h\alpha+\beta(k-h)}\\
&=\sum_{k=0}^{n}\sum_{h=0}^{k}(J^{h\alpha+\beta(k-h)}D^{h\alpha+\beta(k-h)}u(x,t)\\
&\quad -J^{(h+1)\alpha+\beta(k-h)}D^{(h+1)\alpha+\beta(k-h)}u(x,t))\quad \text{using eq. (27)}\\
&=\sum_{k=0}^{n}J^{k\beta}D^{k\beta}u(x,t)-\sum_{h=0}^{n}J^{(h+1)\alpha+\beta(n-h)}D^{(h+1)\alpha+\beta(n-h)}u(x,t)\\
&=u(x,t)+\sum_{k=0}^{n-1}J^{(k+1)\beta}D^{(k+1)\beta}u(x,t)-\sum_{h=0}^{n}J^{(h+1)\alpha+\beta(n-h)}D^{(h+1)\alpha+\beta(n-h)}u(x,t)
\end{aligned}\tag{28}$$

Therefore, from eq. (28), the error term becomes

$$\begin{aligned}
E_n(x,t)&=u(x,t)-\tilde{u}(x,t)\\
&=\sum_{h=0}^{n}J^{(h+1)\alpha+\beta(n-h)}D^{(h+1)\alpha+\beta(n-h)}u(x,t)-\sum_{k=0}^{n-1}J^{(k+1)\beta}D^{(k+1)\beta}u(x,t)\\
&=\sum_{i=0}^{n}J^{(i+1)\alpha+\beta(n-i)}D^{(i+1)\alpha+\beta(n-i)}u(x,t)-\sum_{i=0}^{n-1}J^{(i+1)\beta}D^{(i+1)\beta}u(x,t)\\
&=\sum_{i=0}^{n}\frac{1}{\Gamma((i+1)\alpha+\beta(n-i))}\int_0^t(t-\tau)^{(i+1)\alpha+\beta(n-i)-1}D^{(i+1)\alpha+\beta(n-i)}u(x,\tau)d\tau
\end{aligned}$$

$$-\sum_{i=0}^{n-1}\frac{1}{\Gamma((i+1)\beta)}\int_0^t (t-\tau)^{(i+1)\beta-1}D^{(i+1)\beta}u(x,\tau)d\tau$$

$$=\sum_{i=0}^{n}\frac{D^{(i+1)\alpha+\beta(n-i)}u(x,\xi_1)}{\Gamma((i+1)\alpha+\beta(n-i)+1)}\,t^{(i+1)\alpha+\beta(n-i)}-\sum_{i=0}^{n-1}\frac{D^{(i+1)\beta}u(x,\xi_2)}{\Gamma((i+1)\beta+1)}\,t^{(i+1)\beta},$$

applying integral mean value theorem

$$=\sum_{i=0}^{n-1}\frac{D^{(i+1)\alpha+\beta(n-i)}u(x,\xi_1)}{\Gamma((i+1)\alpha+\beta(n-i)+1)}\,t^{(i+1)\alpha+\beta(n-i)}+\frac{D^{(n+1)\alpha}u(x,\xi_1)}{\Gamma((n+1)\alpha+1)}\,t^{(n+1)\alpha}$$

$$-\sum_{i=0}^{n-1}\frac{D^{(i+1)\beta}u(x,\xi_2)}{\Gamma((i+1)\beta+1)}\,t^{(i+1)\beta}$$

$$\begin{aligned}&=\sum_{i=0}^{n-1}\left[\frac{D^{(i+1)\alpha+\beta(n-i)}u(x,\xi_1)}{\Gamma((i+1)\alpha+\beta(n-i)+1)}\,t^{(i+1)\alpha+\beta(n-i)}-\frac{D^{(i+1)\beta}u(x,\xi_2)}{\Gamma((i+1)\beta+1)}\,t^{(i+1)\beta}\right]\\&\quad+\frac{D^{(n+1)\alpha}u(x,\xi_1)}{\Gamma((n+1)\alpha+1)}t^{(n+1)\alpha}\end{aligned}\tag{29}$$

Using generalized Taylor's series formula eq. (11), eq. (29) becomes

$$\begin{aligned}E_n(x,t)=&u(x,t)-\frac{D^{(n+1)\alpha}u(x,\zeta_1)}{\Gamma((n+1)\alpha+1)}\,t^{(n+1)\alpha}-u(x,t)+\frac{D^{(n+1)\beta}u(x,\zeta_2)}{\Gamma((n+1)\beta+1)}\,t^{(n+1)\beta}\\&+\frac{D^{(n+1)\alpha}u(x,\xi_1)}{\Gamma((n+1)\alpha+1)}\,t^{(n+1)\alpha},\end{aligned}$$

where $0\le\zeta_1,\ \zeta_2\le\max\{\xi_1,\ \xi_2\}$ and $\xi_1,\ \xi_2\to 0+$
This implies

$$\begin{aligned}&\|E_n\|=\|u(x,t)-\tilde{u}(x,t)\|\\&=\sup_{\substack{0\le x\le L\\0\le t\le T}}\left|\frac{D^{(n+1)\beta}u(x,\zeta_2)}{\Gamma((n+1)\beta+1)}\,t^{(n+1)\beta}-\frac{D^{(n+1)\alpha}u(x,\zeta_1)}{\Gamma((n+1)\alpha+1)}\,t^{(n+1)\alpha}+\frac{D^{(n+1)\alpha}u(x,\xi_1)}{\Gamma((n+1)\alpha+1)}\,t^{(n+1)\alpha}\right|<\infty\\&=\sup_{\substack{0\le x\le L\\0\le t\le T}}\left|\frac{D^{(n+1)\beta}u(x,0+)}{\Gamma((n+1)\beta+1)}\,t^{(n+1)\beta}\right|,\quad\text{since }\ \xi_1,\ \xi_2\to 0+\end{aligned}\tag{30}$$

As $n\to\infty,\ \|E_n\|\to 0$

Hence, $u(x,t)$ can be approximated as

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{k} U(h,k-h)t^{h\alpha+(k-h)\beta} \cong \sum_{k=0}^{n}\sum_{h=0}^{k} U(h,k-h)t^{h\alpha+(k-h)\beta} = \tilde{u}(x,t)$$

with the error term given in eq. (30).

Following the similar argument, we may also find the error $\left\|\hat{E}_n\right\| = \|v(x,t) - \tilde{v}(x,t)\|$ for the approximate solution $\tilde{v}(x,t)$.

6 Conclusion

In this article, a new approximate numerical technique Coupled Fractional Reduced differential transform [Saha Ray (2013a, 2013b)] has been proposed for solving nonlinear fractional partial differential equations arising in predator-prey biological population dynamical system. The results thus obtained validate the reliability of the proposed algorithm. It additionally displays that the proposed process is an extraordinarily efficient and strong technique. The main advantage of the proposed method is that it necessitates less amount of computational effort. In later study, it has been planned to use the proposed process for the solution of fractional epidemic model, coupled fractional neutron diffusion equations with delayed neutrons and others physical models with the intention to show the efficiency and wide applicability of the new proposed method.

In view of the author [Bervillier (2012)], there is no difference between Differential Transform Method (DTM) and Taylor Series Method (TSM) both of which (normally) are provided with an analytical continuation via a stepwise procedure, since it is essential to transform the formal series into an approximate solution of the problem (via analytical continuation). The author also wrote in [Bervillier (2012)] that one may then rightly remember the approach as being "*an extended Taylor series method*". Thus, the DTM could, eventually, be named as the Generalized Taylor Series Method (GTSM). In belief of the learned author, "DTM could deserve its name (as a Technique) when it extends the Taylor Series Method to new kinds of expansion (different from a Taylor Series Expansion)." He, additionally, acknowledges that the DTM has allowed an easy generalization of the Taylor Series Method to various derivation procedures. "For example, fractional differential equations have been considered using the DTM extended to the fractional derivative procedure via a modified version of the Taylor series". Despite the fact that there is a controversy in the name of DTM, the author of [Bervillier (2012)] admits that major contribution of the DTM is in the easy generalization of the Taylor Series Method to problems involving fractional derivatives.

Furthermore, it may be stated that Taylor Series Method is used invariably in many mathematical analysis and derivation for the problems of applied science and engineering. Taylor Series Method of order one is commonly known as Euler method. However Euler method has its independent existence. Like that DTM is also self-contained for at least in the application of fractional order calculus and has its own right for its existence.

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Appendix

Proof of theorem 1 (iii):

$$
\begin{aligned}
f(x,y,t) &= u(x,y,t)v(x,y,t)\\
&= \left(\sum_{k=0}^{\infty}\sum_{h=0}^{k} U(h,k-h)t^{h\alpha+(k-h)\beta}\right)\left(\sum_{k=0}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{h\alpha+(k-h)\beta}\right)\\
&= U(0,0)V(0,0)+(U(1,0)V(0,0)+U(0,0)V(1,0))\,t^{\alpha}\\
&\quad +(U(0,1)V(0,0)+U(0,0)V(0,1))\,t^{\beta}+\\
&\quad (U(1,0)V(0,1)+U(0,1)V(1,0)+U(1,1)V(0,0)+U(0,0)V(1,1))\,t^{\alpha+\beta}+\ldots\\
&= \sum_{k=0}^{\infty}\sum_{h=0}^{k}\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h} U(h-l,s)V(l,k-h-s)\right)t^{h\alpha+(k-h)\beta}
\end{aligned}
$$

Hence, $F(h,k-h)=\sum_{l=0}^{h}\sum_{s=0}^{k-h} U(h-l,s)V(l,k-h-s)$.