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New Spectral Solutions of Multi-Term Fractional-Order Initial Value Problems With Error Analysis

W. M. Abd- Elhameed[1,2] **and Y. H. Youssri**[2]

Abstract: In this paper, a new spectral algorithm for solving linear and nonlinear fractional-order initial value problems is established. The key idea for obtaining the suggested spectral numerical solutions for these equations is actually based on utilizing the ultraspherical wavelets along with applying the collocation method to reduce the fractional differential equation with its initial conditions into a system of linear or nonlinear algebraic equations in the unknown expansion coefficients. The convergence and error analysis of the suggested ultraspherical wavelets expansion are carefully discussed. For the sake of testing the proposed algorithm, some numerical examples are considered. The numerical results indicate that the resulting approximate solutions are close to the analytical solutions and they are more accurate than those obtained by some other existing techniques in literature.

Keywords: Wavelets, ultraspherical polynomials, collocation method, fractional-order differential equations

1 Introduction

Fractional calculus is an extension of derivatives and integrals to non-integer orders and has been widely used to model scientific and engineering problems. Fractional-order differential equations have prominent roles in various disciplines. Due to their great importance, they have been investigated by a large number of authors from both theoretical and practical points of view (see, for example, [Diethelm (2010); Brunner, Pedas, and Vainikko (2001); Kilbas, Srivastava, and Trujillo (2006); Podlubny (1998); Kilbas, Marichev, and Samko (1993); Wang, Liu, Chen, Liu, and Liu (2015)]). It is well-known that many physical phenomena in acoustics, damping laws electroanalytical chemistry, neuron modeling, diffusion processing and material sciences (see for example, [Al-Mdallal, Syam, and Anwar (2010); Çenesiz, Keskin, and Kurnaz (2010); Daftardar-Gejji and Jafari (2005)]) are described by

[1] Department of Mathematics, Faculty of Science, University of Jeddah, Jeddah, Saudi Arabia.
[2] Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt.

fractional-order differential equations. Various algorithms are developed for handling different kinds of fractional differential equations. Some of these methods are, Adomian decomposition method [Momani (2007); Jafari and Seifi (2009)], variational iteration method [Sweilam, Khader, and Al-Bar (2007); Das (2009)] and fractional differential transform method [Arikoglu and Ozkol (2009); Erturk, Momani, and Odibat (2008)].

Spectral methods have important roles in many fields of applied science. The main characteristic of spectral methods is that the approximate solutions of differential equations are expressed in terms of truncated series of various orthogonal polynomials. There are three popular techniques for spectral methods, they are the collocation, tau and Galerkin methods (see, for instance [Abd-Elhameed, Doha, and Youssri (2013a,b); Abd-Elhameed (2014); Elgohary, Dong, Junkins, and Atluri (2014); Costabile and Napoli (2015); Abd-Elhameed (2015)]).

The subject of wavelets has recently drawn a great deal of attention from mathematical scientists in various disciplines. Wavelets have been used for solving ordinary and fractional differential equations. For example, a huge number of articles employ Legendre and Chebyshev wavelets for treating ordinary differential equations as well as fractional differential equations (see, for instance [Zhu and Fan (2012); Sadek, Abualrub, and Abukhaled (2007)]). In this paper, we aim to employ ultraspherical wavelets in handling fractional differential equations. A motivation for constructing and employing ultraspherical wavelets is that the Chebyshev and Legendre wavelets can be deduced as special cases of the ultraspherical wavelets.

The ultraspherical polynomials have received considerable attention in recent decades, from both theoretical and practical points of view (see, for example [Elgindy and Smith-Miles (2013a)]). Some authors are interested in employing these polynomials for solving various kinds of differential equations. In this respect, Elgindy and Smith-Miles in [Elgindy and Smith-Miles (2013b)], treated boundary value problems, integral, and integro-differential equations using ultraspherical integration matrices. Moreover, Doha and Abd-Elhameed employed ultraspherical polynomials for solving one and two dimensional second-order differential equations in [Doha and Abd-Elhameed (2002)]. In addition, the same authors in [Doha and Abd-Elhameed (2005)] developed some accurate spectral solutions for treating the parabolic and elliptic partial differential equations based on the ultraspherical tau method.

The main aim of this article is twofold:

- Deriving the ultraspherical wavelets operational matrix of the fractional integration.
- Analyzing efficient spectral wavelets algorithm for treating fractional-order

differential equations via ultraspherical wavelets operational matrices of the fractional integration.

The contents of the paper are arranged as follows. Section 2 is devoted to presenting mathematical preliminaries containing some basic definitions in the fractional calculus theory which are required for establishing our results. Also, some relevant properties of ultraspherical polynomials and their shifted ones are presented and the ultraspherical wavelets are constructed. In Section 3, we investigate in detail the convergence and error analysis of the suggested ultraspherical wavelets expansion. In Section 4, the ultraspherical wavelets operational matrix of the fractional integration is derived. In Section 5, we present and implement an algorithm for solving multi-term fractional-order initial value problems based on employing the constructed ultraspherical wavelets operational matrix. In Section 6, some numerical examples are given to ensure the efficiency, simplicity and applicability of the suggested algorithm. Finally, conclusions are reported in Section 7.

2 Preliminaries

2.1 Some definitions and properties of fractional calculus

In this section, we present some notations, definitions and preliminary facts of the fractional calculus theory which will be useful throughout this article.

Definition 1. The Riemann-Liouville fractional integral operator I^{α} of order α on the usual Lebesgue space $L_1[0,1]$ is defined as

$$I^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \tag{1}$$

The operator I^{α} has the following properties:

$$(i)\, I^{\alpha} I^{\beta} = I^{\alpha+\beta}, \qquad (ii)\, I^{\alpha} I^{\beta} = I^{\beta} I^{\alpha},$$

$$(iii) I^{\alpha}(t-a)^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)}(t-a)^{\nu+\alpha},$$

where $f \in L_1[0,1]$, $\alpha, \beta \geqslant 0$, and $\nu > -1$.

Definition 2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by

$$(D^{\alpha} f)(t) = \left(\frac{d}{dt}\right)^{n} (I^{n-\alpha} f)(t), n-1 \leqslant \alpha < n, \quad n \in \mathbb{N}. \tag{2}$$

Definition 3. The Caputo definition of fractional differential operator is given by

$$(D_*^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau,\ \alpha > 0, t > 0, \tag{3}$$

where $n-1 \leqslant \alpha < n, n \in \mathbb{N}$.

The operator D_*^{α} satisfies the following two basic properties for $n-1 \leqslant \alpha < n$,

$$\begin{aligned} (D_*^{\alpha} I^{\alpha} f)(t) &= f(t), \\ (I^{\alpha} D_*^{\alpha} f)(t) &= f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0^+)}{k!}(t-a)^k, t > 0. \end{aligned}$$

For comprehensive study on the properties of fractional derivatives and integrals, one can see for example, [Podlubny (1998)].

2.2 Some properties of ultraspherical polynomials and their shifted ones

The ultraspherical polynomials $C_n^{(\lambda)}(x)$ (a special type of Jacobi polynomials) associated with the real parameter ($\lambda > -\frac{1}{2}$), are a sequence of orthogonal polynomials defined on (-1,1), with respect to the weight function $w(x) = (1-x^2)^{\lambda-\frac{1}{2}}$. The orthogonality relation is given by

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}}\, C_m^{(\lambda)}(x) C_n^{(\lambda)}(x)\, dx = \begin{cases} \dfrac{\sqrt{\pi}\, n!\, \Gamma(2\lambda)\, \Gamma(\lambda+\frac{1}{2})}{\Gamma(n+2\lambda)\, (n+\lambda)\, \Gamma(\lambda)}, & m = n, \\ 0, & m \neq n. \end{cases} \tag{4}$$

It should be noted here that the ultraspherical polynomials $C_n^{(\lambda)}(x)$ are normalized such that $C_n^{(\lambda)}(1) = 1$. This normalization is characterized by an advantage that the polynomials $C_n^{(0)}(x)$ are identical with the Chebyshev polynomials of the first kind $T_n(x)$, $C_n^{(\frac{1}{2})}(x)$ are the Legendre polynomials $L_n(x)$, and $C_n^{(1)}(x)$ are equal to $(1/(n+1))U_n(x)$, where $U_n(x)$ are the Chebyshev polynomials of the second kind. The polynomials $C_n^{(\lambda)}(x)$ may be generated by using the recurrence relation

$$(n+2\lambda) C_{n+1}^{(\lambda)}(x) = 2(n+\lambda)\, x C_n^{(\lambda)}(x) - n C_{n-1}^{(\lambda)}(x), \quad n = 1, 2, 3, \ldots,$$

with the initial values: $C_0^{(\lambda)}(x) = 1$ and $C_1^{(\lambda)}(x) = x$.
For more properties and relations of ultraspherical polynomials, see for instance [Andrews, Askey, and Roy (1999)].

The shifted ultraspherical polynomials $\tilde{C}_n^{(\lambda)}(x) = C_n^{(\lambda)}(2x-1)$ are a sequence of orthogonal polynomials defined on (0,1), with respect to the weight function $\tilde{w}(x) = (x-x^2)^{\lambda-\frac{1}{2}}$, i.e.

$$\int_0^1 (x-x^2)^{\lambda-\frac{1}{2}}\, \tilde{C}_m^{(\lambda)}(x)\, \tilde{C}_n^{(\lambda)}(x)\, dx = \begin{cases} \dfrac{\pi\, 2^{1-4\lambda}\, \Gamma(n+2\lambda)}{n!\,(n+\lambda)\,(\Gamma(\lambda))^2}, & m=n, \\ 0, & m \neq n. \end{cases} \tag{5}$$

They also may be generated by using the recurrence relation

$$(n+2\lambda)\tilde{C}_{n+1}^{(\lambda)}(x) = 2(n+\lambda)\,(2x-1)\tilde{C}_n^{(\lambda)}(x) - n\tilde{C}_{n-1}^{(\lambda)}(x), \quad n = 1,2,3,\ldots,$$

with the initial values: $\tilde{C}_0^{(\lambda)}(x) = 1$ and $\tilde{C}_1^{(\lambda)}(x) = 2x-1$.

It is worthy to note here that, it is easy to transform all relations and properties of ultraspherical polynomials to give the corresponding relations and properties of the shifted ultraspherical polynomials.

Now, the following integral formula (see, [Andrews, Askey, and Roy (1999)]) is needed

$$\int C_n^{(\lambda)}(x)\, w(x)\, dx = \frac{-2\lambda\,(1-x^2)^{\lambda+\frac{1}{2}}}{n(n+2\lambda)}\, C_{n-1}^{(\lambda+1)}(x), \qquad n \geqslant 1. \tag{6}$$

Also, the following theorem is essential in investigating the convergence analysis for the suggested ultraspherical wavelets expansion.

Theorem 1. *(Bernstein-type inequality) [Giordano and Laforgia (2003)]. The following inequality holds for ultraspherical polynomials:*

$$(\sin\theta)^{\lambda}\, |C_n^{(\lambda)}(\cos\theta)| < \frac{2^{1-\lambda}\,\Gamma(n+\frac{3\lambda}{2})}{\Gamma(\lambda)\,\Gamma(n+1+\frac{\lambda}{2})}, \qquad 0 \leqslant \theta \leqslant \pi,\ 0 < \lambda < 1. \tag{7}$$

2.3 Ultraspherical wavelets

Wavelets constitute a family of functions constructed from dilation and translation of single function called the mother wavelet. When the dilation parameter A and the translation parameter B vary continuously, we have the following family of continuous wavelets:

$$\psi_{A,B}(t) = |A|^{-1/2}\, \psi\left(\frac{t-B}{A}\right) \qquad A, B \in \mathbb{R}, \quad A \neq 0. \tag{8}$$

The ultraspherical wavelets $\psi_{nm}^{(\lambda)}(t) = \psi(k,n,m,\lambda,t)$ are constructed in a way such that they have five arguments: k,n can be assumed to be any positive integer, m is the order for the ultraspherical polynomial, λ is the ultraspherical parameter and t is the normalized time. Explicitly, they are defined on the interval $[0,1]$ as:

$$\psi_{nm}^{(\lambda)}(t) = \begin{cases} 2^{\frac{k}{2}}\,\xi_m \tilde{C}_m^{(\lambda)}\left(2^{k-1}t-n+1\right), & t \in [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}], \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

where $m = 0(1)M-1, \quad n = 1(1)2^{k-1}$, and

$$\xi_m = 2^{\lambda}\,\Gamma(\lambda)\sqrt{\frac{m!\,(m+\lambda)}{2\,\pi\Gamma(m+2\lambda)}}. \tag{10}$$

Remark 1. *It is worthy noting here that $\psi_{nm}^{(\frac{1}{2})}(t)$ is identical to the Legendre wavelets in [Razzaghi and Yousefi (2000); Yousefi (2006)], $\psi_{nm}^{(0)}(t)$ is identical to the first kind Chebyshev wavelets in [Babolian and Fattahzadeh (2007); Yuanlu (2010)] and $\psi_{nm}^{(1)}(t)$ is identical to the second kind Chebyshev wavelets in [Maleknejad, Sohrabi, and Rostami (2007)].*

Now, consider a function $f(t)$ defined on $[0,1]$ and suppose that $f(t)$ may be expanded in terms of ultraspherical wavelets as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}^{(\lambda)}(t), \tag{11}$$

where

$$c_{nm} = \left(f(t), \psi_{nm}^{(\lambda)}(t)\right)_{\tilde{w}} = \int_0^1 (t-t^2)^{\lambda-\frac{1}{2}}\,f(t)\,\psi_{nm}^{(\lambda)}(t)\,dt.$$

Assume that $f(t)$ is approximated in terms of ultraspherical wavelets as

$$f(t) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}^{(\lambda)}(t) = \mathbf{C}^T\,\Psi^{(\lambda)}(t), \tag{12}$$

where $\mathbf{C}$ and $\Psi^{(\lambda)}(t)$ are $2^{k-1}M \times 1$ matrices given by

$$\mathbf{C} = \left[c_{1,0}, c_{1,1}, \ldots, c_{1,M-1}, c_{2,0}, \ldots, c_{2,M-1}, \ldots, c_{2^{k-1},0}, \ldots, c_{2^{k-1},M-1}\right]^T, \tag{13}$$

$$\Psi^{(\lambda)}(t) = \left[\psi_{1,0}^{(\lambda)}, \psi_{1,1}^{(\lambda)}, \ldots, \psi_{1,M-1}^{(\lambda)}, \psi_{2,0}^{(\lambda)}, \ldots, \psi_{2,M-1}^{(\lambda)}, \ldots \psi_{2^{k-1},0}^{(\lambda)}, \ldots, \psi_{2^{k-1},M-1}^{(\lambda)}\right]^T. \tag{14}$$

3 Convergence and error analysis

In this section, we give a comprehensive study on the convergence and error analysis of the suggested ultraspherical wavelets expansion. In this respect, we will state and prove two important theorems, in the first, we follow [Abd-Elhameed and Youssri (2014)] to show that the ultraspherical wavelets expansion of a function $f(x)$ with a bounded second derivative converges uniformly to $f(x)$, and in the second, we give an upper bound for the error (in $L^2_{\tilde{w}}$-norm) of the truncated ultraspherical wavelets expansion.
The following lemma is needed.

Lemma 1. *(see, [Stewart (2012)], p. 742) Let $f(x)$ be a continuous, positive, decreasing function for $x \geqslant n$. If $f(k) = a_k$, provided that $\sum a_n$ is convergent, and $R_n = \sum\limits_{k=n+1}^{\infty} a_k$, then*

$$R_n \leqslant \int_n^\infty f(x)\,dx.$$

Theorem 2. *A function $f(x) \in L^2_{\tilde{w}}[0,1]$, $\tilde{w} = (x-x^2)^{\lambda-\frac{1}{2}}$, $0 < \lambda < 1$ can be expanded as an infinite series of ultraspherical wavelets, which converges uniformly to $f(x)$, given that $|f''(x)| \leqslant L$. Explicitly, the expansion coefficients in* (12) *satisfy the inequality*

$$|c_{nm}| < \frac{4L\,(1+\lambda)^2\,(m+1+\lambda)^2}{(m-2)^4\,n^{\frac{5}{2}}}, \qquad \forall\, n \geqslant 1, m > 2. \tag{15}$$

Proof. If we start with the definition (9) and apply the inner product of $\psi^{(\lambda)}_{nm}(t)$ to both sides of (12), then one can write the coefficients c_{nm} in the form

$$c_{nm} = 2^{\frac{k}{2}}\,\xi_m \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t)\,C^{(\lambda)}_m\left(2^k t - 2n - 1\right) w(2^k t - n)\,dt. \tag{16}$$

If the right hand side of (16) is integrated by parts, then in virtue of relation (6), Eq. (16) is turned into

$$c_{nm} = \frac{2^{\frac{6-k}{2}}\lambda\xi_m}{m(m+2\lambda)} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f'(t) C^{(\lambda+1)}_{m-1}\left(2^k t - 2n - 1\right)(2^k t - n)(1 - 2^k t + n) w(2^k t - n) dt. \tag{17}$$

Repeated use of integration by parts and making use of the substitution: $2^k t-2n-1=\cos\theta$, enable one to write

$$c_{nm}=\frac{\sqrt{2}\,(\lambda)_2\,\xi_m}{2^{2\lambda+\frac{5k-5}{2}}\,(m-1)_2\,(m+2\lambda-1)_2}\int_0^{\pi} f''\left(\frac{1+2n+\cos\theta}{2^k}\right) C_{m-2}^{(\lambda+2)}(\cos\theta)\times (\sin\theta)^{2\lambda+4}\,d\theta. \tag{18}$$

Now, assuming that $m>2$, taking into account the assumption $|f''(t)|\leqslant L$, and with the aid of Theorem 1, we obtain

$$\begin{aligned}
|c_{nm}| &\leqslant \frac{\sqrt{2}\,|(\lambda)(\lambda+1)\,\xi_m|}{2^{2\lambda+\frac{5k-5}{2}}\,(m-1)_2\,(m+2\lambda-1)_2} \\
&\times \int_0^{\pi}\left|f''\left(\frac{1+2n+\cos\theta}{2^k}\right)\right| |C_{m-2}^{(\lambda+2)}(\cos\theta)|\,(\sin\theta)^{2\lambda+4}\,d\theta \\
&< \frac{\sqrt{2}L|\lambda|(1+\lambda)\,\xi_m|}{2^{2\lambda+\frac{5k-5}{2}}\,(m-1)_2\,(m+2\lambda-1)_2}\int_0^{\pi}|C_{m-2}^{(\lambda+2)}(\cos\theta)|\,(\sin\theta)^{2\lambda+4}\,d\theta \\
&< \frac{\sqrt{2\pi}L|\lambda|(1+\lambda)\,|\xi_m|\,\Gamma(m+1+\frac{3\lambda}{2})\,\Gamma(\frac{3+\lambda}{2})}{2^{3\lambda+\frac{5k-5}{2}}\,(m-1)_2\,(m+2\lambda-1)_2\,\Gamma(\lambda+2)\,\Gamma(m+\frac{\lambda}{2})\,\Gamma(2+\frac{\lambda}{2})}.
\end{aligned}$$

Since $\lambda>0$ and $n<2^{k-1}$, therefore with the aid of relation (10), we get

$$\begin{aligned}
|c_{nm}| &< \frac{2L\,|\lambda|\,(1+\lambda)\,\Gamma(\frac{3+\lambda}{2})\,\Gamma(m+1+\frac{3\lambda}{2})\,\sqrt{m!\,(m+\lambda)}}{4^{\lambda}\,\Gamma(2+\frac{\lambda}{2})\,\Gamma(m+\frac{\lambda}{2})\,\sqrt{\Gamma(m+2\lambda)}\,(m-2)^4\,n^{\frac{5}{2}}} \\
&< \frac{4L\,(1+\lambda)^2\,\Gamma(m+1+\frac{3\lambda}{2})\,\sqrt{m!\,(m+\lambda)}}{\Gamma(m+\frac{\lambda}{2})\,\sqrt{\Gamma(m+2\lambda)}\,(m-2)^4\,n^{\frac{5}{2}}} \\
&< \frac{4L\,(1+\lambda)^2\,(m+1+\lambda)^2}{(m-2)^4\,n^{\frac{5}{2}}}.
\end{aligned}$$

This completes the proof of the theorem. □

Note. It should be noted here that, for large values of m and n, and making use of the well known Stirling's formula (see, [Li (2006)]), it can be easily shown that $|c_{nm}|$ is of $\mathscr{O}\left(n^{-\frac{5}{2}}\,m^{-2}\right)$.

Theorem 3. *If f,λ satisfy the hypothesis of Theorem 2, and if we consider the ultraspherical wavelets expansion $f_{k,M}(t)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}^{(\lambda)}(t)$, then the following error estimate (in $L^2_{\tilde{w}}$-norm) is obtained*

$$\|f-f_{k,M}\|_{\tilde{w}} < \frac{(1+\lambda)^2 L(M+\lambda)}{4^k (M-3)^{\frac{7}{2}}}, \quad M>3 \tag{19}$$

Proof. From Eq. (11), and making use of the orthonormality property of $\{\psi_{nm}^{(\lambda)}(t)\}$, we get

$$\|f-f_{k,M}\|^2_{\tilde{w}} = \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty} c^2_{nm}.$$

In virtue of Theorem 2, one can write

$$\|f-f_{k,M}\|^2_{\tilde{w}} < 16L^2(1+\lambda)^4 \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty} \frac{(m+1+\lambda)^4}{(m-2)^8 n^5},$$

and the application of Lemma 1 leads to

$$\begin{aligned}
\|f-f_{k,M}\|^2_{\tilde{w}} &< 16L^2(1+\lambda)^4 \int_{2^{k-1}}^{\infty}\int_{M-1}^{\infty} \frac{(x+1+\lambda)^4}{(x-2)^8 y^5}\,dx\,dy \\
&= \frac{(\lambda+1)^4 L^2 4^{3-2k}\left(15\lambda^2-15\lambda+35\lambda M+21(M-1)M+9\right)}{105(M-3)^7} \\
&< \frac{(1+\lambda)^4 L^2 (M+\lambda)^2}{4^{2k}(M-3)^7},
\end{aligned}$$

and hence

$$\|f-f_{k,M}\|_{\tilde{w}} < \frac{(1+\lambda)^2 L(M+\lambda)}{4^k (M-3)^{\frac{7}{2}}},$$

which completes the proof of the theorem. □

Note. It should be noted here that, for large values of k and M, it can be easily shown that $\|f-f_{k,M}\|_{\tilde{w}}$ is of $\mathscr{O}\left(4^{-k}M^{-\frac{5}{2}}\right)$.

4 Construction of ultraspherical wavelets operational matrix of the fractional integration (UWOMFI)

In this section, we describe in detail the derivation of the shifted ultraspherical wavelets operational matrix of the fractional integration. From now on, we set $m' = 2^{k-1}M$. We select the collocation points t_i to be the zeros of the shifted ultraspherical polynomials of degree m' on the interval $[0,1]$. We define the ultraspherical wavelets matrix $\Phi_{m'\times m'}$ as

$$\Phi_{m'\times m'} = \left[\Psi^{(\lambda)}(t_1), \Psi^{(\lambda)}(t_2), \ldots, \Psi^{(\lambda)}(t_{m'})\right]. \tag{20}$$

Correspondingly, we have

$$\tilde{\mathbf{f}}_{m'} = \left[\tilde{f}(t_1), \tilde{f}(t_2), \ldots \tilde{f}(t_{m'})\right] = \mathbf{C}^T \Phi_{m'\times m'}. \tag{21}$$

Since the shifted ultraspherical wavelets matrix $\Phi_{m'\times m'}$ is an invertible matrix, the ultraspherical wavelets coefficient vector $\mathbf{C}^T$ can be obtained from the relation

$$\mathbf{C}^T = \tilde{\mathbf{f}}_{m'}\,\Phi^{-1}_{m'\times m'}. \tag{22}$$

Now, and if we assume that $f(t)$ can be expanded in terms of the shifted ultraspherical wavelets as in Eq. (12), then the Riemann-Liouville fractional integration in (1) becomes

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t) = \mathbf{C}^T \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * \Psi^{(\lambda)}(t). \tag{23}$$

Thus if $t^{\alpha-1} * f(t)$ can be integrated, then expanded in the shifted ultraspherical wavelets, the Riemann-Liouville fractional integration is solved via the shifted ultraspherical wavelets.

Now, define the following m-set of Block Pulse Function (BPF) on the interval $[0,1)$ as (see, [Zhu and Fan (2012)])

$$b_i(t) = \begin{cases} 1, & \frac{i}{m} \leqslant t < \frac{i+1}{m}, \\ 0, & otherwise. \end{cases} \tag{24}$$

where $i = 0(1)m$. The functions $b_i(t)$ are disjoint and orthogonal, in the sense that

$$b_i(t) b_j(t) = \begin{cases} 0, & i \neq j, \\ b_i(t), & i = j, \end{cases} \quad \text{and} \quad \int_0^1 b_i(t) b_j(t)\, dt = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \tag{25}$$

From the orthogonality property of BPF, it is possible to expand functions in terms of their block pulse series, so for every $f(t) \in [0,1)$, one can write:

$$f(t) \approx \sum_{i=0}^{m-1} f_i b_i(t) = \mathbf{f}^T \mathbf{B}_m(t),$$

where

$$\mathbf{f} = [f_0, f_1, \ldots, f_{m-1}]^T, \ \mathbf{B}_m(t) = [b_0(t), b_1(t), \ldots, b_{m-1}(t)]^T,$$

and

$$f_i = m\int_0^1 f(t)\, b_i(t)\, dt, \quad i = 0(1)m-1.$$

Similarly, the shifted ultraspherical wavelets may be expanded into an m'-term Block Pulse Functions as

$$\Psi^{(\lambda)}(t) = \Phi^{-1}_{m'\times m'} \mathbf{B}_{m'}(t). \tag{26}$$

The Block Pulse operational matrix of the fractional integration F^{α} is given by Kilicman in [Kilicman and Al Zhour (2007)]. This matrix has the following explicit form

$$(I^{\alpha}\mathbf{B}_m)(t) \approx F^{\alpha}\,\mathbf{B}_m(t), \tag{27}$$

where

$$F^{\alpha} = \frac{1}{m^{\alpha}\,\Gamma(\alpha+2)} \begin{pmatrix} 1 & \gamma_1 & \gamma_2 & . & . & . & \gamma_{m-1} \\ 0 & 1 & \gamma_1 & \gamma_2 & . & . & \gamma_{m-1} \\ 0 & 0 & 1 & \gamma_1 & . & . & \gamma_{m-1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 1 & \gamma_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\gamma_k = (k-1)^{\alpha+1} - 2k^{\alpha+1} + (k+1)^{\alpha+1}.$$

Now, for the sake of deriving the shifted ultraspherical wavelets operational matrix of the fractional integration, let

$$(I^{\alpha}\Psi^{(\lambda)})(t) \approx P^{\alpha}_{m'\times m'}\Psi^{(\lambda)}(t), \tag{28}$$

where, the matrix $P^{\alpha}_{m'\times m'}$ is the shifted ultraspherical wavelets operational matrix of the fractional integration. Using Eqs. (26) and (27), we have

$$(I^{\alpha}\Psi^{(\lambda)})(t) \approx P^{\alpha}_{m'\times m'}\Psi^{(\lambda)}(t) = \Phi_{m'\times m'}(I^{\alpha}\mathbf{B}_{m'})(t) \approx \Phi_{m'\times m'}F^{\alpha}\mathbf{B}_{m'}(t), \tag{29}$$

and consequently, Eqs. (28) and (29) lead to

$$P^{\alpha}_{m'\times m'}\Psi^{(\lambda)}(t) = P^{\alpha}_{m'\times m'}\Phi_{m'\times m'}\mathbf{B}_{m'}(t) = \Phi_{m'\times m'}F^{\alpha}\mathbf{B}_{m'}(t), \tag{30}$$

and therefore the ultraspherical wavelets operational matrix of the fractional integration $P^{\alpha}_{m'\times m'}$ is given by

$$P^{\alpha}_{m'\times m'} = \Phi_{m'\times m'}F^{\alpha}\Phi^{-1}_{m'\times m'}. \tag{31}$$

It should be noted that the operational matrix $P^{\alpha}_{m'\times m'}$ contains many zero entries. This special structure, of course reduces the required computations. The calculation for the matrix $P^{\alpha}_{m'\times m'}$ is carried out once and is used to solve fractional order as well as integer order differential equations.

5 A new matrix algorithm for solving multi-term fractional-order differential equation

Consider the one-dimensional multi-term fractional-order differential equation

$$D^{\alpha_1}z(t) + \sum_{i=2}^{N}\varepsilon_i(t)D^{\alpha_i}z(t) = f\left(t, z(t)\right), \qquad t\in[0,1], \tag{32}$$

governed by the initial conditions

$$z^{(i)}(0) = \beta_i, \qquad i = 0(1)n_1 - 1, \tag{33}$$

where

$$n_i - 1 < \alpha_i \leq n_i, \quad n_1 \geqslant n_2 \geqslant \cdots \geqslant n_N, \quad n_1, n_2, \ldots, n_N \in \mathbb{N}, \quad \beta_i \in \mathbb{R},$$

and $\varepsilon_i : [0,1] \to \mathbb{R}, \quad i = 2(1)N, \quad f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a given continuous function. The function $z(t)$ may be approximated by the ultraspherical wavelets as

$$D^{\alpha_1}z(t) \approx \mathbf{Z}^T\Psi^{(\lambda)}(t). \tag{34}$$

Based on Eqs. (4), (29) and (34), we have the following approximations

$$D^{\alpha_j}z(t) \approx \mathbf{Z}^T P^{\alpha_1-\alpha_j}_{m'\times m'}\Psi^{(\lambda)}(t) + \sum_{i=0}^{m'-n_j-1} u^{(i)}(0^+)\frac{t^i}{i!}, \quad j = 2(1)N, \tag{35}$$

$$z(t) \approx \mathbf{Z}^T P^{\alpha_1}_{m' \times m'} \Psi^{(\lambda)}(t) + \sum_{i=0}^{m'-1} z^{(i)}(0^+) \frac{t^i}{i!}, \tag{36}$$

and hence the residual of Eq. (32) takes the form

$$\begin{aligned} R(t) = {} & \mathbf{Z}^T \Psi^{(\lambda)}(t) + \sum_{j=2}^{N} \varepsilon_j(t) \mathbf{Z}^T P^{\alpha_1 - \alpha_j}_{m' \times m'} \Psi^{(\lambda)}(t) + \sum_{j=2}^{N} \varepsilon_j(t) \left(\sum_{i=0}^{m'-n_j-1} z^{(i)}(0^+) \frac{t^i}{i!} \right) \\ & - f\left(t, \mathbf{Z}^T P^{\alpha}_{m' \times m'} \Psi^{(\lambda)}(t) + \sum_{i=0}^{m'-1} z^{(i)}(0^+) \frac{t^i}{i!} \right). \end{aligned} \tag{37}$$

Now, Eq. (37) is enforced to be satisfied exactly at the points $t_j, j = 1(1)m' - n_1$ which are selected to be the first $(m' - n_1)$ roots of the polynomial $C^{(\lambda)}_{m'+1}(t)$, then we have

$$R(t_j) = 0, \qquad j = 1(1)m' - n_1. \tag{38}$$

Moreover, the initial conditions (33) yield

$$\frac{d^r}{dt^r} \left(\mathbf{Z}^T P^{\alpha}_{m' \times m'} \Psi^{(\lambda)}(t) + \sum_{i=0}^{m'-1} z^{(i)}(0^+) \frac{t^i}{i!} \right)_{t=0} = \beta_r, \qquad r = 0(1)n_1 - 1. \tag{39}$$

Eqs. (38) together with Eqs. (39) constitute m' nonlinear equations in the expansion coefficients, c_{nm}, which can be solved with the aid of the well-known Newton's iterative method.

6 Numerical examples

In this section, the ultraspherical wavelets collocation method (UWCM) which employs the operational matrix of fractional integration is applied for handling some numerical examples accompanied with some comparisons hoping to demonstrate the efficiency and applicability of the proposed algorithm.

Example 1. *Consider the following nonlinear initial value problem (see, [Pedas and Tamme (2014)]):*

$$(D^{0.5}z)(t) = z^2(t) + \frac{\sqrt{t}}{\Gamma(1.5)} - t^2, \qquad z(0) = 0, \quad t \in [0,1]. \tag{40}$$

The exact solution of (40) *is* $z(t) = t$. *We apply UWCM to Eq.* (40) *for the case corresponding to* $k = 1, M = 2\,(m' = 2)$ *and* $\lambda = \frac{1}{2}$, *The initial condition and the* *evaluation of the residual of Eq.* (40) *at the collocation point* $x_0 = \frac{1}{2}$, *yield the following two equations*

$$\frac{\xi^2}{16} - \sqrt{\frac{\xi}{\pi}} + 2\sqrt{\frac{\xi}{\pi}}c_{1,1} - \left(c_{1,0} + \left(\frac{\xi}{2} - 1\right)c_{1,1}\right)^2 = 0,$$
$$c_{1,0} - c_{1,1} = 0,$$

where $\xi = 2 - \sqrt{2}$. *This system can be solved to give*

$$c_{1,0} = c_{1,1} = \frac{1}{2},$$

and consequently

$$z(t) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2t-1 \end{pmatrix} = t,$$

which is the exact solution.

Example 2. *Consider the following nonlinear initial value problem (see, [Sweilam, Khader, and Al-Bar (2007); Gejji and Jafari (2007); Saadatmandi and Dehghan (2010); Kazem, Abbasbandy, and Kumar (2013); Kazem (2013)]):*

$$D^3 z(t) + D^{\frac{5}{2}} z(t) + z^2(t) = t^4, \qquad z(0) = z'(0) = 0, z''(0) = 2, \qquad t \in [0,1]. \quad (41)$$

The exact solution of (41) *is* $z(t) = t^2$. *If UWCM is applied on Eq.* (41), *for the case corresponding to* $k = 1, M = 4\,(m' = 4)$, *and* $\lambda = 1$, *then we have the following approximations*

$$D^3 z(t) \approx \mathbf{Z}^T \Psi^{(1)}(t),$$
$$D^{\frac{5}{2}} z(t) \approx \mathbf{Z}^T P^{\frac{1}{2}}_{4\times4} \Psi^{(1)}(t),$$
$$z(t) \approx \mathbf{Z}^T P^{3}_{4\times4} \Psi^{(1)}(t) + t^2,$$

and hence the residual of Eq. (41) *is given by*

$$R(t) = \mathbf{Z}^T \Psi^{(1)}(t) + \mathbf{Z}^T P^{\frac{1}{2}}_{4\times4} \Psi^{(1)}(t) + \left(\mathbf{Z}^T P^{3}_{4\times4} \Psi^{(1)}(t) + t^2\right)^2 - t^4. \quad (42)$$

If $R(t)$ *is enforced to vanish at the collocation point* $x_1 = \frac{2-\sqrt{3}}{4}$, *then the resulting equation jointly with the three equations resulted from the satisfaction of the initial conditions, yield a system of four equations whose solution is*

$$c_{1,0} = \frac{5}{16},\ c_{1,1} = \frac{1}{4},\ c_{1,2} = \frac{1}{16},\ c_{1,3} = 0,$$

and consequently

$$z(t) = \begin{pmatrix} \frac{5}{16} & \frac{1}{4} & \frac{1}{16} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 4t-2 \\ 16t^2-16t+3 \\ 64t^3-96t^2+40t-4 \end{pmatrix} = t^2,$$

which is the exact solution.

Example 3. *Consider the following linear initial value problem (see, [Doha, Bhrawy, and Ezz-Eldien (2011); Doha, Bhrawy, Baleanu, and Ezz-Eldien (2013)]):*

$$D^2 z(t) + D^{\frac{3}{2}} z(t) + z(t) = g(t), \qquad z(0)=0, z'(0)=\gamma, \qquad t \in [0,1], \tag{43}$$

where $g(t)$ is chosen such that the exact solution of (43) is $z(t) = \sin(\gamma t)$. In Table 1, we display the maximum absolute error E resulted from the application of UWCM for the case $k = 2$, with various choices of M, γ and λ. In addition, Table 2 illustrates a comparison between the results obtained by UWCM with those obtained by using the following two methods:

- *Shifted Chebyshev tau method (SCT) in [Doha, Bhrawy, and Ezz-Eldien (2011)].*
- *Shifted Jacobi tau method (SJT) in [Doha, Bhrawy, Baleanu, and Ezz-Eldien (2013)].*

The results in Table 2 show that our algorithm is more accurate if compared with the two methods developed in [Doha, Bhrawy, and Ezz-Eldien (2011)] and [Doha, Bhrawy, Baleanu, and Ezz-Eldien (2013)].

Table 1: Maximum absolute error of Example 3

	$\gamma=1$			$\gamma=4\pi$		
M	$\lambda=0$	$\lambda=\frac{1}{2}$	$\lambda=1$	$\lambda=0$	$\lambda=\frac{1}{2}$	$\lambda=1$
2	$1.314\,.10^{-3}$	$1.228\,.10^{-3}$	$1.148\,.10^{-3}$	$2.152\,.10^{-2}$	$3.514\,.10^{-2}$	$2.912\,.10^{-2}$
4	$6.062\,.10^{-9}$	$7.068\,.10^{-9}$	$7.899\,.10^{-9}$	$3.712\,.10^{-7}$	$5.661\,.10^{-7}$	$6.725\,.10^{-7}$
8	$2.360\,.10^{-14}$	$8.113\,.10^{-15}$	$8.351\,.10^{-15}$	$7.291\,.10^{-11}$	$9.147\,.10^{-12}$	$2.982\,.10^{-12}$

Example 4. *Consider the following linear initial value problem (see, [Doha, Bhrawy, and Ezz-Eldien (2011)]):*

$$D^{\frac{5}{2}} z(t) - 3D^{\frac{2}{3}} z(t) = g(t), \qquad z(0)=1, z'(0)=\gamma, z''(0)=\gamma^2, \qquad t \in [0,1], \tag{44}$$

Table 2: Comparison between the best errors of Example 3

	$\gamma = 1$		
	SCT [Doha, Bhrawy, and Ezz-Eldien (2011)] $N = 64$	SJT [Doha, Bhrawy, Baleanu, and Ezz-Eldien (2013)] $N = 32$	UWCM $m' = 16$
E	$2.4\ .10^{-11}$	$7.1\ .10^{-10}$	$8.1\ .10^{-15}$
	$\gamma = 4\pi$		
E	$4.8\ .10^{-8}$	$1.4\ .10^{-6}$	$2.9\ .10^{-12}$

where $g(t)$ is chosen such that the exact solution of (44) *is $z(t) = \exp(\gamma t)$. In Table 3, we list the maximum absolute errors E by using UWCM for the case $k = 3$ with various choices of M, γ and λ. Moreover, in Table 4, we give a comparison between the present method with the shifted Chebyshev tau method (SCT) obtained in [Doha, Bhrawy, and Ezz-Eldien (2011)]. In addition, in Figure 1, we illustrate the exact and numerical wavelets solutions for the case corresponding to $\gamma = 6, k = 3, M = 1$ and for various values of λ. The results in Table 4 show that the error resulted from the application of our method are smaller than those obtained if SCT method in [Doha, Bhrawy, and Ezz-Eldien (2011)] is applied.*

Table 3: Maximum absolute error of Example 4

	$\gamma = 1$			$\gamma = 6$		
M	$\lambda = 0$	$\lambda = \frac{1}{2}$	$\lambda = 1$	$\lambda = 0$	$\lambda = \frac{1}{2}$	$\lambda = 1$
1	$6.06\ .10^{-3}$	$2.54\ .10^{-3}$	$1.22\ .10^{-3}$	$7.52\ .10^{-1}$	$1.14\ .10^{-1}$	$9.57\ .10^{-2}$
2	$8.43\ .10^{-8}$	$5.67\ .10^{-8}$	$4.37\ .10^{-8}$	$5.24\ .10^{-6}$	$5.34\ .10^{-6}$	$2.71\ .10^{-7}$
3	$5.16\ .10^{-10}$	$7.58\ .10^{-10}$	$7.64\ .10^{-10}$	$2.14\ .10^{-8}$	$8.27\ .10^{-8}$	$7.62\ .10^{-9}$

Table 4: Comparison best errors of Example 4

	$\gamma = 1$	
	SCT [Doha, Bhrawy, and Ezz-Eldien (2011)] $N = 64$	UWCM $m' = 12$
E	$1.6\ .10^{-5}$	$5.2\ .10^{-10}$
	$\gamma = 6$	
E	$6.5\ .10^{-5}$	$7.6\ .10^{-9}$

Example 5. *Consider the following linear initial value problem (see, [Bhrawy,*

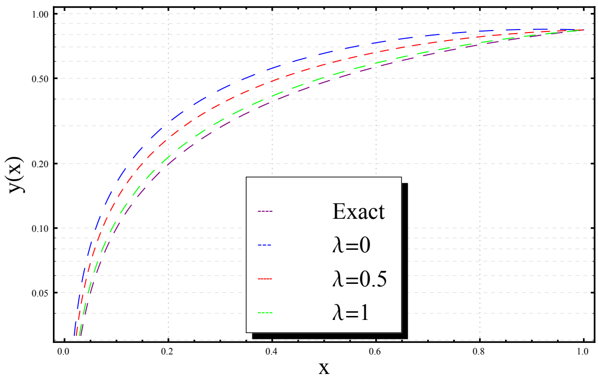

Figure 1: Different solutions of Example 4.

Alofi, and Ezz-Eldien (2011); Doha, Bhrawy, Baleanu, and Ezz-Eldien (2013)]):

$$D^2 z(t) + \sin t\, D^{\frac{1}{2}} z(t) + t\, z(t) = g(t), \qquad z(0) = z'(0) = 0, \qquad t \in [0,1], \tag{45}$$

where

$$g(t) = t^9 - t^8 + 56t^6 - 42t^5 + \sin t \left(\frac{32768}{6435} t^{\frac{15}{2}} - \frac{2048}{429} t^{\frac{13}{2}} \right).$$

The exact solution for (45) *is* $z(t) = t^8 - t^7$. *In Table, 5 we introduce the maximum absolute error* E *resulted from the application of UWCM for the case* $k = 1, M = 8\,(m' = 8)$ *with various choices of* λ*, while in Table 6, we give a comparison between the best errors obtained from the application of UWCM with those obtained by the following two methods*

- *Quadrature shifted Legendre tau method (Q-SLT) in [Bhrawy, Alofi, and Ezz-Eldien (2011)].*
- *Quadrature shifted Jacobi tau method (Q-SJT) in [Doha, Bhrawy, Baleanu, and Ezz-Eldien (2013)].*

Moreover, in Figure 2, we illustrate the exact and numerical wavelets solutions in case of $k = 1, M = 8$ *and for various values of* λ.

Remark 2. *The results of Tables 3 and 5 ensure that the results corresponding to the first kind of Chebyshev wavelets expansion (in case of* $\lambda = 0$*) are not always better than the other expansions (see, [Doha and Abd-Elhameed (2005)] and [Light (1986)]).*

Table 5: Maximum absolute error of Example 5

λ	-0.49	-0.25	0.00	0.25	0.50	0.75	1.00
E	$5.2\,.10^{-16}$	$2.0\,.10^{-16}$	$3.4\,.10^{-15}$	$5.9\,.10^{-16}$	$4.7\,.10^{-16}$	$1.6\,.10^{-16}$	$1.2\,.10^{-16}$

Table 6: Comparison between the best errors of Example 5

Method	E
UWCM	$1.2\,.10^{-16}$
Q-SLT [Bhrawy, Alofi, and Ezz-Eldien (2011)]	$4.5\,.10^{-16}$
Q-SJT [Doha, Bhrawy, Baleanu, and Ezz-Eldien (2013)]	$8.8\,.10^{-16}$

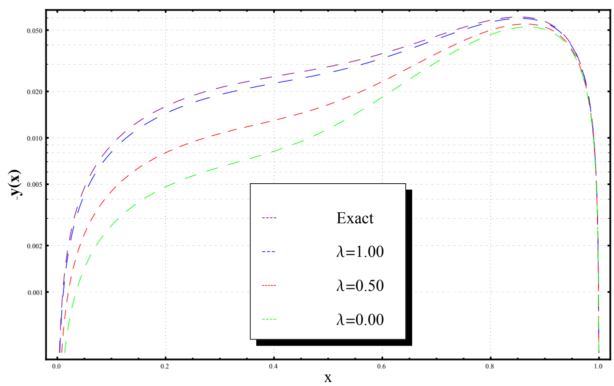

Figure 2: Different solutions of Example 5

Example 6. *Consider the following nonlinear initial value problem (see, [Doha, Bhrawy, Baleanu, and Ezz-Eldien (2013)]):*

$$
\begin{aligned}
&aD^{2.2}z(t)+bD^{\alpha_2}z(t)+cD^{\alpha_1}z(t)+e|z(t)|^3=f(t),\\
&z^{(i)}(0)=0, \quad i=0,1,2, \quad t\in[0,1],
\end{aligned}
\tag{46}
$$

where

$$
f(t)=\frac{2at^{0.8}}{\Gamma(1.8)}+\frac{2bt^{3-\alpha_2}}{\Gamma(4-\alpha_2)}+\frac{2ct^{3-\alpha_1}}{\Gamma(4-\alpha_1)}+\frac{et^9}{27}.
$$

The exact solution of (46) *is* $z(t)=\frac{t^3}{3}$. *We apply UWCM to Eq.* (46), *for the case corresponding to* $k=1, \quad M=3\,(m'=3)$ *and* $\lambda=\frac{1}{2}$. *In this case*

$$
z_{1,3,\frac{1}{2}}(t)=c_0\tilde{L}_0(t)+c_1\tilde{L}_1(t)+c_2\tilde{L}_2(t)+c_3\tilde{L}_3(t),
$$

where $\tilde{L}_i(t)$ *is the well-known shifted Legendre polynomial of degree i on* $[0,1]$ *and* $c_i = \sqrt{2}\,\xi_i\, c_{1,i}$, $i = 0,1,2$. *We solve Eq.* (46) *for* $a = b = c = e = 1$, $\alpha_2 = 1.25$, $\alpha_1 = 0.75$. *(see, [Doha, Bhrawy, Baleanu, and Ezz-Eldien (2013)]). In this case, we have the following approximations*

$$
\begin{aligned}
&D^{2.2}z(t) \approx \mathbf{Z}^T \Psi^{(\frac{1}{2})}(t),\\
&D^{1.25}z(t) \approx \mathbf{Z}^T P^{0.95}_{3\times 3} \Psi^{(\frac{1}{2})}(t),\\
&D^{0.75}z(t) \approx \mathbf{Z}^T P^{1.45}_{3\times 3} \Psi^{(\frac{1}{2})}(t),\\
&z(t) \approx \mathbf{Z}^T P^{2.2}_{3\times 3} \Psi^{(\frac{1}{2})}(t).
\end{aligned}
$$

The residual of Eq. (46) *is given by*

$$
R(t) = \mathbf{Z}^T \Psi^{(\frac{1}{2})}(t) + \mathbf{Z}^T P^{0.95}_{3\times 3} \Psi^{(\frac{1}{2})}(t) + \mathbf{Z}^T P^{1.45}_{3\times 3} \Psi^{(\frac{1}{2})}(t) + |\mathbf{Z}^T P^{2.2}_{3\times 3} \Psi^{(\frac{1}{2})}(t)|^3 - f(t). \tag{47}
$$

If Eq. (47) *is collocated at the first root of* $\tilde{L}_4(t)$*, i.e, at* $t_1 = \frac{35-\sqrt{35\left(15+2\sqrt{30}\right)}}{70}$*, then we get*

$$
\begin{aligned}
&(c_0 - 0.86\,c_1 + 0.6094\,c_2 - 0.30014\,c_3)^3 + 67.9679\,c_0 - 72.0141\,c_1\\
&+ 99.8085\,c_2 - 174.575\,c_3 - 0.26967 = 0.
\end{aligned} \tag{48}
$$

Moreover, the use of the initial conditions yield

$$
c_0 - c_1 + c_2 - c_3 = 0, \tag{49}
$$

$$
2\,c_1 - 6\,c_2 + 12\,c_3 = 0, \tag{50}
$$

$$
12\,c_2 - 60c_3 = 0. \tag{51}
$$

The system of equations (48)-(51) *can be solved to give*

$$
c_0 = \frac{1}{12}, \quad c_1 = \frac{3}{20}, \quad c_2 = \frac{1}{12}, \quad c_3 = \frac{1}{60},
$$

and consequently

$$
z(t) = \begin{pmatrix} \frac{1}{12} & \frac{3}{20} & \frac{1}{12} & \frac{1}{60} \end{pmatrix} \begin{pmatrix} 1 \\ 2t-1 \\ 6t^2-6t+1 \\ 20t^3-30t^2+12t-1 \end{pmatrix} = \frac{t^3}{3},
$$

which is the exact solution.

7 Conclusions

In this paper, we have presented a new algorithm for obtaining some numerical spectral solutions for multi-term fractional-order initial value problems. The derivation of this algorithm is essentially based on constructing the ultraspherical wavelets operational matrix of the fractional integration. The convergence and error analysis of the suggested expansion is carefully investigated. One of the main advantages of the presented algorithm are its availability for application on both linear and non linear fractional-order initial value problems. Another advantage of the developed algorithm is its high accuracy since accurate approximate solutions can be achieved by using a few number of terms of the ultraspherical wavelets expansion.

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