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Probability Density Transitions in the FitzHugh-Nagumo Model with Lévy Noise

Xu Yong[1,2], Feng Jing[1], Xu Wei[1] and Gu Rencai[1]

Abstract: In this paper, bifurcation analysis and numerical simulations are performed on the FitzHugh-Nagumo system in the presence of Lévy stable noise. The stationary probability density functions are obtained to examine the influences of noise intensity and stability index. Results show that under the influences of noise intensity and stability index, the dynamic of the FitzHugh-Nagumo model can be well characterized through the concept of stochastic bifurcation, consisting in qualitative changes of the stationary probability distribution. Then, the mean passage time between the resting and action state is investigated as functions of noise intensity and stability index of the external signal by means of numerical simulations. Dependences of the results on the parameters of Lévy noise are discussed to find the different mechanisms compared with Gaussian case.

Keywords: Lévy noise, FitzHugh-Nagumo model, firing time, stochastic bifurcation, stationary probability density, mean first passage time.

1 Introduction

The investigation about influences of various external perturbations on complex system dynamics has attracted large attention over the past decade. A number of issues have emphasized the constructive role of noise sources [Hennequin (2004); Ushakov et al. (2005); Mankin et al. (2006)]. Starting from the need to understand how nonlinear systems evolve in the presence of noise, varieties of different phenomena have been discovered [Bondareva, Zmievskaya and Levchenko (2008); Ghosh, Barik and Ray (2005); Xu et al. (2014); Yue, Xu and Yuan (2013)]. Stochastic bifurcation, which has been developed to study the changes of existence and stability of limiting distributions, is one of the interesting phenomena induced by noise. Nowadays, large number of investigations has been devoted to study changes in the dynamics of nonlinear systems through the concept of stochastic

[1] Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, China.
[2] Corresponding author.

bifurcations [Chiarella et al. (2008); Zakharova et al. (2010)]. The mean first passage time (MFPT), which is defined as the average time that the walker used to reach some given final node for the first time from a given initial node [Risken (1996)], has also received a considerable amount of attention. As a basic quantitative measure of the transportation efficiency, it is widely applied to describe various dynamic features such as exit problems, switching time, activation rates and so on [Hofmann and Ivanyuk(2003); McKenzie, Lewis and Merrill (2009)].

In neuronal systems, noise arises from kinds of different sources in neurons. Understanding how the noise influences the dynamical property is very important for one to comprehend biological characters in theoretical neuroscience. A broad spectrum of phenomena due to noise has been discovered in different neuronal networks [Jia et al. (2012); Kitajima and Kurths (2005); Tuckwell, Rodriguez and Wan (2003); Wu et al. (2013)]. One of the most famous neuronal models is the FitzHugh-Nagumo (FHN) model for the conduction of electrical impulses along a nerve fiber [FitzHugh (1961)]. As a simplification of the Hodgkin–Huxley model proposed by Hodgkin and Huxley in 1952, FHN model was developed to investigate the response of cylindrical cells to external electric fields. The investigation of effects of various external perturbations in FHN system has attracted large attention in these years. In [Pikovsky and Kurths (1997)], the dynamics of the excitable FHN system under external noisy driving was examined. The response time of a neuron in the presence of a strong periodic driving in the stochastic FitzHugh–Nagumo model was studied in [Pankratova, Polovinkin and Spagnolo (2005)]. The FHN neural model driven by two multiplicative noises and one additive noise was also investigated [Tang et al. (2008)]. It can thus be seen, that fluctuation plays a significant role in neurons.

Despite the broad interest of this problem, few people paid attention to the effects of non-Gaussian noise in the FHN system. Stochastic fluctuations associated with an FHN system were always assumed to be Gaussian cases, since the distributions of most random excitation can be approximated by Gaussian distribution with the foundation of central limit theorem. However, Gaussian noise can only model those small fluctuations and many experimental results offer strong evidence that noise source which is more frequently encountered in practice could be non-Gaussian in distribution. The Lévy stable distribution with $0 < \alpha < 2$ is a more appropriate choice when one considers realistic phenomena, where complexity, non-uniformity and long-range correlations play a role [Dybiec, Gudowska-Nowak and Sokolov (2007); Edwards et al. (2007); Chechkin et al. (2006); Dubkov, Spagnolo and Uchaikin (2008)]. Recently, effects of non-Gaussian Lévy noise in nonlinear systems have attracted growing attention in physics, biology, engineering, natural science and social science [Bartumeus et al. (2005); Krlín, Paprok and Svoboda

(2008); Majumdar and Zi (2008); Ponomarev, Denisov and Hänggi (2011); Xu et al. (2013); Xu et al. (2015); Zeng and Xu (2010)]. Despite that, an FHN model with Lévy noise has never been discussed in the literature and we aim to fill up the research gap in this paper. With this in mind, the purpose of the present paper is to study the stochastic bifurcation in the FHN system with non-Gaussian Lévy noise. Besides, we will focus on transport properties of the FHN system which can be obtained as the average of first passage times.

2 The system model

Consider the following one-dimensional FHN neural system [Alarcón, Pérez-Madrid and Rubí (1998)] driven by Lévy stable noise

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v(a-v)(v-1) - \frac{b}{\gamma}v + \zeta(t), \tag{1}$$

where v is the variable representing the neuron membrane voltage, $0 < a < 1$ is essentially the threshold value, b and γ are positive constants, $\zeta(t)$ is Lévy stable noise, which is derivative of the Lévy stable motion $L(t)$ and $L(t)$ can be see a generalized Wiener processes.

Lévy distributions $L_{\alpha,\beta}(\zeta;\sigma,\mu)$ correspond to a four-parametrical family of the probability density functions characterized by their Fourier transforms $\Phi(k)$ (characteristic functions of the distributions), that is $\Phi(k) = F(L_{\alpha,\beta}(\zeta;\sigma,\mu)) = \int_{-\infty}^{+\infty} d\zeta e^{ik\zeta} L_{\alpha,\beta}(\zeta;\sigma,\mu)$[Janicki and Weron (1994)], then

$$\Phi(k) = \exp\left[-\sigma^{\alpha}|k|^{\alpha}\left(1 - i\beta sgn(k)\tan\frac{\pi\alpha}{2}\right)\right], \tag{2}$$

for$\alpha \in (0,1)\cup(1,2]$ and

$$\Phi(k) = \exp\left[-\sigma|k|\left(1 + i\beta sgn(k)\frac{2}{\pi}\ln|k|\right)\right]. \tag{3}$$

for $\alpha = 1$. Here the parameter $\alpha(0 < \alpha \leq 2)$ characterizes the asymptotic tail of the Lévy distribution $L_{\alpha,\beta}(\zeta;\sigma,\mu)$ for $\alpha < 2$ as $L_{\alpha,\beta}(\zeta;\sigma,\mu) \sim |\zeta|^{-\alpha-1}$ with $|\zeta| << 1$. The parameter β $(-1 \leq \beta \leq 1)$ is the skewness parameter defining the degree of asymmetry of the distribution, μ $(-\infty \leq \mu \leq \infty)$ is the center or location parameter which denotes the mean value of the distribution when $\alpha > 1$. σ represents the generalized diffusion coefficient and $D = \sigma^{\alpha}$ is the noise intensity. The probability distributions for different values of α and β are given in Fig. 1 (a). Fig. 1 (b) shows a sample trajectory generated by McCuUoch algorithm with $\alpha = 1.75, \beta = 0.0, \mu = 0.0, \sigma = 1.0$. In this paper, we consider the symmetric Lévy distribution case with $\beta = 0.0$, $\mu = 0$ and $1 \leq \alpha \leq 2$.

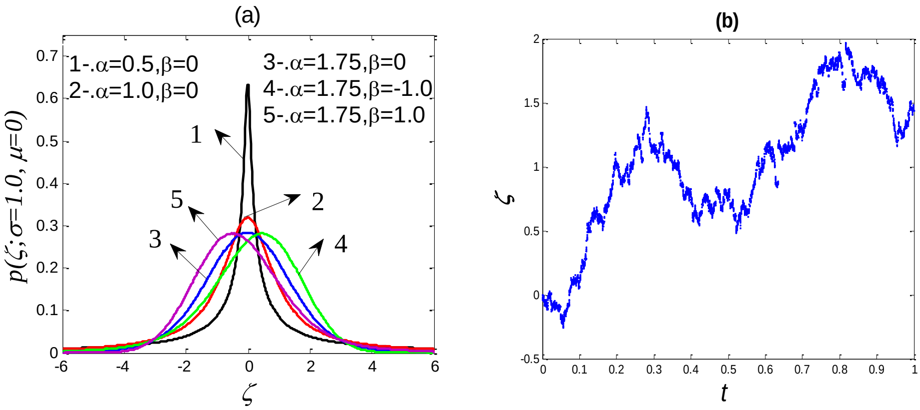

Figure 1: (a) Probability density function of Lévy stable noisefor fixed values of σandμ, and different values of $\alpha$and β; (b) A sample trajectory of Lévy stable noise ($\alpha = 1.75, \beta = 0.0$).

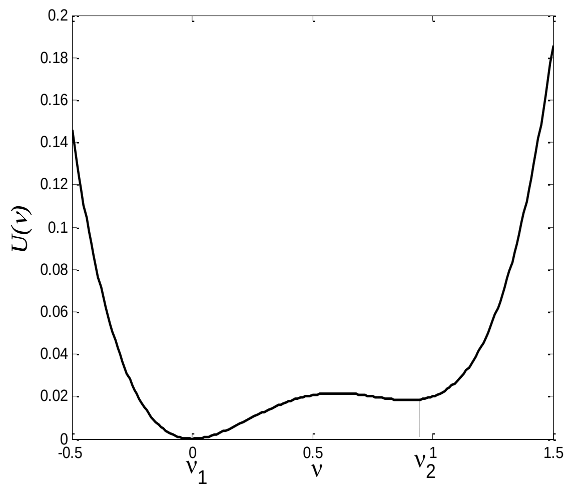

Figure 2: The potential$U(v)$for$a = 0.5, b = 0.04, \gamma = 1.0$.

The potential function of system (1) is

$$U(v) = \frac{1}{4}v^4 - \frac{a+1}{3}v^3 + \frac{a+b/\gamma}{2}v^2. \tag{4}$$

If $\frac{b}{\gamma} < \left(\frac{a-1}{2}\right)^2$, the potential function have two stable points $v_1 = 0$, $v_2 = \frac{a+1+\sqrt{(a-1)^2-4b/\gamma}}{2}$

and an unstable state $v_u = \frac{a+1-\sqrt{(a-1)^2-4b/\gamma}}{2}$. In Fig.2, we show the potential whose shape varies with the parameters. In our paper, we set $a = 0.5, b = 0.04, \gamma = 1.0$ and it can clearly see that the potential has two stable states. The positions of the potential minima, corresponding to v_1 and v_2, are regarded as the resting state and the excited state of the neuron, respectively.

3 Phase transitions in the FHN system with Lévy stable noise

The corresponding fractional Fokker-Plank equation (FFPE) of system (1) is [Benson, Wheatcraft and Meerschaert(2000)]:

$$\frac{\partial}{\partial t}P(v,t) = -\frac{\partial}{\partial v}F(v)P(v,t) + D\frac{\partial^\alpha P(v,t)}{\partial |v|^\alpha}, \tag{5}$$

where $F(v) = v(a-v)(v-1) - \frac{b}{\gamma}v = -v^3 + (a+1)v^2 - (a+\frac{b}{\gamma})v$. The Riesz space fractional derivative $\partial^\alpha/\partial|x|^\alpha$ is defined through the Weyl fractional operator as [Risken (1996)]:

$$\frac{\partial^\alpha P(v,t)}{\partial |x|^\alpha} = -\frac{D_+^\alpha P(v,t) + D_-^\alpha P(v,t)}{2\cos(\pi\alpha/2)}, \tag{6}$$

where

$$D_+^\alpha P(v,t) = \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{d^2x}\int_{-\infty}^{x}\frac{P(\eta,t)d\eta}{(x-\eta)^{\alpha-1}}, \tag{7}$$

and

$$D_-^\alpha P(v,t) = \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{d^2x}\int_{x}^{\infty}\frac{P(\eta,t)d\eta}{(x-\eta)^{\alpha-1}}. \tag{8}$$

Concerning the exact analytical solution of Eq. (5) is very difficult to get, we esort to the Grünwald-Letnikov scheme [Zeng and Xu (2010)] to numerically solve Eq. (5). In our simulation, we set the time step $\Delta t = 0.01$ and the initial conditions $P(v,0) = \delta(v)$. In another way, the numerical solutions could also be obtained by order-4 stochastic Runge-Kutta algorithm as the following form:

$$\begin{aligned} &k_1 = \Delta t \cdot F(v_n), k_2 = \Delta t \cdot F(v_n + k_1/2), \\ &k_3 = \Delta t \cdot F(v_n + k_2/2), k_4 = \Delta t \cdot F(v_n + k_3), \\ &v_{n+1} = v_n + (k_1 + 2k_2 + 2k_3 + k_4)/6 + \Delta t^{1/\alpha} \cdot \zeta_n, \end{aligned} \tag{9}$$

where ζ_n is Lévy distributed random variable with noise intensity D and stability index α. Take the initial conditions $t_0 = 0, v(0) = 0.0$ and the time step $\Delta t = 0.01$

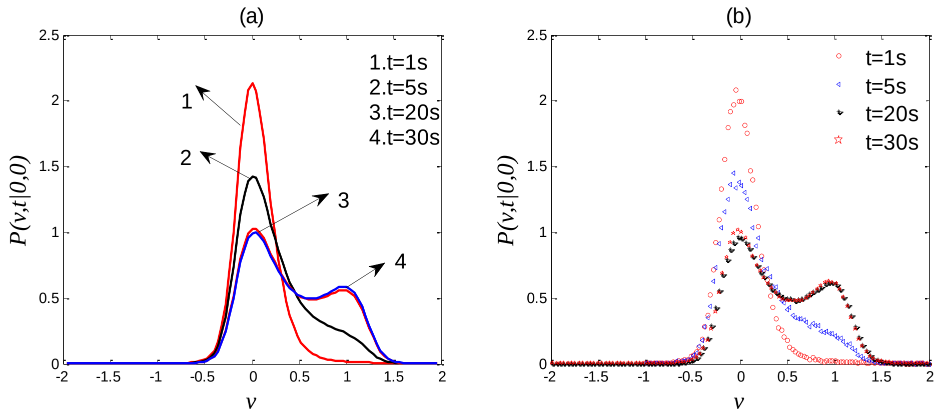

Figure 3: The probability density $P(v,t)$ at different times for $\alpha = 1.75, D = 0.05$ of the noisy FHN neural system. (a) from using Grünwald-Letnikov scheme to solve the spatial FFPE (5); (b) Monte Carlo experiments results from Eq.(1).

in numerical calculations, and then the probability density function $p(v,t)$ can be obtained by Monte-Carlo method with simulation times $N = 10^5$.

The probability density functions $P(v,t)$ at different times for $\alpha = 1.75, D = 0.05$ of the noisy FHN neural system with system parameters $a = 0.5, b = 0.04, \gamma = 1.0$ are given in Fig. 3. Where Fig. 3 (a) is obtained by using Grünwald-Letnikov scheme to solve Eq. (5), while the Monte-Carlo simulation results are depicted in Fig. 3 (b). It is clear that the presented results demonstrate an excellent agreement between the two different approaches, which shows the validity of the Grünwald-Letnikov scheme. From Fig. 3, we can found that all the systems preserve the states after the time $t = 20s$. Therefore, it can be regarded as stable, i.e. $P_{st}(v) = P(v,20)$.

We demonstrate the stationary probability density $P_{st}(v)$ versus different noise intensity D in Fig. 4. One can observe that, for fixed $\alpha = 1.8$ in Fig. 4(a), the probability distribution has two peaks for small noise intensity, at the rest state and the excited state, respectively. The peak on the left is much higher, indicating that the neuron system stays at the rest state most of the time. When we increase the value of noise intensity D, both the peak values of probability distribution $P_{st}(v)$ will decrease, and the stationary probability density $P_{st}(v)$ gradually undergoes a succession of a phase transition from bimodal to unimodal. Finally, the system can only be found at the rest state and we can say that the system is locked at the rest state. We come to the same conclusion in Fig.4(b) when $\alpha = 1.6$ and we can say that the noise intensity of Lévy noise plays a negative role in firing neurons.

The stationary probability density $P_{st}(v)$ versus different stability index is shown in

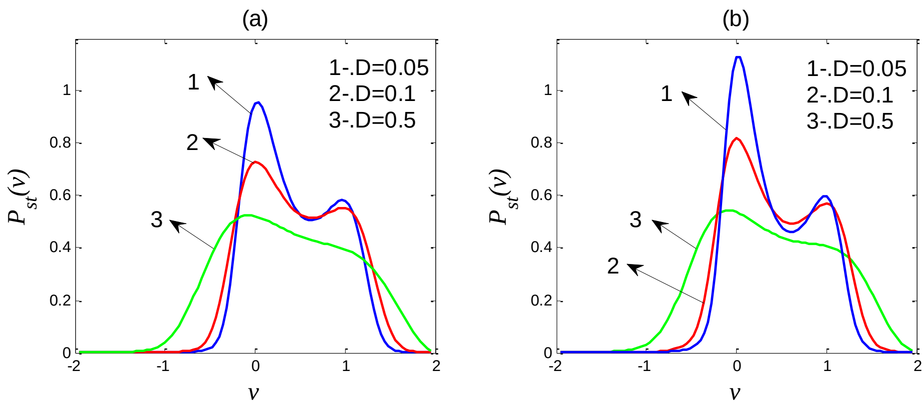

Figure 4: The probability density $P_{st}(v)$ of the noisy FHN neural system versus different value of noise intensity$D$ (a) $\alpha = 1.8$ (b) $\alpha = 1.6$.

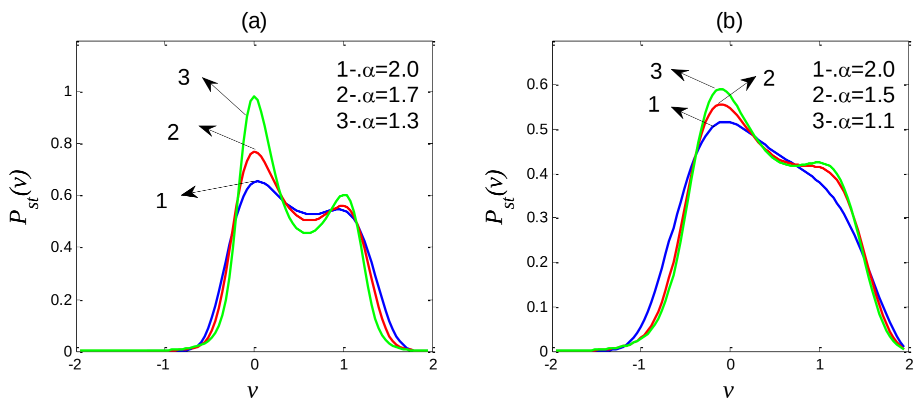

Figure 5: The probability density $P_{st}(v)$ of the noisy FHN neural system versus different value of stability index α (a) $D = 0.1$; (b) $D = 0.5$.

Fig. 5. In Fig. 5 (a), the intensity D is fixed at 0.1 and α changes. From which, we note that $P_{st}(v)$ is bimodal distribution but the gap between the left peak and the right peak is not large for $\alpha = 2.0$. Decreasing the stability index α, the bimodality of $P_{st}(v)$ does not change, which means there is no transition occurs. However, the peaks of $P_{st}(v)$ become more precipitous and the gap between two peaks becomes much larger. Fix $D = 0.5$, the stationary probability density $P_{st}(v)$ versus different index α is shown in Fig. 5 (b). We could find that the probability distribution $P_{st}(v)$ is unimodal in the Gaussian case, see curve 1 in Fig.5(b); and for $\alpha = 1.1$, $P_{st}(v)$ convert to bimodal distribution, as curve 3 in Fig. 5(b) shows. Therefore, there is a

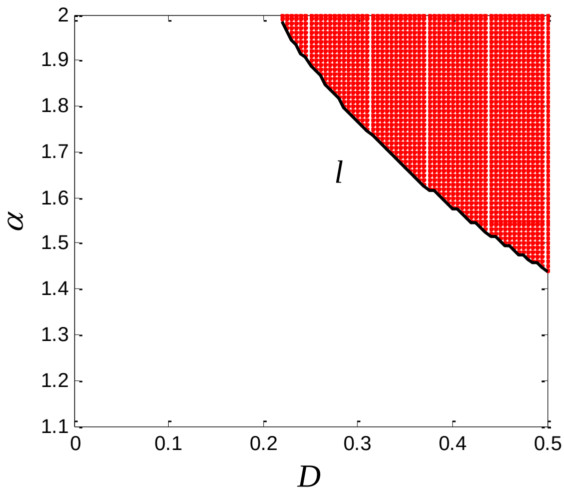

Figure 6: Bifurcation diagram of the system (1) in the parameter plane (D,α). The stationary probability density is bimodal in the tinted region and unimodal in the gap region. The line l corresponds to phase transition.

phase transition happens in the noisy FHN neural system during the stability index α decrease from 2.0 to 1.1 and it also show compelling evidence that there are noticeable differences in dynamics under Gaussian and non-Gaussian Lévy noise.

Form analyses above, we can get the conclusion that both the change of noise intensity D and stability index α could induce phase transitions. The bifurcation diagram of the system (1) in the parameter plane (D,α) is given in Fig.6. The stationary probability density is bimodal in the tinted region and unimodal in the gap region. The line l is the boundary of the tinted region, which corresponding to phase transition. One can clearly observe that both noise intensity and stability index could be regarded as relevant control parameters for phase transitions. $P_{st}(v)$ is unimodal for any value of α if $D < 0.22$. Increasing the intensity D, the bimodality region becomes broader. Similarly, $P_{st}(v)$ always presents a single peak for small values of stability index and the bimodality region appears when $\alpha > 1.44$. With the increase of the stability index, the bimodality region becomes broader and reaches the maximum in the Gaussian case. Additionally, it could be worth noticing that when the stationary probability density $P_{st}(v)$ is bimodal distribution, the left peak is large than the right peak, which is agree with the fact that cell neuron system keeps in the resting state in most time.

4 Mean First Passage Time

In this section, we focus on the passage time between the resting and action state. The problem has been extensively studied for the Gaussian process. However, if the stochastic driving obeys the general Lévy stable distribution which different from the Gaussian, the transport properties of these dynamic systems will also be different.

For the purpose of analysis, we will compute the MFPT which is defined by the first time of the particle jumps from one potential well to another and compare the results. To evaluate how Lévy noise affects the activation time of the system, we first use an approximation of the potential by a linear slope with a re?ecting barrier placed at the resting state v_1 and absorbing barrier at action v_2 to obtain the time in forward direction. When the parameters $a = 0.5$, $b = 0.04$, $\gamma = 1.0$, $v_1 = 0$, $v_2 = 0.9$. Taking the time step $\Delta t = 0.001$, and average over $N = 10000$ different noise realizations. Then, we compute the MFPT from v_2 to v_1 in order to study the influences of Lévy noise on the time that is used to transport from action state to resting state. The simulative results of MFPT and some analyses are presented as follows.

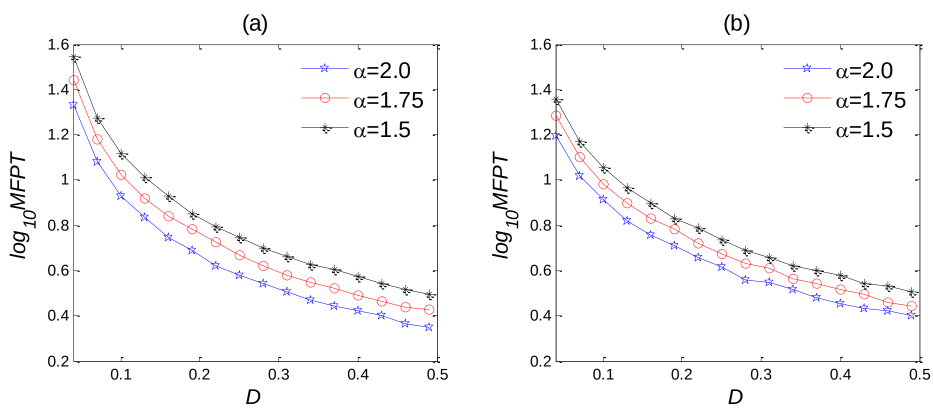

Figure 7: Mean first passage time as a function of D for $\alpha = 2.0, 1.75$ and 1.5.(a) $v_1 \rightarrow v_2$; (b) $v_2 \rightarrow v_1$. The mean first passage time decrease with the increase of the noise intensity in two directions.

In Fig. 7(a) and 8(a), it can be seen that the mean activation time decreases due to the increases of noise intensity and stability index. That means the two parameters could facilitate the transition from the resting state to the action one. Therefore, they can speed up the activation of the system. As shown in Fig. 7(b) and 8(b), we find that when the system is initially at action state, the transition from the action state to the rest one would become much easier with the increase of both noise intensity and stability index.

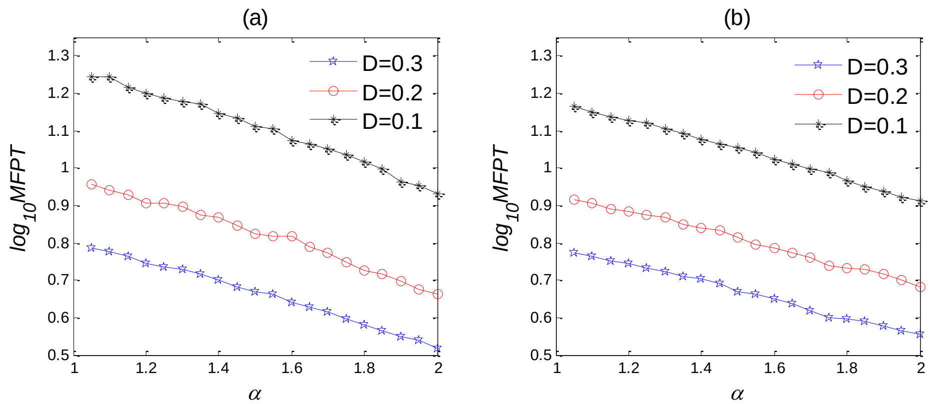

Figure 8: Mean first passage time as a function of α for $D = 0.1, 0.2$ and 0.3. (a) $v_1 \to v_2$; (b) $v_2 \to v_1$. Increasing the stability index, the mean first passage time decreases in two directions.

Looking at the Fig. 8 in more detail, both the time in two directions reach minimum for the Gaussian case corresponds to $\alpha = 2.0$. That is to say, as $\alpha \to 2.0$, large jumps gradually disappear and the mean first passage time becomes smaller. The analysis reveals that large jumps play negative roles in the transitions of the system (1).

From Fig. 7 and Fig. 8, one can notice that the changing relation of $\log_{10} MFPT$ and α is nearly linear, while the changing relation of $\log_{10} MFPT$ and D is nonlinear, and the decreasing rate of MFPT gradually decrease as noise intensity increasing. Therefore, both the reduction of noise intensity D and stability index α could make the MFPT decline, but there are quilt different.

5 Conclusions

The paper has discussed the phase transitions and the mean first passage time of the one-dimensional FHN neural system driven by Lévy stable noise. We have investigated the interaction mechanism of noise intensity and stability index. The stationary probability density functions are obtained by the technique of Grünwald-Letnikov scheme, and the Monte Carlo method is used to test the validity of the numerical scheme, which indicates the effectiveness of the numerical method. According to the change of the stationary probability density $P_{st}(v)$, the transition behaviors are researched, and bifurcation diagram of the system in the parameters plane (D, α) is given in the paper. We find that the decrease of the stability index α of Lévy noise can make the peak value of probability density increases, even can make $P_{st}(v)$ split from unimodal to bimodal distribution. That is to say, not only the noise intensity D but also the stability index α can be seen as bifurcation parameter. In addition, the effects of D and α on MFPT of the system in two directions are studied. The results demonstrate that when α is fixed, the MFPT decreases with the increase of D. Thus, the increase of noise intensity can shorten the change-over time of the cell neurons resting and excited state and accelerate neurons discharge rhythms. Therefore, the existence of the Lévy stable noise has the positive sense to the transmission of Neurons information. Besides, if α is fixed, the MFPT presents increasing trend with the decrease of α. That is, the decrease of Lévy stable index α can make the transition of cell neurons between two states be more difficult, which is bad for the transition of cell neurons. So, if we want to speed up the transmission speed of cell neurons information, we should avoid rather fluctuant external stimuli.

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