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A High-Order Accurate Wavelet Method for Solving Three-Dimensional Poisson Problems

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Abstract: Based on the approximation scheme for a L^2-function defined on a three-dimensional bounded space by combining techniques of boundary extension and Coiflet-type wavelet expansion, a modified wavelet Galerkin method is proposed for solving three-dimensional Poisson problems with various boundary conditions. Such a wavelet-based solution procedure has been justified by solving five test examples. Numerical results demonstrate that the present wavelet method has an excellent numerical accuracy, a fast convergence rate, and a very good capability in handling complex boundary conditions.

Keywords: Coiflet-type wavelet; Galerkin method; three-dimensional Poisson equation; mixed boundary conditions.

1 Introduction

The Poisson problem, as a typical elliptic partial differential equation, plays a central role in mathematics, theoretical physics, mechanics and other fields, such as electromagnetics [Heise and Kuhn (1996)], fluid dynamics [Vuik, Segal, and Meijerink (1999)], plasma physics [Feng and Sheng (2015)], and electrical power network modeling [Howle and Vavasis (2005)]. It has so broad applications that researchers have to frequently find a numerical solution of the Poisson equation [Doha (1989); Wordelman, Aluru, and Ravaioli (2000); Feng and Sheng (2015)].

In the past few decades, a number of numerical methods have been proposed to solve the Poisson equations. For example, Mittal and Gahlaut (1987) introduced the second- and fourth-order finite difference schemes for solving Poisson equations. Doha (1989) developed a Chebyshev spectral method to study Poisson problems. A fourth-order compact difference scheme for solving the three-dimensional Poisson

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equation on a cubic domain has been proposed by Ge (2010). Barad and Colella (2005) used a local refinement finite volume method to solve the Poisson equation. And a Haar wavelet method for solving two- and three-dimensional Poisson problems has been derived by Shi, Cao, and Chen (2012). Although these numerical methods are effectively applied to the solution of Poisson problems, researchers have been trying to find a more high-order accurate method to solve numerically Poisson problems, which allows for using lower computational cost with respect to a low-order accurate method to ensure a similar accuracy [Zhang, (1998); Shi, Cao, and Chen (2012); Feng and Sheng (2015); Bardazzi, Lugni, Antuono, Graziani, and Faltinsen (2015)].

In our recent work [Liu, Zhou, Wang, and Wang (2013); Liu, Wang, and Zhou (2013); Liu, Wang, and Zhou (2013); Liu, Zhou, Zhang, and Wang (2014)], we have developed the modified wavelet Galerkin methods for solutions of one- and two-dimensional nonlinear problems, which have shown an excellent numerical accuracy and a fast convergence rate. For instance, the order of convergence of such wavelet algorithm for two-dimensional Bratu-like equations is about 5 [Liu, Zhou, Wang, and Wang (2013)]. And for the Burgers' equation, this wavelet method also has a convergence rate of order 5, and shows a much better accuracy than many other existing methods [Liu, Zhou, Zhang, and Wang (2014)], such as the multiquadric method [Hon and Mao (1998)], the multiquadric quasi-interpolation method [Chen and Wu (2006)], the cubic B-spline quasi-interpolation method [Zhu and Wang (2009)], the multiquadric-RBF method [Xie and Li (2013)], the weighted average differential quadrature method [Jiwari, Mittal, and Sharma (2013)], the lattice Boltzmann method [Gao, Le, and Shi (2013)], and the B-spline finite element method [Kutluay, Esen, and Dag (2004)].

In the present study, based on the modified wavelet Galerkin methods respectively for the solution of one- and two-dimensional nonlinear boundary value problems [Liu, Zhou, Wang, and Wang (2013); Liu, Wang, and Zhou (2013)], we propose a wavelet approximation scheme for three-dimensional bounded functions based on techniques of boundary extension and Coiflet-type wavelet expansion, which can eliminate the undesired oscillating error near boundary points due to function value jump [Liu, Zhou, Wang, and Wang (2013)]. Then a wavelet-based solution procedure for three-dimensional Poisson problems is derived in detail. At last, a comparison between the present solutions and those obtained by using other existing numerical methods is made to demonstrate the effectiveness of the proposed wavelet method.

2 Wavelet approximation of an interval-bounded L^2-function

Following the theory of wavelet based multiresolution analysis, a set of scaling bases for three-dimensional space can be directly extended by the tensor products of one-dimensional wavelet bases [Meyer (1992); Ray and Gupta (2014)]. Therefore based on our previous work [Liu, Zhou, Wang, and Wang (2013)], for a function $f(x,y,z) \in L^2[01]^3$, we have

$$f(x,y,z) \approx P^{j_x,j_y,j_z} f(x,y,z) = \sum_{k=0}^{2^{j_x}} \sum_{l=0}^{2^{j_y}} \sum_{n=0}^{2^{j_z}} f(\frac{k}{2^{j_x}}, \frac{l}{2^{j_y}}, \frac{n}{2^{j_z}}) \varphi_{j_x,k}(x) \varphi_{j_y,l}(y) \varphi_{j_z,n}(z) \quad (1)$$

in which j_x, j_y and j_z are the decomposition level respectively in the x, y and z directions, and the modified one-dimensional wavelet basis

$$\varphi_{j,k}(x) = \begin{cases} \sum_{i=-9}^{-1} T_{0,k}(\frac{i}{2^j}) \phi(2^j x - i + 7) + \phi(2^j x - k + 7) & k \in [0,3] \\ \phi(2^j x - k + 7) & k \in [4, 2^j - 4] \\ \sum_{i=2^j+1}^{2^j+6} T_{1,2^j-k}(\frac{i}{2^j}) \phi(2^j x - i + 7) + \phi(2^j x - k + 7) & k \in [2^j - 3, 2^j] \end{cases} . \quad (2)$$

Here $\phi(x)$ is the generalized Coiflet-type orthogonal scaling function with first order moment $M_1 = 7$ and number of vanishing moment $\beta = 6$ of the corresponding wavelet function, which is developed by Wang [Wang (2001)]. And in Eq. (2), expressions

$$T_{0,k}(x) = \sum_{i=0}^{3} \frac{p_{0,i,k}}{i!} x^i, \quad T_{1,k} = \sum_{i=0}^{3} \frac{p_{1,i,k}}{i!} (x-1)^i \quad (3)$$

where coefficients $p_{0,i,k}$ and $p_{1,i,k}$ of numerical differentiation are determined by relationships $\mathbf{P_0} = \{2^{-ij} p_{0,i,k}\}$, $\mathbf{P}_1 = \{2^{-ij} p_{1,i,k}\}$, and matrixes [Liu, Zhou, Wang, and Wang (2013); Wang (2014)]

$$\mathbf{P_0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -11/6 & 3 & -3/2 & 1/3 \\ 2 & -5 & 4 & -1 \\ -1 & 3 & -3 & 1 \end{bmatrix}, \quad \mathbf{P}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 11/6 & -3 & 3/2 & -1/3 \\ 2 & -5 & 4 & -1 \\ 1 & -3 & 3 & -1 \end{bmatrix}. \quad (4)$$

Following the error analysis for this wavelet approximation in one- and two-dimensional problems accomplished by Liu et al. [Liu, Zhou, Zhang, and Wang (2014); Liu, Wang, and Zhou (2013)] and the theory of multiresolution analysis [Meyer (1992)], we can similarly obtain

$$\left\| \frac{\partial^{n+m+l} f(x)}{\partial x^n \partial y^m \partial z^l} - \frac{\partial^{n+m+l} P^{j_x,j_y,j_z} f(x)}{\partial x^n \partial y^m \partial z^l} \right\|_{L^2[0,1]^3} \leq C 2^{-\gamma} \tag{5}$$

in which exponent $\gamma = \min\{j_x(\beta - n), j_y(\beta - m), j_z(\beta - l)\}$, constants C depends on the smoothness and boundary extension property of $f(x, y, z)$, and n, m, l are non-negative integers satisfying $n, m, l < \beta = 6$.

3 Solution of the three-dimensional poisson equation

In this section, we will propose a modified Galerkin method based on the wavelet approximation (2) to solve the three-dimensional Poisson equation in Cartesian coordinates with the Robin type boundary conditions, as follows:

$$\frac{\partial^2 p}{\partial x_1^2} + \frac{\partial^2 p}{\partial x_2^2} + \frac{\partial^2 p}{\partial x_3^2} = f(x_1, x_2, x_3), \;\; x_1, x_2, x_3 \in [0,1] \tag{6a}$$

$$a_i p(x_1, x_2, x_3) + b_i \frac{\partial p}{\partial x_i} = g_i, x_i = 0, i = 1,2,3 \tag{6b}$$

$$c_i p(x_1, x_2, x_3) + d_i \frac{\partial p}{\partial x_i} = h_i, x_i = 1, i = 1,2,3 \tag{6c}$$

in which a_i, b_i, c_i, d_i, $i = 1,2,3$ are constants, and g_i, h_i, f are known functions.

Following the wavelet approximation (1), the unknown function $p(x_1, x_2, x_3)$ and the source function $f(x_1, x_2, x_3)$ can be approximated respectively as

$$p(x_1, x_2, x_3) \approx \sum_{k=0}^{2^{j_1}} \sum_{l=0}^{2^{j_2}} \sum_{n=0}^{2^{j_3}} p(\frac{k}{2^{j_1}}, \frac{l}{2^{j_2}}, \frac{n}{2^{j_3}}) \varphi_{j_1,k}(x_1) \varphi_{j_2,l}(x_2) \varphi_{j_3,n}(x_3) \tag{7}$$

$$f(x_1, x_2, x_3) \approx \sum_{k=0}^{2^{j_1}} \sum_{l=0}^{2^{j_2}} \sum_{n=0}^{2^{j_3}} f(\frac{k}{2^{j_1}}, \frac{l}{2^{j_2}}, \frac{n}{2^{j_3}}) \varphi_{j_1,k}(x_1) \varphi_{j_2,l}(x_2) \varphi_{j_3,n}(x_3). \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (6a), yields

$$\sum_{k=0}^{2^{j_1}} \sum_{l=0}^{2^{j_2}} \sum_{n=0}^{2^{j_3}} p(\frac{k}{2^{j_1}}, \frac{l}{2^{j_2}}, \frac{n}{2^{j_3}}) [\frac{d^2 \varphi_{j_1,k}(x_1)}{dx_1^2} \varphi_{j_2,l}(x_2) \varphi_{j_3,n}(x_3)$$

$$+ \varphi_{j_1,k}(x_1) \frac{d^2 \varphi_{j_2,l}(x_2)}{dx_2^2} \varphi_{j_3,n}(x_3) + \varphi_{j_1,k}(x_1) \varphi_{j_2,l}(x_2) \frac{d^2 \varphi_{j_3,n}(x_3)}{dx_3^2}]. \tag{9}$$

$$\approx \sum_{k=0}^{2^{j_1}} \sum_{l=0}^{2^{j_2}} \sum_{n=0}^{2^{j_3}} f(\frac{k}{2^{j_1}}, \frac{l}{2^{j_2}}, \frac{n}{2^{j_3}}) \varphi_{j_1,k}(x_1) \varphi_{j_2,l}(x_2) \varphi_{j_3,n}(x_3)$$

Multiplying both sides of Eq. (9) by $\varphi_{j_1,k_0}(x_1)\varphi_{j_2,l_0}(x_2)\varphi_{j_3,n_0}(x_3)$, $k_0 = 1,2,\ldots,2^{j_1}-1$, $l_0 = 1,2,\ldots,2^{j_2}-1$, $n_0 = 1,2,\ldots,2^{j_3}-1$, respectively and perform integration over the region $[0,\ 1]^3$, gives

$$\mathbf{AP} \approx \mathbf{BF} \tag{10}$$

in which matrixes $\mathbf{A} = \{a_{o_0q} = \Gamma^{j_1,2}_{k,k_0}\Gamma^{j_2,0}_{l,l_0}\Gamma^{j_3,0}_{n,n_0} + \Gamma^{j_1,0}_{k,k_0}\Gamma^{j_2,2}_{l,l_0}\Gamma^{j_3,0}_{n,n_0} + \Gamma^{j_1,0}_{k,k_0}\Gamma^{j_2,0}_{l,l_0}\Gamma^{j_3,2}_{n,n_0}\}$, $\mathbf{B} = \{b_{o_0q} = \Gamma^{j_1,0}_{k,k_0}\Gamma^{j_2,0}_{l,l_0}\Gamma^{j_3,0}_{n,n_0}\}$, and vectors $\mathbf{P} = \{p_q = p(k/2^{j_1}, l/2^{j_2}, n/2^{j_3})\}^{\mathrm{T}}$, $\mathbf{F} = \{f_q = f(k/2^{j_1}, l/2^{j_2}, n/2^{j_3})\}^{\mathrm{T}}$, where the subscripts $o_0 = (2^{j_1}-1)(2^{j_2}-1)(n_0-1) + (2^{j_1}-1)(l_0-1) + k_0 - 1$, $q = (2^{j_1}+1)(2^{j_2}+1)n + (2^{j_1}+1)l + k$, and $k = 0,1,\ldots,2^{j_1}$, $l = 0,1,\ldots,2^{j_2}$, $n = 0,1,\ldots,2^{j_3}$,$k_0 = 1,2,\ldots,2^{j_1}-1$, $l_0 = 1,2,\ldots,2^{j_2}-1$, $n_0 = 1,2,\ldots,2^{j_3}-1$. Here, the generalized connection coefficients $\Gamma^{j,n}_{k,l} = \int_0^1 d^n \varphi_{j,k}(x)/dx^n \varphi_{j,l}(x)dx$ can be obtained exactly by using the procedure suggested by Wang [Wang (2001)], and the expression of the modified scaling basis have been given by Eq. (2).

On the other hand, substituting Eq. (7) into boundary conditions (6b), we have

$$a_i p(x_1,x_2,x_3) + b_i \sum_{k=0}^{2^{j_1}}\sum_{l=0}^{2^{j_2}}\sum_{n=0}^{2^{j_3}} p\left(\frac{k}{2^{j_1}}, \frac{l}{2^{j_2}}, \frac{n}{2^{j_3}}\right)\frac{\partial \varphi_{j_1,k}(x_1)\varphi_{j_2,l}(x_2)\varphi_{j_3,n}(x_3)}{\partial x_i} \approx g_i, x_i = 0, i = 1,2,3. \tag{11}$$

Assigning $x_2 = 0, 1/2^{j_2}, \ldots 1$, $x_3 = 0, 1/2^{j_3}, \ldots 1$ for $i = 1$, $x_1 = 1/2^{j_1}, 2/2^{j_1}, \ldots 1 - 1/2^{j_1}$, $x_3 = 0, 1/2^{j_3}, \ldots 1$ for $i = 2$, and $x_1 = 1/2^{j_1}, 2/2^{j_1}, \ldots 1 - 1/2^{j_1}$, $x_2 = 1/2^{j_2}$, $2/2^{j_2}, \ldots 1 - 1/2^{j_2}$ for $i = 3$, respectively, yields

$$\mathbf{A}_1\mathbf{P} \approx \mathbf{G}_1 \tag{12a}$$

$$\mathbf{A}_2\mathbf{P} \approx \mathbf{G}_2 \tag{12b}$$

$$\mathbf{A}_3\mathbf{P} \approx \mathbf{G}_3 \tag{12c}$$

where matrixes $\mathbf{A}_1 = \{a_{1,o_1q} = a_1\delta_{k0}\delta_{ll_1}\delta_{nn_1} + b_1\Lambda^{j_1,1}_{k,0}\Lambda^{j_2,0}_{l,l_1}\Lambda^{j_3,0}_{n,n_1}\}$, $\mathbf{A}_2 = \{a_{2,o_2q} = a_2\delta_{kk_2}\delta_{l0}\delta_{nn_2} + b_2\Lambda^{j_1,0}_{k,k_2}\Lambda^{j_2,1}_{l,0}\Lambda^{j_3,0}_{n,n_2}\}$, $\mathbf{A}_3 = \{a_{3,o_3q} = a_3\delta_{kk_3}\delta_{ll_3}\delta_{n0} + b_3\Lambda^{j_1,0}_{k,k_3}\Lambda^{j_2,0}_{l,l_3}\Lambda^{j_3,1}_{n,0}\}$, and the vectors $\mathbf{G}_1 = \{g_{1,o_1} = g_1(l_1/2^{j_2}, n_1/2^{j_3})\}^{\mathrm{T}}$, $\mathbf{G}_2 = \{g_{2,o_2} = g_2(k_2/2^{j_2}, n_2/2^{j_3})\}^{\mathrm{T}}$, $\mathbf{G}_3 = \{g_{3,o_3} = g_3(k_3/2^{j_2}, l_3/2^{j_3})\}^{\mathrm{T}}$, in which δ_{kl} is the Kronecker delta function, subscripts $o_1 = (2^{j_3}+1)l_1 + n_1$, $o_2 = (2^{j_1}-1)n_2 + k_2 - 1$, $o_3 = (2^{j_1}-1)(l_3-1) + k_3 - 1$, $k_2, k_3 = 1,2,\ldots,2^{j_1}-1$, $l_1 = 0,1,\ldots,2^{j_2}$, $l_3 = 1,2,\ldots,2^{j_2}-1$,$n_1, n_2 = 0,1,\ldots,2^{j_3}$, and the coefficients $\Lambda^{j,n}_{k,l} = d^n\varphi_{j,k}(x)/dx^n|_{x=l/2^j}$ also can be obtained exactly by using the procedure suggested by Wang [Wang (2001)] and the modified scaling basis (2).

Similarly, based on boundary conditions (6c) one can obtain

$$\mathbf{B}_1\mathbf{P} \approx \mathbf{H}_1 \tag{13a}$$

$$\mathbf{B}_2\mathbf{P} \approx \mathbf{H}_2 \tag{13b}$$

$$\mathbf{B}_3\mathbf{P} \approx \mathbf{H}_3 \tag{13c}$$

where matrixes $\mathbf{B}_1 = \{b_{1,o_1q} = c_1\delta_{k2^{j_1}}\delta_{ll_1}\delta_{nn_1} + d_1\Lambda^{j_1,1}_{k,2^{j_1}}\Lambda^{j_2,0}_{l,l_1}\Lambda^{j_3,0}_{n,n_1}\}$, $\mathbf{B}_2 = \{b_{2,o_2q} = c_2\delta_{kk_2}\delta_{l2^{j_2}}\delta_{nn_2} + d_2\Lambda^{j_1,0}_{k,k_2}\Lambda^{j_2,1}_{l,2^{j_2}}\Lambda^{j_3,0}_{n,n_2}\}$, $\mathbf{B}_3 = \{b_{3,o_3q} = c_3\delta_{kk_3}\delta_{ll_3}\delta_{n2^{j_3}} + d_3\Lambda^{j_1,0}_{k,k_3}\Lambda^{j_2,0}_{l,l_3}\Lambda^{j_3,1}_{n,2^{j_3}}\}$, and the vectors $\mathbf{H}_1 = \{h_{1,o_1} = h_1(l_1/2^{j_2}, n_1/2^{j_3})\}^{\mathrm{T}}$, $\mathbf{H}_2 = \{h_{2,o_2} = h_2(k_2/2^{j_2}, n_2/2^{j_3})\}^{\mathrm{T}}$, $\mathbf{H}_3 = \{h_{3,o_3} = h_3(k_3/2^{j_2}, l_3/2^{j_3})\}^{\mathrm{T}}$.
By solving simultaneously Eqs. (10), (12) and (13), we can obtain the nodal values of unknown function $p(k/2^{j_1}, l/2^{j_2}, n/2^{j_3})$, $k = 0,1,\ldots,2^{j_1}$, $l = 0,1,\ldots,2^{j_2}$, $n = 0,1,\ldots,2^{j_3}$, which can be used to reconstruct $p(x_1, x_2, x_3)$ in terms of Eq. (7). We note that this wavelet solution is valid for the Poisson equation (6a) with almost all the classic types of boundary conditions. For example in Eq. (6b), parameters $b_i = 0$ represent the Dirichlet boundary conditions, $a_i = 0$ represent the Neumann boundary conditions, and arbitrary values of a_i and b_i represent the general Robin boundary conditions. Moreover by using the similar algorithm, we also can obtain the wavelet solutions of the one- and two-dimensional Poisson problems.

4 Numerical examples

In the following, we will demonstrate the efficiency and accuracy of the proposed wavelet method by numerically solving Poisson equations with various boundary conditions. To effectively evaluate the performance of the present method, we consider the maximum absolute error L_∞, mean absolute error L_1, relative error norm L_2 and order of convergence $R_{\infty,1}$, which are, respectively, defined as

$$L_\infty = \max_k\{|u_k^{num} - u_k^{exact}|\} \tag{14}$$

$$L_1 = \sum_k |u_k^{num} - u_k^{exact}|/N \tag{15}$$

$$L_2 = \int_\Omega (u^{mum} - u^{exact})^2 d\Omega \Big/ \int_\Omega (u^{exact})^2 d\Omega \tag{16}$$

$$R_{\infty,1} = \frac{\log[L_{\infty,1}(N_2)/L_{\infty,1}(N_1)]}{\log[N_1^{1/D}/N_2^{1/D}]} \tag{17}$$

in which N and D are the number of grid points and the spatial dimension of the problem, respectively [Barad and Colella (2005); Atluri and Zhu (1998)].

Example 1 We consider the one-dimensional Poisson equation with Dirichlet boundary conditions as follows:

$$\begin{cases} \dfrac{\partial^2 p}{\partial x^2} = 4[(1-2x)^2\pi^2 - 2]\sin(2\pi x) + 16\pi(1-2x)\cos(2\pi x) \\ p(0) = p(1) = 0, x \in [0,1] \end{cases} \tag{18}$$

whose exact solution is $p(x) = -(1-2x)^2\sin(2\pi x)$ [Gibou, Fedkiw, Cheng, and Kang (2002)].

Table 1: Mean absolute error L_1 and order of convergence R_1 for problem (18).

Present			FDM [Gibou (2002)]		
Grid N	L_1-error	Order R_1	Grid N	L_1-error	Order R_1
16	1.351E-04	—	40	4.422E-04	—
32	6.756E-06	4.32	80	1.132E-04	1.97
64	2.986E-07	4.50	160	2.736E-05	2.04

Table 2: Maximum absolute error L_∞ and order of convergence R_∞ for problem (18).

Present			FDM [Gibou (2002)]		
Grid N	L_∞-error	Order R_∞	Grid N	L_∞-error	Order R_∞
16	7.801E-04	—	40	9.236E-04	—
32	6.578E-05	3.57	80	2.654E-04	1.79
64	4.532E-06	3.86	160	7.306E-05	1.86

Tables 1 and 2 show the errors of numerical solutions of the one-dimensional Poisson problem (18) given by the proposed wavelet method with various values of grid points N. It can be seen from Tables 1 and 2 that results obtained using the present wavelet method with less number of grid points N has a much better numerical accuracy than those given by the finite difference method (FDM) [Gibou, Fedkiw, Cheng, and Kang (2002)].

Example 2 Consider the two-dimensional Laplace equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0, \quad x, y \in [0,2] \tag{19}$$

subjected to the Dirichlet boundary conditions which are extracted from the exact solution $p(x,y) = 3x^2y + 3xy^2 - x^3 - y^3$ [Atluri and Zhu (1998); Zhu, Zhang and Atluri (1998)].

Figure 1 shows the relative error norm L_2 of numerical solutions of Eq. (19), which are obtained respectively by using the local boundary integral equation method (LBIE) [Zhu, Zhang, and Atluri (1998)], the meshless Local Petrov-Galerkin method (MLPG) [Atluri and Zhu (1998)], and the present wavelet method. We see from Figure 1 that the present wavelet solution is very accurate and almost independent of the number of grid points N, which is different from those given by LBIE and MLPG whose order of convergence is about 7.5 [Atluri and Zhu (1998); Zhu, Zhang, and Atluri (1998)]. The reason for this phenomenon may be the fact that the wavelet expansion (1) can exactly characterize the theoretical solution of Eq. (19), since Eq. (1) is a completely accurate representation of the polynomial whose order is below the vanishing moment $\beta = 6$ of the wavelet function we use [Meyer (1992); Wang (2001); Liu, Zhou, Zhang, and Wang (2014)]. And the very slight error of the proposed solution shown in Figure 1 may be caused by rounding errors which are not specially handled in this study.

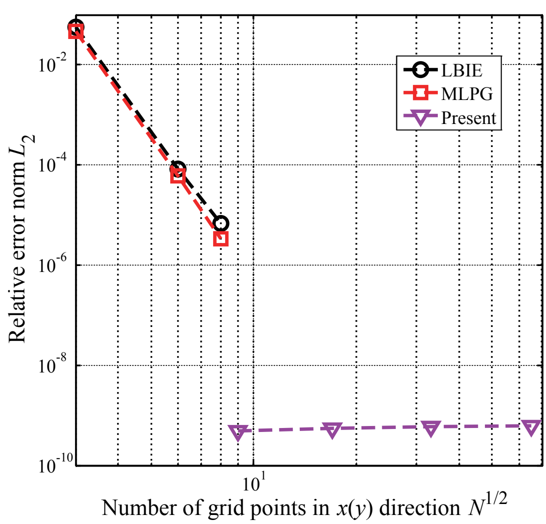

Figure 1: Relative error norm L_2 of numerical solutions of Eq. (19) obtained respectively by using LBIE, MLPG and the present wavelet method.

Example 3 Consider the two-dimensional Poisson equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -8\pi^2 \sin(2\pi x)\sin(2\pi y), \quad x, y \in [0,1] \tag{20}$$

subjected respectively to the Dirichlet boundary conditions [Barad and Colella (2005)]

$$p(0,y) = p(1,y) = p(x,0) = p(x,1) = 0 \tag{21}$$

and the mixed boundary conditions

$$\begin{cases} \frac{\partial p}{\partial x} = -2\pi\sin(2\pi y), x = 0 \\ p + \frac{\partial p}{\partial x} = -2\pi\sin(2\pi y), x = 1 \\ p(x,0) = p(x,1) = 0 \end{cases} . \tag{22}$$

The exact solution of this problem is $p(x,y) = \sin(2\pi x)\sin(2\pi y)$.

Table 3: Error L_1 and order of convergence R_1 for Eq. (20) with boundary conditions (21).

Grid N	Present		TFVD [Barad (2005)]		FFVD [Barad (2005)]	
	L_1-error	Order R_1	L_1-error	Order R_1	L_1-error	Order R_1
16×16	3.795E-05	—	—	—	—	—
32×32	7.566E-07	5.65	—	—	—	—
64×64	1.394E-08	5.76	1.075E-04	—	1.361E-07	—
128×128	2.710E-10	5.68	2.644E-05	2.02	8.490E-09	4.00
256×256	6.920E-12	5.29	6.562E-06	2.01	5.302E-10	4.00
512×512	2.589E-12	1.42	1.635E-06	2.00	3.312E-11	4.00

Table 4: Error L_∞ and order of convergence R_∞ for Eq. (18) with boundary conditions (21).

Grid N	Present		TFVD [Barad (2005)]		FFVD [Barad (2005)]	
	L_∞-error	Order R_∞	L_∞-error	Order R_∞	L_∞-error	Order R_∞
16×16	1.959E-04	—	—	—	—	—
32×32	6.868E-06	4.83	—	—	—	—
64×64	2.217E-07	4.95	2.306E-04	—	3.182E-07	—
128×128	7.002E-09	4.98	5.457E-05	2.08	1.970E-08	4.01
256×256	2.199E-10	4.99	1.330E-05	2.04	1.228E-09	4.00
512×512	7.396E-12	4.89	3.286E-06	2.02	7.652E-11	4.00

Tables 3 and 4 respectively show the comparisons of the mean absolute error L_1 and the maximum absolute error L_∞ between solutions obtained respectively by using different numerical methods for the two-dimensional Poisson equation (20) with the Dirichlet boundary conditions (21). It can be seen from Tables 3 and 4 that the present wavelet solutions are more accurate than those given by both of the two-order finite volume method (TFVD) and the four-order finite volume method (TFVD) [Barad and Colella (2005)]. And from Tables 3 and 4, we also can find that the order of convergence $R_{\infty,1}$ of the proposed wavelet method is about 5, which obviously exceeds the order of convergence of the finite volume method [Barad and Colella (2005)]. In Figure 2, we show the relation between the errors and the number of grid points N for the two-dimensional Poisson equation (20) with the mixed boundary conditions (22). It can been seen from Figure 1 that the present wavelet method for solving Poisson problems with complex boundary conditions is also very accurate and efficient, in which the order of convergence $R_{\infty,1} \approx 4$ and the mean absolute error $L_1 \approx 2.27 \times 10^{-7}$ for the number of grid points $N = 128 \times 128$.

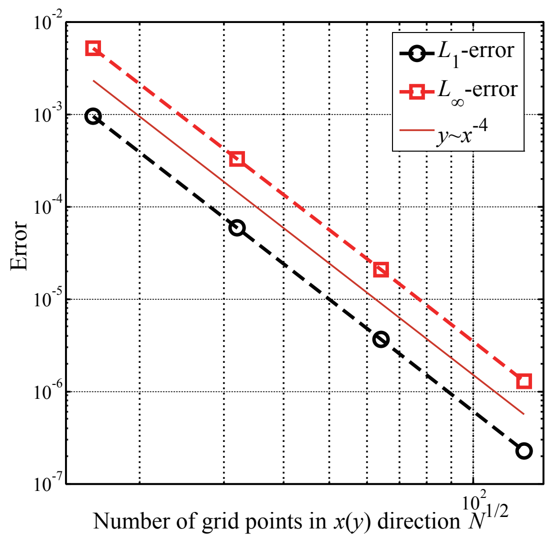

Figure 2: Mean absolute error L_1 and maximum absolute error L_∞ of wavelet solutions of Eq. (20) with mixed boundary conditions (22) as a function of the number of grid points in $x(y)$ direction $N^{1/2}$.

Example 4 Consider the three-dimensional Poisson equation with Dirichlet boundary conditions

$$\begin{cases} \dfrac{\partial^2 p}{\partial x^2} + \dfrac{\partial^2 p}{\partial y^2} + \dfrac{\partial^2 p}{\partial z^2} = -3\pi^2 \sin(\pi x)\sin(\pi y)\sin(\pi z) \\ p(x,y,z) = 0, \quad x,y,z = 0,1, \quad x,y,z \in [0,1] \end{cases} \tag{23}$$

which has the exact solution $p(x,y,z) = \sin(\pi x)\sin(\pi y)\sin(\pi z)$ [Zhang (1998)].

Table 5: Maximum absolute error L_∞ and order of convergence R_∞ for problem (23).

Grid N	Present		FOS [Zhang (1998)]		CDS [Zhang (1998)]		HWM [Shi (2012)]	
	L_∞-error	Order R_∞	L_∞-error	Order R_∞	L_∞-error	Order R_∞	L_∞-error	Order R_∞
8×8×8	1.89E-04	—	2.35E-04	—	1.30E-02	—	1.35E-04	—
16×16×16	6.71E-06	4.82	1.43E-05	4.04	3.22E-03	2.01	3.55E-05	1.92
32×32×32	2.20E-07	4.93	9.04E-07	3.98	8.04E-04	2.00	—	—

Table 5 lists the maximum absolute error L_∞ and order of convergence R_∞ of the numerical solutions for the three-dimensional Poisson problem (21) obtained respectively by using the proposed wavelet method and other existing methods. The results listed in Table 5 show clearly that the present wavelet solutions are more accurate than those given respectively by the fourth-order compact scheme (FOS) [Zhang (1998)], the central difference scheme (CDS) [Zhang (1998)], and the Haar wavelet method (HWM) [Shi, Cao, and Chen (2012)].

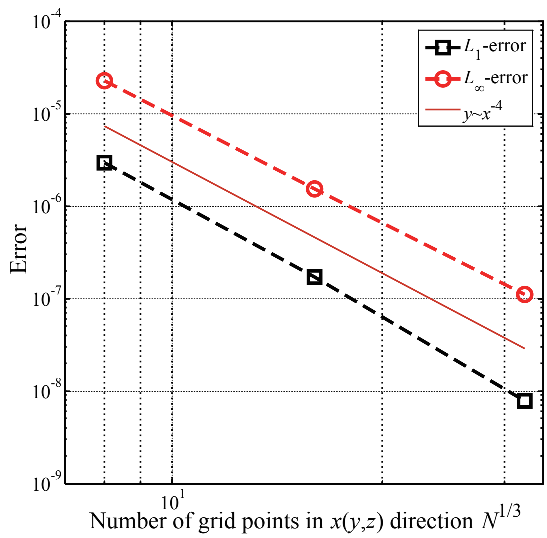

Figure 3: Mean absolute error L_1 and maximum absolute error L_∞ of wavelet solutions as a function of the number of grid points in $x(y, z)$ direction $N^{1/3}$.

Example 5 Consider the three-dimensional Poisson equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = e^x + e^y + e^z, x, y, z \in [0, 1] \tag{24}$$

subjected to the Dirichlet boundary conditions extracted from the exact solution $p(x, y, z) = e^x + e^y + e^z$.

Figure 3 shows the mean absolute error L_1 and maximum absolute error L_∞ of the proposed wavelet solutions as a function of the number of grid points N. From Figure 3, we can find out that the present wavelet method has a good accuracy and efficiency for solving the three-dimensional Poisson problem (24), where the order of convergence $R_{\infty,1} \approx 4$ and the mean absolute error L_1 can reach 7.830×10^{-9} when the number of grid points $N = 32 \times 32 \times 32$.

5 Conclusion

In this paper, an approximation scheme for a L^2-function defined on a three-dimensional bounded space by combining techniques of boundary extension and Coiflet-type wavelet expansion is introduced. Based on such approximation scheme, we proposed a modified wavelet Galerkin method for the solution of Poisson equations with various boundary conditions. By numerically solving the one-, two- and three-dimensional Poisson problems, results demonstrate that the proposed wavelet has a much better accuracy and convergence rate than many methods developed so far, and has a good capability in dealing with mixed boundary conditions.

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