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Kernel-Based Local Meshless Method for Solving Multi-Dimensional Wave Equations in Irregular Domain

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Abstract: This work explores the application of kernel based local meshless method for solving multi-dimensional wave equations in irregular domain. The method is tested for various types of boundary conditions in irregular shaped domain. The method is capable of solving multi-dimension large scaled problems in complex shaped domain.

Keywords: Kernel functions, local meshless scheme, wave equation, complex shaped domain.

1 Introduction

The construction of robust and efficient numerical methods for solving large-scale wave problems in computational physics is an important and challenging research topic. The wave equation govern many physical problems such as the stress wave in an elastic solid, water wave propagation in water bodies, and sound wave propagation in a medium. The kernel-based numerical methods introduced in Hardy (1971) for scattered data fitting problems. It was shown in Micchelli (1984) that the system matrix for the MQ kernel is invertible. The kernels (RBFs) was first used for solving partial differential equations in the year 1990 [Kansa (1990a,b)]. In this original work the fluid mechanics problems were solved by approximating the derivatives by the derivative of MQ kernel functions directly. The convergence theory of kernel based approach was proved in Schaback (2007). During the past three decades a large number of robust kernel-based approximation methods have been developed [Atluri and Zhu (1998); Buhmann (2003); Wendland (2004); Fasshauer and Zhang (2007); Voller, Vertnik, and Šarler (2006); Sarra and Kansa (2009); Beatson, Davydov, and Levesley (2010); Georgoulis, Levesley, and Subhan (2013); Levesley and Sun (2005); Chen, Fu, and Chen (2014)], These methods have effectively used for solving many engineering and science problems [Elgohary, Dong, Junkins,

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and Atluri (2014); Dong, Alotaibi, Mohiuddine, and Atluri (2014); Sladek, Stanak, Han, Sladek, and Atluri (2013); Shibahara and Atluri (2011); Šarler (2005); Yao, Šarler, and Chen (2011); Yao, Sarler, et al. (2010); Fu, Chen, and Gu (2014); Lin, Chen, Chen, and Jiang (2013); Haq, Uddin, and Islam (2009); Uddin (2013); Uddin and Haq (2013)]. The main advantage of using the kernels (RBFs) for the solution of PDEs is its simplicity, applicability to various PDEs, and effectiveness in dealing with multi-dimensional problems and complicated domains. In most global kernel-based methods the differentiation matrices are unsymmetric as well as dense. Due to the high resolution for large amount of data points it becomes difficult to solve the problem with global kernel based method. Many robust numerical approximation methods have been developed to overcome this difficulty some of them are the transforms based methods and the multipole approaches [Greengard and Strain (1991); Cherrie, Beatson, and Newsam (2002); Gumerov and Duraiswami (2007)], the domain decomposition methods [Beatson, Light, and Billings (2001); Kansa and Hon (2000); Li and Hon (2004)], the partition of unity methods [Wendland (2002)], the greedy algorithms [Schaback and Wendland (2000); Hon, Schaback, and Zhou (2003); Ling and Schaback (2008)], the multilevel methods [Fasshauer (1999); Fasshauer and McCourt (2012)], and the use of locally supported kernel functions [Wendland (1995); Floater and Iske (1996)]. An other alternative approach to overcome this difficulty was developed by Tolstykh [Tolstykh (2000)], here local kernel interpolants in small domains centered around each node is used to form differentiation weights. This idea has been used to construct various types of local kernel based approximate methods and has been applied successfully to a wide range of problems. These include convection diffusion [Chandhini and Sanyasiraju (2007); Stevens, Power, Lees, and Morvan (2009); Šarler and Vertnik (2006); Sarra (2012)], incompressible Navier-Stokes [Chinchapatnam, Djidjeli, Nair, and Tan (2009); Shan, Shu, and Lu (2008); Shu, Ding, and Yeo (2003)], elliptic equations [Tolstykh and Shirobokov (2003); Wright and Fornberg (2006)] and [Wong, Hon, Li, Chung, and Kansa (1999); Xiao and McCarthy (2003); Brown, Ling, Kansa, and Levesley (2005)]. In the present work we used the same idea to construct local kernel based numerical scheme for solving multi-dimensional wave equations in irregular domain.

2 Local meshless numerical scheme for time dependent PDEs

In this section, we describe a local RBF approximation method for

$$\frac{\partial u(x,t)}{\partial t} = \mathscr{L}u(x,t), \quad x \in \Omega \subset \mathbb{R}^d, \quad t > 0. \tag{1}$$

2.1 Spatial approximation

In multivariate scattered data interpolation, we always need to recover an unknown function $u : \mathbb{R}^d \to \mathbb{R}$ from a given set of N function values $\{u(x_1), \ldots, u(x_N)\} \subset \mathbb{R}$. Where the scattered centers $x_1, \ldots, x_N \in \Omega$ and $\Omega \subset \mathbb{R}^d$ is arbitrary shaped domain and the centers can be chosen any where in the domain. In the local RBF approximation method, at each center $x_i \in \Omega$, the local interpolant takes the form

$$u(x_i,t) = \sum_{x_j \in \Omega_i} \lambda_j(t)\phi(\|x_i - x_j\|), \tag{2}$$

where $\lambda^i = [\lambda_1, \ldots, \lambda_n]$ is a vector of expansion coefficients, $\|x_i - x_j\|$, denotes the Euclidean distance between two centers x_i and x_j, $\phi(r)$ is a kernel function defined for $r \geq 0$ and $\Omega_i \subset \Omega$ is a local domain corresponding to each node x_i and contains n nearest centers around the center x_i. The N number of $n \times n$ linear systems in matrix form are given by

$$\mathbf{u}^i = \mathbf{A}^i \lambda^i, \quad i = 1, 2, \ldots, N, \tag{3}$$

where the entries of the matrix $\mathbf{A}^i$ are $a^i_{kj} = \phi(\|x_k - x_j\|), x_k, x_j \in \Omega_i$, the matrix $\mathbf{A}^i$ is called the interpolation matrix, and each system has to be solved for the expansion coefficients. In order to approximate the linear differential operator $\mathscr{L}u(x,t)$, we have

$$\mathscr{L}u(x_i,t) = \sum_{x_j \in \Omega_i} \lambda_j(t)\mathscr{L}\phi(\|x_i - x_j\|). \tag{4}$$

The expression in (4) may be given as a dot product of two vectors,

$$\mathscr{L}u(x_i,t) = \mathbf{v}^i \cdot \lambda^i, \tag{5}$$

where λ^i is the $n \times 1$ vector of expansion coefficients, and $\mathbf{v}^i$ is $1 \times n$ vector with entries

$$\mathbf{v}^i = \mathscr{L}\phi(\|x_i - x_j\|), \quad x_j \in \Omega_i. \tag{6}$$

To eliminate the expansion coefficients, we have from equation (3)

$$\lambda^i = (\mathbf{A}^i)^{-1}\mathbf{u}^i, \tag{7}$$

we substitute the values of λ^i from (7) in (5) to get,

$$\mathscr{L}u(x_i,t) = \mathbf{v}^i(\mathbf{A}^i)^{-1}\mathbf{u}^i = \mathbf{w}^i\mathbf{u}^i, \tag{8}$$

where,

$$\mathbf{w}^i = \mathbf{v}^i(\mathbf{A}^i)^{-1}, \tag{9}$$

is the weight corresponding to center x_i. Hence for all centers locations, we have

$$\mathscr{L}u = \mathbf{W}\mathbf{u}, \tag{10}$$

where $\mathbf{W}$ is $N \times N$ sparse differentiation matrix, each row of the matrix $\mathbf{W}$ contains n non-zeros elements and $N-n$ zeros elements, where n is the number of nodes in each local domain Ω_i.

2.2 Temporal approximation

After spatial local RBF approximation, we obtained the following system of ODEs

$$\frac{\partial u}{\partial t} = F(u). \tag{11}$$

In our case $F(\mathbf{u}) = \mathbf{W}\mathbf{u}$. To discretize in time we can use any ODE solver like ode113, ode45 from Matlab. The starting vector will be the initial solution u_0. ode45 is based on an explicit Runge-Kutta (4,5) formula, the Dormand-Prince pair [Dormand and Prince (1980)]. It is a one-step solver for computing $u(t_n)$, and it needs only the solution at the immediately preceding time point, $u(t_{n-1})$. In general, ode45 is the best function to apply as a first try for most problems. ode113 is a variable order Adams-Bashforth-Moulton PECE solver [Shampine and Gordon (1975)]. It may be more efficient than ode45 for stringent tolerances and when the ODE file function is particularly expensive to evaluate. ode113 is a multistep solver; it normally requires the solution at several preceding time points to compute the current solution. A good ODE solver will automatically select a reasonable time step δt and detect stiffness of the ODE system. For this ODE computation we have used a fourth-order Runge-Kutta method and selected the time step δt manually.

$$\mathbf{r_1} = \mathbf{F}(\mathbf{u^n}), \quad \mathbf{r_2} = \mathbf{F}(\mathbf{u} + \frac{\delta t}{2}\mathbf{r_1}), \quad \mathbf{r_3} = \mathbf{F}(\mathbf{u} + \frac{\delta t}{2}\mathbf{r_2}), \quad \mathbf{r_4} = \mathbf{F}(\mathbf{u} + \delta t\mathbf{r_3}),$$
$$\mathbf{u^{n+1}} = \mathbf{u^n} + \frac{\delta t}{6}(\mathbf{r_1} + 2\mathbf{r_2} + 2\mathbf{r_3} + \mathbf{r_4}). \tag{12}$$

2.3 Stability of the local meshless numerical scheme

In the present local meshless method of lines our numerical scheme is given by

$$\mathbf{u}_t = \mathbf{W}\mathbf{u}, \tag{13}$$

here the time-dependent PDE is transformed into a system of ODEs in time. The method of lines refers to the idea of solving the coupled system of ODEs by a finite difference method in t (e.g Runge-Kutta, etc.). The numerical stability of the method of lines is investigated by a rule of rhumb. *The method of lines is stable if the eigenvalues of the (linearized) spatial discretization operator, scaled by δt, lie in the stability region of the time-discretization operator* [Trefethen (2000)].

The stability region is a part of a complex plane consisting of those eigenvalues for which the technique produce a bounded solution. In the present meshless method of lines our numerical scheme is given in (11). We can investigate the stable and unstable eigenvalue spectrum for the given model by computing the eigenvalues of the matrix $\mathbf{W}$, scaled by δt.

2.4 Choosing a good value of shape parameter

A variety of kernel functions are available in the literature. In our computation we used the multiquadrics, $\phi(r) = \sqrt{1+c^2r^2}$. As usual these RBFs contain a shape parameter c and solution accuracy greatly depends on this parameter. A variety of methods are available for choosing good value of shape parameter [Carlson and Foley (1991); Foley (1994); Rippa (1999); Trahan and Wyatt (2003); Fasshauer (2007); Scheuerer (2011); Uddin (2014)]. A condition number may be used to quantify the sensitivity to perturbations of a linear system, and to estimate the accuracy of a computed solution. The conditioning results require that in order for the system matrix to be well-conditioned that the shape parameter and minimum separation distance be large. Obviously, both situations cannot occur at the same time. This observation has been referred to as the uncertainty principle [Schaback (1995)]. Using this fact, the smallest errors occur when the condition number of the system matrix is approximately kept in the range $10^{13} < \kappa < 10^{15}$. The system matrix is decomposed as $\mathbf{U},\mathbf{S},\mathbf{V} = svd(\mathbf{A}^i)$. Here svd is the singular value decomposition of the interpolation matrix $\mathbf{A}^i$. $\mathbf{U}$, $\mathbf{V}$ are $n \times n$ orthogonal matrices and $\mathbf{S}$ is $n \times n$ diagonal matrix contains the n singular values of $\mathbf{A}^i$, and $\kappa = \max(S)/\min(S)$ is the condition number of the matrix $\mathbf{A}^i$. When an acceptable value of shape parameter is returned by the above algorithm, then the svd is used to compute $(\mathbf{A}^i)^{-1} = (\mathbf{USV^T})^{-1} = VS^{-1}U^T$ [see Trefethen and Bau III (1997)]. Note that for orthogonal matrices the inverse of the matrix is equal to its transpose. Consequently, we can compute the weights $\mathbf{w}^i$ in (9).

3 Numerical experiments

In this section, we apply the local meshless approximation method described in the above section to check the accuracy and validity of the present method for the wave

equations given by.

$$u_{tt}(x,y,t) = \alpha^2 \nabla^2 u(x,y,t),\ x,y \in \Omega, t > 0, \tag{14}$$

which may be transformed into a system of two partial differential equations given by

$$u_t = v,\ v_t = \alpha^2 \nabla^2 u(x,y,t), x,y \in \Omega, t > 0, \tag{15}$$

where α is speed of the wave.

3.1 The 2d membrane vibrating problem

We consider the 2D membrane vibration problem (14), which has an exact solution [Young, Gu, and Fan (2009)]

$$u(x,y,t) = \frac{64}{\pi^6} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^3 n^3} \sin(m\pi x) \sin(n\pi y) \cos(\sqrt{m^2+n^2}\pi\alpha t), \tag{16}$$

To solve the problem, we used the initial conditions $u(x,y,0) = f(x,y)$, and $u_t(x,y,0) = g(x,y,0)$, along with the boundary conditions $u = 0$. The $N = 625$ uniformly distributed centers in Ω, and $5 \leq n \leq 20$ nodes in Ω_i are used for solving the vibrating problem (14). The obtained results in the form of the L_∞ error norm, the condition number κ of the interpolation matrix, the shape parameter c of the given kernel, and the computer time in seconds are shown in Table 1 and Figure 2.

Table 1: Numerical results with different stencil sizes n, corresponding to Problem 3.1: when $N = 625$, $t = 1$, $\delta t = 0.0001$, $\alpha = 1$, domain $[0,1]^2$.

	n	L_∞	κ	c	C.time(s)
RK4(MQ)	5	3.8266e-004	4.8829e+013	0.0200	2.004643
	10	2.9402e-004	5.3690e+013	0.5400	2.057774
	15	4.3191e-005	4.2786e+013	1.0800	2.563979
	20	2.9205e-005	1.5692e+013	1.6200	2.567946
	50	5.8947e-006	1.5551e+013	2.9800	4.839784
	60	6.7504e-006	1.1923e+013	3.2300	6.006623
	70	1.0156e-005	1.5810e+013	3.3400	7.372028

3.2 Wave propagation problem in two-dimensional rectangular domain

We consider the wave propagation problem in a rectangular domain. The initial hump displacement is selected at the center of the domain. The smooth wave front

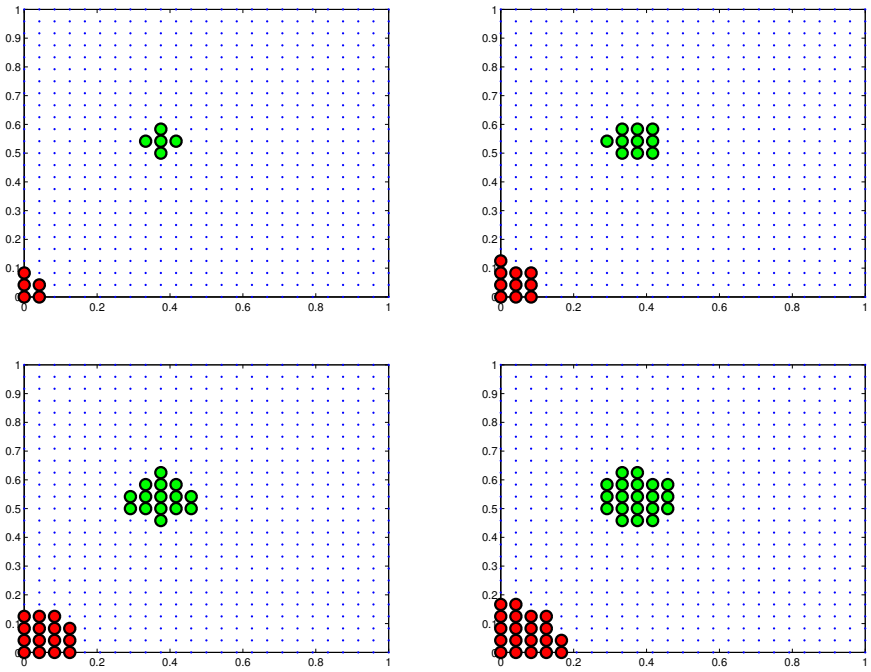

Figure 1: Centers distributions and a typical stencil associated with a boundary center (red) and an interior center (green), corresponding to Problem 3.1

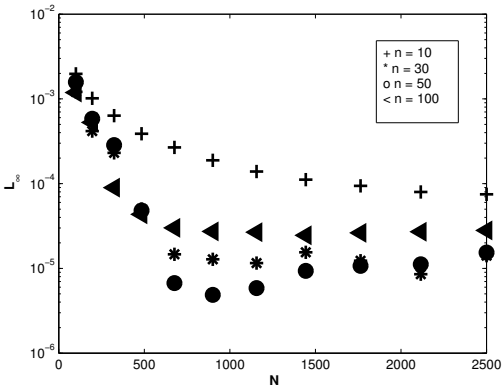

Figure 2: L_∞ error norm versus number of nodes N in the domain $[0,1]^2$ for a given stencil size n.

propagates towards the boundaries with a speed $\alpha = 1.5$. The hump shape of the initial displacement and the zero initial velocities $u(x,y,0) = e^{(-4/25)(x^2+y^2)}$, and

$u_t(x,y,0)=0$, and the boundary condition $u=0$ are used in the computation. The $N=1600$ uniformly distributed centers in domain $\Omega=[-10,10]^2$ and $n=50$ nodes in the local domain Ω_i are used. The approximate solution by the present method in the form of wave propagation are shown in Figure 3.

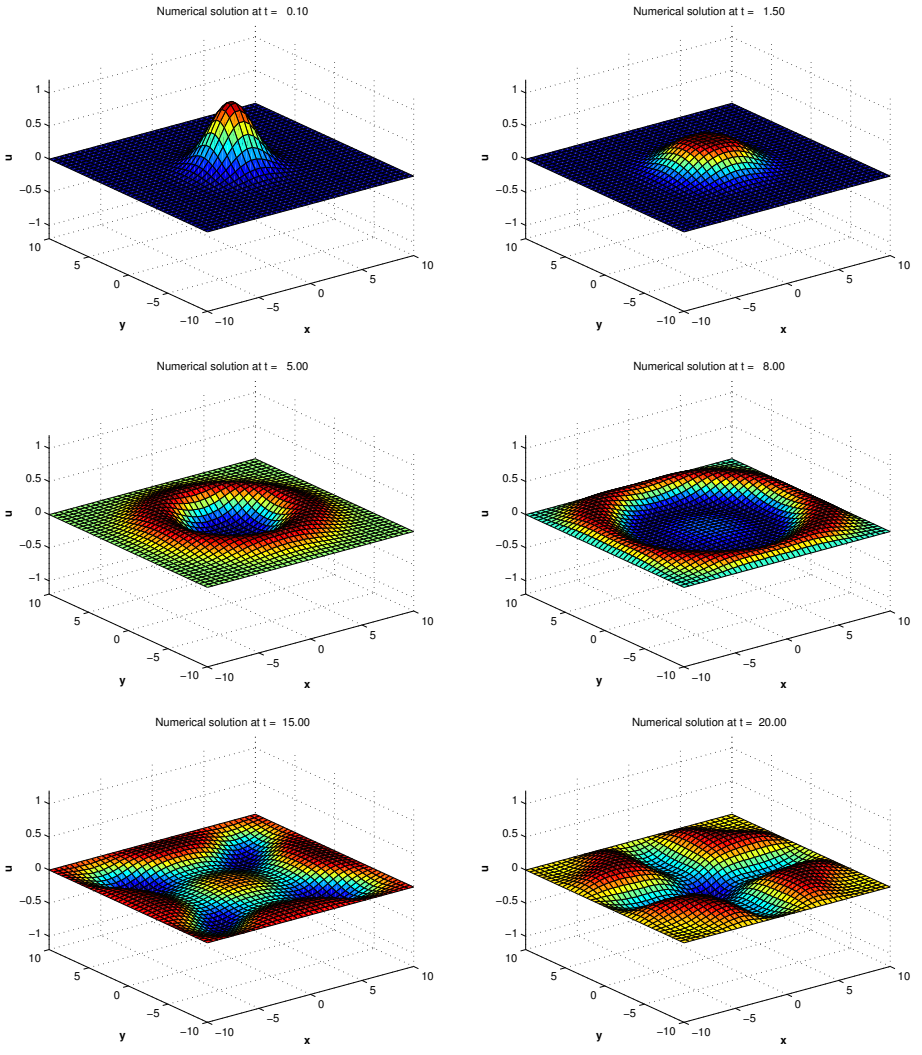

Figure 3: The evolution of wave propagation problem in rectangular domain with zero Dirichlet boundary conditions, corresponding to Problem 3.2.

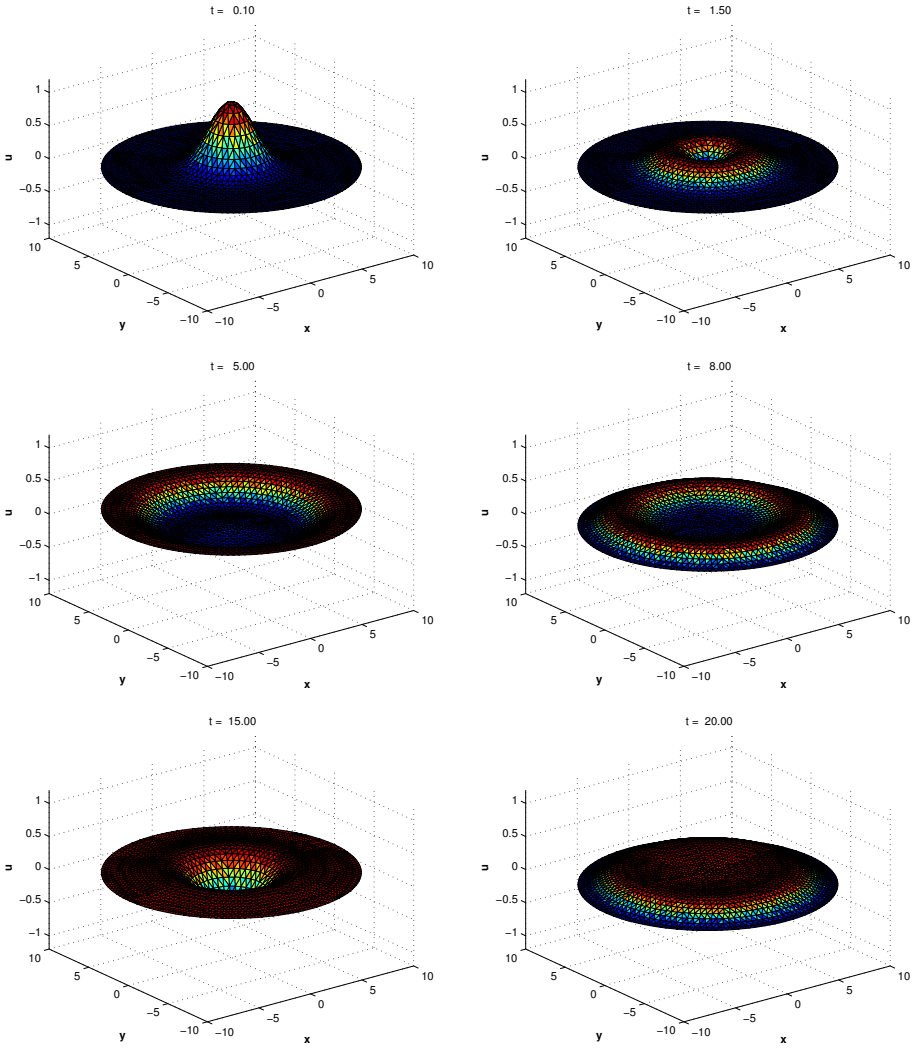

Figure 4: The evolution of wave propagation problem in circular domain with zero Neumann boundary conditions, corresponding to Problem 3.3.

3.3 Wave propagation problem in two-dimensional circular domain

Next we consider the wave propagation problem in a circular domain. The initial hump displacement is selected at the center of the domain. The smooth wave front propagates towards the boundaries with a wave speed of $\alpha = 1.5$. The computational domain is selected as circular of radius 10 units. The the initial displace-

ment and the zero initial velocities are selected as $u(x,y,0) = e^{(-4/25)(x^2+y^2)}$, and $u_t(x,y,0) = 0$, along with the no flux boundary condition $\nabla u.\mathbf{n} = 0$ in the computations, where $\mathbf{n}$ is a unit normal vector to the circular boundary of the circular domain. The solution is again advanced in time by fourth-order Runge-Kutta scheme with time step size $\delta t = 0.001$. We used $N = 1600$ uniformly distributed centers in Ω and $n = 50$ in Ω_i. The obtained results in form of wave propagation are shown in Figure 4.

3.4 Wave propagation problem in two-dimensional L-shaped domain

Here we consider the local meshless scheme for a more complicated wave propagation problem. In this problem we consider the L-shaped domain with non smooth boundary. We choose initial Gaussian hump displacement with a center at the point (-5,5). The computational domain is discretized with $N = 1541$ equally spaced centers and the stencil size $n = 50$ are shown in Fig. 6. The initial displacement and the zero initial velocity are selected as $u(x,y,0) = e^{(-4/25)(x^2+y^2)}$, and $u_t(x,y,0) = 0$ respectively. The boundary condition is selected as $u = 0$ on the boundary $\partial\Omega$. The local meshless scheme accurately simulate the wave propagation in the L-shaped domain with a wave speed of $\alpha = 2$. The simulation of wave propagation in the L-shaped domain is shown Fig. 6 and the results of the present local meshless scheme is well agreed with work of Young, Gu, and Fan (2009).

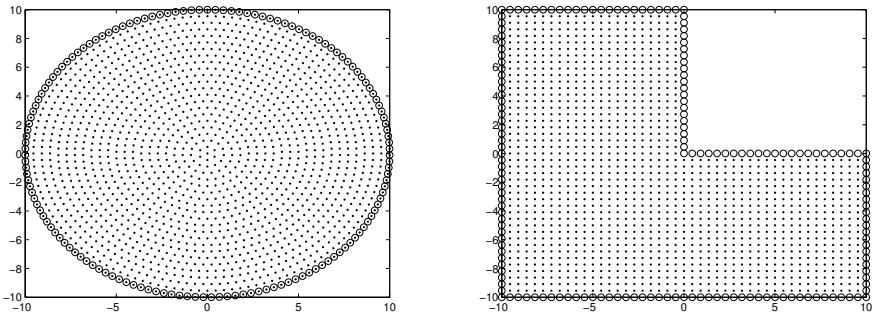

Figure 5: Centers distributions in circular and L-shaped domain, corresponding to Problems 3.3 and 3.4 respectively.

4 Conclusions

In this work, the local kernel-based approximation scheme for solving time-dependent wave equations is applied. Due to the radial kernels the present local scheme has a

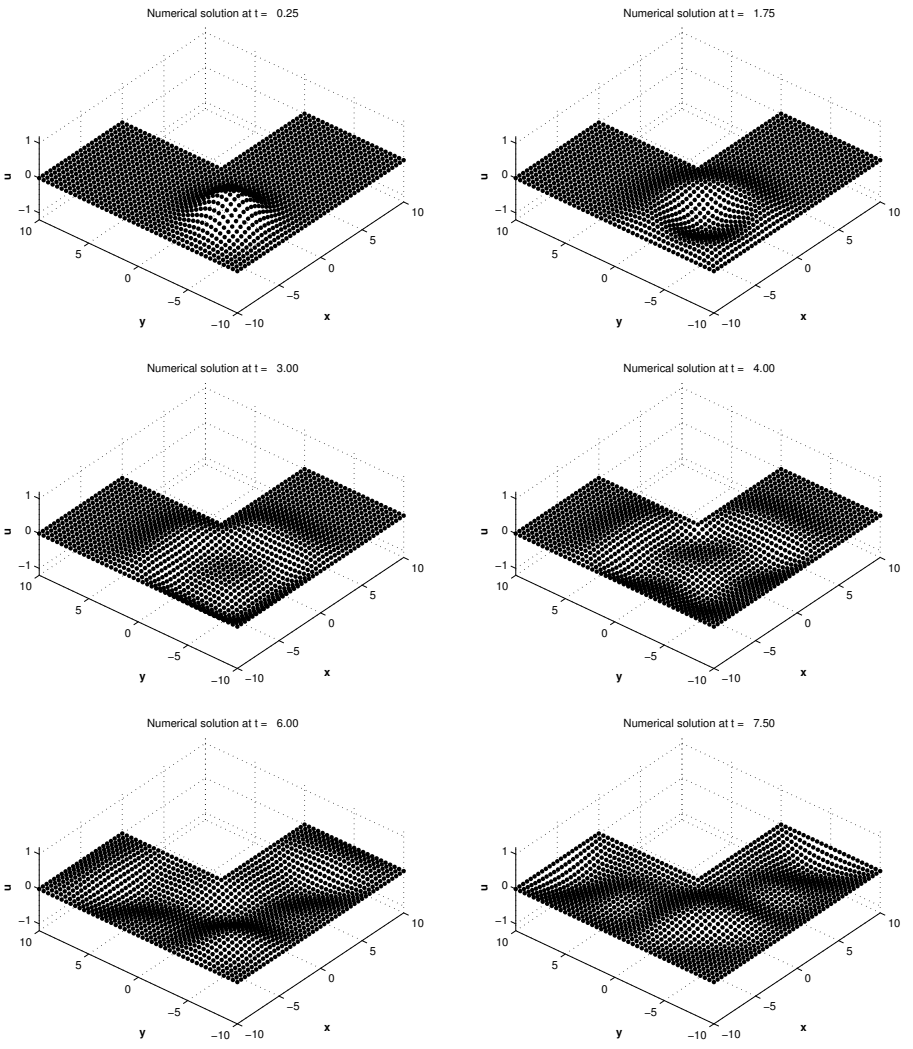

Figure 6: The evolution of wave propagation problem in L-shaped domain with zero Dirichlite boundary conditions, corresponding to Problem 3.4.

great flexibility to solve multi-dimensional problems with arbitrary shaped domain. For large scale problems it is not possible to apply the global radial basis functions method which results a dense differentiation matrices. On the other hand the present local kernel based scheme has the capability of solving large scale problems because of solving small size linear systems in the local domain. The present lo-

cal scheme is equally applicable for large scale time-dependent multi-dimensional problems with irregular shaped domain.

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