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Numerical Solutions of Fractional System of Partial Differential Equations By Haar Wavelets

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Abstract: In this paper, time fractional one dimensional coupled KdV and coupled modified KdV equations are solved numerically by Haar wavelet method. Proposed method is new in the sense that it doesn't use fractional order Haar operational matrices. In the proposed method $L1$ discretization formula is used for time discretization where fractional derivatives are Caputo derivative and spatial discretization is made by Haar wavelets. L_2 and L_∞ error norms for various initial and boundary conditions are used for testing accuracy of the proposed method when exact solutions are known. Numerical results which produced by the proposed method for the problems under consideration confirm the feasibility of Haar wavelet method combined with $L1$ discretization formula.

Keywords: Haar wavelet method, Fractional coupled KdV equation, Fractional coupled MKdV equation, Linearization, Numerical solution.

1 Introduction

In this paper we will consider time fractional coupled KdV (FCKdV) equation

$$\begin{aligned} D_t^{\alpha} u &= 6auu_x + 2bvv_x + au_{xxx}, \quad 0 < \alpha \leq 1 \\ D_t^{\beta} v &= -3uv_x - v_{xxx}, \quad 0 < \beta \leq 1 \end{aligned} \tag{1}$$

where a and b are constants and time fractional coupled modified KdV (FCMKdV) equation

$$\begin{aligned} D_t^{\alpha} u &= \frac{1}{2}u_{xxx} - 3u^2u_x + \frac{3}{2}v_{xx} + 3(uv)_x - 3\lambda u_x, \quad 0 < \alpha \leq 1 \\ D_t^{\beta} v &= -v_{xxx} - 3vv_x - 3u_xv_x + 3u^2v_x + 3\lambda v_x, \quad 0 < \beta \leq 1 \end{aligned} \tag{2}$$

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where

$$D_t^{\alpha} f(t) = \frac{\partial^{\alpha}}{\partial t^{\alpha}} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n f(\tau)}{\partial t^n} d\tau \ \ n-1 < \alpha < n$$

is the fractional derivative in the Caputo's sense [Kilbas, Srivastava, and Trujillo (2006); Podlubny (1999)], here n is an integer. When $\alpha = \beta = 1$, Eq. (1) corresponds to classical coupled KdV equation which was first introduced by Hirota and Satsuma (1981) in 1981 and describes interaction of two long waves with different dispersion relations. When $\alpha = \beta = 1$, Eq. (2) corresponds to integer order coupled modified KdV equation [Fan (2000, 2001, 2002)].

In recent years, fractional differential equations have been broadly used in different branches of physics and engineering [West, Bolognab, and Grigolini (2003)]. For example; nonlinear oscillation of earthquake [He (1999)], diffusion in a certain type of porous medium [Podlubny (1999)], phenomenons occurred in electromagnetics, acoustics electrochemistry and material science [Podlubny (1999); Chechkin, Gorenflo, Sokolov, and Gonchar (2003); Miller and Ross (1993); Samko, Kilbas, and Marichev (1993)], can be well modeled by the aid of fractional derivative. Many researchers focus their attention to fractional differential equation due to suitable applications of fractional differential equations in different fields of engineering and science [Agrawal (2002)]. Even though there are some techniques for getting analytical solutions of the fractional differential equations [Kilbas, Srivastava, and Trujillo (2006); Agrawal (2002)], usually their exact solutions are not known. Therefore various numerical techniques devised by many researchers for obtaining approximate solutions of fractional differential equations. Finite difference method [Sun, Chen, and Li (2012); Meerschaert, Scheffler, and Tadjeran (2006); Yuste (2006); Yuste and Acedo (2005); Quintana-Murillo and Yuste (2011); Sweilam, Khader, and Mahdy (2012); Celik and Duman (2012)], finite element method [Sun, Chen, and Sze (3 Sep 2011); Esen, Ucar, Yagmurlu, and Tasbozan (2013)], second kind Chebyshev wavelets method with differential operator matrix and the product operation matrix (Chen, Sun, Li, and Fu, 2013), Bernstein polynomials operational matrixes method [Chen, Liu, Li, and Sun (2014)], homotopy analysis method [Hashim and Abdulaziz (2009)], generalized differential transform method [Momani and Odibat (2007); Odibat and Momani (2008); L.J.Cun and H.G.Lin (2010)], adomian decomposition method [Hosseini (2006); EI-Kalla (2008); Yong and Hong-Li (2008)] and variational iteration method [Odibat (2010)] are some of these techniques.

Recently, for getting numerical solutions of differential equations, a lot of techniques related to Haar wavelet based methods are developed. We can put in order some of studies specially devoted to fractional differential equations as follows. Fractional Volterra and Fredholm integral equations are solved by Lepik

(2009). Wu (2009), solved some fractional partial differential equations by fractional order Haar wavelet operational matrix. Li and Zhao (2010), solved Bagley-Torvik, Ricatti and composite fractional oscillation equations with Haar wavelet operational matrix of the fractional order. Mujeeb [Rehman and Khan (2013)] et al., employed Haar wavelets for obtaining numerical solutions of boundary value problems for linear fractional partial differential equations. Ray and Patra (2013) solved fractional order nonlinear oscillatory Van der Pol system with Haar wavelet operational matrix. Nonlinear fractional ordinary differential equations are solved by Haar wavelet and a quasilinearization technique by Saeed and Rehman (2013). Wang, Ma, and Meng (2014) solved some fractional partial differential equations by Haar wavelet operational matrix. Neutron point kinetics equation is solved with Haar wavelet operational method by Patra and Ray (2014b,a). Variable coefficient fractional differential equations are solved by M.Yi and Huang (2014) with Haar wavelet operational matrix of fractional order. Most of studies aforementioned use Haar wavelet operational matrix of fractional order for obtaining numerical solutions of fractional differential equations. As an alternative approach, in this study we propose a relatively new and simple technique for obtaining numerical solutions of fractional differential equations which consists of Haar wavelets and $L1$ discretization formula.

In this paper, to get numerical solutions of systems (1) and (2), we have applied Haar wavelet method with $L1$ discretization formula. The paper organized as follows; In Section 2, Haar wavelets are introduced. Time and spatial discretizations are described in Section 3. Numerical results are given in Section 4 and finally the paper concluded in Section 5.

2 Haar wavelets

Haar wavelets were first introduced by Hungarian mathematician Alfred Haar in 1910 which are the simplest of possible wavelets with compact support. They are piecewise constant functions which form an orthonormal system on the interval $[0,1)$ in the space of square integrable functions. Because of their discontinuity they can not be directly used in solution process of differential equations. To alleviate this situation, Chen and Hsiao (1997) used the integral method, in which the highest derivative appeared in the differential equation is expanded into Haar series. In recent years, Haar wavelets are used widely to obtain numerical solutions of differential equations and are favored by researchers because of their simplicity and desirable computational features. The ith Haar wavelet is defined as follows for $x \in [0,1]$

$$h_i(x) = \begin{cases} 1, & \text{for } x \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ -1, & \text{for } x \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right] \\ 0, & \text{elsewhere} \end{cases} \tag{3}$$

where $m = 2^j$, $j = 0,1,\ldots,J$ and $k = 0,1,\ldots,m-1$ is dilation parameter and translation parameter, respectively. The index of h_i in Eq. (3) can be found by relation $i = m+k+1$. For the lowest values of $m = 1$, $k = 0$, we have $i = 2$ and the greatest value of i will be $i = 2M = 2^{J+1}$; where J is the maximum resolution of the wavelet. For $i = 1$ we have Haar scaling function

$$h_1(x) = \begin{cases} 1, & \text{for } x \in [0,1) \\ 0, & \text{elsewhere} \end{cases}.$$

Any function $u(x) \in L^2[0,1)$ can be expanded into Haar series as

$$u(x) = \sum_{i=1}^{\infty} c_i h_i(x),$$

where c_i can be found by

$$c_i = 2^j \int_0^1 u(x)h_i(x)dx, \quad i = 2^j + k,\ j \geq 0,\ 0 \leq k < 2^j.$$

In practice, for approximating a function $u(x) \in L^2[0,1)$, finite terms of Haar series are needed, hence one may write

$$\begin{aligned} u(x) &= \sum_{i=1}^{2M} c_i h_i(x) = c_{(2M)}^T h_{(2M)}(x), \\ c_{(2M)}^T &= [c_1, c_2, \ldots, c_{(2M)}] \\ h_{(2M)}(x) &= [h_1(x), h_2(x), \ldots, h_{(2M)}(x)]^T \end{aligned}$$

where $M = 2^j$ and T denotes transpose.

While using Haar wavelet method for solving any order partial differential equation one needs to following integrals in solution process.

$$\begin{aligned} p_{i,1}(x) &= \int_0^x h_i(x)dx \\ p_{i,n+1}(x) &= \int_0^x p_{i,n}(x)dx, \quad n = 1,2,3,\ldots \end{aligned}$$

The first three integrals can be calculated from Eq. (3) as follows;

$$p_{i,1}(x) = \begin{cases} x-\zeta_1, & \text{for } x \in [\zeta_1, \zeta_2) \\ \zeta_3 - x, & \text{for } x \in [\zeta_2, \zeta_3] \\ 0, & \text{elsewhere} \end{cases} \tag{4}$$

$$p_{i,2}(x) = \begin{cases} \dfrac{(x-\zeta_1)^2}{2}, & \text{for } x \in [\zeta_1, \zeta_2) \\ \dfrac{1}{4m^2} - \dfrac{(\zeta_3 - x)^2}{2}, & \text{for } x \in [\zeta_2, \zeta_3) \\ \dfrac{1}{4m^2}, & \text{for } x \in [\zeta_3, 1] \\ 0, & \text{elsewhere} \end{cases} \tag{5}$$

$$p_{i,3}(x) = \begin{cases} \dfrac{(x-\zeta_1)^3}{6}, & \text{for } x \in [\zeta_1, \zeta_2) \\ \dfrac{x-\zeta_2}{4m^2} - \dfrac{(\zeta_3 - x)^3}{6}, & \text{for } x \in [\zeta_2, \zeta_3) \\ \dfrac{x-\zeta_2}{4m^2}, & \text{for } x \in [\zeta_3, 1] \\ 0, & \text{elsewhere} \end{cases} \tag{6}$$

where ζ_1, ζ_2 and ζ_3 defined as follow.

$$\zeta_1 = \frac{k}{m}, \quad \zeta_2 = \frac{k+0.5}{m}, \quad \zeta_3 = \frac{k+1}{m}.$$

3 Discretization scheme for the equations

3.1 Time discretization for FCKdV

In this section we start with the discretization of the fractional time derivative that appears in Eq. (1) in the Caputo's sense by $L1$ formula [Oldham and Spanier (1974)]

$$\left.\frac{\partial^\gamma u(t)}{\partial t^\gamma}\right|_{t_n} = \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=0}^{n-1} \eta_k^\gamma \left[u(t_{n-k}) - u(t_{n-1-k})\right] + O(\Delta t) \tag{7}$$

where

$$\eta_k^\gamma = (k+1)^{1-\gamma} - k^{1-\gamma}, \qquad 0 < \gamma \leq 1.$$

We also take time averages of the other terms of Eq. (1), as follows

$$\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}(u_{n+1}-u_n)+\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{n-1}\eta_k^{\alpha}[u_{n-k+1}-u_{n-k}]$$
$$=\begin{cases}\frac{6a}{2}[(uu_x)_{n+1}+(uu_x)_n]\\+\frac{2b}{2}[(vv_x)_{n+1}+(vv_x)_n]\\+\frac{a}{2}[(u_{xxx})_{n+1}+(u_{xxx})_n]\end{cases}$$
$$\frac{(\Delta t)^{-\beta}}{\Gamma(2-\beta)}(v_{n+1}-v_n)+\frac{(\Delta t)^{-\beta}}{\Gamma(2-\beta)}\sum_{k=1}^{n-1}\eta_k^{\beta}[v_{n-k+1}-v_{n-k}]$$
$$=\begin{cases}-\frac{3}{2}[(uv_x)_{n+1}+(uv_x)_n]\\-\frac{1}{2}[(v_{xxx})_{n+1}+(v_{xxx})_n]\end{cases}$$

We apply Rubin Graves linearization [Rubin and Graves (1975)] formula $u_{n+1}(u_x)_n$ $+u_n(u_x)_{n+1}-(uu_x)_n$ to nonlinear term $(uu_x)_{n+1}$. Similar linearization is made for $(vv_x)_{n+1}$ and $(uv_x)_{n+1}$ terms. Hence we get

$$G_1u_{n+1}-3a[u_{n+1}(u_x)_n+u_n(u_x)_{n+1}]-b[v_{n+1}(v_x)_n+v_n(v_x)_{n+1}]-\frac{a}{2}(u_{xxx})_{n+1}=$$
$$G_1u_n-G_1\sum_{k=1}^{n-1}\eta_k^{\alpha}[u_{n-k+1}-u_{n-k}]+\frac{a}{2}(u_{xxx})_n$$
$$G_2v_{n+1}+\frac{3}{2}[u_{n+1}(v_x)_n+u_n(v_x)_{n+1}]+\frac{1}{2}(v_{xxx})_{n+1}=$$
$$G_2v_n-G_2\sum_{k=1}^{n-1}\eta_k^{\beta}[v_{n-k+1}-v_{n-k}]-\frac{1}{2}(v_{xxx})_n \tag{8}$$

with initial conditions

$$u_0=f(x),\quad v_0=g(x),\quad x\in[a,b]$$

and boundary conditions

$$u_{n+1}(0)=f_1(t_{n+1}),\quad u_{n+1}(1)=f_2(t_{n+1}),\quad (u_x)_{n+1}(1)=f_3(t_{n+1}),\quad n=0,1,\dots,N-1$$
$$v_{n+1}(0)=g_1(t_{n+1}),\quad v_{n+1}(1)=g_2(t_{n+1}),\quad (v_x)_{n+1}(1)=g_3(t_{n+1}),\quad n=0,1,\dots,N-1 \tag{9}$$

where u_{n+1} and v_{n+1} are the solutions of the Eq.(8) at the $(n+1)$th time step, Δt denotes the step size in time so that $t_n = n \times \Delta t$ and

$$G_1 = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}, \quad G_2 = \frac{(\Delta t)^{-\beta}}{\Gamma(2-\beta)}.$$

3.2 Time discretization for FCMKdV

Again the Caputo derivative appeared in Eq. (2) is discretized by $L1$ formula as made in subsection and time averages used for some of the terms of Eq. (2) as follows

$$\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}(u_{n+1}-u_n) + \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{n-1}\eta_k^{\alpha}\left[u_{n-k+1}-u_{n-k}\right]$$
$$= \begin{cases} \dfrac{\left[(u_{xxx})_{n+1}+(u_{xxx})_n\right]}{2.2} - \dfrac{3\left[\left(u^2u_x\right)_{n+1}+\left(u^2u_x\right)_n\right]}{2} \\ +\dfrac{3\left[(v_{xx})_{n+1}+(v_{xx})_n\right]}{2.2} + 3\left[(uv)_x\right]_n - 3\lambda\,(u_x)_n \end{cases}$$
$$\frac{(\Delta t)^{-\beta}}{\Gamma(2-\beta)}(v_{n+1}-v_n) + \frac{(\Delta t)^{-\beta}}{\Gamma(2-\beta)}\sum_{k=1}^{n-1}\eta_k^{\beta}\left[v_{n-k+1}-v_{n-k}\right]$$
$$= \begin{cases} \dfrac{-\left[(v_{xxx})_{n+1}+(v_{xxx})_n\right]}{2} - \dfrac{3\left[(vv_x)_{n+1}+(vv_x)_n\right]}{2} \\ -3\,(u_xv_x)_n + 3\,(u_n)^2\,(v_x)_n + 3\lambda\,(v_x)_n \end{cases}$$

In the above equation by using linearizations $\left(u^2u_x\right)_{n+1} = 2u_{n+1}u_n(u_x)_n + u_nu_n(u_x)_{n+1}$ $-2u_nu_n(u_x)_n$ and $(vv_x)_{n+1} = v_{n+1}\,(v_x)_n + v_n\,(v_x)_{n+1} - (vv_x)_n$ we get

$$G_1u_{n+1} - \frac{(u_{xxx})_{n+1}}{2.2} - \frac{3\,(v_{xx})_{n+1}}{2.2} + \frac{3\left[2u_{n+1}u_n(u_x)_n + u_nu_n(u_x)_{n+1} - u_nu_n(u_x)_n\right]}{2}$$
$$= G_1u_n - G_1\sum_{k=1}^{n-1}\eta_k^{\alpha}\left[u_{n-k+1}-u_{n-k}\right] + \frac{(u_{xxx})_n}{2.2} + \frac{3\,(v_{xx})_n}{2.2} + 3\left[(uv)_x\right]_n - 3\lambda\,(u_x)_n$$
$$G_2v_{n+1} + \frac{(v_{xxx})_{n+1}}{2} + \frac{3\left[v_{n+1}\,(v_x)_n + v_n(v_x)_{n+1}\right]}{2}$$
$$= G_2v_n - G_2\sum_{k=1}^{n-1}\eta_k^{\beta}\left[v_{n-k+1}-v_{n-k}\right] - \frac{(v_{xxx})_n}{2} - 3\,(u_xv_x)_n + 3\,(u_n)^2\,(v_x)_n + 3\lambda\,(v_x)_n \tag{10}$$

with initial conditions

$$u_0 = f(x), \quad v_0 = g(x), \quad x \in [a,b]$$

and boundary conditions

$$u_{n+1}(0) = f_1(t_{n+1}),\ u_{n+1}(1) = f_2(t_{n+1}),\ (u_x)_{n+1}(1) = f_3(t_{n+1}),\quad n = 0,1,\ldots,N-1$$
$$v_{n+1}(0) = g_1(t_{n+1}),\ v_{n+1}(1) = g_2(t_{n+1}),\ (v_x)_{n+1}(1) = g_3(t_{n+1}),\quad n = 0,1,\ldots,N-1 \tag{11}$$

where u_{n+1} and $v_{n+1}$are the solutions of the Eq. (10) at the $(n+1)$th time step, Δt denotes the step size in time so that $t_n = n \times \Delta t$ and

$$G_1 = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}, \qquad G_2 = \frac{(\Delta t)^{-\beta}}{\Gamma(2-\beta)}$$
$$\eta_k^\alpha = (k+1)^{1-\alpha} - k^{1-\alpha}, \quad \eta_k^\beta = (k+1)^{1-\beta} - k^{1-\beta} \quad 0 < \alpha,\beta \le 1.$$

3.3 Space discretization by Haar wavelets

In this subsection we describe discretization of spatial derivatives that appeared in Eqs. (1), (2). We start with the highest derivative by Haar wavelets. To do so we assume

$$(u_{xxx})_{n+1}(x) = \sum_{i=1}^{2M} c_i h_i(x). \tag{12}$$

Integrating Eq. (12) with respect to x from 0 to x, we get the following equation

$$(u_{xx})_{n+1}(x) = (u_{xx})_{n+1}(0) + \sum_{i=1}^{2M} c_i p_{i,1}(x). \tag{13}$$

In Eq. (13), $(u_{xx})_{n+1}(0)$ is unknown so to find it, we need to integrate Eq. (13) from 0 to 1. After that, using boundary conditions (9) we get

$$(u_x)_{n+1}(1) - (u_x)_{n+1}(0) = (u_{xx})_{n+1}(0) + \sum_{i=1}^{2M} c_i p_{i,2}(1)$$
$$(u_{xx})_{n+1}(0) = f_3(t_{n+1}) - (u_x)_{n+1}(0) - \sum_{i=1}^{2M} c_i p_{i,2}(1). \tag{14}$$

Substituting (14) into Eq. (13) results in the following equation

$$(u_{xx})_{n+1}(x) = \sum_{i=1}^{2M} c_i p_{i,1}(x) + f_3(t_{n+1}) - (u_x)_{n+1}(0) - \sum_{i=1}^{2M} c_i p_{i,2}(1). \tag{15}$$

Now, if we integrate Eq. (15) from 0 to x we get

$$(u_x)_{n+1}(x) = (u_x)_{n+1}(0) + \sum_{i=1}^{2M} c_i p_{i,2}(x) + x\left(f_3(t_{n+1}) - (u_x)_{n+1}(0)\right) - x\sum_{i=1}^{2M} c_i p_{i,2}(1). \tag{16}$$

In Eqs. (14), (15) and (16),$(u_x)_{n+1}(0)$ term is unknown. So to find $(u_x)_{n+1}(0)$ term we integrate Eq. (16) from 0 to 1 and use boundary conditions (9). Therefore we have

$$(u_x)_{n+1}(0) = 2\left[f_2(t_{n+1}) - f_1(t_{n+1}) - \frac{1}{2}f_3(t_{n+1}) - \sum_{i=1}^{2M} c_i p_{i,3}(1) + \frac{1}{2}\sum_{i=1}^{2M} c_i p_{i,2}(1)\right]$$

Now by plugging the calculated value of $(u_x)_{n+1}(0)$ into Eq. (16) we obtain

$$(u_x)_{n+1}(x) = 2\left[f_2(t_{n+1}) - f_1(t_{n+1}) - \frac{1}{2}f_3(t_{n+1}) - \sum_{i=1}^{2M} c_i p_{i,3}(1) + \frac{1}{2}\sum_{i=1}^{2M} c_i p_{i,2}(1)\right] \\ (1-x) + x\left(f_3(t_{n+1})\right) + \sum_{i=1}^{2M} c_i p_{i,2}(x) - x\sum_{i=1}^{2M} c_i p_{i,2}(1) \tag{17}$$

Finally, integrating (17) from 0 to x, we obtain

$$(u)_{n+1}(x) = 2\left[f_2(t_{n+1}) - f_1(t_{n+1}) - \frac{1}{2}f_3(t_{n+1}) - \sum_{i=1}^{2M} c_i p_{i,3}(1) + \frac{1}{2}\sum_{i=1}^{2M} c_i p_{i,2}(1)\right] \\ \left(x - \frac{x^2}{2}\right) + \frac{x^2}{2}\left(f_3(t_{n+1})\right) + \sum_{i=1}^{2M} c_i p_{i,3}(x) - \frac{x^2}{2}\sum_{i=1}^{2M} c_i p_{i,2}(1) + f_1(t_{n+1}) \tag{18}$$

If we summarize, we have

$$(u_{xxx})_{n+1}(x) = \sum_{i=1}^{2M} c_i h_i(x),$$

$$(u_{xx})_{n+1}(x) = \sum_{i=1}^{2M} c_i p_{i,1}(x) + f_3(t_{n+1}) - \sum_{i=1}^{2M} c_i p_{i,2}(1) \\ -2\left[f_2(t_{n+1}) - f_1(t_{n+1}) - \frac{1}{2}f_3(t_{n+1}) - \sum_{i=1}^{2M} c_i p_{i,3}(1) + \frac{1}{2}\sum_{i=1}^{2M} c_i p_{i,2}(1)\right],$$

$$(u_x)_{n+1}(x) = 2\left[f_2(t_{n+1}) - f_1(t_{n+1}) - \frac{1}{2}f_3(t_{n+1}) - \sum_{i=1}^{2M} c_i p_{i,3}(1) + \frac{1}{2}\sum_{i=1}^{2M} c_i p_{i,2}(1)\right]$$

$$\cdot(1-x)+x\left(f_3(t_{n+1})\right)+\sum_{i=1}^{2M}c_ip_{i,2}(x)-x\sum_{i=1}^{2M}c_ip_{i,2}(1), \tag{19}$$

$$(u)_{n+1}(x)=2\left[f_2(t_{n+1})-f_1(t_{n+1})-\frac{1}{2}f_3(t_{n+1})-\sum_{i=1}^{2M}c_ip_{i,3}(1)+\frac{1}{2}\sum_{i=1}^{2M}c_ip_{i,2}(1)\right]$$

$$\left(x-\frac{x^2}{2}\right)+\frac{x^2}{2}\left(f_3(t_{n+1})\right)+\sum_{i=1}^{2M}c_ip_{i,3}(x)-\frac{x^2}{2}\sum_{i=1}^{2M}c_ip_{i,2}(1)+f_1(t_{n+1}).$$

Similarly, we have

$$(v_{xxx})_{n+1}(x)=\sum_{i=1}^{2M}d_ih_i(x),$$

$$(v_{xx})_{n+1}(x)=\sum_{i=1}^{2M}d_ip_{i,1}(x)+g_3(t_{n+1})-\sum_{i=1}^{2M}d_ip_{i,2}(1)$$

$$-2\left[g_2(t_{n+1})-g_1(t_{n+1})-\frac{1}{2}g_3(t_{n+1})-\sum_{i=1}^{2M}d_ip_{i,3}(1)+\frac{1}{2}\sum_{i=1}^{2M}d_ip_{i,2}(1)\right],$$

$$(v_x)_{n+1}(x)=2\left[g_2(t_{n+1})-g_1(t_{n+1})-\frac{1}{2}g_3(t_{n+1})-\sum_{i=1}^{2M}d_ip_{i,3}(1)+\frac{1}{2}\sum_{i=1}^{2M}d_ip_{i,2}(1)\right]$$

$$\cdot(1-x)+x\left(g_3(t_{n+1})\right)+\sum_{i=1}^{2M}d_ip_{i,2}(x)-x\sum_{i=1}^{2M}d_ip_{i,2}(1), \tag{20}$$

$$(v)_{n+1}(x)=2\left[g_2(t_{n+1})-g_1(t_{n+1})-\frac{1}{2}g_3(t_{n+1})-\sum_{i=1}^{2M}d_ip_{i,3}(1)+\frac{1}{2}\sum_{i=1}^{2M}d_ip_{i,2}(1)\right]$$

$$\left(x-\frac{x^2}{2}\right)+\frac{x^2}{2}\left(g_3(t_{n+1})\right)+\sum_{i=1}^{2M}d_ip_{i,3}(x)-\frac{x^2}{2}\sum_{i=1}^{2M}d_ip_{i,2}(1)+g_1(t_{n+1}).$$

For FCKdV system, if we substitute Eqs. (19), (20) into Eq. (8) and discretize the results at the collocation points $x_l=\frac{l-0.5}{2M}$, $l=1,2,\ldots,2M$ we obtain following system of equations

$$\begin{aligned}\mathbf{A}_{l,i}\mathbf{c}_i+\mathbf{B}_{l,i}\mathbf{d}_i&=G_1u_n-G_1\eta^{\alpha}-G_1S_1+3a\left[(u_x)_nS_1+u_nP_1\right]\\&\quad+b\left[(v_x)_nS_2+v_nP_2\right]+\frac{a}{2}(u_{xxx})_n\\\mathbf{D}_{l,i}\mathbf{c}_i+\mathbf{E}_{l,i}\mathbf{d}_i&=G_2v_n-G_1\eta^{\beta}-G_2S_2-\frac{3}{2}\left[(v_x)_nS_1+u_nP_2\right]-\frac{1}{2}(v_{xxx})_n\end{aligned} \tag{21}$$

where

$$\mathbf{A}_{l,i}=\left(2\left[-p_{i,3}(1)+\frac{1}{2}p_{i,2}(1)\right]\left(x_l-\frac{x_l^2}{2}\right)+p_{i,3}(x_l)-\frac{x_l^2}{2}p_{i,2}(1)\right)\left(G_1-3a.(u_x)_n\right)$$

$$
\begin{aligned}
&- 3a.u_n\left(2\left[-p_{i,3}(1)+\frac{1}{2}p_{i,2}(1)\right](1-x_l)+p_{i,2}(x_l)-x_l p_{i,2}(1)\right)-\frac{a}{2}h_i(x_l),\\
\mathbf{B}_{l,i} =& -b\left(\left(2\left[-p_{i,3}(1)+\frac{1}{2}p_{i,2}(1)\right]\left(x_l-\frac{x_l^2}{2}\right)+p_{i,3}(x_l)-\frac{x_l^2}{2}p_{i,2}(1)\right)(v_x)_n\right)\\
&- bv_n\left(2\left[-p_{i,3}(1)+\frac{1}{2}p_{i,2}(1)\right](1-x_l)+p_{i,2}(x_l)-x_l p_{i,2}(1)\right),\\
\mathbf{D}_{l,i} =& \frac{3}{2}(v_x)_n\left(2\left[-p_{i,3}(1)+\frac{1}{2}p_{i,2}(1)\right]\left(x_l-\frac{x_l^2}{2}\right)+p_{i,3}(x_l)-\frac{x_l^2}{2}p_{i,2}(1)\right),\\
\mathbf{E}_{l,i} =& G_2\left(2\left[-p_{i,3}(1)+\frac{1}{2}p_{i,2}(1)\right]\left(x_l-\frac{x_l^2}{2}\right)+p_{i,3}(x_l)-\frac{x_l^2}{2}p_{i,2}(1)\right)\\
&+\frac{3}{2}u_n\left(2\left[-p_{i,3}(1)+\frac{1}{2}p_{i,2}(1)\right](1-x_l)+p_{i,2}(x_l)-x_l p_{i,2}(1)\right)+\frac{1}{2}h_i(x_l),\\
\eta^\alpha =& \sum_{k=1}^{n-1}\eta_k^\alpha\left[u_{n-k+1}-u_{n-k}\right], \quad \eta^\beta=\sum_{k=1}^{n-1}\eta_k^\beta\left[v_{n-k+1}-v_{n-k}\right]\\
S_1 =& 2\left[f_2(t_{n+1})-f_1(t_{n+1})-\frac{1}{2}f_3(t_{n+1})\right]\left(x_l-\frac{x_l^2}{2}\right)+\frac{x_l^2}{2}\left(f_3(t_{n+1})\right)+f_1(t_{n+1}),\\
S_2 =& 2\left[g_2(t_{n+1})-g_1(t_{n+1})-\frac{1}{2}g_3(t_{n+1})\right]\left(x_l-\frac{x_l^2}{2}\right)+\frac{x_l^2}{2}\left(g_3(t_{n+1})\right)+g_1(t_{n+1}),\\
P_1 =& 2\left[f_2(t_{n+1})-f_1(t_{n+1})-\frac{1}{2}f_3(t_{n+1})\right](1-x_l)+x_l\left(f_3(t_{n+1})\right),\\
P_2 =& 2\left[g_2(t_{n+1})-g_1(t_{n+1})-\frac{1}{2}g_3(t_{n+1})\right](1-x_l)+x_l\left(g_3(t_{n+1})\right),
\end{aligned}
$$

$\mathbf{c}_i$ and $\mathbf{d}_i$ are wavelet coefficients. By solving Eq. (21), wavelet coefficients $\mathbf{c}_i$ and $\mathbf{d}_i$ can be calculated successively. Then by plugging these coefficients into the Eqs. (19) and (20), the numerical solutions can be constructed.

For FCMKdV system, if we substitute Eqs. (19), (20) into Eq. (10) and discretize the results at the collocation points $x_l=\frac{l-0.5}{2M}$, $l=1,2,\ldots,2M$ we obtain following system of equations

$$
\begin{aligned}
\mathbf{A}_{l,i}\mathbf{c}_i+\mathbf{B}_{l,i}\mathbf{d}_i =& G_1u_n-G_1\eta^\alpha+\left(-G_1-3u_n(u_x)_n\right)S_1-\frac{3}{2}(u_n)^2\left[P_1-(u_x)_n\right]\\
&+\frac{3}{2.2}R_1+\frac{3}{2.2}(v_{xx})_n+3((uv)_x)_n-3\lambda(u_x)_n+\frac{1}{2.2}(u_{xxx})_n\\
\mathbf{0}_{l,i}\mathbf{c}_i+\mathbf{E}_{l,i}\mathbf{d}_i =& G_2v_n-G_1\eta^\beta+\left(-G_2-\frac{3}{2}(v_x)_n\right)S_2-\frac{3}{2}v_nP_2-\frac{1}{2}(v_{xxx})_n\\
&-3(u_x)_n(v_x)_n+3(u_n)^2(v_x)_n+3\lambda(v_x)_n
\end{aligned} \tag{22}
$$

where

$$\mathbf{A}_{l,i} = \left(2\left[-p_{i,3}(1)+\frac{1}{2}p_{i,2}(1)\right]\left(x_l-\frac{x_l^2}{2}\right)+p_{i,3}(x_l)-\frac{x_l^2}{2}p_{i,2}(1)\right)(G_1+3u_n(u_x)_n)$$
$$+\frac{3}{2}.u_n\left(2\left[-p_{i,3}(1)+\frac{1}{2}p_{i,2}(1)\right](1-x_l)+p_{i,2}(x_l)-x_l p_{i,2}(1)\right)-\frac{1}{2.2}h_i(x_l)$$
$$\mathbf{B}_{l,i} = -\frac{3}{2.2}\left(p_{i,1}(x)-p_{i,2}(1)\right)$$
$$\mathbf{E}_{l,i} = \left(G_2+\frac{3}{2}(v_x)_n\right)\left(2\left[-p_{i,3}(1)+\frac{1}{2}p_{i,2}(1)\right]\left(x_l-\frac{x_l^2}{2}\right)+p_{i,3}(x_l)-\frac{x_l^2}{2}p_{i,2}(1)\right)$$
$$+\frac{3}{2}v_n\left(2\left[-p_{i,3}(1)+\frac{1}{2}p_{i,2}(1)\right](1-x_l)+p_{i,2}(x_l)-x_l p_{i,2}(1)\right)+\frac{1}{2}h_i(x_l)$$
$$\mathbf{0}_{l,i} = \text{ Zero matrix}$$
$$\eta^{\alpha} = \sum_{k=1}^{n-1}\eta_k^{\alpha}\left[u_{n-k+1}-u_{n-k}\right], \quad \eta^{\beta} = \sum_{k=1}^{n-1}\eta_k^{\beta}\left[v_{n-k+1}-v_{n-k}\right]$$
$$S_1 = 2\left[f_2(t_{n+1})-f_1(t_{n+1})-\frac{1}{2}f_3(t_{n+1})\right]\left(x_l-\frac{x_l^2}{2}\right)+\frac{x_l^2}{2}\left(f_3(t_{n+1})\right)+f_1(t_{n+1}),$$
$$S_2 = 2\left[g_2(t_{n+1})-g_1(t_{n+1})-\frac{1}{2}g_3(t_{n+1})\right]\left(x_l-\frac{x_l^2}{2}\right)+\frac{x_l^2}{2}\left(g_3(t_{n+1})\right)+g_1(t_{n+1}),$$
$$P_1 = 2\left[f_2(t_{n+1})-f_1(t_{n+1})-\frac{1}{2}f_3(t_{n+1})\right](1-x_l)+x_l\left(f_3(t_{n+1})\right),$$
$$P_2 = 2\left[g_2(t_{n+1})-g_1(t_{n+1})-\frac{1}{2}g_3(t_{n+1})\right](1-x_l)+x_l\left(g_3(t_{n+1})\right),$$
$$R_1 = g_3(t_{n+1})-2\left[g_2(t_{n+1})-g_1(t_{n+1})-\frac{1}{2}g_3(t_{n+1})\right],$$

$\mathbf{c}_i$ and $\mathbf{d}_i$ are wavelet coefficients. By solving Eq. (22), wavelet coefficients $\mathbf{c}_i$ and $\mathbf{d}_i$ can be calculated successively. Again by plugging these coefficients into the Eqs. (19) and (20), the numerical solutions can be constructed.

3.4 Error analysis

To analyze the convergence of the method, we use the asymptotic expansion of Eq. (18) as given in Kumar and Pandit (2014), the resulting equation is as follows

$$u(x) = 2\left[f_2(t_{n+1})-f_1(t_{n+1})-\frac{1}{2}f_3(t_{n+1})-\sum_{i=1}^{\infty}c_i p_{i,3}(1)+\frac{1}{2}\sum_{i=1}^{\infty}c_i p_{i,2}(1)\right]\left(x-\frac{x^2}{2}\right)$$
$$+\frac{x^2}{2}\left(f_3(t_{n+1})\right)+\sum_{i=1}^{\infty}c_i p_{i,3}(x)-\frac{x^2}{2}\sum_{i=1}^{\infty}c_i p_{i,2}(1)+f_1(t_{n+1})$$

Lemma 1. Suppose that $u(x) \in L^2(R)$ with $\left|\frac{\partial u(x)}{\partial x}\right| \leq K$, $\forall x \in (0,1)$; $K > 0$ and $u(x) = \sum_{i=1}^{\infty} c_i h_i(x)$. Then $|c_i| \leq K 2^{(-3j-2)/2}$ [Ray (2012)].

Lemma 2. Let $u(x) \in L^2(R)$ be a continuous function defined in $(0,1)$. Then the error norm at J th level satisfies the following inequality

$$\left\|E_j\right\|^2 \leq \frac{K^2}{12} 2^{-2J}$$

where $\left|\frac{\partial u(x)}{\partial x}\right| \leq K$, $\forall x \in (0,1)$; $K > 0$, M is a positive number related to the J th level resolution of the wavelet given by $M = 2^J$ [Ray (2012)].

Theorem: Suppose that $u(x)$ is exact and $u_{2M}(x)$ is approximate solution of the Eq. (18), then

$$\left\|E_j\right\| = \|u(x) - u_{2M}(x)\| \leq \frac{\sqrt{C} K 2^{-3(2^J)-1}}{1 - 2^{-3/2}}$$

Proof. See Kumar and Pandit (2014)

Similar approach is valid for $v_{2M}(x)$. It is clear from above equation that the level of resolution J of the Haar wavelet is inversely proportional to the error bound. Therefore the accuracy of the method increases as we increase the level of resolution J.

4 Numerical Examples

Free software package GNU Octave is used in this study for numerical computations and Matplotlib package is used [Hunter (2007)] for graphical outputs. In order to check accuracy of the proposed methods we considered the error norms L_2 and L_∞ defined by

$$L_2 = \sqrt{\Delta x \sum_{i=1}^{2M} |u_i^{\text{exact}} - u_i^{\text{num}}|^2}$$

$$L_\infty = \max_i \left|u_i^{\text{exact}} - u_i^{\text{num}}\right|.$$

4.1 Example 1

Firstly, we consider following initial conditions for the Eq. (1)

$$u(x,0) = 2\lambda^2 \text{sech}^2(\xi), \qquad v(x,0) = \frac{1}{2\sqrt{w}} \text{sech}(\xi)$$

and the boundary conditions

$$u(x_1,t) = u(x_2,t) = u_x(x_2,t) = 0 \quad t \in [0,T]$$
$$v(x_1,t) = v(x_2,t) = v_x(x_2,t) = 0 \quad t \in [0,T].$$

This problem have the following exact solution [Hirota and Satsuma (1981)] for $\alpha = \beta = 1$.

$$u(x,t) = 2\lambda^2 \operatorname{sech}^2(\xi), \qquad v(x,t) = \frac{1}{2\sqrt{w}}\operatorname{sech}(\xi)$$

where

$$\xi = \lambda(x - \lambda^2 t) + \frac{1}{2\log(w)}, \qquad w = \frac{-b}{8(4a+1)\lambda^4}.$$

We solve the problem for the various values of $2M$ in the interval $-25 \leq x \leq 25$ and tabulated the L_2, L_∞ error norms when $\alpha = \beta = 1$ in Table 1. We can easily see from the table that when we increase the collocation points the error norms decrease and they are sufficiently small for $t = 0.1$. To see the accuracy of the Haar wavelet method we depicted the evolution of numerical solutions of u and v at $\alpha = \beta = 1$ in Fig. 1. also for $\alpha = 1/2$, $\beta = 1/3$ in Fig. 2. The comparison of these figures shows that these results are almost same and this verifies the accuracy of Haar wavelet method.

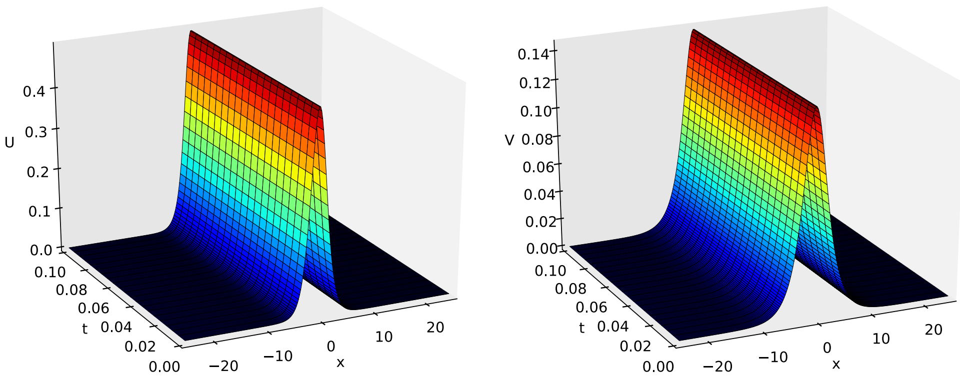
Table 1: Numerical results for $\Delta t = 0.005$, $\lambda = 0.5$, $a = -0.125$, $b = -3$ and $\alpha = \beta = 1$ at $t = 0.1$

$2M$	$L_2(u)$	$L_2(v)$	$L_\infty(u)$	$L_\infty(v)$
128	1.122876e-005	6.881984e-006	6.828475e-005	3.964352e-005
256	2.774834e-006	1.737210e-006	1.650130e-005	9.694622e-006
512	6.950655e-007	5.226886e-007	4.149853e-006	2.450309e-006

4.2 Example 2

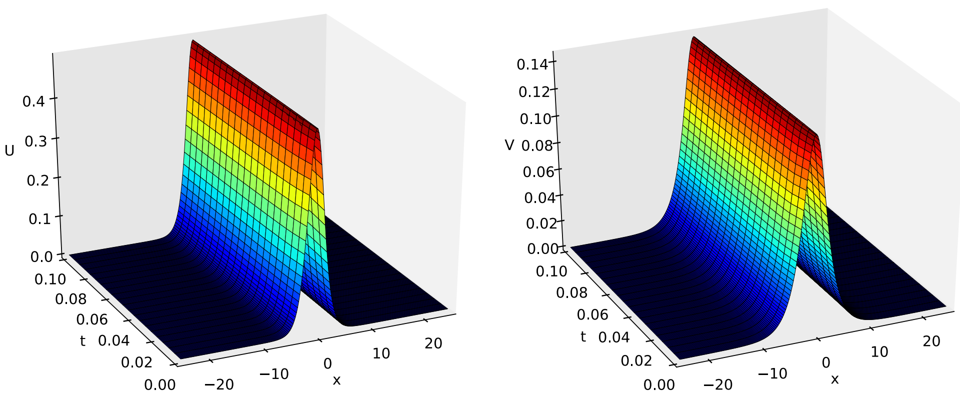
Secondly, we consider the Eq. (1) with following initial conditions

$$u(x,0) = -\frac{1+a}{3+6a}k^2 + 4k^2\frac{e^{k\xi}}{\left(1+e^{k\xi}\right)^2}, \qquad v(x,0) = \sqrt{\frac{-24a}{b}}\frac{k^2 e^{k\xi}}{\left(1+e^{k\xi}\right)^2}$$



(a) (b)

Figure 1: Exact solutions for $\alpha = \beta = 1$, $\Delta t = 0.005$ and $2M = 256$ at $t = 0.1$



(a) (b)

Figure 2: Numerical solutions for $\alpha = 1/2$, $\beta = 1/3$, $\Delta t = 0.005$ and $2M = 256$ at $t = 0.1$

where $\xi = x - t\frac{ak^2}{1+2a}$, $a \neq 1/2$, $ab < 0$ and k is an arbitrary constant. The exact solutions for the special case where $\alpha = \beta = 1$ is given by Lu and Wang (1999) as follows.

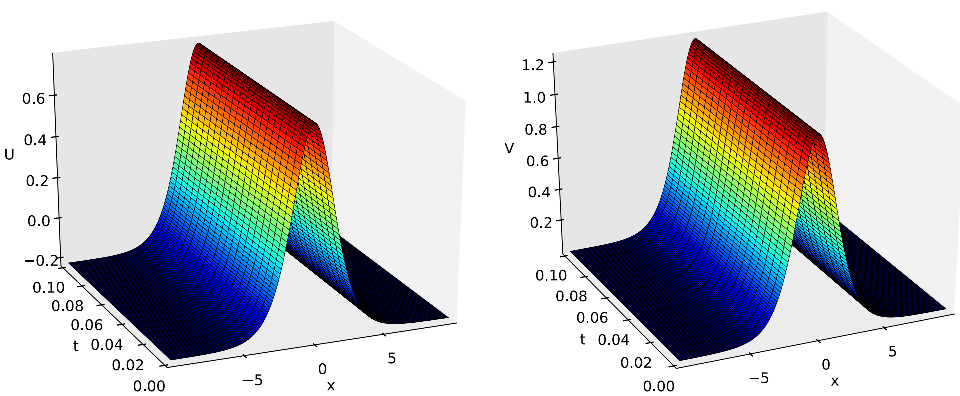
$$u(x,t) = -\frac{1+a}{3+6a}k^2 + 4k^2\frac{e^{k\xi}}{\left(1+e^{k\xi}\right)^2}, \qquad v(x,t) = \sqrt{\frac{-24a}{b}}\frac{k^2 e^{k\xi}}{\left(1+e^{k\xi}\right)^2}.$$

Boundary conditions are taken from exact solutions, computer simulations for this problem are done with parameters $k = 1$, $a = 1$ and $b = -1$ in the interval $-10 \leq$

$x \leq 10$. L_2 and L_∞ error norms for $\alpha = \beta = 1$ at different time levels are tabulated in Table 2. We see from the table that as t gets larger so as the error norms. We depicted the exact solutions of the problem for $\alpha = 1$, $\beta = 1$ in Fig. 3., and to see the evolution of numerical solutions of u and v for fractional derivatives we also depicted them for $\alpha = 1/2$, $\beta = 1/3$ in Fig. 4. Comparison of these figures also proves that the present method gives results with high accuracy.

Table 2: Numerical results for $2M = 128$, $\Delta t = 0.005$, $k = 1$, $a = 1$, $b = -1$ and $\alpha = \beta = 1$ at different times.

t	$L_2(u)$	$L_2(v)$	$L_\infty(u)$	$L_\infty(v)$
0.1	4.2024e-005	5.2887e-005	1.5864e-004	2.0719e-004
0.5	2.1815e-004	2.1387e-004	6.7778e-004	7.3855e-004
1.0	2.1078e-003	3.6879e-004	9.1166e-003	1.1608e-003



(a) (b)

Figure 3: Exact solutions for $\alpha = \beta = 1$, $\Delta t = 0.005$ and $2M = 256$ at $t = 0.1$

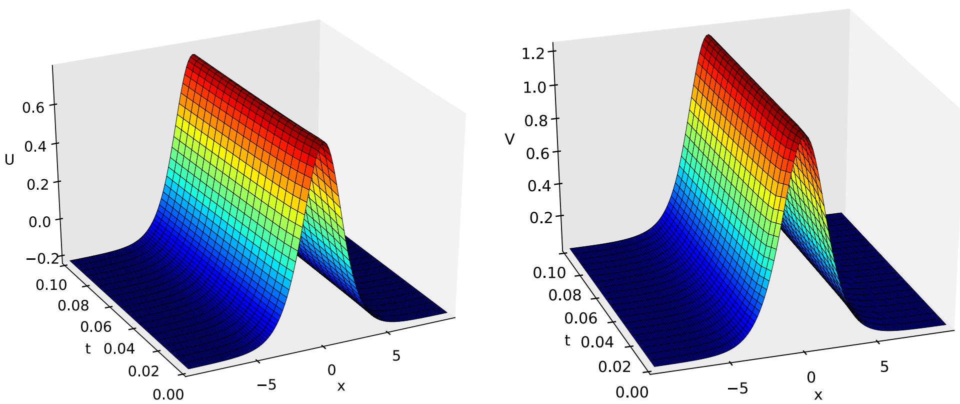
4.3 Example 3

Finally, we consider Eq. (2) with initial conditions

$$u(x,0) = \frac{b}{2k} + k\tanh(kx), \qquad v(x,0) = \frac{\lambda}{2}\left(1 + \frac{k}{b}\right) + b\tanh(kx).$$

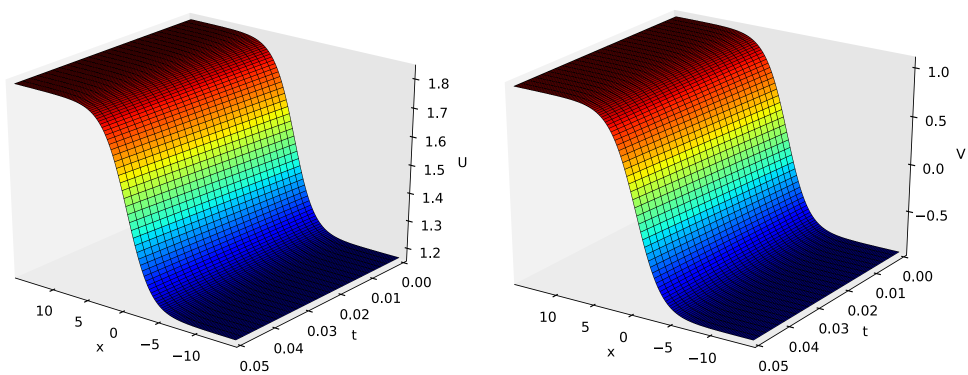
The exact solutions of this problem for $\alpha = \beta = 1$ given by Fan (2000, 2001, 2002) as follows

$$u(x,t) = \frac{b}{2k} + k\tanh(k\xi), \qquad v(x,t) = \frac{\lambda}{2}\left(1 + \frac{k}{b}\right) + b\tanh(k\xi)$$



(a) (b)

Figure 4: Numerical solutions for $\alpha = 1/2$, $\beta = 1/3$, $\Delta t = 0.005$ and $2M = 256$ at $t = 0.1$



(a) (b)

Figure 5: Exact solutions for $\alpha = \beta = 1$, $\Delta t = 0.001$ and $2M = 256$ at $t = 0.05$

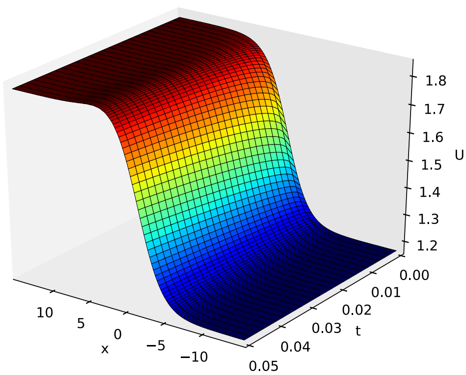
where

$$\xi = x + \frac{1}{4}\left(-4k^2 - 6\lambda + \frac{6k\lambda}{b} + \frac{3b^2}{k^2}\right)t, \quad k \neq 0, \quad b \neq 0.$$

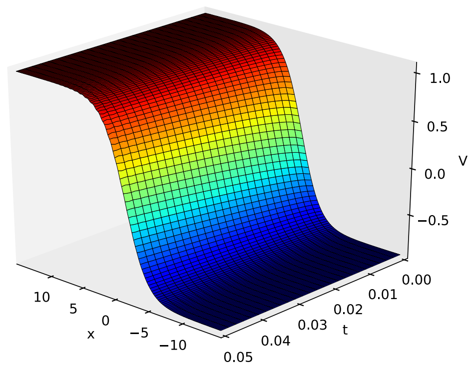
The boundary conditions taken from exact solutions. Numerical simulation of this problem are done for $b = 1$, $\lambda = 0.1$ and $k = 1/3$ in the interval $-15 \leq x \leq 15$ for growing times, L_2 and L_∞ error norms are tabulated when $\alpha = \beta = 1$ in Table 3. We see from the table the error norms are sufficiently small. We depicted the exact solutions for $\Delta t = 0.001$, $\alpha = 1$, $\beta = 1$ at $t = 0.05$ in Fig. 5. also we depicted the numerical solutions for $\alpha = 1/2$, $\beta = 2/3$ in Fig. 6. Comparison of these figures proves that high accuracy results can be achieved by using Haar wavelet method.

Table 3: Numerical results for $\Delta t = 0.005$, $k = 1/3$, $\lambda = 0.1$, $b = 1$, $2M = 128$ and $\alpha = \beta = 1$.

	$L_2(u)$	$L_2(v)$	$L_\infty(u)$	$L_\infty(v)$
$t = 0.05$	3.5712e-004	1.1651e-003	1.3252e-003	3.0867e-003
$t = 0.1$	8.9437e-004	2.2636e-003	4.4984e-003	5.9635e-003



(a)



(b)

Figure 6: Numerical solutions for $\alpha = 1/2$, $\beta = 2/3$, $\Delta t = 0.001$ and $2M = 256$ at $t = 0.05$

5 Conclusion

In this paper, we applied Haar wavelet method integrated with $L1$ discretization formula in the Caputo's sense to time fractional nonlinear coupled partial differential equations with various initial and boundary conditions. The main idea of the proposed method is using $L1$ formula for fractional time derivatives and Haar wavelets for spatial derivatives. This approach gives a system of algebraic equations which can be solved easily with suitable methods. Also the proposed method is novel in the sense that it doesn't use fractional order Haar operational matrices which is a more complex method when compared with the proposed method. The numerical results are quite good in the cases where the exact solutions are known. The proposed method can easily handle similar fractional system of partial differential equations.

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