

Buckley-Leverett Analysis for Transient Two-phase Flow in Fractal Porous Medium

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Abstract: Analysis of Buckley-Leverett solution in fractal porous medium does prediction of water saturation profile a favor. On the approximation that porous medium consists of a bundle of tortuous capillaries, a physical conceptual Buckley-Leverett model of transient two-phase flow in fractal porous medium is developed based on the fractal characteristics of pore size distribution. The relationship between water saturation and distance is presented according to Buckley-Leverett solution, and the proposed Buckley-Leverett expression is the function of fractal structural parameters (such as pore fractal dimension, tortuosity fractal dimension, maximum and minimum diameters of capillaries) and fluid properties (such as viscosity, contact angle and interfacial tension) in fractal porous medium. The sensitive parameters that impact on Buckley-Leverett expression are formulated and their sensitivities on water saturation file are discussed.

Keywords: fractal theory, transient two-phase flow, porous medium, fractional flow Buckley-Leverett analysis.

1 Introduction

Petroleum reservoir engineering problems especially in porous media are known to be inherently nonlinear. Consequently, solutions to the complete multiphase flow equations have been principally attempted with numerical methods. However, simplified forms of the problem were solved some 60 years ago, when the Buckley-Leverett formulation was introduced. The Buckley-Leverett displacement theory was applied to petroleum reservoirs engineering consisting of a finite number of layers [Snyder and Ramey (1967)]. Internal consistency requires that the parameters should be corrected for the removal of interfacial tension because capillary pressure was ignored in the Buckley-Leverett analysis [Spanos, De La Cruz and

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Hube (1986)]. Mohsen (1985) found that a mass-conserving front would be located farther down the flow direction for a nonzero initial condition and discussed the implications of this finding for error analysis in comparing numerical solutions to the analytical one. Larsen, Kviljo and Litlehamar (1990) examined the relationship between backflow of water from the invaded zone and changed in skin owing to reduced water saturation and the associated change in mobility for homogeneous reservoirs with Buckley-Leverett methods. Langtangen, Tveito and Winther (1992) studied the simultaneous one-dimensional flow of water and oil in a heterogeneous medium modeled by the Buckley-Leverett equation. A variety of heterogeneity profiles were studied with Buckley-Leverett methods. Capillary heterogeneity significantly affects the saturation distributions, which closely follow the heterogeneity variation [Chang and Yortsos (1992)]. The geothermal saturation wave speed under all conditions is formally identical with the Buckley-Leverett wave speed when the latter is written as the saturation derivative of a volumetric flow [Young (1993)]. A Buckley-Leverett-type analytical solution for one dimensional immiscible displacement was presented in a linear composite porous medium [Wu, Pruess and Chen (1993)]. Frid (1995) solved the initial boundary-value problem for the regularized Buckley-Leverett system, which described the flow of two immiscible incompressible fluids through a porous medium. The measurement of three-phase relative permeability using the extension of the Buckley-Leverett theory has both sound theoretical and experimental bases [Siddiqui, Hicks and Grader (1996)]. An experimental investigation was made to test the validity of the three-phase extension of the Buckley-Leverett (B-L) theory using three immiscible liquids [Siddiqui, Hicks and Grader (1996)]. Method of characteristics (MOC) solutions to the three-phase Buckley-Leverett problem was presented with and without gravity [Guzman and Fayers (1997)]. Terez and Firoozabadi (1999) examined water injection in water-wet fractured porous media and its modeling using the Buckley-Leverett theory. A stochastic analysis of immiscible two-phase flow with Buckley-Leverett displacement was presented in heterogeneous reservoirs [Dongxiao and Tchelepi (1999)]. Kaasschieter (1999) derived an entropy inequality from a regularization procedure, where the physical capillary pressure term is added to the Buckley-Leverett equation. An extension of the Buckley-Leverett (BL) equation describing two-phase flow in porous media was discussed by Van Duijn, Peletier and Pop (2007). Mustafiz, Mousavizadegan and Islam (2008) used a semi-analytical technique and the Adomian decomposition method (ADM) to unravel the true nature of the one-dimensional, two-phase flow. Sumnu-Dindoruk and Dindoruk (2008) have solved the resulting nonisothermal two-phase convective flow equation in porous media analytically, including a tracer component. A new approach was proposed to the mathematical modeling of the Buckley-Leverett system, which describes two-phase flows in porous media [Chemetov and Neves (2013)]. Wang and Kao (2013)

extended the second and third order classical central schemes for the hyperbolic conservation laws to solve the modified Buckley-Leverett (MBL) equation which is of pseudo-parabolic type. Wang and Kao (2014) numerically verified that the convergence rate is consistent with the theoretical derivation by Buckley-Leverett (BL) equation.

It's well known that the microstructures of real porous medium are usually disordered and extremely complicated. Fortunately, it has been shown that the microstructures of porous medium have the self-similarity and fractal characteristics [Katz and Thompson (1985); Xiao, Jiang, and Chen (2010)]. The pore spaces of several sandstones are fractal geometries by using the scanning electron microscopy and optical data [Katz and Thompson (1985)]. The pore-size distribution fractal model of unsaturated soils was derived and Hydraulic conductivity and soil-water diffusivity of unsaturated soils could be expressed by two fractal parameters [Xu and Dong (2004)]. Completely multi-scale configurations can guide the flowing in porous medium based on the application of tree-shaped fractal structures [Lorente and Bejan (2006)]. Deinert, Dathe, and Parlange (2008) presented that the relationship between capillary pressure and saturation in a fractal porous medium and exhibited a power-law. A fractal model for the relative permeability of porous medium was presented by assuming that porous medium consists of a bundle of tortuous capillaries, but the effects of capillary pressure was not considered [Yu and Liu (2004)]. Based on [Yu and Liu (2004)], the relative permeability of unsaturated porous medium embedded with a fractal-like tree branched networks was studied by considering the capillary pressure [Wang and Yu (2011)]. Xu, Qiu, Yu, and Jiang (2013) investigated the relative permeability in unsaturated porous medium by assuming that all capillaries with the radius less than a critical radius are saturated, and the others with the radius larger than the critical radius are unsaturated. Recently, a generalized fractal model was developed for spontaneous imbibition in porous medium with shapes of pores included [Cai, Perfect, Cheng, and Hu (2014)]. Based on the fractal characteristics of pore size distribution and on the approximation that porous medium consists of a bundle of tortuous capillaries a relative permeability model for transient two-phase flow in fractal porous medium was derived [Tan, Li, and Liu (2014)]. For fractal systems, the pore or solid may be fractal. For pore fractal, mass fractal and pore-solid fractal, recent review work is helpful to understand [Cai, Luo, and Ye (2015)].

But it is difficult to apply Buckley-Leverett equation to solve petroleum reservoir engineering problems in porous medium saturated with multiphase fluid because the microstructures of real porous medium are usually disordered and extremely complicated. However, quite a few scholars have done excellent job to study petroleum reservoir engineering problems using Buckley-Leverett equation and re-

search the microstructure of porous media with fractal theory in recent few decades. Thus, we pay close attention to Buckley-Leverett model based on fractal theory in porous media.

In this thesis, we continue a line of research on Buckley-Leverett analysis for transient two-phase flow in fractal porous media based on fractional flow analysis which is our previous work entitled “Analysis of the fractional flow for transient two-phase flow in fractal porous media” (under review). The purpose of the present paper is to derive an analytical expression for Buckley-Leverett solution based on the fractal geometry theory for porous medium, and it is expected the proposed fractal expression for the front position to compute water saturation profile.

This paper is organized as follows. In Sect. 2, the relevant theoretical bases which mainly include velocity and flow rate of transient two-phase flow in a single capillary are introduced. In Sect. 3, on the approximation that porous media consists of a bundle of tortuous capillaries, an analytical expression for Buckley-Leverett solution is derived based on the fractal characteristics of pore size distribution. The results and discussion are shown in Sect. 4. Eventually, we summarize the conclusions in Sect. 5.

2 Theoretical Bases

We approximate that porous media is comprised of a bundle of tortuous capillaries with different diameters. According to this approximation, a single fractal capillary with transient two-phase flow is proposed here, as shown in Fig. 1. The fractal capillary is only saturated with oil (red) initially, but later water (blue) intrudes into the capillary and displaces oil with a constant pressure difference between points a and b . Therefore, the fractal capillary is separated by a two-phase flow interface at point c . The constant pressure difference between points a and b is ΔP . The fractal capillary diameter, the straight distance of the capillary, and the straight distance between points a and c are λ , L , and X respectively.

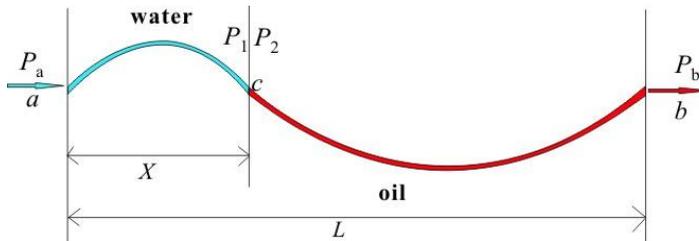


Figure 1: A single fractal capillary with transient two-phase flow.

Tan, Li, and Liu (2014) presented a theoretical analytical expression for the transient two-phase flow velocity in a single capillary

$$v = \frac{\lambda^2(\Delta P + P_C)}{32[(\mu_w - \mu_o)X_T + \mu_o L_T]} \quad (1)$$

where μ_w and μ_o are the viscosity of water and oil, respectively. λ is the capillary diameter, X_T is the actual length of the capillary between points a and c , and L_T is the actual length of the capillary. ΔP is the constant pressure difference between points a and b , and P_C is capillary pressure.

The fractal scaling law for tortuous capillaries in porous medium is given by Yu and Cheng (2002)

$$L_T = L^{D_T} \lambda^{1-D_T} \quad (2)$$

and

$$X_T = X^{D_T} \lambda^{1-D_T} \quad (3)$$

where D_T is the tortuosity fractal dimension.

The capillary pressure function can be expressed as:

$$P_C = \frac{4\sigma \cos \theta}{\lambda} \quad (4)$$

where σ is surface tension, θ is contact angle.

Substituting Eqs. (1)–(2) into Eq. (4) the expression of two-phase flow velocity v can be obtained as:

$$v = \frac{\lambda^{1+D_T} \Delta P + 4\lambda^{D_T} \sigma \cos \theta}{32[(\mu_w - \mu_o)X^{D_T} + \mu_o L^{D_T}]} \quad (5)$$

From Eq. (5), we can get the expression of the two-phase flow rate q_t , in a single tortuous capillary:

$$q_t = \frac{\pi \lambda^2}{4} v = \frac{\lambda^{3+D_T} \pi \Delta P + 4\lambda^{2+D_T} \pi \sigma \cos \theta}{128[(\mu_w - \mu_o)X^{D_T} + \mu_o L^{D_T}]} \quad (6)$$

In Eq. (6), when $\sigma = 0$, $X = L$, the two-phase flow rate can be regarded as single-phase flow rate, q_s , which is expressed as

$$q_s = \frac{\lambda^{3+D_T} \pi \Delta P}{128\mu_w L^{D_T}} \quad (7)$$

Equation (7) is the same as the expression of the single-phase flow rate in a single fractal capillary by Yu and Cheng (2002).

It is known that

$$v = \frac{dX_T}{dt} \tag{8}$$

Combining Eq. (5) and Eq (8), we get the expression of the two-phase flow velocity in a single capillary

$$\frac{dX_T}{dt} = \frac{\lambda^{1+D_T} \Delta p + 4\lambda^{D_T} \sigma \cos \theta}{32[(\mu_w - \mu_o) X^{D_T} + \mu_o L^{D_T}]} \tag{9}$$

From Eq. (3), we get

$$dX_T = d(X^{D_T} \lambda^{1-D_T}) = D_T X^{D_T-1} \lambda^{1-D_T} dX \tag{10}$$

Substituting Eq. (10) into Eq. (9), we can yield

$$32D_T X^{D_T-1} \lambda^{1-D_T} [(\mu_w - \mu_o) X^{D_T} + \mu_o L^{D_T}] dX = (\lambda^{1+D_T} \Delta p + 4\lambda^{D_T} \sigma \cos \theta) dt \tag{11}$$

Taking an integration of Eq. (11) with initial condition $t=0$ and $X=0$, and after rearranging, we get the relationship between the two-phase interface position, X , and relevant displace time t

$$X^{2D_T} + \frac{2\mu_o L^{D_T}}{\mu_w - \mu_o} X^{D_T} - \frac{(\lambda^{2D_T} \Delta p + 4\lambda_r^{2D-1} \sigma \cos \theta) t}{16(\mu_w - \mu_o)} = 0 \tag{12}$$

From Eq. (12) we can see that when $X=0$ and $t=0$ the capillary is only saturated with oil. $0 < X < L$ represents the flow regime in capillary is transient two-phase flow. When $X = L$, oil is completely displaced by water. Substituting $X = L$ into Eq. (12), the expression for completely displacing time, t_d , can be written as

$$t_d = \frac{16(\mu_w + \mu_o)L^{2D_T}}{\lambda^{2D_T} \Delta p + 4\lambda^{2D_T-1} \sigma \cos \theta} \tag{13}$$

Eq. (13) reveals that the completely displacing time is greatly affected by fractal parameters and fluid properties. Since the fractal porous medium is assumed to be comprised of a bundle of tortuous capillaries with different diameters, we only discuss the influence of the capillary diameter on the completely displacing time. Assuming other parameters (viscosities of water and oil, tortuosity fractal dimension, the difference pressure, contact angle and interfacial tension) are constant, a

bigger capillary diameter corresponds to a smaller value of displacement time. That is to say, it will take less time for water to displace oil completely in a capillary with bigger diameter.

Here we define the critical capillary diameter, λ_{cr} , to be the capillary diameter [Xu, Qiu, Yu, and Jiang (2013)], at which oil is just right displaced by water completely at a given time. Substituting $t_d = t$ into Eq. (13), we obtain the following equation for the critical diameter as:

$$\lambda_{cr}^{2D_T} + \frac{4\sigma \cos \theta}{\Delta P} \lambda_{cr}^{2D_T-1} - \frac{16(\mu_w + \mu_o)L^{2D_T}}{\Delta P t} = 0 \quad (14)$$

From Eq. (14), we can see that the critical capillary diameter is affected by fluid properties (such as contact angle, viscosities, pressure difference and interfacial tension) and the fractal dimension D_T . Here, we particularly emphasize on analyzing how time impacts the critical capillary diameter. It is easily to see that the critical capillary diameter becomes smaller with the increase of displacement time and more and more capillaries will change its flow regime from transient two-phase flow to single-phase flow. Thus, if $\lambda \geq \lambda_{cr}$, oil is displaced by water completely and water flows out point b of the capillary. Otherwise, oil is not displaced by water completely and oil flows out point b of the capillary.

3 A Model for Buckley-Leverett solution

It has been shown that the cumulative size-distribution of capillary sizes which are greater than or equal to the capillary diameter, λ , follows the fractal scaling law [Yu and Cheng (2002)]:

$$N(l \geq \lambda) = \left(\frac{\lambda_{max}}{\lambda} \right)^{D_f} \quad (15)$$

where λ_{max} is the maximum diameter of capillary, and D_f is pore fractal dimension. Generally, $0 < D_f < 2$ denotes two dimensional space, and $0 < D_f < 3$ refers to three dimensional space.

Since there are numerous pores in a sample porous medium, Eq. (15) can be approximately considered as a continuous and differentiable function. Then, taking derivative with respect to diameter in Eq. (15) yields the number of pores whose diameters are within the infinitesimal range of λ and $\lambda + d\lambda$.

$$-dN = D_f \lambda_{max}^{D_f} \lambda^{-(D_f+1)} d\lambda \quad (16)$$

Based on the approximation that porous medium consist of a bundle of tortuous fractal capillaries with variable diameters and on the fractal characteristics of pore

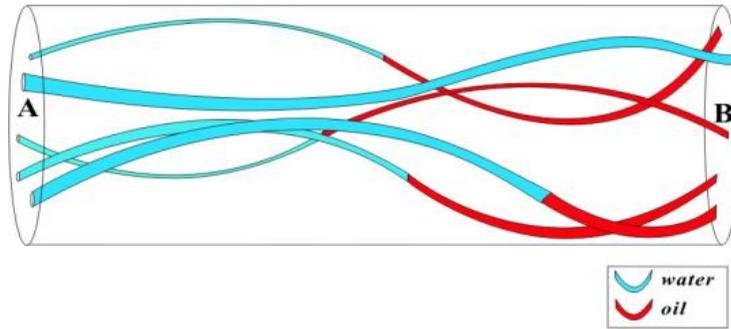


Figure 2: Transient two-phase flow in a porous medium with tortuous capillaries at a given time.

size distribution in porous medium, we can derive Buckley-Leverett solution model of transient two-phase flow in a fractal porous medium as shown in Fig. 2.

As is shown in Fig. 2, water flows out of cross-section *B* in fractal capillaries whose diameters are larger than the critical capillary diameter and it is single-phase flow in these capillaries. On the contrary, if capillary diameter is smaller than the critical capillary diameter, oil flows out of cross-section *B* (or water cannot flow out cross-section *B*) and it is transient two-phase flow.

We can get the total flow rate, Q by summing up the flow rates through all the capillaries at the cross-section *B*.

The total flow rate of water, Q_w , at across-section *B* can be obtained by integrating Eq. (14) from critical capillary diameter to maximum diameter.

$$\begin{aligned}
 Q_w &= - \int_{\lambda_{cr}}^{\lambda_{max}} q_{sw} dN \\
 &= \frac{\pi D_f \Delta P \lambda_{max}^{3+D_T}}{128 \mu_w L^{D_T}} \int_{\lambda_{cr}}^{\lambda_{max}} \lambda^{2+D_T-D_f} d\lambda \\
 &= \frac{\pi D_f \Delta P \lambda_{max}^{D_f}}{128 \mu_w L^{D_T} (3 + D_T - D_f)} \left[\lambda_{max}^{3+D_T-D_f} - \lambda_{cr}^{3+D_T-D_f} \right]
 \end{aligned} \tag{17}$$

While oil flows out of across-section *B* in fractal capillaries whose diameters are smaller than the critical capillary diameter. Because oil is not completely displaced by water, flow pattern is still two-phase flow The total flow rate of oil, Q_o at section plane *B* can be obtained by integrating Eq. (13) from minimum diameter to critical

capillary diameter.

$$\begin{aligned}
 Q_o &= - \int_{\lambda_{\min}}^{\lambda_{cr}} q_t dN \\
 &= \frac{\pi D_f \lambda_{\max}^{D_f} \Delta P}{128} \int_{\lambda_{\min}}^{\lambda_{cr}} \frac{\lambda^{2+D_T-D_f}}{(\mu_w - \mu_o) X^{D_T} + \mu_o L^{D_T}} d\lambda \\
 &\quad + \frac{\pi D_f \lambda_{\max}^{D_f} \sigma \cos \theta}{32} \int_{\lambda_{\min}}^{\lambda_{cr}} \frac{\lambda^{1+D_T-D_f}}{(\mu_w - \mu_o) X^{D_T} + \mu_2 L^{D_T}} d\lambda
 \end{aligned}
 \tag{18}$$

where λ_{cr} is the critical capillary diameter. Based on the fractal theory, the total pore volume in the fractal porous medium can be expressed as

$$\begin{aligned}
 V_P &= - \int_{\lambda_{\min}}^{\lambda_{\max}} \frac{\pi \lambda^2}{4} L_T dN \\
 &= \frac{\pi D_f L^{D_T} \lambda_{\max}^{D_f}}{4(3 - D_T - D_f)} \left[\lambda_{\min}^{3-D_T-D_f} - \lambda_{\max}^{3-D_T-D_f} \right]
 \end{aligned}
 \tag{19}$$

As is seen from Fig. 2, the pore volume of the fractal porous medium saturated with water is composed of two portions: the whole capillary volume whose flow regime is single-phase flow and the back part of capillary volume whose flow regime is transient two-phase flow.

$$\begin{aligned}
 V_w &= - \int_{\lambda_{\min}}^{\lambda_{cr}} \frac{\pi \lambda^2}{4} X_T dN - \int_{\lambda_{cr}}^{\lambda_{\max}} \frac{\pi \lambda^2}{4} L_T dN \\
 &= \frac{\pi D_f \lambda_{\max}^{D_f}}{4} \int_{\lambda_{\min}}^{\lambda_{cr}} X^{D_T} \lambda^{2-D_T-D_f} d\lambda \\
 &\quad + \frac{\pi D_f L^{D_T} \lambda_{\max}^{D_f}}{4(3 - D_T - D_f)} \left[\lambda_{cr}^{3-D_T-D_f} - \lambda_{\max}^{3-D_T-D_f} \right]
 \end{aligned}
 \tag{20}$$

The pore volume of the fractal porous medium saturated with oil is only the front part of capillary volume whose flow regime is transient two-phase flow.

$$\begin{aligned}
 V_o &= - \int_{\lambda_{\min}}^{\lambda_{cr}} \frac{\pi \lambda^2}{4} (L_T - X_T) dN \\
 &= \frac{\pi D_f L^{D_T} \lambda_{\max}^{D_f}}{4(3 - D_T - D_f)} \left[\lambda_{cr}^{3-D_T-D_f} - \lambda_{\min}^{3-D_T-D_f} \right] \\
 &\quad - \frac{\pi D_f \lambda_{\max}^{D_f}}{4} \int_{\lambda_{\min}}^{\lambda_{cr}} X^{D_T} \lambda^{2-D_T-D_f} d\lambda
 \end{aligned}
 \tag{21}$$

Combining Eqs. (19)-(21), saturation of water, S_w , and saturation of oil S_o can be respectively expressed as:

$$S_w = \frac{V_w}{V_P} = \frac{\frac{(3-D_T-D_f)}{L^{D_T}} \int_{\lambda_{min}}^{\lambda_{cr}} X^{D_T} \lambda^{2-D_T-D_f} d\lambda + \left(\lambda_{max}^{3-D_T-D_f} - \lambda_{cr}^{3-D_T-D_f} \right)}{\left(\lambda_{max}^{3-D_T-D_f} - \lambda_{min}^{3-D_T-D_f} \right)} \quad (22)$$

$$S_o = \frac{V_o}{V_P} = \frac{\left(\lambda_{cr}^{3-D_T-D_f} - \lambda_{min}^{3-D_T-D_f} \right) - \frac{(3-D_T-D_f)}{L^{D_T}} \int_{\lambda_{min}}^{\lambda_{cr}} X^{D_T} \lambda^{2-D_T-D_f} d\lambda}{\left[\lambda_{max}^{3-D_T-D_f} - \lambda_{min}^{3-D_T-D_f} \right]} \quad (23)$$

In Eqs. (20) and (23), when $X = L$ and $\lambda_{cr} = \lambda_{max}$, oil is completely displaced by water, so $S_w = 1$ and $S_o = 0$. While $X = 0$ and $\lambda_{cr} = \lambda_{min}$ in Eqs. (20) and (23), the fractal porous medium is only saturated with oil, thus $S_w = 0$ and $S_o = 1$.

Fractional flow of water, f_w , can be defined as the proportion of flow rate of water in the total flow rate. So fractional flow of water can be expressed as

$$f_w = \frac{Q_w}{Q_w + Q_o} \quad (24)$$

Substituting Eqs (17) and (18) into Eq. (24), fractional flow of water can be expressed as:

$$f_w(S_w) = \frac{\frac{\Delta P}{\mu_w L^{D_T(3+D_T-D_f)}} \left(\lambda_{max}^{3+D_T-D_f} - \lambda_{cr}^{3+D_T-D_f} \right)}{\frac{\Delta P}{\mu_w L^{D_T(3+D_T-D_f)}} \left(\lambda_{max}^{3+D_T-D_f} - \lambda_{cr}^{3+D_T-D_f} \right) + \Delta P \int_{\lambda_{min}}^{\lambda_{cr}} \frac{\lambda^{2+D_T-D_f}}{(\mu_w - \mu_o) X^{D_T + \mu_o L^{D_T}}} d\lambda + 4\sigma \cos\theta \int_{\lambda_{min}}^{\lambda_{cr}} \frac{\lambda^{1+D_T-D_f}}{(\mu_w - \mu_o) X^{D_T + \mu_o L^{D_T}}} d\lambda} \quad (25)$$

From Eq. (25), we can see that fractional flow of water relates to not only fractal structural parameters (such as tortuosity fractal dimension, pore fractal dimension, maximum and minimum diameters of capillaries) but also fluid properties (such as contact angle, viscosity and interfacial tension). That is to say, both fractal porous medium and relevant fluid properties may have important impacts on fractional flow of water. Fractional flow of water is the proportion of flow rate of water in the total flow rate. So if water dominates the flow in a fractal porous medium, the fractional flow value of water is larger. More details will be shown in Sect. 4

In this paper, we use weighted method to differentiate fractional flow of water. The

derivative of fractional flow of water can be expressed as:

$$\begin{aligned}
 f'_w(S_w) &= \frac{\partial f_{wj}}{\partial S_{wj}} \\
 &= \left[\frac{f_{wj} - f_{wj-1}}{S_{wj} - S_{wj-1}} (S_{wj+1} - S_{wj}) + \frac{f_{wj+1} - f_{wj}}{S_{wj+1} - S_{wj}} (S_{wj} - S_{wj-1}) \right] / (S_{wj+1} - S_{wj-1})
 \end{aligned}
 \tag{26}$$

where f_{wj} is the fractional flow of water at the j th time step. f_{wj-1} is the fractional flow of water at the $(j-1)$ th time step. f_{wj+1} is the fractional flow of water at the $(j+1)$ th time step. S_{wj} is the saturation of water at the j th time step. S_{wj-1} is the saturation of water at the $(j-1)$ th time step. S_{wj+1} is the saturation of water at the $(j+1)$ th time step.

Eq. (26) represents first order-derivative of fractional flow of water with respect to independent variable saturation of water. $f'_w(S_w)$ is the rate of change in fractional flow of water with saturation of fluid and it can help us to determine the frontal advance equation.

From Eq. (24), we can get

$$f_w = \frac{v_w}{v_w + v_o}
 \tag{27}$$

So

$$v_w = (v_w + v_o) f_w = v_t f_w
 \tag{28}$$

where v_t is the total velocity of water and oil

Differentiating Eq. (28) with respect to x , we yield

$$\frac{\partial v_w}{\partial x} = v_t \frac{\partial f_w}{\partial x} = v_t \frac{df_w}{dS_w} \frac{\partial S_w}{\partial x}
 \tag{29}$$

$$\frac{\partial v_{wx}}{\partial x} = -\phi \frac{\partial S_w}{\partial t}
 \tag{30}$$

Combining Eq. (29) and Eq. (30), we can get the following expression after arranging:

$$\frac{v_t}{\phi} \frac{df_w}{dS_w} = - \frac{\frac{\partial S_w}{\partial t}}{\frac{\partial S_w}{\partial x}}
 \tag{31}$$

If we ignore the compressibility of rock and fluid, the total velocity is constant and f_w is the function of S_w when the flow regime is steady. i.e $S_w=C$ and $dS_w=0$

$$dS_w = \frac{\partial S_w}{\partial x} dx + \frac{\partial S_w}{\partial t} dt
 \tag{32}$$

After arranging, we can get the following expression:

$$\frac{dx}{dt} = - \frac{\frac{\partial S_w}{\partial t}}{\frac{\partial S_w}{\partial x}} \tag{33}$$

Combining Eq.(32) and Eq.(33)

$$\frac{dx}{dt} = \frac{v_t df_w}{\phi dS_w} \tag{34}$$

Eq. (34) is single phase flow equal saturation plane moving equation which is also called Buckley-Leverett equation.

It is known that

$$v_t = q(t)/A \tag{35}$$

where $q(t)$ is the feed water rate of injection.

Substituting Eq. (35) into Eq. (34), we get

$$\frac{dx}{dt} = \frac{q(t) df_w}{\phi A dS_w} \tag{36}$$

Taking an integration of Eq. (36), we yield

$$\int_{x_0}^x dx = \frac{1}{\phi A} \frac{df_w}{dS_w} \int_0^t q(t) dt \tag{37}$$

where x is the beginning location of transient two phase flow, and x_0 is any location of transient two phase flow.

From Eq. (37), we can see that

$$x - x_0 = \frac{1}{\phi A} \frac{df_w}{dS_w} \int_0^t q(t) dt \tag{38}$$

Based on the fractal theory, the total area of pore can be expressed as the following:

$$A = - \int_{\lambda_{\min}}^{\lambda_{\max}} \frac{\pi \lambda^2}{4} dN = \frac{\pi D_f}{4(2-D_f)} \lambda_{\max}^2 \left[1 - \left(\frac{\lambda_{\min}}{\lambda_{\max}} \right)^{2-D_f} \right] \tag{39}$$

and porosity can be expressed as follows[Yu and Li (2001)]:

$$\phi = \left(\frac{\lambda_{\min}}{\lambda_{\max}} \right)^{2-D_f} \tag{40}$$

Substituting Eqs. (39) and (40) into Eq. (38), we get the expression of water saturation profile:

$$x_f - x_0 = \frac{df_w}{dS_w} \frac{4(2 - D_f) \int_0^t q(t) dt}{\pi D_f \lambda_{max}^2 \left(\frac{\lambda_{min}}{\lambda_{max}}\right)^{2-D_f} \left[1 - \left(\frac{\lambda_{min}}{\lambda_{max}}\right)^{2-D_f}\right]} \tag{41}$$

4 Results and Discussion

Figure 3 shows a typical plot of fractional flow and its derivative curve based on the predictive model of Eq. (25) and Eq. (26). Figure 3 shows an increase in fractional flow of water $f_w(S_w)$ when the saturation of water, S_w , goes up

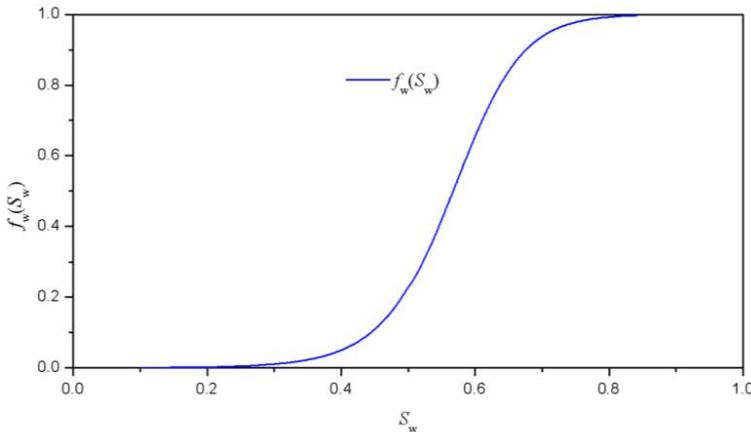


Figure 3: Fractional flow curve.

$(\mu_w=1 \times 10^{-3} Pa.s, \mu_o=1.2 \times 10^{-3} Pa.s, \sigma=0.06 N/m, \theta=0.5, L=1 \times 10^{-2} m, \lambda_{max}=1.5 \times 10^{-4} m, \lambda_{min}=0.2 \times 10^{-7} m, \Delta P=5 \times 10^5 Pa, D_T=1.14, D_f=1.7)$

The determination of the water saturation at the front is shown graphically in Fig. 4.

Using the expression for the front position, and plotting water saturation vs. distance, we get the Fig. 5.

Clearly, the plot of saturation is showing an impossible physical situation, since we have two saturations at each x -position. However, this is a result of the discontinuity in the saturation function, and the Buckley-Leverett solution to this problem is to modify the plot by defining a saturation discontinuity at x_f and balancing of the areas ahead of the front and below the curve, as shown in Fig. 6.

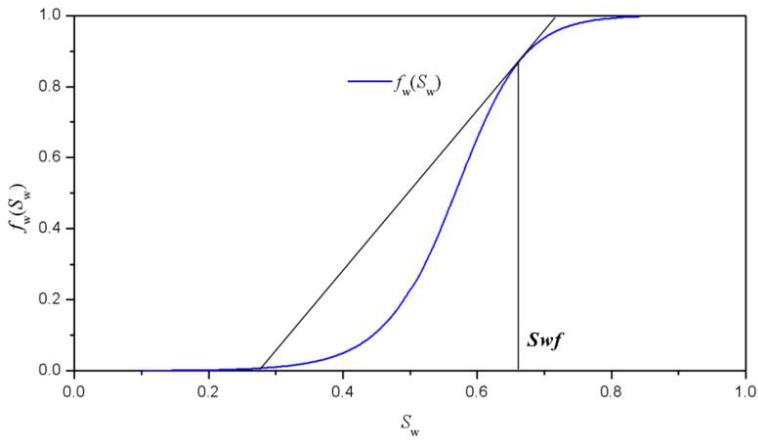


Figure 4: Determination of water saturation at the front.

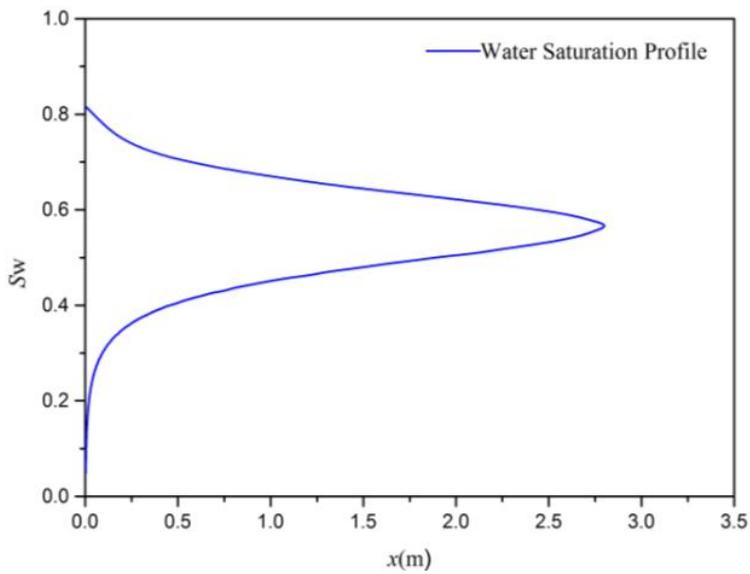


Figure 5: Computed water saturation profile.

The final saturation profile thus becomes Fig. 7.

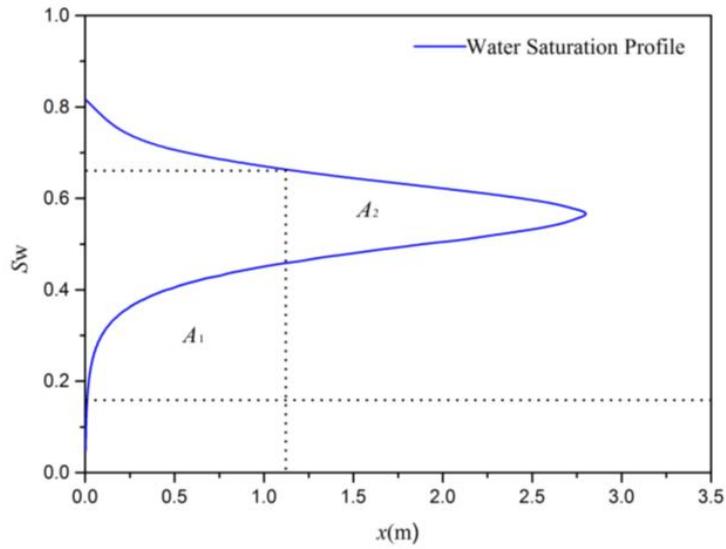


Figure 6: Balancing of areas.

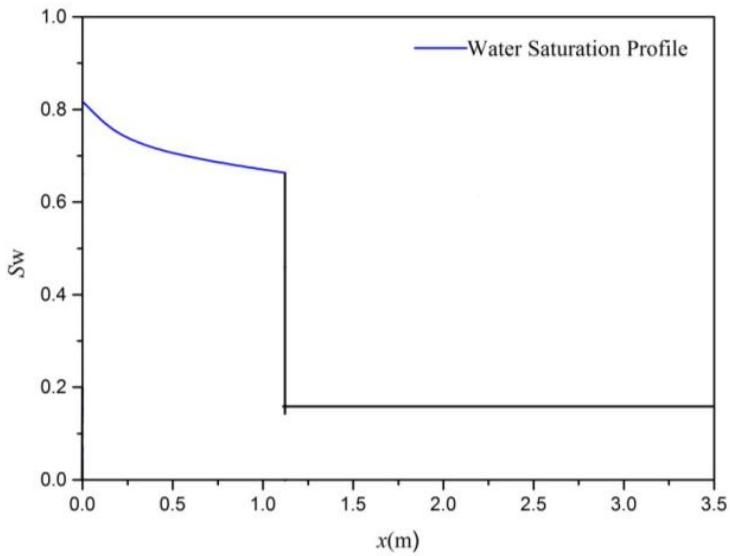


Figure 7: Final water saturation profile.

From Eq. (41), we can see that water saturation profile is affected by fractal structural parameters (such as pore fractal dimension, tortuosity fractal dimension, maximum and minimum diameters of capillaries) and fluid properties (such as viscosity, contact angle and interfacial tension) in fractal porous medium. we emphatically analysis the affection of pore fractal dimension, tortuosity fractal dimension, maximum and minimum diameters of capillaries on water saturation profile.

Figures 8-11 show that water saturation vs. distance at different pore fractal dimensions, D_f , and tortuosity fractal dimension, D_T , respectively.

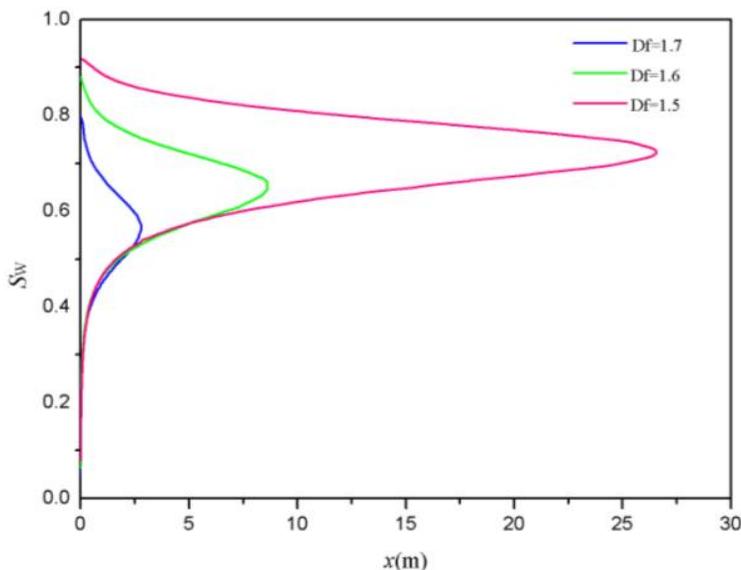


Figure 8: Computed water saturation profile affected by D_f .

D_f represents the cross-sectional distribution of pores in porous medium. The heterogeneity of porous medium increases and the distributions of fluids 1 and 2 in fractal porous medium become more complex with the increase of D_f . Thus the value of two-phase position is smaller at the same water saturation with the increase of D_f as is shown in Fig. 9. This implies it becomes harder for water to displace oil completely with the increase of D_f .

D_T represents the tortuosity of capillaries in porous medium. The flow path length of fluid in a porous medium increases with the increase of D_T , which indicates a higher flow resistance in a fractal porous medium. Thus the heterogeneity of a porous medium enhances and the distributions of oil and water in fractal porous medium become more complicated. Thus the value of two-phase position is smaller

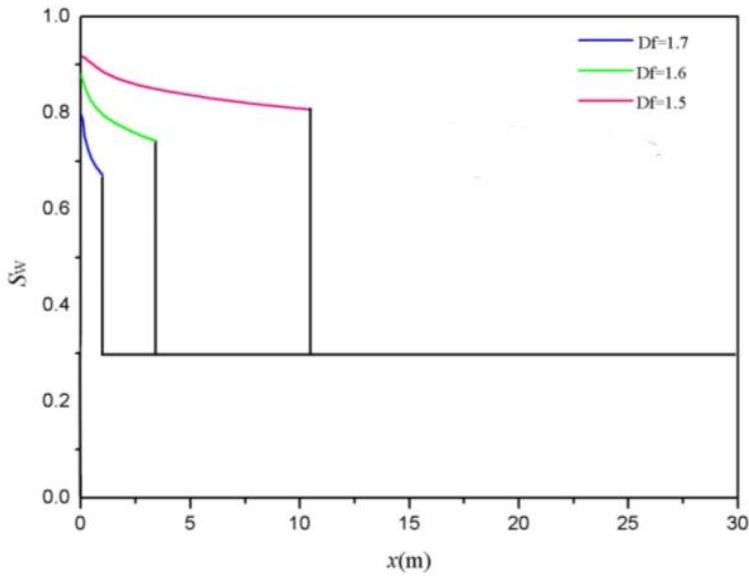


Figure 9: final water saturation profile affected by D_f .
 ($\mu_w=1 \times 10^{-3} Pa.s$, $\mu_o=1.2 \times 10^{-3} Pa.s$, $\sigma=0.06 N/m$, $\theta=0.5$, $L=1 \times 10^{-2} m$,
 $\lambda_{max}=1.5 \times 10^{-4} m$, $\lambda_{min}=0.2 \times 10^{-7} m$, $\Delta P=5 \times 10^5 Pa$, $D_T=1.14$)

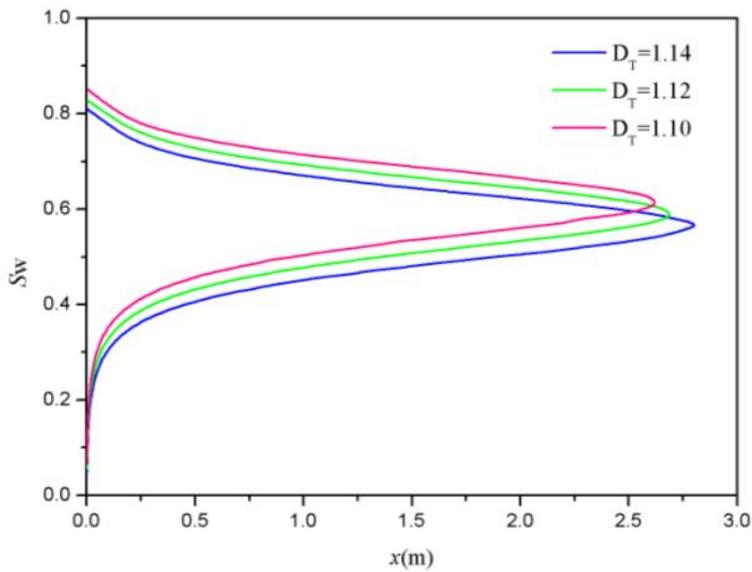


Figure 10: Computed water saturation profile affected by D_T .

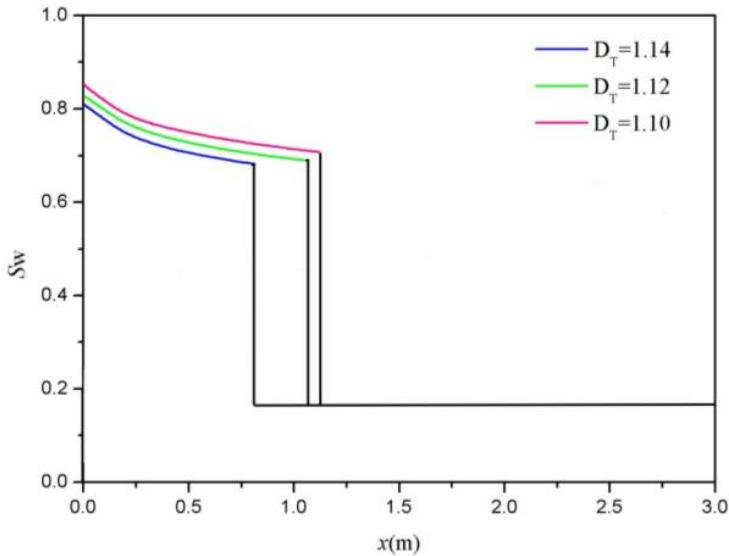


Figure 11: Final water saturation profile affected by D_T .

($\mu_w=1 \times 10^{-3} Pa.s$, $\mu_o=1.2 \times 10^{-3} Pa.s$, $\sigma=0.06 N/m$, $\theta=0.5$, $L=1 \times 10^{-2} m$, $\lambda_{max}=1.5 \times 10^{-4} m$, $\lambda_{min}=0.2 \times 10^{-7} m$, $\Delta P=5 \times 10^5 Pa$, $D_f=1.7$)

at the same water saturation with the increase of D_T as is shown in Fig. 11. This implies it becomes harder for water to displace oil completely with the increase of D_T .

Based on Eq. (41), figures 12-15 show that water saturation vs. distance at different minimum pore diameter, λ_{min} , and maximum pore diameter λ_{max} , respectively.

For a given λ_{min} , fractal porous medium is better in permeability if the value of λ_{max} is bigger. Because of higher value of permeability, the position of two-phase is bigger at the same water saturation with the increase of λ_{max} as is shown in Fig. 13. This implies it becomes more easily for water to displace oil completely with the increase of λ_{max} .

Similarly, fractal porous medium is better in permeability if the value of λ_{min} is bigger. Because of higher value of permeability, the position of two-phase is bigger at the same water saturation with the increase of λ_{max} as is shown in Fig. 15. This implies it becomes more easily for water to displace oil completely with the increase of λ_{min} .

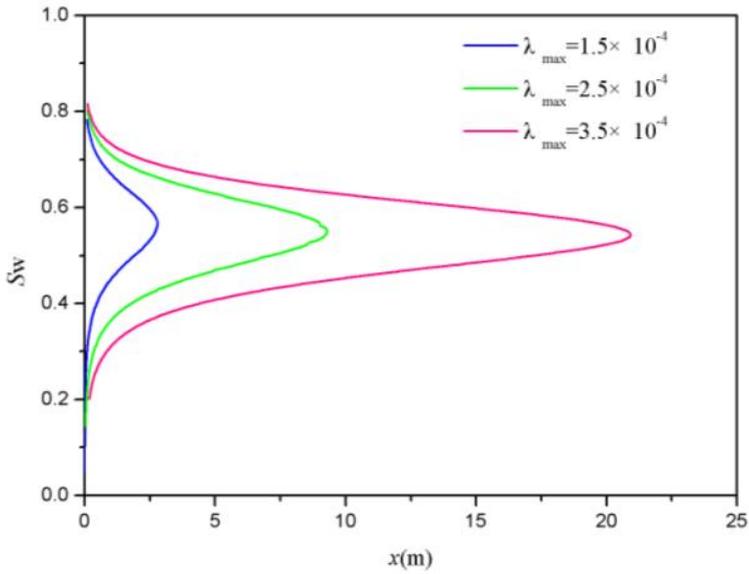


Figure 12: Computed water saturation profile affected by λ_{max} .

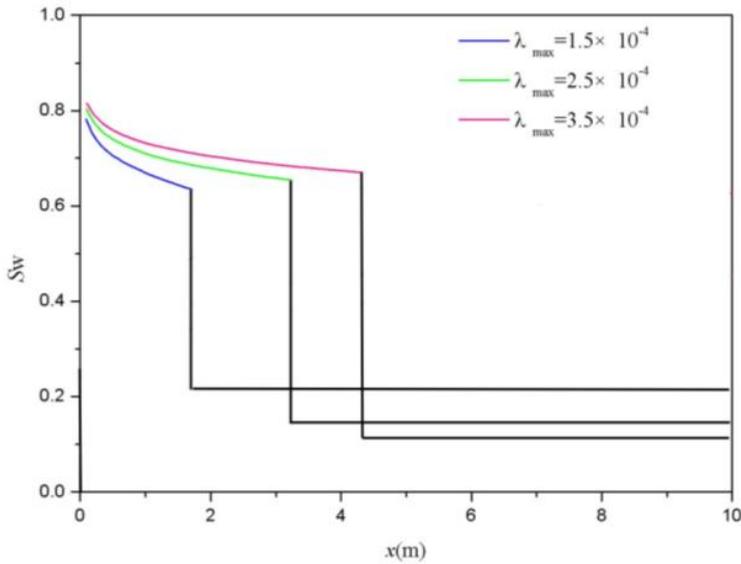


Figure 13: Final water saturation profile affected by λ_{max} .
 ($\mu_w=1 \times 10^{-3} Pa.s$, $\mu_o=1.2 \times 10^{-3} Pa.s$, $\sigma=0.06 N/m$, $\theta=0.5$, $L=1 \times 10^{-2} m$,
 $\lambda_{min}=0.2 \times 10^{-7} m$, $\Delta P=5 \times 10^5 Pa$, $D_T=1.14$, $D_f=1.7$).

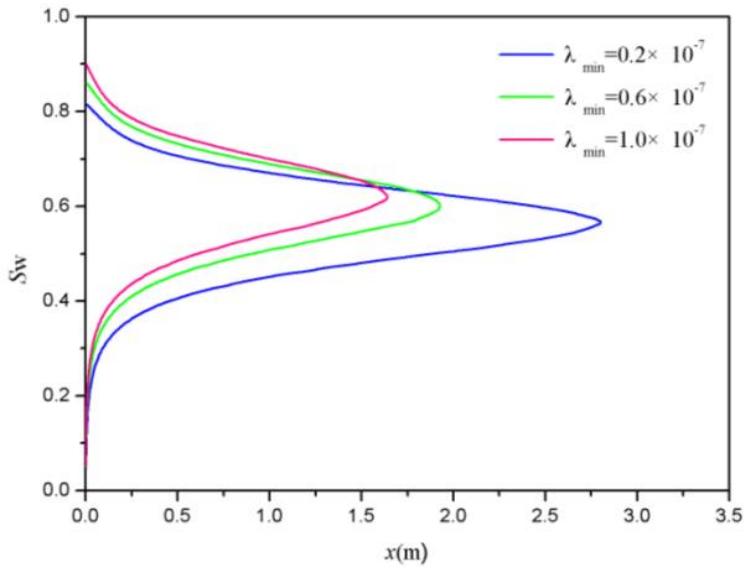


Figure 14: Computed water saturation profile affected by λ_{min} .

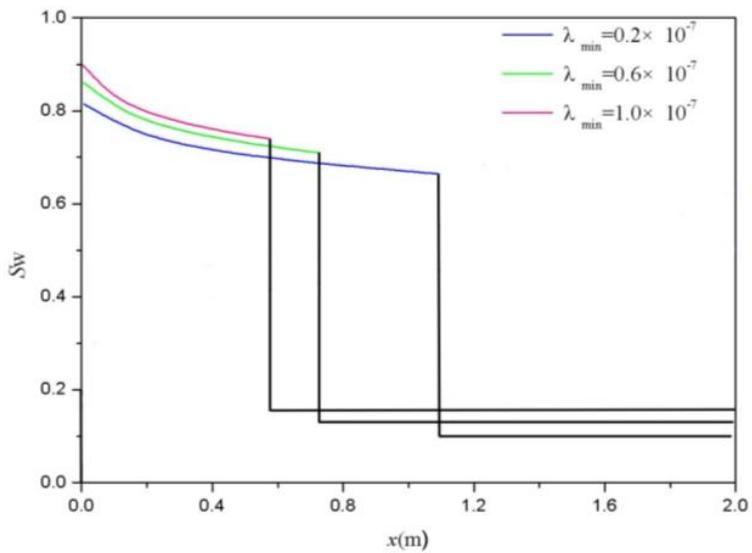


Figure 15: Final water saturation profile affected by λ_{min} .
 $(\mu_w=1 \times 10^{-3} Pa.s, \mu_o=1.2 \times 10^{-3} Pa.s, \sigma=0.06 N/m, \theta=0.5, L=1 \times 10^{-2} m$
 $\lambda_{max}=1.5 \times 10^{-4} m, \Delta P=5 \times 10^5 Pa, D_T=1.14, D_f=1.7).$

5 Conclusions

On the approximation that a porous medium consist of a bundle of tortuous capillaries with different diameters, we have derived a Buckley-Leverett model based on the fractal characteristics of pore size distribution in porous medium. The proposed model connects any location of transient two phase flow with the structural parameters (e.g. tortuosity fractal dimension, pore fractal dimension, and maximum and minimum pore diameters) and fluid properties (e.g. interfacial tension, contact angle, viscosities and so on) of fractal porous medium. In this work, we emphatically exhibit the process how to predict the fluid saturation profile based on fractal theory and Buckley-Leverett analysis.

Some conclusions have been obtained as follows:

1. Based on the fractional flow model using fractal theory, we derive Buckley-Leverett solution to analyze water saturation profile for transient two-phase flow in porous medium.
2. With the increase of D_f and D_T , the value of two-phase position is bigger when at the same water saturation. The permeability of porous media is better and it will take a shorter time for water to completely displace oil.
3. The permeability of porous media is better with the increase of λ_{max} and λ_{min} , the value of two-phase position is bigger at the same water saturation. It's much easier for water to completely displace oil.

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