

On Collision Local Time of Two Independent Subfractional Brownian Motions

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Abstract: We study the existence of collision local time of two independent subfractional Brownian motions with different coefficients in $(-\frac{1}{2}, \frac{1}{2})$ using an alternative expression. We prove that the collision local time is a Hida distribution based on the canonical framework of white noise analysis, and get chaos expansions. Finally, we show that the collision local time exists in (L^2) under appropriate conditions.

Keywords: Subfractional Brownian motion, Collision local time, Hida distribution.

1 Introduction

Stochastic processes can make models in Biology, Physics, Engineering and so on, which have become an important tool to master for scientific and technological workers. In this paper, we mainly consider a self-similar Gaussian process - subfractional Brownian motion (sfBm). Let $S_t^{k_i}$ ($i = 1, 2$) be two independent subfractional Brownian motions (sfBms) with different parameter $k_i \in (-\frac{1}{2}, \frac{1}{2})$ on \mathbb{R}^d . For each $i = 1, 2$, $S_t^{k_i}$ is a centered Gaussian process with representation

$$S_t^{k_i} = \frac{1}{c(k_i)} \int_{\mathbb{R}} [(t-s)_+^{k_i} + (t+s)_-^{k_i} - 2(-s)_+^{k_i}] dW_s, \quad (1)$$

where $c(k_i) = [2(\int_0^\infty ((1+s)^{k_i} - s^{k_i})^2 + \frac{1}{2k_i+1})]^{1/2}$, $a_+ = \max\{a, 0\}$, $a_- = \max\{-a, 0\}$ and W is a Brownian motion (Bm), which means that sfBm is an extension of Bm, a rather special class of self-similar Gaussian process.

The object of study in this paper will be collision local time of S^{k_1} and S^{k_2} , which

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is formally defined as

$$L_k = \int_I \delta(S_t^{k_1} - S_t^{k_2}) dt, \quad (2)$$

where $\delta(x)$ is a Dirac delta function, $I = [0, T]$ and $T > 0$. The intuitive idea of the local time L_k for two processes $S_t^{k_i}$ ($i = 1, 2$) is that L_k characterizes collision time during the interval $[0, T]$.

For $k_i = 0$ ($i = 1, 2$), processes S^{k_1} and S^{k_2} are classical Brownian motions (Bms). The local time of Bm has been studied by many authors. In recent years, some authors focus on the research on fractional integral process and related problems, e.g. fractional Brownian motion (fBm), due to its interesting properties and its applications in various scientific areas such as telecommunications, turbulence and finance [Biagini et. al (2008)]. Others have studied the numerical solution of fractional-integral differential equations [Chen et. al (2014); Wang et. al (2015)]. In general, sfBm is intermediate between Bm and fBm. The local times of sfBm have been studied by many authors as well, e.g. [Liu et. al (2012)] for the intersection case and [Yan et. al (2010)] for the collision case, where authors have proved that the local time is smooth in the sense of Meyer and Watanabe.

In spite of sfBm has many properties analogous of fBm such as self-similar and long-rang dependent, sfBm has non-stationary increments and weakly correlated in comparing with fBm. On account of the complexity structure of sfBm, people pay little attention on these process. Owing to sfBm be not semimartingale (or Markov process), many classical methods in stochastic analysis can not deal with the problems of sfBm. If we can get a continuous version of sfBm, it will be effective to study the existence local time of sfBm through white noise analysis approach. White noise is an original acoustic concept. In engineering technology, engineers often use the term of white noise to represent a kind of random disturbance in the dynamic system. For a long time, in order to give strict and reasonable mathematical meaning of white noise, Hida put forward infinite dimensional distribution theory similar to Schwart distribution theory -white noise analysis. Now white noise analysis becomes an effective method to deal with the problems of infinite dimensional space.

In this paper, motivated by [Oliveira et. al (2011); Liu et. al (2012)], we give an alternative expression of sfBm by using odd extension and fractional integrals operators, and study the existence of the collision local time of two independent sfBms with the different coefficients in $(-\frac{1}{2}, \frac{1}{2})$. We prove that the collision time is a Hida distribution and belongs to (L^2) , respectively. The paper is organized as follows. In Section 2, we provide some background material from white noise analysis. In Section 3, we present the main results and their demonstrations.

2 White noise analysis

In this section, we briefly recall some notions and facts in white noise analysis, for details see Refs [Bender (2003); Biagin et. al (2008); Oliveira et. al (2011)].

The first real Gelfand triple is $\mathcal{S}_{2d}(\mathbb{R}) \subset L^2(\mathbb{R}, \mathbb{R}^{2d}) \subset \mathcal{S}_{2d}^*(\mathbb{R})$, where $\mathcal{S}_{2d}(\mathbb{R})$ and $\mathcal{S}_{2d}^*(\mathbb{R})$ are the Schwartz spaces of the vector-valued test functions and tempered distributions, respectively. Denote the norm $|\cdot|$ in $L^2(\mathbb{R}, \mathbb{R}^{2d})$ and the dual pairing between $\mathcal{S}_{2d}^*(\mathbb{R})$ and $\mathcal{S}_{2d}(\mathbb{R})$ by $\langle \cdot, \cdot \rangle$, respectively. We consider two-independent d-tuple of Gaussian white noises $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$, where $\mathbf{w}_i = (w_{i,1}, \dots, w_{i,d})$. For every test function $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2)$ on $\mathcal{S}_{2d}(\mathbb{R})$, $\mathbf{f}_i \in \mathcal{S}_d(\mathbb{R})$, the characteristic function of vector-valued white noise \mathbf{w} is given by $C(\mathbf{f}) = E(e^{i\sum_{k=1}^2 \langle w_k, f_k \rangle}) = e^{-\frac{1}{2} \langle \mathbf{f}, \mathbf{f} \rangle}$.

Introduce the following notations:

$$\mathbf{n} = (n_1, \dots, n_d), \quad n = \sum_{i=1}^d n_i, \quad \mathbf{n}! = \prod_{i=1}^d n_i!$$

Let $(L^2) \equiv L^2(\mathcal{S}_{2d}^*(\mathbb{R}), d\mu)$ be the Hilbert space of square integrable functionals with respect to Lebesgue measure μ on $\mathcal{S}_{2d}^*(\mathbb{R})$. By the Wiener-Itô-Segal isomorphism theorem, we have chaos expansion

$$f(\mathbf{w}) = \sum_{\mathbf{m} \in \mathbb{N}^d} \sum_{\mathbf{k} \in \mathbb{N}^d} \langle : \mathbf{w}_1^{\otimes \mathbf{m}} : \otimes : \mathbf{w}_2^{\otimes \mathbf{k}} : , \mathbf{f}_{\mathbf{m}, \mathbf{k}} \rangle,$$

for each $f \in (L^2)$.

Let $\Gamma(A)$ be the second quantization of A , where A is defined by $(\mathbf{A}g)_i(t) = (-\frac{d^2}{dt^2} + t^2 + 1)g_i(t)$. For each integer p , let (\mathcal{S}_p) be the completion of $\text{Dom}\Gamma(A)^p$ with respect to the Hilbert norm $\|\cdot\|_p = \|\Gamma(A)^p \cdot\|_0$. Let $(\mathcal{S}) = \bigcap_{p \geq 0} (\mathcal{S}_p)$ be the projective limit of $\{(\mathcal{S}_p) \mid p \geq 0\}$ and be $(\mathcal{S})^* = \bigcup_{p \geq 0} (\mathcal{S}_{-p})$ the inductive limit of $\{(\mathcal{S}_{-p}) \mid p \geq 0\}$, respectively. Thus, there is the second Gelfand triple: $(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$. Elements of (\mathcal{S}) (resp. $(\mathcal{S})^*$) are called Hida testing (resp. generalized) functionals. For $\mathbf{f} \in \mathcal{S}_{2d}(\mathbb{R}, \mathbb{R}^{2d})$, the S -transform is defined by $S\Phi(\mathbf{f}) = \ll \Phi, : \exp \langle \cdot, \mathbf{f} \rangle : \gg$.

Definition 2.1. A mapping $G : \mathcal{S}_{2d}(\mathbb{R}) \rightarrow \mathbb{C}$ is called a U -functional if

- (1) $G(\lambda \mathbf{f}_1 + \mathbf{f}_2)$ is entire in λ for any pair $\mathbf{f}_i \in \mathcal{S}_{2d}(\mathbb{R}, \mathbb{R}^{2d})$ of test functions;
- (2) $|G(\mathbf{z}\mathbf{f})| \leq C_1 \exp\{C_2 |z|^2 |A^p \mathbf{f}|_2^2\}$ with $C_i, p > 0$, for any complex z .

Lemma 2.2. Let $\{G_k\}_{k \in \mathbb{N}}$ denote a sequence of U -functional with following properties:

- (1) for all $\mathbf{f} \in \mathcal{S}_{2d}(\mathbb{R}, \mathbb{R}^{2d})$, $\{G_k(\mathbf{f})\}_{k \in \mathbb{N}}$ is a Cauchy sequence;
- (2) there exist C_i and p such that $|G_k(\mathbf{z}\mathbf{f})| \leq C_1 \exp\{C_2 |z|^2 |A^p \mathbf{f}|_2^2\}$ uniformly in \mathbb{R} .

Then, there is a unique $\Phi \in (\mathcal{S})^*$ such that $S^{-1}G_k$ converges strongly to Φ .

Lemma 2.3. Let $(\Omega, \mathfrak{B}, \mu)$ be a measure space, and let Φ_λ be a mapping defined on Ω with values in $(\mathcal{S})^*$. We assume S -transform of Φ

- (1) is an μ -measurable function of λ for $\mathbf{f} \in \mathcal{S}_{2d}(\mathbb{R}, \mathbb{R}^{2d})$; and

(2) obeys a U-functional estimate

$$|S\Phi_\lambda(zf)| \leq C_1(\lambda) \exp\{C_2(\lambda) |z|^2 |A^p f|_2^2\}$$

for some fixed p and for $C_1 \in L^1(\mu)$, $C_2 \in L^\infty(\mu)$.

Then

$$\int_\Omega \Phi_\lambda d\mu(\lambda) \in (\mathcal{S})^* \quad \text{and} \quad S(\int_\Omega \Phi_\lambda d\mu(\lambda))(f) = \int_\Omega (S\Phi_\lambda)(f) d\mu(\lambda).$$

3 Collision local time

In this section, our main aim is to study the existence of the collision local time of two independent sfBms $S^{k_i} = \{S^{k_i}(t), t \geq 0\}$ ($i = 1, 2$) as (2). From now, we always approximate the Dirac delta function by the heat kernel

$$p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} \exp\{-\frac{x^2}{2\varepsilon}\}.$$

For any $\varepsilon > 0$, we define

$$L_{k,\varepsilon} = \int_I p_\varepsilon(S^{k_1}(t) - S^{k_2}(t)) dt. \tag{3}$$

To get our main results, we give an alternative expression of sfBm S_t^k using fractional integrals operators I_\pm^k and odd extension.

Lemma 3.1. Let $k \in (-\frac{1}{2}, \frac{1}{2})$ be given. Subfractional Brownian motion S_t^k has a continuous version of $\langle \cdot, \frac{1}{c(k)} I_-^k \mathbb{I}_{[0,t]}^o \rangle$, where $\mathbb{I}_{[0,t]}^o$ denotes the odd extension of $\mathbb{I}_{[0,t]}$ and $c(k) = [2(\int_0^\infty ((1+s)^k - s^k)^2 + \frac{1}{2k+1})]^{1/2}$.

proof: For $f : \mathbb{R}^+ \rightarrow \mathbb{R}$, we define its odd extension $f^o(x)$ in [Tudor (2003)] as follows:

$$\begin{cases} -f(-x), & x < 0 \\ f(x), & x \geq 0. \end{cases} \tag{4}$$

For given $\alpha \in (0, 1)$, we obtain

$$(I_-^\alpha f)(x) \equiv \frac{1}{\Gamma(\alpha)} \int_x^\infty f(t)(t-x)^{\alpha-1} dt = \frac{1}{\Gamma(\alpha)} \int_0^\infty f(x+t)t^{\alpha-1} dt,$$

$$(I_+^\alpha f)(x) \equiv \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x f(t)(x-t)^{\alpha-1} dt = \frac{1}{\Gamma(\alpha)} \int_0^\infty f(x-t)t^{\alpha-1} dt,$$

if the integrals exist for all $x \in \mathbb{R}$. For an arbitrary parameter $k \in (0, \frac{1}{2})$, using fractional integrals of Weyl's type an alternative representation of S_t^k is given by

$$S_t^k = \frac{\Gamma(k+1)}{c(k)} \int_{\mathbb{R}} I_-^k(\mathbb{I}_{[0,t]}^o)(s) dW_s, \tag{5}$$

where W is a Wiener process. Integrand in (5) yields

$$I_-^k(\mathbb{I}_{[0,t]}^o)(s) = \frac{1}{\Gamma(k+1)} [(t-s)_+^k + (t+s)_-^k - 2(-s)_+^k]. \tag{6}$$

On the other hand, apart from fractional derivatives I_{\pm}^{α} , we shall also use fractional derivatives operators. For $\alpha \in (0, 1)$ and $\varepsilon > 0$, we have

$$(D_{\pm, \varepsilon}^{\alpha} f)(x) \equiv \frac{\alpha}{\Gamma(1-\alpha)} \int_{\varepsilon}^{\infty} \frac{f(x) - f(x \mp t)}{t^{\alpha+1}} dt.$$

The fractional derivatives of Marchaud’s type are given by $D_{\pm}^{\alpha} f \equiv \lim_{\varepsilon \rightarrow 0+} D_{\pm, \varepsilon}^{\alpha} f$, if the integrals exist almost surely. In this case, the continuous version of S_t^k is $\langle \cdot, \frac{1}{c(k)} D_{\pm}^{-k} \mathbb{I}_{[0,t]}^{\circ} \rangle$. Using $I^{\alpha} = D^{-\alpha}$, we obtain the same form of continuous version. From what we said above, we can safely see that a sfBm is given by a continuous version of $\langle \cdot, \frac{1}{c(k)} I_{\pm}^k \mathbb{I}_{[0,t]}^{\circ} \rangle$, for $k \in (0, \frac{1}{2})$.

The case $k \in (-\frac{1}{2}, 0)$ can be considered similarly.

The following lemma is very useful to prove our main results. [Bender (2003); Drumond et al. (2008)] have given the similar estimate in discussion the local time of fBm, respectively.

Lemma 3.2. *Let $k \in (-\frac{1}{2}, \frac{1}{2})$ and $f \in \mathcal{S}_1(\mathbb{R})$ be given. Then there exists a non-negative constant C_k such that*

$$\left| \int_{\mathbb{R}} f(x) \frac{1}{c(k)} (I_{\pm}^k \mathbb{I}_{[s,t]}^{\circ})(x) dx \right| \leq C_k |t - s| \|f\|, \tag{7}$$

where $c(k) = [2(\int_0^{\infty} ((1+s)^k - s^k)^2 + \frac{1}{2k+1})]^{\frac{1}{2}}$, C_k is some constant independent of f and $\|f\| \equiv \sup_{x \in \mathbb{R}} |f(x)| + \sup_{x \in \mathbb{R}} |f'(x)|$.

proof: Recall that $\{S_t^k\}$ has the average representation

$$S_t^k = \frac{1}{c(k)} \int_{\mathbb{R}} [(t-s)_+^k + (t+s)_-^k - 2(-s)_+^k] dW_s, \tag{8}$$

where $\{W_t\}$ is a Bm and $c(k) = [2(\int_0^{\infty} ((1+s)^k - s^k)^2 + \frac{1}{2k+1})]^{\frac{1}{2}}$. By Lemma 3.1, (8) becomes

$$S_t^k = \frac{\Gamma(k+1)}{c(k)} \int_{\mathbb{R}} I_{\pm}^k (\mathbb{I}_{[0,t]}^{\circ})(s) dW_s. \tag{9}$$

For $s \leq t$ and given $f \in \mathcal{S}_1(\mathbb{R})$, we obtain

$$\begin{aligned} & \int_{\mathbb{R}} f(x) \frac{1}{c(k)} (I_{\pm}^k \mathbb{I}_{[s,t]}^{\circ})(x) dx \\ &= \frac{1}{c(k)} \left[\int_0^{\infty} f(-x) (I_{\pm}^k \mathbb{I}_{[s,t]}^{\circ})(x) dx + \int_0^{\infty} f(x) (I_{\pm}^k \mathbb{I}_{[s,t]}^{\circ})(x) dx \right] \\ &= \frac{1}{c(k)} \int_0^{\infty} (f(-x) + f(x)) (I_{\pm}^k \mathbb{I}_{[s,t]}^{\circ})(x) dx. \end{aligned}$$

Since I_-^k and I_+^k are dual operators, note that

$$\begin{aligned} & \frac{1}{c(k)} \left[\int_0^\infty (f(-x) + f(x))(I_-^k \mathbb{I}_{[s,t]})(x) dx \right] \\ &= \frac{1}{c(k)} \left[\int_s^t (I_+^k (f(-x) + f(x)))(x) dx \right] \\ &\leq \frac{1}{c(k)} |t - s| \sup_{x \in \mathbb{R}^+} |I_+^k (f(-x) + f(x))|. \end{aligned}$$

Next we will discuss the estimation of the integral for different k .

Step 1. For $0 < k < \frac{1}{2}$, it has

$$\begin{aligned} & \int_0^\infty |f(u)| |x - u|^{k-1} du \\ &\leq \int_{\mathbb{R}} |f(u)| |x - u|^{k-1} du \\ &= \int_{|u-x|<1} |f(u)| |x - u|^{k-1} du + \int_{|u-x|\geq 1} |f(u)| |x - u|^{k-1} du \\ &\equiv \Delta_1 + \Delta_2, \end{aligned}$$

where

$$\begin{aligned} \Delta_1 &\equiv \int_{|u-x|<1} |f(u)| |x - u|^{k-1} du \\ &< \max_{x \in \mathbb{R}} |f(x)| \int_{|u-x|<1} |x - u|^{k-1} du \\ &= \max_{x \in \mathbb{R}} |f(x)| \int_{-1}^1 |u|^{k-1} du \\ &= \frac{2}{k} \max_{x \in \mathbb{R}} |f(x)|. \end{aligned}$$

For the second part, we have

$$\Delta_2 \leq \int_{|u-x|\geq 1} |f(u)| du \leq \|f\|_{L^1(\mathbb{R})}.$$

As similar techniques in the proof of Lemma 6 in [Drumond et. al (2008)] and applying Schwartz inequality, another integral becomes

$$\int_{|u-x|\geq 1} |f(u)| |u - x|^{k-1} du.$$

Therefore

$$\int_0^\infty |f(u)| |u - x|^{k-1} du \leq \frac{2}{k} \max_{x \in \mathbb{R}} |f(x)| + \|f\|_{L^1(\mathbb{R})}.$$

Next using similar methods, we estimate

$$\begin{aligned} & \int_0^\infty |f(-u)| | -u - x |^{k-1} du \\ & \leq \int_{|u+x|<1} |f(-u)| |u+x|^{k-1} du + \int_{|u+x|\geq 1} |f(-u)| |u+x|^{k-1} du \\ & \equiv \Delta_3 + \Delta_4. \end{aligned}$$

At the same time, we obtain

$$\begin{aligned} \Delta_3 &= \int_{|u+x|<1} |f(-u)| |u+x|^{k-1} du \\ &\leq \max\{f(-x)\} \int_{-1}^1 |u|^{k-1} du \\ &= \frac{2}{k} \max_{x \in \mathbb{R}} |f(-x)|, \end{aligned}$$

and

$$\begin{aligned} \Delta_3 &= \int_{|u+x|<1} |f(-u)| |u+x|^{k-1} du \\ &\leq \max\{f(-x)\} \int_{-1}^1 |u|^{k-1} du \\ &= \frac{2}{k} \max_{x \in \mathbb{R}} |f(-x)|, \end{aligned}$$

For fixed $x \in \mathbb{R}$ and $0 < k < \frac{1}{2}$, we have

$$\begin{aligned} & \int_0^\infty |f(u)| |u-x|^{k-1} du + \int_0^\infty |f(-u)| | -u - x |^{k-1} du \\ & \leq \frac{2}{k} \max_{x \in \mathbb{R}} |f(x)| + \|f\|_{L^1(\mathbb{R})} + \frac{2}{k} \max_{x \in \mathbb{R}} |f(-x)| + \|f\|_{L^1(\mathbb{R})} \\ & \leq 2\left(\frac{2}{k} \max_{x \in \mathbb{R}} |f(x)| + \|f\|_{L^1(\mathbb{R})}\right). \end{aligned}$$

Hence

$$\begin{aligned} & \int_{\mathbb{R}} f(x) \frac{1}{c(k)} (I_{-}^k \mathbb{I}_{[s,t]}^o)(x) dx \\ & \leq C_{k,1} |t-s| \left(\frac{2}{k} \max_{x \in \mathbb{R}} |f(x)| + \|f\|_{L^1(\mathbb{R})}\right), \end{aligned}$$

where constant $C_{k,1} = 2(c(k)\Gamma(k))^{-1}$ depending on k .

Step 2. For $-\frac{1}{2} < k < 0$, by the proof of Theorem 2.3 in [Bender (2003)] again, we

have

$$\sup_{x \in \mathbb{R}^+} |(I_+^k f)(x)| \leq C_{k,2} \frac{-2}{k} \sup_{x \in \mathbb{R}^+} |f(x)| + \frac{1}{k+1} \sup_{x \in \mathbb{R}^+} |f'(x)|, x > 0,$$

$\sup_{x \in \mathbb{R}^+} |(I_+^k f)(-x)| \leq C_{k,3} \frac{-2}{k} \sup_{x \in \mathbb{R}^+} |f(-x)| + \frac{1}{k+1} \sup_{x \in \mathbb{R}^+} |f'(-x)|, x > 0,$
 where $C_{k,2}$ and $C_{k,3}$ are both constants dependent on k . Thus we have shown that there exists a constant $C_{k,4}$ such that

$$\sup_{x \in \mathbb{R}} |(I_+^k f)(x)| \leq C_{k,4} \frac{-2}{k} \sup_{x \in \mathbb{R}} |f(x)| + \frac{1}{k+1} \sup_{x \in \mathbb{R}} |f'(x)|, x \in \mathbb{R}.$$

Therefore, for an arbitrary parameter $k \in (-\frac{1}{2}, \frac{1}{2})$, we obtain

$$\int_{\mathbb{R}} f(x) \frac{1}{c(k)} (I_{-}^k \mathbb{I}_{[s,t]}^o)(x) dx \leq C_k |t - s| \|f\|,$$

where $\|f\| \equiv \sup_{x \in \mathbb{R}} |f(x)| + \sup_{x \in \mathbb{R}} |f'(x)|$ and C_k is a constant independent f . □

Theorem 3.3. For each $k_1, k_2 \in (-\frac{1}{2}, \frac{1}{2})$ and every positive integer $d \geq 1$ such that $(\min\{k_1, k_2\} + \frac{1}{2})d < 1$, the collision local time of two independent sfBMs S^{k_1} and S^{k_2} given by

$$L_{k,\varepsilon} \equiv \int_I dt p_\varepsilon(S^{k_1}(t) - S^{k_2}(t)) = \int_I dt \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{d}{2}} \exp\left\{-\frac{(S^{k_1}(t) - S^{k_2}(t))^2}{2\varepsilon}\right\},$$

is a Hida distribution. Moreover, $L_{k,\varepsilon}$ has the following chaos expansion

$$L_{k,\varepsilon} = \sum_m \sum_l \langle : w_1^{\otimes m} : \otimes : w_2^{\otimes l} :, \mathbf{G}_{m,l} \rangle,$$

where the kernel functions

$$\begin{aligned} & \mathbf{G}_{m,l}(u_1, \dots, u_m, v_1, \dots, v_l) \\ &= (-1)^l \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \left(-\frac{1}{2}\right)^{\frac{m+l}{2}} \frac{1}{\left(\frac{m+l}{2}\right)!} \frac{(m+l)!}{m!l!} \int_I dt \left(\frac{1}{\varepsilon + (2-2^{2k_1})t^{2k_1+1} + (2-2^{2k_2})t^{2k_2+1}}\right)^{\frac{d+m+l}{2}} \\ & \cdot \prod_{i=1}^m \left(\frac{1}{c(k_1)} I_{-}^{k_1} \mathbb{I}_{[0,t]}^o\right)(u_i) \prod_{j=1}^l \left(\frac{1}{c(k_2)} I_{-}^{k_2} \mathbb{I}_{[0,t]}^o\right)(v_j) \end{aligned}$$

for $m+k \neq 0$, $m+l$ even, and zero otherwise.

proof: In order to prove the result, we only apply Lemma 2.3 to the S -transform of the integral with respect to Lebesgue measure dt on I .

Denote

$$\Phi_{k,\varepsilon}(\mathbf{w}_1, \mathbf{w}_2) \equiv \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{d}{2}} \exp\left\{-\frac{(S^{k_1}(t) - S^{k_2}(t))^2}{2\varepsilon}\right\}.$$

By the definition of S -transform, we know that

$$S\Phi_{k,\varepsilon}(\mathbf{f}) = \left(\frac{1}{2\pi[\varepsilon + (2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1}]} \right)^{\frac{d}{2}} \cdot \exp\left\{ -\frac{(\int_{\mathbb{R}} d\mathbf{s}\mathbf{f}(s)[(\frac{1}{c(k_1)}I_{[0,t]}^{k_1}) (s) - (\frac{1}{c(k_2)}I_{[0,t]}^{k_2}) (s)])^2}{2[\varepsilon + (2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1}]} \right\},$$

For any complex number z and $\mathbf{f} \in \mathcal{S}_d(\mathbb{R})$, it implies

$$|S\Phi_{k,\varepsilon}(z\mathbf{f})| \leq \left(\frac{1}{2\pi[\varepsilon + (2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1}]} \right)^{\frac{d}{2}} \cdot \exp\left\{ \frac{|z|^2 \int_{\mathbb{R}} d\mathbf{s}\mathbf{f}(s)[(\frac{1}{c(k_1)}I_{[0,t]}^{k_1}) (s) - (\frac{1}{c(k_2)}I_{[0,t]}^{k_2}) (s)]^2}{2[\varepsilon + (2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1}]} \right\}.$$

To check the boundary condition in Lemma 2.3, introduce norm $\|\cdot\|$ in $\mathcal{S}_d(\mathbb{R})$ defined by

$$\|\mathbf{f}\| \equiv \left(\sum_{i=1}^d (\sup_{x \in \mathbb{R}} |f_i(x)| + \sup_{x \in \mathbb{R}} |f'_i(x)|)^2 \right)^{\frac{1}{2}}, \quad \mathbf{f} = (f_1, \dots, f_d) \in \mathcal{S}_d(\mathbb{R}).$$

By Lemma 3.2 we obtain

$$\begin{aligned} & \left| \int_{\mathbb{R}} d\mathbf{s}\mathbf{f}(s) \left(\frac{1}{c(k_1)}I_{[0,t]}^{k_1} (s) - \int_{\mathbb{R}} d\mathbf{s}\mathbf{f}(s) \left(\frac{1}{c(k_2)}I_{[0,t]}^{k_2} (s) \right) \right)^2 \right| \\ & \leq 2 \left\{ \left(\int_{\mathbb{R}} d\mathbf{s}\mathbf{f}(s) \left(\frac{1}{c(k_1)}I_{[0,t]}^{k_1} (s) \right) \right)^2 + \left(\int_{\mathbb{R}} d\mathbf{s}\mathbf{f}(s) \left(\frac{1}{c(k_2)}I_{[0,t]}^{k_2} (s) \right) \right)^2 \right\} \\ & \leq 2(C_{k_1}^2 t^2 + C_{k_2}^2 t^2) \|\mathbf{f}\|^2 \\ & \leq C_{k,0} t^2 \|\mathbf{f}\|^2, \end{aligned}$$

where $C_{k,0}$, C_{k_1} and C_{k_2} are not negative constants.

So

$$|S\Phi_{k,\varepsilon}(z\mathbf{f})| \leq \left(\frac{1}{2\pi[\varepsilon + (2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1}]} \right)^{\frac{d}{2}} \cdot \exp\left\{ \frac{|z|^2 C_{k,0} t^2 \|\mathbf{f}\|^2}{2[\varepsilon + (2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1}]} \right\},$$

where the first part is integrable on I if $(\min\{k_1, k_2\} + \frac{1}{2})d < 1$, and the second part is bound.

To get the kernel functions of $G_{\mathbf{m},1}$, we consider the S -transform of $L_{H,\varepsilon}$. Comparing with the general form of the chaos expansion, we find the kernel functions. \square

On account of the local time, it is desirable to be regularized by sequences of Gaussian, and renormalized functions of local time need be subtracted since the functions in high dimensions fail to exist without subtractions [Nualart et. al (2007)]. We are now ready to prove that collision local time L_k is as well as its subtracted counterpart $L_k^{(N)}$, where $N \geq 0$.

Proposition 3.4. For each $k_1, k_2 \in (-\frac{1}{2}, \frac{1}{2})$, every positive integer $d \geq 1$ and $N \geq 0$ such that $(d + 2N)\min\{k_1, k_2\} + \frac{d}{2} - N < 1$, the Bochner integer

$$L_k^{(N)} \equiv \int_I dt \delta^{(N)}(S^{k_1}(t) - S^{k_2}(t))$$

is a Hida distribution.

proof: Let us denote the truncated exponential series by $\exp_N(x) \equiv \sum_{n=N}^{\infty} \frac{x^n}{n!}$. For each $t > 0$, the Bochner integral

$$\delta(S^{k_1}(t) - S^{k_2}(t)) \equiv \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} d\lambda \exp\{i\lambda(S^{k_1} - S^{k_2})\} \tag{10}$$

is a Hida distribution. Its S -transform is given by

$$S(\delta(S^{k_1}(t) - S^{k_2}(t)))(\mathbf{f}) \equiv \left(\frac{1}{2\pi[(2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1}]} \right)^{\frac{d}{2}} \cdot \exp\left\{-\frac{(\int_{\mathbb{R}} ds \mathbf{f}(s) [(\frac{1}{c(k_1)} I_{-}^{k_1} \mathbb{I}_{[0,t]}^o)(s) - (\frac{1}{c(k_2)} I_{-}^{k_2} \mathbb{I}_{[0,t]}^o)(s)])^2}{2[(2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1}]}\right\}$$

for all $\mathbf{f} \in \mathcal{S}_d(\mathbb{R})$.

In fact, since $S^{k_1}(t)$ and $S^{k_2}(t)$ are independent of sfBms, then

$$\begin{aligned} & S e^{i\lambda(S^{k_1}(t) - S^{k_2}(t))}(\mathbf{f}) \\ &= E(e^{i\lambda \langle \mathbf{w}_1 + \mathbf{f}, \frac{1}{c(k_1)} I_{-}^{k_1} \mathbb{I}_{[0,t]}^o \rangle}) E(e^{-i\lambda \langle \mathbf{w}_2 + \mathbf{f}, \frac{1}{c(k_2)} I_{-}^{k_2} \mathbb{I}_{[0,t]}^o \rangle}) \\ &= \exp\left\{-\frac{1}{2} |\lambda|^2 ((2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1})\right\} \\ & \quad \cdot \exp\left\{i\lambda \left(\int_{\mathbb{R}} ds \mathbf{f}(s) \left[\left(\frac{1}{c(k_1)} I_{-}^{k_1} \mathbb{I}_{[0,t]}^o\right)(s) - \left(\frac{1}{c(k_2)} I_{-}^{k_2} \mathbb{I}_{[0,t]}^o\right)(s)\right]\right)\right\}. \end{aligned}$$

We can verify that the S -transform of the integrand satisfies the conditions of Lemma 2.3.

Hence the S -transform of $\delta^{(N)}$ is given by

$$S(\delta^{(N)}(S^{k_1}(t) - S^{k_2}(t)))(\mathbf{f}) \equiv \left(\frac{1}{2\pi((2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1})}\right)^{\frac{d}{2}} \cdot \exp_N\left\{-\frac{(\int_{\mathbb{R}} ds \mathbf{f}(s) [(\frac{1}{c(k_1)} I_{-}^{k_1} \mathbb{I}_{[0,t]}^o)(s) - (\frac{1}{c(k_2)} I_{-}^{k_2} \mathbb{I}_{[0,t]}^o)(s)])^2}{(2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1}}\right\},$$

which is a measurable function. To prove the bounded condition, we consider

$$\begin{aligned} & |S(\delta^{(N)}(S^{k_1}(t) - S^{k_2}(t)))(z\mathbf{f})| \\ & \leq \left(\frac{1}{2\pi((2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1})} \right)^{\frac{d}{2}} \\ & \quad \cdot \exp_N\{C_{k,0} |z|^2 \frac{\|\mathbf{f}\|^2}{(2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1}}\}, \end{aligned}$$

estimating the function \exp_N by

$$\begin{aligned} & \exp_N\{C_{k,0} |z|^2 \frac{t^2}{(2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1}} \|\mathbf{f}\|^2\} \\ & \leq \left(\frac{t^2}{(2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1}} \right)^N \exp\{C_{k,0} |z|^2 \|\mathbf{f}\|^2\}. \end{aligned}$$

Then $\frac{t^{2N}}{((2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1})^{\frac{d}{2} + N}}$ is integral on I if $(d + 2N)\min\{k_1, k_2\} + \frac{d}{2} - N < 1$. □

Theorem 3.5. For each $k_1, k_2 \in (-\frac{1}{2}, \frac{1}{2})$, every positive integer $d \geq 1$ and $N \geq 0$ satisfied $(d + 2N)\min\{k_1, k_2\} + \frac{d}{2} - N < 1$, the truncated local times $L_{k,\varepsilon}^{(N)}$ converges strongly to truncated local times $L_k^{(N)}$ in $(\mathcal{S})^*$ when ε tends to zero.

proof: By S-transform of $L_{k,\varepsilon}^{(N)}$ given by

$$\begin{aligned} SL_{k,\varepsilon}^{(N)}(\mathbf{f}) &= \int_I dt \left(\frac{1}{2\pi(\varepsilon + (2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1})} \right)^{\frac{d}{2}} \\ & \quad \cdot \exp_N\left\{ - \frac{(\int_{\mathbb{R}} ds \mathbf{f}(s) [(\frac{1}{c(k_1)} I_{-}^{k_1} \mathbb{I}_{[0,t]}^o)(s) - (\frac{1}{c(k_2)} I_{-}^{k_2} \mathbb{I}_{[0,t]}^o)(s)])^2}{2[\varepsilon + (2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1}]} \right\}, \end{aligned}$$

and

$$\begin{aligned} & |S(p_\varepsilon(S^{k_1}(t) - S^{k_2}(t)))(z\mathbf{f})| \\ & \leq \left(\frac{1}{2\pi} \right)^{\frac{d}{2}} \frac{t^N}{((2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1})^{\frac{d}{2} + N}} \exp\{C_{k,0} |z|^2 \|\mathbf{f}\|^2\}, \end{aligned}$$

it is to see that $L_{k,\varepsilon}^{(N)}$ is a Hida distribution. For each complex number z and $\mathbf{f} \in \mathcal{S}_d(\mathbb{R})$, we have

$$\begin{aligned} & |SL_{k,\varepsilon}^{(N)}(z\mathbf{f})| \\ & \leq \left(\frac{1}{2\pi} \right)^{\frac{d}{2}} \exp\{C_{k,0} |z|^2 \|\mathbf{f}\|^2\} \int_I dt \frac{t^N}{[(2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1}]^{\frac{d}{2} + N}}. \end{aligned}$$

On the other hand, there is

$$SL_k^{(N)}(\mathbf{f}) = \int_I dt \left(\frac{1}{2\pi((2-2^{2k_1})t^{2k_1+1} + (2-2^{2k_2})t^{2k_2+1})} \right)^{\frac{d}{2}} \cdot \exp_N \left\{ - \frac{(\int_{\mathbb{R}} ds \mathbf{f}(s) [(\frac{1}{c(k_1)} I_{[0,t]}^{k_1}) (s) - (\frac{1}{c(k_2)} I_{[0,t]}^{k_2}) (s)])^2}{2[(2-2^{2k_1})t^{2k_1+1} + (2-2^{2k_2})t^{2k_2+1}]} \right\}.$$

It follows from the above inequality with $z = 1$ and by a dominated criterion, the $SL_{k,\varepsilon}^{(N)}(\mathbf{f})$ converges to $SL_k^{(N)}(\mathbf{f})$, when ε tends to zero. Applying Lemma 2.2, we obtain the required convergence. \square

To study the existence of local time of sfBms in (L^2) , we need verify the following lemmas.

Lemma 3.6. For each $u \in (\frac{1}{2}, 1)$, $f(k)$ is a decreasing function of $k \in (-\frac{1}{2}, \frac{1}{2})$, where

$$f(k) = (2-2^{2k})^2 \left(\frac{s}{t}\right)^{2k+1} - \left(1 + \left(\frac{s}{t}\right)^{2k+1} - \frac{1}{2} \left[\left(1 + \frac{s}{t}\right)^{2k+1} + \left|1 - \frac{s}{t}\right|^{2k+1} \right] \right)^2. \tag{11}$$

proof: Making change variance $u = \frac{s}{t}$, (11) can be written as

$$f(k) = (2-2^{2k})^2 u^{2k+1} - \left(1 + u^{2k+1} - \frac{1}{2} \left[(1+u)^{2k+1} + |1-u|^{2k+1} \right] \right)^2.$$

We discuss the properties of $f(k)$. The derivative of $f(k)$ is given by

$$\begin{aligned} f'(k) &= -2^{2k+1}(2-2^{2k}) \ln 4 u^{2k+1} + 2(2-2^{2k})^2 u^{2k+1} \ln u \\ &- 2 \left(1 + u^{2k+1} - \frac{1}{2} \left[(1+u)^{2k+1} + (1-u)^{2k+1} \right] \right) 2u^{2k+1} \ln u \\ &- 2 \left(1 + u^{2k+1} - \frac{1}{2} \left[(1+u)^{2k+1} + (1-u)^{2k+1} \right] \right) (1+u)^{2k+1} \ln(1+u) \\ &+ 2 \left(1 + u^{2k+1} - \frac{1}{2} \left[(1+u)^{2k+1} + (1-u)^{2k+1} \right] \right) (1-u)^{2k+1} \ln(1-u). \end{aligned}$$

Introduce the following notations

$$\begin{aligned} A_1 &\equiv -2^{2k+1}(2-2^{2k}) \ln 4 u^{2k+1} < 0, \quad A_2 \equiv 2(2-2^{2k})^2 u^{2k+1} \ln u < 0, \\ A_3 &\equiv -2 \left(1 + u^{2k+1} - \frac{1}{2} \left[(1+u)^{2k+1} + (1-u)^{2k+1} \right] \right) 2u^{2k+1} \ln u > 0, \\ A_4 &\equiv -2 \left(1 + u^{2k+1} - \frac{1}{2} \left[(1+u)^{2k+1} + (1-u)^{2k+1} \right] \right) (1+u)^{2k+1} \ln(1+u) < 0, \\ A_5 &\equiv 2 \left(1 + u^{2k+1} - \frac{1}{2} \left[(1+u)^{2k+1} + (1-u)^{2k+1} \right] \right) (1-u)^{2k+1} \ln(1-u) < 0. \end{aligned}$$

In order to verify $f'(k) < 0$, we prove

$$B \equiv [-u^{2k+1} \ln u - (1+u)^{2k+1} \ln(1+u)] + [-u^{2k+1} \ln u + (1-u)^{2k+1} \ln(1-u)] < 0.$$

Now we show that the first part is negative. In fact

$$\begin{aligned}
 B_1 &\equiv u^{2k+1} \ln u + (1+u)^{2k+1} \ln(1+u) \\
 &= u^{2k+1} \ln u + \sum_{n=0}^{2k+1} C_{2k+1}^n u^{2k+1-n} \ln(1+u) \\
 &= u^{2k+1} \ln u + u^{2k+1} \ln(1+u) + \sum_{n=1}^{2k+1} u^{2k+1-n} \ln(1+u).
 \end{aligned}$$

Notice that $y = \ln u$ is an increasing function, and there exist

$$u^{2k+1} \ln u + u^{2k+1} \ln(1+u) = u^{2k+1} (\ln u + \ln(1+u)) > 0,$$

$$\sum_{n=1}^{2k+1} u^{2k+1-n} \ln(1+u) > 0.$$

Therefore $B_1 > 0$. For $u \in (\frac{1}{2}, 1)$, we get $u > 1-u$. For $\alpha \equiv 2k+1 \in (0, 2)$, $y = u^\alpha$ is also an increasing function. Then

$$B_2 \equiv u^{2k+1} \ln u - (1-u)^{2k+1} \ln(1-u) \geq (1-u)^{2k+1} \ln \frac{u}{1-u} > 0.$$

Note that B is negative. Thus $f'(k) < 0$, which means that $f(k)$ is a decreasing function of $k \in (-\frac{1}{2}, \frac{1}{2})$. □

Lemma 3.7. For each $k \in (-\frac{1}{2}, \frac{1}{2})$ and $\varepsilon > 0$, there exists $0 < \Delta_5 < 1$, where

$$\Delta_5 \equiv \frac{(t^{2k_1+1} + t'^{2k_1+1} - \frac{1}{2}[(t+t')^{2k_1+1} + |t-t'|^{2k_1+1}] + t^{2k_2+1} + t'^{2k_2+1} - \frac{1}{2}[(t+t')^{2k_2+1} + |t-t'|^{2k_2+1}])^2}{(\varepsilon + (2-2^{2k_1})t^{2k_1+1} + (2-2^{2k_2})t'^{2k_2+1})(\varepsilon + (2-2^{2k_1})t'^{2k_1+1} + (2-2^{2k_2})t^{2k_2+1})}.$$

proof: For $k \in (0, \frac{1}{2})$, according to results in [Bojdecki(2004)], there is $C_h(s, t) < R_h(s, t)$, where $C_h(s, t)$ and $R_h(s, t)$ denote the covariance of sfBm and fBm, respectively. We obtain

$$0 < \Delta_5 < \frac{\frac{1}{2}[t^{2k_1+1} + t'^{2k_1+1} - |t-t'|^{2k_1+1}] + \frac{1}{2}[t^{2k_2+1} + t'^{2k_2+1} - |t-t'|^{2k_2+1}]}{((2-2^{2k_1})t^{2k_1+1} + (2-2^{2k_2})t'^{2k_2+1})((2-2^{2k_1})t'^{2k_1+1} + (2-2^{2k_2})t^{2k_2+1})} \leq 1.$$

On the other hand, for $k \in (-\frac{1}{2}, 0)$, we rewrite the Δ_5 as follows

$$\begin{aligned}
 \Delta_5 &= \frac{(E[S_t^{k_1} S_{t'}^{k_1}] + E[S_t^{k_2} S_{t'}^{k_2}])^2}{(\varepsilon + \text{Var}(S_t^{k_1}) + \text{Var}(S_t^{k_2}))(\varepsilon + \text{Var}(S_{t'}^{k_1}) + \text{Var}(S_{t'}^{k_2}))} \\
 &= \frac{(E[S_t^{k_1} S_{t'}^{k_1}] + E[S_t^{k_2} S_{t'}^{k_2}])^2}{(\varepsilon + E[(S_t^{k_1})^2] + E[(S_t^{k_2})^2])(\varepsilon + E[(S_{t'}^{k_1})^2] + E[(S_{t'}^{k_2})^2])} \\
 &< \frac{(E[S_t^{k_1} S_{t'}^{k_1}] + E[S_t^{k_2} S_{t'}^{k_2}])^2}{(E[(S_t^{k_1})^2] + E[(S_t^{k_2})^2])(E[(S_{t'}^{k_1})^2] + E[(S_{t'}^{k_2})^2])},
 \end{aligned}$$

for $\forall \varepsilon > 0$. To finish the proof, we need verify

$$(E[S_t^{k_1} S_{t'}^{k_1}] + E[S_t^{k_2} S_{t'}^{k_2}])^2 \leq (E[(S_t^{k_1})^2] + E[(S_t^{k_2})^2])(E[(S_{t'}^{k_1})^2] + E[(S_{t'}^{k_2})^2]).$$

In fact, the last inequality above is tested by following fact $(xy + zy)^2 \leq (x^2 + z^2)(y^2 + h^2)$, for $x, y, z, h \in \mathbb{R}^+$. □

Lemma 3.6 and Lemma 3.7 imply the following theorem

Theorem 3.8. For any pair of $k_1, k_2 \in (-\frac{1}{2}, \frac{1}{2})$ and every positive integer $d \geq 1$ such that $d < \frac{1}{2(k_1 \vee k_2) + 1}$, $L_{k, \varepsilon}$ converges to L_k in (L^2) as ε tends to zero.

proof: By Theorem 3.3 and Proposition 3.4, we consider chaos expansion of $L_{k_1, k_2, \varepsilon}, L_{k_1, k_2} \in (L^2)$. We need show that the sums

$$\sum_{\mathbf{m}} \sum_{\mathbf{l}} \mathbf{m}!! \cdot | \mathbf{G}_{\mathbf{k}_1, \mathbf{k}_2, \varepsilon, \mathbf{m}, \mathbf{l}} |_{(L^2(\mathbb{R}))^{\otimes(m+l)}}^2, \sum_{\mathbf{m}} \sum_{\mathbf{l}} \mathbf{m}!! \cdot | \mathbf{G}_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{m}, \mathbf{l}} |_{(L^2(\mathbb{R}))^{\otimes(m+l)}}^2,$$

converge, when ε tends to zero.

Step 1. Let us consider the convergence of the first sum in (L^2) .

$$\begin{aligned} I_1 &\equiv \sum_{\mathbf{m}} \sum_{\mathbf{l}} \mathbf{m}!! \cdot | \mathbf{G}_{\mathbf{m}, \mathbf{l}, \varepsilon} |_{(L^2(\mathbb{R}))^{\otimes(m+l)}}^2 \\ &= \sum_{\mathbf{m}} \sum_{\mathbf{l}} \mathbf{m}!! \cdot \left(\frac{1}{2\pi}\right)^d \frac{(-1)^{m+3l}}{\left(\frac{m+l}{2}\right)!^2} \left(\frac{1}{2}\right)^{m+l} \left(\frac{\mathbf{m}+1}{\mathbf{m}!!}\right)^2 \int_0^T dt \int_0^T dt' \\ &\cdot \prod_{j=1}^d \left(\frac{1}{(\varepsilon + (2-2^{2k_1})t^{2k_1+1} + (2-2^{2k_2})t^{2k_2+1})(\varepsilon + (2-2^{2k_1})t'^{2k_1+1} + (2-2^{2k_2})t'^{2k_2+1})} \right)^{m_j+l_j+1} \\ &\cdot \left\langle \frac{1}{c(k_1)} I_{-}^{k_1} \mathbb{I}_{[0,t]}^o, \frac{1}{c(k_1)} I_{-}^{k_1} \mathbb{I}_{[0,t']}^o \right\rangle^{m_j} \left\langle \frac{1}{c(k_2)} I_{-}^{k_2} \mathbb{I}_{[0,t]}^o, \frac{1}{c(k_2)} I_{-}^{k_2} \mathbb{I}_{[0,t']}^o \right\rangle^{l_j}. \end{aligned}$$

For $i = 1, 2$, there is

$$\left\langle \frac{1}{c(k_i)} I_{-}^{k_i} \mathbb{I}_{[0,t]}^o, \frac{1}{c(k_i)} I_{-}^{k_i} \mathbb{I}_{[0,t']}^o \right\rangle = t^{2k_i+1} + t'^{2k_i+1} - \frac{1}{2} [(t+t')^{2k_i+1} + |t-t'|^{2k_i+1}],$$

Hence

$$\begin{aligned} I_1 &= \left(\frac{1}{2\pi}\right)^d \int_0^T dt \int_0^T dt' \\ &\cdot \prod_{j=1}^d \left(\frac{1}{(\varepsilon + (2-2^{2k_1})t^{2k_1+1} + (2-2^{2k_2})t^{2k_2+1})(\varepsilon + (2-2^{2k_1})t'^{2k_1+1} + (2-2^{2k_2})t'^{2k_2+1})} \right)^{-\frac{1}{2}} \\ &\cdot \sum_{n=0}^{\infty} \frac{1}{4^n n!} \sum_{n_1!, \dots, n_d!} \frac{n!}{n_1! \cdots n_d!} \prod_{j=1}^d \frac{(2n_j)!}{n_j!} \\ &\cdot \left(\frac{1}{(\varepsilon + (2-2^{2k_1})t^{2k_1+1} + (2-2^{2k_2})t^{2k_2+1})(\varepsilon + (2-2^{2k_1})t'^{2k_1+1} + (2-2^{2k_2})t'^{2k_2+1})} \right)^{n_j} \\ &\cdot (t^{2k_1+1} + t'^{2k_1+1} - \frac{1}{2} [(t+t')^{2k_1+1} + |t-t'|^{2k_1+1}]) \\ &+ (t^{2k_2+1} + t'^{2k_2+1} - \frac{1}{2} [(t+t')^{2k_2+1} + |t-t'|^{2k_2+1}])^{2n_j}. \end{aligned}$$

As similar as the proof of Theorem 12 in [Oliveira et. al (2011)] and by Lemma 3.7, the integrand of I_1 can be rewritten as

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \prod_{i=0}^{n-1} \left(-\frac{1}{2} - i\right) \Delta_5^n,$$

where $\Delta_5 \equiv \frac{(t^{2k_1+1} + t'^{2k_1+1} - \frac{1}{2}[(t+t')^{2k_1+1} + |t-t'|^{2k_1+1}] + t^{2k_2+1} + t'^{2k_2+1} - \frac{1}{2}[(t+t')^{2k_2+1} + |t-t'|^{2k_2+1}])^2}{(\varepsilon + (2-2^{2k_1})t^{2k_1+1} + (2-2^{2k_2})t'^{2k_2+1})(\varepsilon + (2-2^{2k_1})t'^{2k_1+1} + (2-2^{2k_2})t^{2k_2+1})}$.

Comparing with the Taylor expansion of the function $[1 - \Delta_5]^{-\frac{d}{2}}$, I_1 becomes

$$\begin{aligned} & \left(\frac{1}{2\pi}\right)^d \int_0^T \int_0^T dt dt' [(\varepsilon + (2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t'^{2k_2+1}) \\ & \cdot (\varepsilon + (2 - 2^{2k_1})t'^{2k_1+1} + (2 - 2^{2k_2})t^{2k_2+1}) - [t^{2k_1+1} + t'^{2k_1+1} - \frac{1}{2}[(t+t')^{2k_1+1} \\ & + |t-t'|^{2k_1+1}] + t^{2k_2+1} + t'^{2k_2+1} - \frac{1}{2}[(t+t')^{2k_2+1} + |t-t'|^{2k_2+1}]]^d \\ & < +\infty, \end{aligned}$$

which implies that $L_{k_1, k_2, \varepsilon} \in (L^2)$.

Step 2. Let us consider the convergence of the second sum in (L^2) .

Similarly, we take $\varepsilon = 0$ and obtain $G_{k_1, k_2, \mathbf{m}, \mathbf{l}}$ equal to $G_{k_1, k_2, \varepsilon, \mathbf{m}, \mathbf{l}}$. As a result, we obtain

$$\begin{aligned} I_2 & \equiv \sum_{\mathbf{m}} \sum_{\mathbf{l}} \mathbf{m}!! \mathbf{l}! | \mathbf{G}_{\mathbf{m}, \mathbf{l}} |_{(L^2(\mathbb{R}))^{\otimes(m+l)}}^2 \\ & = \left(\frac{1}{2\pi}\right)^d \int_0^T \int_0^T dt dt' [((2 - 2^{2k_1})t^{2k_1+1} + (2 - 2^{2k_2})t'^{2k_2+1})((2 - 2^{2k_1})t'^{2k_1+1} \\ & + (2 - 2^{2k_2})t^{2k_2+1}) - [t^{2k_1+1} + t'^{2k_1+1} - \frac{1}{2}[(t+t')^{2k_1+1} + |t-t'|^{2k_1+1}] \\ & + t^{2k_2+1} + t'^{2k_2+1} - \frac{1}{2}[(t+t')^{2k_2+1} + |t-t'|^{2k_2+1}]]^d. \end{aligned}$$

When replace δ function with Gaussian sequence, similar techniques in [Nualart et. al (2007); Oliveira et. al (2011)] allow us to consider singular points.

It is easily to see that

$$\begin{aligned} & ((2-2^{2k_1})t^{2k_1+1} + (2-2^{2k_2})t'^{2k_2+1})((2-2^{2k_1})t'^{2k_1+1} + (2-2^{2k_2})t^{2k_2+1}) - [t^{2k_1+1} + t'^{2k_1+1} \\ & - \frac{1}{2}[(t+t')^{2k_1+1} + |t-t'|^{2k_1+1}] + t^{2k_2+1} + t'^{2k_2+1} - \frac{1}{2}[(t+t')^{2k_2+1} + |t-t'|^{2k_2+1}]]^2 \\ & \geq (2-2^{2k_1})(tt')^{2k_1+1} + (2-2^{2k_2})(tt')^{2k_2+1} - (t^{2k_1+1} + t'^{2k_1+1} - \frac{1}{2}[(t+t')^{2k_1+1} \\ & + |t-t'|^{2k_1+1}])^2 + (t^{2k_2+1} + t'^{2k_2+1} - \frac{1}{2}[(t+t')^{2k_2+1} + |t-t'|^{2k_2+1}])^2. \end{aligned}$$

Denote

$$\varphi_k(s, t) \equiv (2 - 2^{2k})^2(st)^{2k+1} - (s^{2k+1} + t^{2k+1} - \frac{1}{2}[(s + t)^{2k+1} + |s - t|^{2k+1}])^2,$$

which is a homogeneous function with respect to s and t with the order $4k + 2$. For $0 < s < t < T$,

$$\begin{aligned} \varphi_k(s, t) &= t^{4k+2}((2 - 2^{2k})^2(\frac{s}{t})^{2k+1} - (1 + (\frac{s}{t})^{2k+1} - \frac{1}{2}[(1 + \frac{s}{t})^{2k+1} + |1 - \frac{s}{t}|^{2k+1}])^2) \\ &\equiv t^{4k+2}f(k). \end{aligned}$$

Using Fubini's theorem and the fact

$$\lambda^{-\frac{d}{2}} = \frac{1}{\Gamma(\frac{d}{2})} \int_0^{+\infty} dz e^{-\lambda z} z^{\frac{d}{2}-1},$$

and taking $T = 1$ for simplicity, the multiple integral in I_2 becomes

$$I_3 \equiv \frac{1}{\Gamma(\frac{d}{2})} \int_0^\infty dz z^{\frac{d}{2}-1} \int_0^1 \int_0^t dt dt' e^{-z(\varphi_{k_1}(t,t') + \varphi_{k_2}(t,t'))}. \tag{12}$$

As for all $z \in [0, 1]$, the integral in (12) is convergent in a neighborhood of zero. By Lemma 3.6, we get the following fact

$$\begin{aligned} &\varphi_{k_1}(t, t') + \varphi_{k_2}(t, t') \\ &= t^{4k_1+2}(2 - 2^{2k_1})^2[(\frac{t'}{t})^{2k_1+1} - (1 + (\frac{t'}{t})^{2k_1+1} - \frac{1}{2}[(1 + \frac{t'}{t})^{2k_1+1} + |1 - \frac{t'}{t}|^{2k_1+1}])^2] \\ &+ t^{4k_2+2}(2 - 2^{2k_2})^2[(\frac{t'}{t})^{2k_2+1} - (1 + (\frac{t'}{t})^{2k_2+1} - \frac{1}{2}[(1 + \frac{t'}{t})^{2k_2+1} + |1 - \frac{t'}{t}|^{2k_2+1}])^2] \\ &\equiv t^{4k_1+2}(2 - 2^{2k_1})^2 f(k_1) + t^{4k_2+2}(2 - 2^{2k_2})^2 f(k_2) \\ &\geq 2\min\{(2 - 2^{2k_1})^2, (2 - 2^{2k_2})^2\} t^{4(k_1 \vee k_2)+2} f(k_1 \vee k_2). \end{aligned}$$

Therefore

$$I_3 \leq \int_1^\infty z^{\frac{d}{2}-1} \int_0^1 \int_0^t dt dt' \exp\{-2z(\min\{(2 - 2^{2k_1})^2, (2 - 2^{2k_2})^2\}) t^{4(k_1 \vee k_2)+2} f(k_1 \vee k_2)\}.$$

Comparing with the homogeneity properties of

$$g_k(t, t') \equiv \exp\{-2(\min\{(2 - 2^{2k_1})^2, (2 - 2^{2k_2})^2\}) t^{4(k_1 \vee k_2)+2} f(k_1 \vee k_2)\}$$

with respect to t and t' , the further estimation is obtained

$$\begin{aligned}
 I_3 &\leq \int_1^\infty z^{\frac{d}{2}-1} \int_0^1 \int_0^t dt dt' e^{-zg_k(t,t')} \\
 &= \int_1^\infty dz z^{\frac{d}{2}-1-\frac{1}{2(k_1 \vee k_2)+1}} \left(\int_0^{z^{\frac{1}{4(k_1 \vee k_2)+2}}} \int_0^x dy dx e^{-zg_k(x,y)} \right).
 \end{aligned}$$

Using that $\{(x,y) : 0 < x < z^{\frac{1}{4(k_1 \vee k_2)+2}}, 0 < y < x\} \subset \{(x,y) : x^2 + y^2 \leq 2z^{\frac{1}{4(k_1 \vee k_2)+2}}\}$ and making a polar change of coordinates, we get

$$\int_0^{z^{\frac{1}{4(k_1 \vee k_2)+2}}} \int_0^x dy dx e^{-g_k(x,y)} \leq \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}z^{\frac{1}{4(k_1 \vee k_2)+2}}} dr d\theta r \exp\{-r^{4(k_1 \vee k_2)+2} g_k(\theta)\}.$$

After taking $x = r^{4(k_1 \vee k_2)+1} g_k(\theta)$, the last integral is equal to

$$\begin{aligned}
 &\int_0^{z^{\frac{1}{4(k_1 \vee k_2)+2}}} \int_0^x dy dx e^{-g_k(x,y)} \\
 &= \frac{1}{4(k_1 \vee k_2) + 1} \int_0^{\frac{\pi}{4}} d\theta g_k(\theta)^{\frac{1}{4(k_1 \vee k_2)+2}} \gamma((4(k_1 \vee k_2) + 2)^{-1}, 2^{4(k_1 \vee k_2)+2} z g_k(\theta)),
 \end{aligned}$$

where $\gamma(\alpha, x) = \int_0^x e^{-y} y^{\alpha-1} dy$. Applying Lemma 2 in [Nualart et. al (2007)], for all $\varepsilon < \alpha, \alpha > 0$ and $x > 0$, we obtain $\gamma(\alpha, x) \leq K(\alpha)x^\varepsilon$, where $K(\alpha) \equiv \frac{1}{\alpha} \vee \Gamma(\alpha)$ and $\Gamma(\alpha) = \gamma(\alpha, +\infty)$.

Therefore, for $\varepsilon < \frac{1}{2(k_1 \vee k_2)+1}$,

$$\begin{aligned}
 I_3 &\leq K^2 \left(\frac{1}{2(k_1 \vee k_2) + 1} \right)^{\frac{2^\varepsilon[2(k_1 \vee k_2)+1]}{4(k_1 \vee k_2) + 2}} \\
 &\cdot \int_1^{+\infty} dz z^{\frac{d}{2}-1-\frac{1}{2(k_1 \vee k_2)+1}+\varepsilon} \int_0^{\frac{\pi}{4}} d\theta g_k(\cos\theta, \sin\theta)^{\varepsilon-\frac{1}{2(k_1 \vee k_2)+1}}. \tag{13}
 \end{aligned}$$

The integral in z converges when $\varepsilon < \min\{\frac{1}{2(k_1 \vee k_2)+1}, \frac{d}{2} - \frac{1}{2(k_1 \vee k_2)+1}\}$. Another integral with respect to θ is also convergence. Thus (13) is convergence. \square

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