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New Insights on Energy Conserved Planar Motion

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Abstract: The planar motion of a particle within an arbitrary potential field is considered. The particle is additionally subject to an external force wherein the applied thrust-acceleration is constrained to remain normal to the velocity vector. The system is thus non-conservative but since the thrust force is non-working, the total energy is a conserved quantity. Under this setting, a major result of fundamental importance is established in this paper: that the flight direction angle (more precisely, the sine of the angle between the position and velocity vectors) is shown to always satisfy a linear first-order differential equation with variable coefficients that depend upon the underlying potential function. As a consequence, an analytical solution for the flight direction angle can be obtained directly in terms of the particle's distance from the center of the field for a significant number of special cases for the potential function. In the case of J_2 perturbed spacecraft motion within equatorial orbits, the problem is reduced to that of solving an incomplete elliptic integral. Another important implication of the main result established here is that motion problems subject to velocity-normal thrusting can always be reduced to the study of equivalent single degree-of-freedom conservative systems with an effective potential function. The paper concludes with various examples of both academic and practical interest including the study of bounded two-body Keplerian orbits and hodograph interpretetions

Keywords: Energy conserved spacecraft motion, flight direction angle, velocity-normal thrusting, effective potential, hodograph interpretation.

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1 Introduction

The problem of continuous thrusting in the two-body problem has rich history and has been extensively studied for spacecraft applications. However, analytical solutions to these problems are available only for very few special cases. For example, some of the earliest work was done by Tsien (1953) using circumferential thrust for escaping from an initial circular orbit. The classical problem of spacecraft motion subject to a constant radial thrust is remarkable in the sense that it is fully integrable; accordingly, it has been extensively investigated [Prussing (1998); Akella (2000); Akella and Broucke (2002)]. The problem of tangential thrust acceleration also allows for some analytical solutions as established by Benney (1958), as well as an exponential sinusoid solution approximations for many revolution transfers and interplanetary trajectories. The problem of continuous thrusting in the direction perpendicular to the velocity vector has however received very limited attention, albeit the fact that energy remains a conserved quantity for these classes of problems. Notable exceptions are recent work by Hernandez and Akella (2015) wherein initial circular orbits are considered with a focus on mission design and orbit transfer analysis. An immediate consequence of thrusting normal to the velocity vector is that, no matter how high the acceleration magnitude, trajectories always remain bounded so long as thrusting commences from initially bounded orbits. Energy conservation in this class of problems is reminiscent of the constant-radial-thrust acceleration problem wherein angular momentum is the conserved quantity rather than the energy. In the radial problem, there is one additional integral of motion; therefore, the problem can be solved analytically [Akella and Broucke (2002)]. On the other hand, a full analytical solution is not possible for the case thrusting normal to the velocity, which has only one known constant.

The main contribution of this paper is that the flight direction angle, i.e., the angle between the position and velocity vectors satisfies a linear first-order ordinary differential equation in terms of the radial distance. This remarkable result holds for arbitrary potential fields. As a consequence, an effective potential can be interpreted to reduce the original system from two to a single degree of freedom system. Our problem can therefore be shown to be equivalent to the one-dimensional motion of a unit point mass in the central force field subject to the velocity-normal thrust acceleration.

The reminder of the paper is organized as follows. In Sec. II, the equations of motion are derived in both inertial and body-fixed rotating coordinates. The first order linear differential equation govering the flight direction angle is also established in this section. Several special case examples for potential functions are shown in Sec. III together with a discussion on the effective potential formulation. We finalize the paper with some concluding remarks in Sec. IV.

2 Coordinate Frames and Problem Statement

Consider the planar motion of a point-mass object described by position vector $\mathbf{r} = [x_1,\ x_2]^T$, velocity vector $\mathbf{v} = [\dot{x}_1,\ \dot{x}_2]^T$ and potential energy $W(r)$ where $r = \|\mathbf{r}\|_2$, the radial distance from the origin of an inertial frame. The inertial frame is taken to have a basis $\mathscr{I} = \{\hat{\mathbf{i}}, \hat{\mathbf{j}}\}$. Additionally, we consider a body-fixed frame $\mathscr{S} = \{\hat{\mathbf{i}}_t, \hat{\mathbf{i}}_n\}$, where $\hat{\mathbf{i}}_t$ is the unit vector in the velocity direction and $\hat{\mathbf{i}}_n$ is the unit vector that is normal to the instantaneous velocity direction. For this study, it needs to be noted that a constant external acceleration vector $\mathbf{u}$ is assumed to be acting along the $\hat{\mathbf{i}}_n$-direction.

2.1 Equation of Motion

The general equations of planar motion subject to external perturbation acceleration $\mathbf{u}$ in Cartesian coordinates are

$$\dot{\mathbf{r}} = \mathbf{v}, \quad \dot{\mathbf{v}} = -\frac{\partial W(r)}{\partial \mathbf{r}} + \mathbf{u} \tag{1}$$

where as stated already, $\mathbf{u}$ is the thrust-acceleration applied normal to the velocity direction; i.e., $\mathbf{u}^T\mathbf{v} = 0$. The thrust acceleration is assumed to be parameterized through

$$\mathbf{u} = \sigma A\,[\cos\phi,\ \sin\phi]^T \tag{2}$$

where $\sigma = \pm\,1$ (sign depends in the direction of the applied thrust), A is the constant thrust acceleration parameter, and ϕ is the angle between the unit-vector $\hat{\mathbf{i}}$ and the direction of the thrust acceleration. The convection adopted here is that $\sigma = 1$ when the thrust acceleration is along the $+\hat{\mathbf{i}}_n$ direction (firing "inwards"); and $\sigma = -1$ when the thrust acceleration is along the $-\hat{\mathbf{i}}_n$ direction (firing "outwards"). Given this choice of the thrust acceleration direction, motion remains planar but angular momentum defined by $\mathbf{h} = \mathbf{r}\times\mathbf{v}$ is no longer constant with time.

On the other hand, since the thrust is applied perpendicular to the velocity vector, the total energy defined by

$$E = W(r) + \frac{v^2}{2} \tag{3}$$

is a conserved quantity, where $v = \|\mathbf{v}\|$. This can be readibly confirmed by taking the derivative of E with respect time along trajectories defined by Eq. 1, such that,

$$\dot{E} = \left(\frac{\partial W}{\partial \mathbf{r}}\right)^T \mathbf{v} + \mathbf{v}^T\left[-\frac{\partial W}{\partial \mathbf{r}} + \mathbf{u}\right] = 0 \tag{4}$$

An immediate consequence of rearranging Eq. 3 is that velocity magnitude is dependent only on radial distance, i.e.,

$$v^2 = 2\left[E - W(r)\right] \tag{5}$$

Let s be the arc length of the path measured relative to the origin of the inertial frame $\mathscr{I}$. Suppose the velocity vector $\mathbf{v}$ makes an angle θ with the inertial $\hat{\mathbf{i}}$ direction. Let ρ denote the radius of curvature of the path, i.e., $\rho = ds/d\theta$. Then, kinematics leads to the velocity vector given by

$$\mathbf{v} = v\hat{\mathbf{i}}_t = \frac{ds}{dt}\hat{\mathbf{i}}_t \tag{6}$$

and the acceleration vector

$$\mathbf{a} = \frac{dv}{dt}\hat{\mathbf{i}}_t + \frac{v^2}{\rho}\hat{\mathbf{i}}_n = v\frac{dv}{ds}\hat{\mathbf{i}}_t + \frac{v^2}{\rho}\hat{\mathbf{i}}_n \tag{7}$$

From the governing equations of motion in Eq. 1, the total acceleration experienced by the body resolved in the moving frame $\mathscr{S}$ is given by

$$\mathbf{a} = -\frac{\partial W}{\partial r}\,\hat{\mathbf{i}}_r + \sigma A\,\hat{\mathbf{i}}_n \tag{8}$$

where $\hat{\mathbf{i}}_r$ is the unit vector along the $\mathbf{r}$ direction (i.e., $\hat{\mathbf{i}}_r = \mathbf{r}/r$). In terms of the flight direction angle γ, i.e., the angle between the position vector $\mathbf{r}$ and velocity vector $\mathbf{v}$, we have the kinematic identity,

$$\hat{\mathbf{i}}_r = \cos\gamma\,\hat{\mathbf{i}}_t - \sin\gamma\,\hat{\mathbf{i}}_n \tag{9}$$

It should be noted that $\gamma \in [0,\pi)$ by definition. Combining Eq. 1, Eq. 6, and Eq. 9, it can be established that

$$\frac{dr}{ds} = \cos\gamma \tag{10}$$

Next, comparing terms from Eq. 7 and Eq. 8, it follows that

$$\frac{dv}{dt} = -\frac{\partial W}{\partial r}\cos\gamma \tag{11}$$

and

$$\frac{v^2}{\rho} = \sigma A + \left(\frac{\partial W}{\partial r}\right)\sin\gamma \tag{12}$$

Straightforward calculus provides the useful identity

$$\frac{d^2r}{ds^2} = \frac{1}{2}\frac{d}{dr}\left(\frac{dr}{ds}\right)^2 \tag{13}$$

Defining the function β as

$$\beta \equiv \sin\gamma \tag{14}$$

and substituting the following Bernoulli formula for the inverse of the radius curvature ρ [Battin (1999)],

$$\frac{1}{\rho} = \left[\frac{1}{r}\sin^2\gamma - \frac{d^2r}{ds^2}\right]\frac{1}{\sin\gamma} \tag{15}$$

in Eq. 12 results in

$$v^2 r\frac{d^2r}{ds^2} = -\sigma A\beta r + \left[v^2 - r\left(\frac{\partial W}{\partial r}\right)\right]\beta^2 \tag{16}$$

After performing some straightforward algebra, we obtain the following first-order linear ordinary differential equation governing β as given by

$$\frac{d\beta}{dr} = -\frac{\left[v^2 - r\left(\frac{\partial W}{\partial r}\right)\right]}{v^2 r}\beta + \frac{\sigma A}{v^2} \tag{17}$$

wherein $v^2 = 2\left[E - W(r)\right]$ needs to be interpreted from Eq. 5. The establishment of this "Fundamental Equation" governing the flight direction angle (specifically, $\sin\gamma$) is the major result of this paper. Given the fact that Eq. 17 is a linear differential equation in terms of radial distance r, the important implication is that $\sin\gamma$ can always be directly expressed as a function of radial distance r for any potential function $W(r)$.

3 The Flight Direction Angle and Interpretation of the Effective Potential Function

As was shown in the foregoing section, an analytical solution for the flight direction angle can be obtained in terms of the particle's distance from the center of the field. There is yet another important consequence to this result. Recall that the energy constant of motion in Eq. 3 can also be expressed as

$$\begin{aligned} E &= \frac{1}{2}\left[v^2\cos^2\gamma + v^2\sin^2\gamma\right] + W(r) \\ &= \frac{1}{2}\left[\dot{r}^2 + v^2\sin^2\gamma\right] + W(r) \\ &= \frac{1}{2}\dot{r}^2 + \left[\frac{1}{2}v^2\sin^2\gamma + W(r)\right] \end{aligned} \tag{18}$$

Using Eq. 5 in Eq. 18 results in

$$E = \frac{1}{2}\dot{r}^2 + \left[E\sin^2\gamma + \left(1-\sin^2\gamma\right)W(r)\right] \tag{19}$$

which allows for the interpretation of an effective potential function

$$W_{\text{eff}}(r) = \left[E\sin^2\gamma + \left(1-\sin^2\gamma\right)W(r)\right] \tag{20}$$

such that the energy constant $E = \dot{r}^2/2 + W_{\text{eff}}(r)$ corresponds to the motion of an equivalent single-degree-of-freedom conservative system.

The reminder of this section will consider various special cases for the potential function $W(r)$ to further illustrate the current discussion.

3.1 Linear Harmonic Oscillator

The potential function for linear harmonic motion in the plane is defined by $W(r) = (\omega^2 r^2)/2$ where $\omega > 0$ is the constant associated with the unforced oscillation frequency. Substituting this particular expression for $W(r)$ for v^2 in Eq. 5 and subsequently in Eq. 17 results in

$$\frac{d\beta}{dr} = -\frac{2(E-\omega^2 r^2)}{r(2E-\omega^2 r^2)}\beta + \frac{\sigma A}{(2E-\omega^2 r^2)} \tag{21}$$

Allowing for initial conditions $r(0) = r_0$ and $\beta(r_0) = \beta_0$ for the flight-direction angle, a closed-form solution of Eq. 21 can be written as

$$\beta(r) = -\frac{\sigma A}{\omega^2 r} + \left(\frac{\beta_0\omega^2 r_0 + \sigma A}{\omega^2 r}\right)\sqrt{\frac{2E-\omega^2 r_0^2}{2E-\omega^2 r^2}} \tag{22}$$

which can be easily verified by substitution in Eq. 21. It can be seen that Eq. 22 represents the analytical solution for $w(r) \equiv \sin\gamma(r)$.

3.2 Kepler 2-Body Motion

For the case of two-body motion, the potential energy is given by the expression $W(r) = -\mu/r$ with μ being the gravitational constant. Using this expression in Eq. 5 presents

$$v^2 = 2\left(E + \frac{\mu}{r}\right) \tag{23}$$

which can be substituted within Eq. 17 to provide the following analytical solution for the flight direction angle, specifically for $\beta (= \sin\gamma)$; i.e.,

$$\begin{aligned}\beta(r) = {} & \beta_0 \frac{\sqrt{r_0}\sqrt{Er_0+\mu}}{\sqrt{r}\sqrt{Er+\mu}} \\ & + \frac{3\mu^2\sigma A}{8E^{\frac{5}{2}}\sqrt{r}\sqrt{Er+\mu}} \ln\left(\frac{\sqrt{Er+\mu}+\sqrt{Er}}{\sqrt{Er_0+\mu}+\sqrt{Er_0}}\right) \\ & + \frac{\sigma A}{8E^2}\left[(2Er-3\mu) - \frac{\sqrt{r_0}\sqrt{Er_0+\mu}}{\sqrt{r}\sqrt{Er+\mu}}(2Er_0-3\mu)\right] \end{aligned} \tag{24}$$

wherein the initial conditions $r(0) = r_0$ and $\beta(r_0) = \beta_0$ had been applied. It should be stated that a special case of this particular result was discussed earlier by Hernandez and Akella (2015) and Hernandez (2014), wherein an initial circular orbit was assumed (more specifically, $\mu = 1$, $r_0 = 1$, $\beta_0 = 1$). However, the result established here in Eq. 24 generalizes the analytical solution for the flight direction angle for arbitrary bounded initial Keplerian orbits. Given the fact that the flight-direction angle is an explicit function of radial distance r from Eq. 24, an extremely elegant interpretation for intial non-circular orbits can be made within the hodograph plane through Figure 1. Specifically, it needs to be noted that for true Keplerian motion,

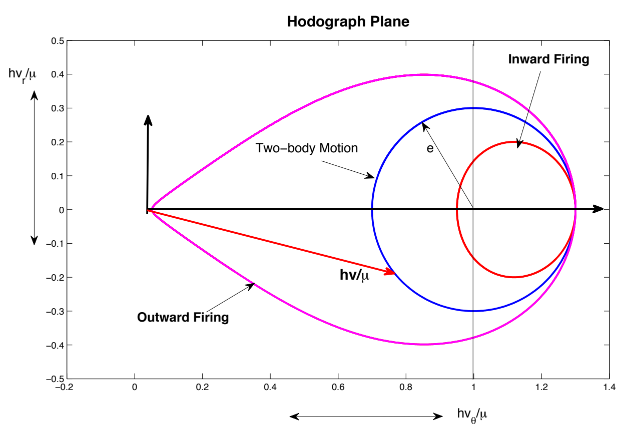

Figure 1: The hodograph interpretation for initial non-circular orbits (eccentricity, $e > 0$).

i.e., with $A = 0$, the hodograph representation for the velocity vector in Fig. 1 follows the classical result of being a circle having radius equaling the eccentricity e of the initial orbit with the center of the circle at $(1,0)$. On the other hand, when thrusting is introduced ($A \neq 0$), the hodograph circle is seen to deform into an "oval" shape with inward thrusting ($\sigma = +1$), and a "teardrop" with outward thrusting ($\sigma = -1$).

3.3 J_2 *Perturbed Equatorial Orbits*

The final special case analyzed here corresponds to J_2 perturbed motion for equatorial earth orbits. In the absence of thrusting, an analytical solution for this problem was obtained by Jezewski (1983) in terms of elliptic integrals. We now consider motion subject to constant acceleration continuous thrusting along a direction normal to the velocity vector. The potential function is given by

$$W(r) = -\frac{\mu}{r} - \frac{J_0}{3r^3} \tag{25}$$

wherein the constant J_0 is given by

$$J_0 = \frac{3}{2}\mu J_2 r_e^2$$

with J_2 being the perturbation coefficient due to Earth's oblateness and r_e is the equatorial radius of the Earth. The solution for the flight direction angle from Eq. 17 in this case reduces to

$$\beta(r) = \beta(0)\frac{\sqrt{r}\sqrt{3Er_0^3+3\mu r_0^2+J_0}}{\sqrt{r_0}\sqrt{3Er^3+3\mu r^2+J_0}} + \frac{3}{2}\frac{\sigma A\sqrt{r}}{\sqrt{3Er^3+3\mu r^2+J_0}}M(r,r_0) \tag{26}$$

where $M(r,r_0)$ is an elliptical integral defined by

$$M(r,r_0) = \int_{r_0}^{r} \frac{x^{\frac{5}{2}}}{3Ex^3+3\mu x^2+J_0}dx \tag{27}$$

4 Conclusions

This paper establishes a fundamental result for the flight direction angle in terms of radial distance in the case of planar motion subject to constant acceleration continuous thrusting that is constrained to a direction normal to the instantenous velocity vector. Energy is a conserved quantity as a consequence of this choice of thrust direction. The sine of the flight direction angle is shown to satisfy a first-order linear ordinary differential equation. This result holds for arbitrary potential functions. An interesting corollory is that an effective potential function can be described for a single degree-of-freedom equivalent system.

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