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# Atomic Exponential Basis Function Eup($x, \omega$) - Development and Application

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**Abstract:** This paper presents exponential Atomic Basis Functions (ABF), which are called $Eup(x, \omega)$. These functions are infinitely differentiable finite functions that unlike algebraic $up(x)$ basis functions, have an unspecified parameter - frequency $\omega$. Numerical experiments show that this class of atomic functions has good approximation properties, especially in the case of large gradients (Gibbs phenomenon). In this work, for the first time, the properties of exponential ABF are thoroughly investigated and the expression for calculating the value of the basis function at an arbitrary point of the domain is given in a form suitable for implementation in numerical analysis. Application of these basis functions is shown in the function approximation example. The procedure for determining the best frequencies, which gives the smallest approximation error in terms of the least squares method, is presented.

**Keywords:** Exponential atomic basis function, Fourier transform, compact support, frequency.

## 1 Introduction

A special task in all numerical methods is the choice of basis functions. The most common engineering problems are determined on the irregular area and have complex boundary conditions and external action. The indentedness of the domain almost excludes the practical application of conventional basis functions (algebraic and trigonometric polynomials) [Zienkiewicz, Taylor, and Zhu (2013)].

A number of meshless methods have been developed for solving engineering problems. Among a few others, prominent meshless discretization techniques include the Meshless Local Petrov-Galerkin (MLPG) Method. Various MLPG methods were compared and shown to be promising contenders to the FEM in [Atluri and Shen (2002)]. Remarkable successes of the MLPG method have been reported

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in solving the convection-diffusion problems [Lin and Atluri (2000)], for elastostatic problems [Atluri, Han, and Rajendran (2004)], for elasto-dynamic problems [Han and Atluri (2004)], and for atomistic/continuum simulation [Shen and Atluri (2005)]. The MLPG method provides the flexibility in the choice of the test and trial functions, and therefore makes it possible to construct various meshless implementations, by combining different trial and test functions. Meshless methods that are based on radial basis functions (RBFs) have recently gained much attention in many different applications in numerical analysis. Some applications using RBFs for heat transfer problems and solution of the Navier-Stokes equations were reported in [Mai-Duy (2004)], the numerical simulation of two-phase flow in porous media in [Iske and Käser (2005)], dealing with transport phenomena in [Šarler (2005)]. The concept of wavelet analysis was introduced in applied mathematics in the late 1980s and recently there is a growing interest in developing wavelet-based numerical algorithms in both the uniform and adaptive node distribution schemes for the solution of partial differential equations (PDEs). Libre, Emdadi, Kansa, Shekarchi, and Rahimian (2009) developed a wavelet based adaptive scheme for solving nearly singular potential PDEs over irregularly shaped domains.

To obtain high-quality numerical solutions, it is important that the length of the basis function support (area of the nonzero values) be small with regard to the entire domain of the considered problem (compact support) and the basis functions be sufficiently smooth so that their linear combination gives a good approximation of the function from the associated solution space of the boundary value problem. For example, the required high smoothness of the approximate solutions calls into question the efficiency of spline or wavelet basis functions [Prenter (1989); Vasilyev and Paolucci (1997)] because the continuity of derivation of approximate solutions, i.e., fluxes, more commonly represents a physically significant result than the basic variable that is being observed.

Therefore, a finite basis function of unlimited smoothness with a small support that does not depend on the type and degree of the boundary value problem should be chosen.

In this paper, we address this class of atomic basis functions [Rvachev and Rvachev (1971); Rvachev and Rvachev (1979)], their properties and the method of their use. In 1971, Rvachev and Rvachev defined for the first time atomic basis functions (ABFs) as solutions of a particular type of differential-functional equations and opened the way for their use in numerical analysis. Atomic functions are finite, infinitely differentiable basis functions that have the advantage of practical application of splines (compact support) and at the same time the property of universality, which is characteristic of algebraic and trigonometric polynomials. Atomic basis functions can be classified into three groups: algebraic, exponential and trigonometric functions. Atomic functions of algebraic type $up(x)$ and $Fup_n(x)$ are the most detailed that have been studied [Gotovac (1986); Gotovac and Kozulić (2002); Kolodyazhny and Rvachev (2007); Rvachev and Rvachev (1979)]. In fact, the operations for calculating the values of atomic functions at arbitrary points seem quite complex and inconvenient for numerical applications. This is the most likely reason that they are poorly represented in the analysis despite their good approximation properties and that the number of authors who use them in their numerical models is not large. However, for practical solving of engineering problems, it is enough to calculate the values of a basic function at a small number of points; then, with specific formulas, the values of all required derivations, integrals, scalar products of selected basis functions with derivatives, elementary functions, etc. can be quickly and accurately calculated [Gotovac and Kozulić (2002)]. Applications of Fup basis functions, as the most commonly used ABFs, are shown in problems of signal processing [Kravchenko, Rvachev, and Rvachev (1995); Kravchenko, Basarab, and Perez-Meana (2001)], initial value problems [Gotovac and Kozulić (2002)] and problems of mathematical physics [Gotovac and Gotovac (2009)]. The authors of this article have worked intensively on the development and application of ABFs of algebraic type in solving problems of structural mechanics and have therefore demonstrated their significant potential compared to conventional procedures with finite elements [Gotovac and Kozulić (2000); Kozulić and Gotovac (2011)]. Research has led to the development of the effective adaptive Fup collocation method, which was successfully implemented in problems of fluid mechanics and groundwater hydraulics [Gotovac, Andričević, and Gotovac (2007); Kozulić, Gotovac, and Gotovac (2007); Gotovac, Kozulić, and Gotovac (2010)].

In modern numerical analysis, algebraic basis functions are almost exclusively used, although most physical and engineering problems do not have solutions from this vector space.

In analysing physical problems whose solutions are not from the class of algebraic polynomials, there is a need for basis functions that can better describe the solution function, that is, those that will belong to the chosen vector space. The idea of choosing basis functions that correspond to the class of solution whose problems we are solving is long established [Rvachev and Rvachev (1971); Gotovac (1986)] but rarely implemented in practice. Engineering problems that exhibit large local gradients and singularities require exponential basis functions. Classical examples are the advective-dispersion (diffusion) equation and the heat conduction equation, which describe transfer of mass and energy, respectively. To obtain quality numerical solutions of such problems, the application of B-splines of exponential type is suitable. These basis functions have not been sufficiently explored, and to date, they are very rarely used in numerical analysis [Kadalbajoo and Patidar (2002);

Konovalov and Kravchenko (2014); McCartin (1981); Radunović (2008)].

Encouraged by the good results achieved by ABFs of algebraic type, we have come to the conclusion that it is worthwhile to explore ABFs of exponential type and to bring them into numerically suitable form. This is the main task of this paper.

The following sections describe the basic (mother) ABF of algebraic and exponential type. Section 2 shows a procedure for generating the function $up(x)$, its derivatives and the basic properties in a manner that is suitable for definition and derivation from the mother basis function $Eup(x,\omega)$. In Section 3, the mother exponential basis function $Eup(x,\omega)$ is derived together with its derivatives, important properties and procedure for use. Implementation of the ABF $Eup(x,\omega)$ in numerical approximations of the given function is presented in Section 4. Finally, conclusions are given in Section 5.

## 2 The mother function up$(\xi)$ of algebraic Abfs

### *2.1 Definition and basic properties of the function up$(\xi)$*

The common characteristic of all ABFs is the possibility of effectively constructing their Fourier Transformation (FT) - image. The function values (the original) and also all associated values required for practical application can be calculated from the FT. The procedure will be illustrated in the most studied ABF - $up(\xi)$ using certain similarities with the B-spline.

Knowing $B_0(\xi)$ spline

$$B_0(\xi) = \begin{cases} 1 & for\ \xi \in [-1/2, 1/2] \\ 0 & else \end{cases}$$

and its FT $f_0(t)$

$$f_0(t) = \int_{-\infty}^{+\infty} B_0(\xi) \cdot e^{it\xi} d\xi = \int_{-1/2}^{+1/2} 1 \cdot e^{it\xi} d\xi = \frac{\sin(t/2)}{t/2} \qquad (1)$$

algebraic spline $B_n(\xi)$ of arbitrary degree $n$ and its FT $f_n(t)$ can be constructed in the following form:

$$\begin{aligned} B_n(\xi) &= \underbrace{B_0(\xi) * B_0(\xi) * \ldots * B_0(\xi)}_{n+1} \\ f_n(t) &= \underbrace{f_0(t) \cdot f_0(t) \cdot \ldots \cdot f_0(t)}_{n+1} \end{aligned} \qquad (2)$$

The function $B_n(\xi)$ corresponds to the convolution of $(n+1)\,B_0(\xi)$ splines, so its support is a union of all supports of the convolution factors with the individual length $h_0 = 1$. Thus, $h_n = (n+1) \cdot h_0$.

Obviously, as the degree of polynomial $n$ increases, the length of the function support increases too, and when $n \to \infty$, the corresponding length of the support $h_n \to \infty$.

A modified form of the expression (1) is used for the function $up(\xi)$ in [Gotovac (1986); Gotovac and Kozulić (2002)] in a way that $B_0(\xi)$ is summarized to half of its support length $(h_0/2)$; thus, a second member in the convolution is obtained, and then, this second member is again compressed to half of its support length $(h_0/4)$ and so on.

$$up(\xi) = B_0(\xi) * B_0(2\xi) * \ldots * B_0(2^k\xi) * \ldots * B_0(2^\infty\xi) \tag{3}$$

From the Paley-Wiener theorem [Gotovac and Kozulić (2002)] in the form $\int_{-\infty}^{+\infty} B_0 (2^n\xi)d\xi = 1$, it follows that the ordinates of each additional member are doubled.

$$B_0(2^k\xi) = \begin{cases} 2^k & for\ \xi \in \left[-2^{-k-1}, 2^{-k-1}\right] \\ 0 & else \end{cases}$$

Support of the function $up(\xi)$ is the union of infinitely many segments, and yet, its length is finite

$$h_{up} = \sum_{k=0}^{\infty} \frac{1}{2^k} = 2 \quad \to \quad \text{supp}\ up(\xi) = [-1, 1]$$

In [Gotovac (1986)], it is shown that the support length can be presented as a measure of the set of all binary rational points $2^{-k}, k = 0, 1, \ldots, \infty$, whereas all other points of the support like $\pm 1/3, \pm 4/7, \pm\sqrt{2}/2, \pm\pi/8$, etc. form a set whose measure is - *empty set*. So, it is a compact support.

The consequence of repeated compression of the starting $B_0(\xi)$ spline to half its previous length increases the algebraic polynomial degree as shown in Fig. 1. The FT of the basic atomic function $up(\xi)$ according to (1), (2) and (3) is

$$F_0(t) = \prod_{j=1}^{\infty} \frac{\sin(t/2^j)}{t/2^j} \tag{4}$$

From (4), using numerical procedures, it is possible to determine function values and the derivatives *approximately* according to formula

$$up(\xi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-it\xi} \cdot F_0(t)dt$$

However, on the set of binary rational points

$$\xi_{br} = -1 + k2^{-n},\ n \in N,\ k = 1, \ldots, 2^{n+1} \tag{5}$$

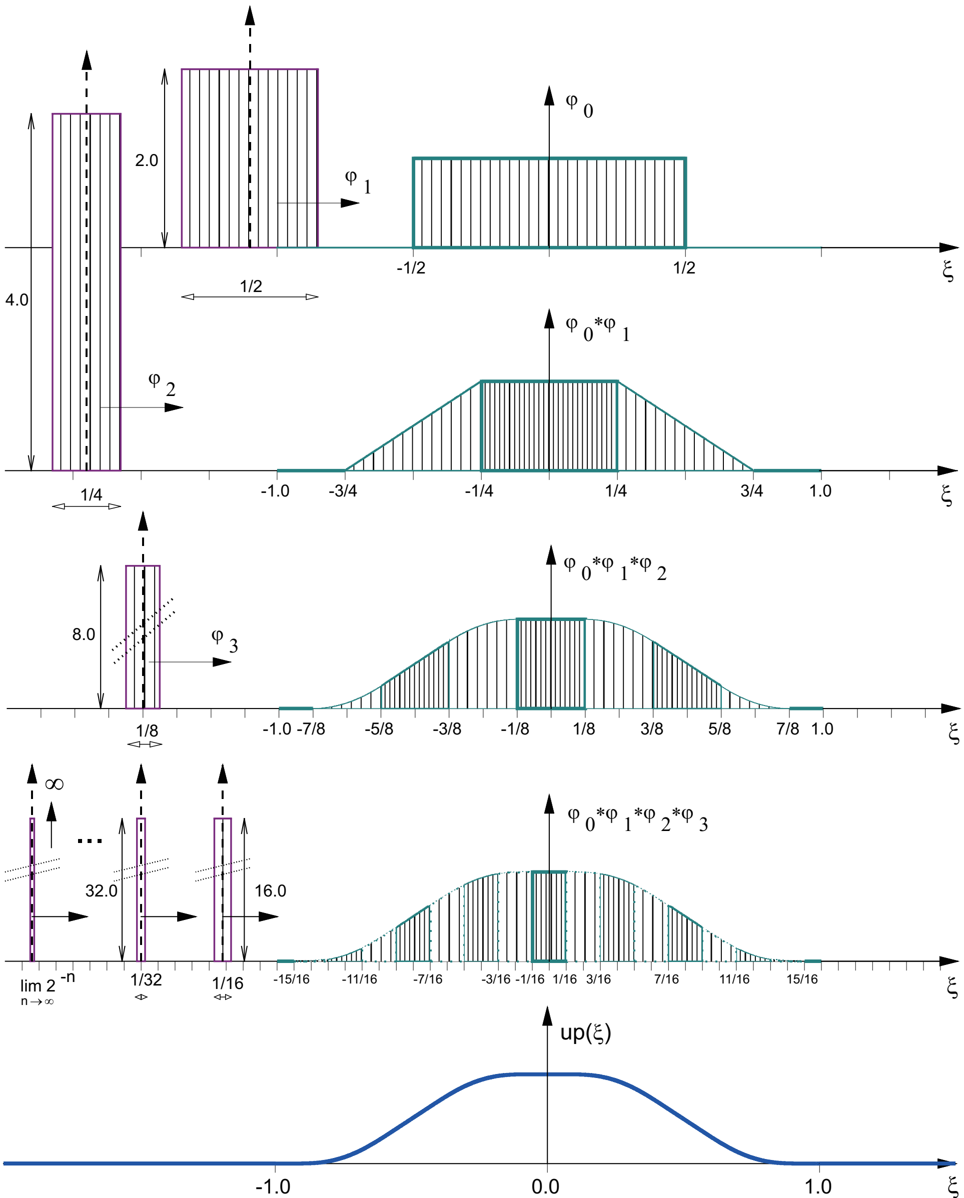



Figure 1: Basis function $up(\xi)$ generation

The values and derivatives of the function $up(\xi)$ can be determined in the form of rational numbers – i.e., *exactly*. The function values at other points of the support are calculated with computer precision.

About ABF $up(\xi)$ can be referred to as the perfect spline, which is differentiable an infinite number of times, although it is not an analytical function in any point of its support. Additionally, the finiteness is more expressive than in spline functions, and the smoothness is less than that for conventional basic functions such as algebraic and trigonometric polynomials.

The mother ABF $up(\xi)$ maintains a good property of finiteness of B-splines and also possesses the important property of algebraic and trigonometric polynomials – the universality of the vector space $UP$ that they form.

### 2.2 Differential functional equation for the function $up(\xi)$

The Fourier transform in the form of (4) can be converted to a form that is more suitable for describing the properties of the function $up(\xi)$. $F(t/2)$ is calculated from (4), and then the left side of equation (4) is divided by $F(t/2)$, and the right side is divided by $\prod_{j=2}^{\infty} \frac{\sin(t/2^j)}{t/2^j}$, which gives

$$F_0(t) = \frac{\sin(t/2)}{t/2} \cdot F_0(t/2) \tag{6}$$

From the FT of function $up(\xi)$ written in the form (6), it is obvious that $up(\xi)$ possesses the quality of crushing (fragmenting); that is, any part of it contains the whole function (holographic effect). If $\sin(t/2) = \left(e^{it/2} - e^{-it/2}\right)/(2i)$ is substituted in (6) and the resulting equation is multiplied by $(-1)$, after arranging, follows

$$-it \cdot F_0(t) = e^{it/2} \cdot F_0(t/2) - e^{-it/2} \cdot F_0(t/2) \tag{7}$$

Using the inverse operator $\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-it\xi}\,dt\right)$ for all members of the equation (7), a differential functional equation of ABF $up(\xi)$ is obtained in the final form:

$$up'(\xi) = 2 \cdot up(2\xi + 1) - 2 \cdot up(2\xi - 1) \tag{8}$$

On the left side of equation (8), there is a linear differential operator with constant coefficients, and on the right side, there is a linear combination of compressed and shifted ABFs $up(\xi)$.

Fig. 2 shows the function $up(\xi)$ and its first derivative. It is evident that it is an even function and that its support is supp $up(\xi) = [-1, 1]$.

The support of the function $up(\xi)$ is composed of two unit length characteristic intervals $\Delta\xi_0$.

Characteristic points $\xi_k$ are the boundary points of characteristic intervals (the point of the origin and end points of the support $\xi = \pm 1$).

The function value at the origin $up(0) = 1$ is a consequence of the normed condition selection, which determines the value of the integral $\int_{-\infty}^{+\infty} up(\xi)d\xi = \int_{-1}^{1} up(\xi)d\xi = 1$ in the original domain or the value of the FT at the origin $F_0(0) = 1$ in the image domain.

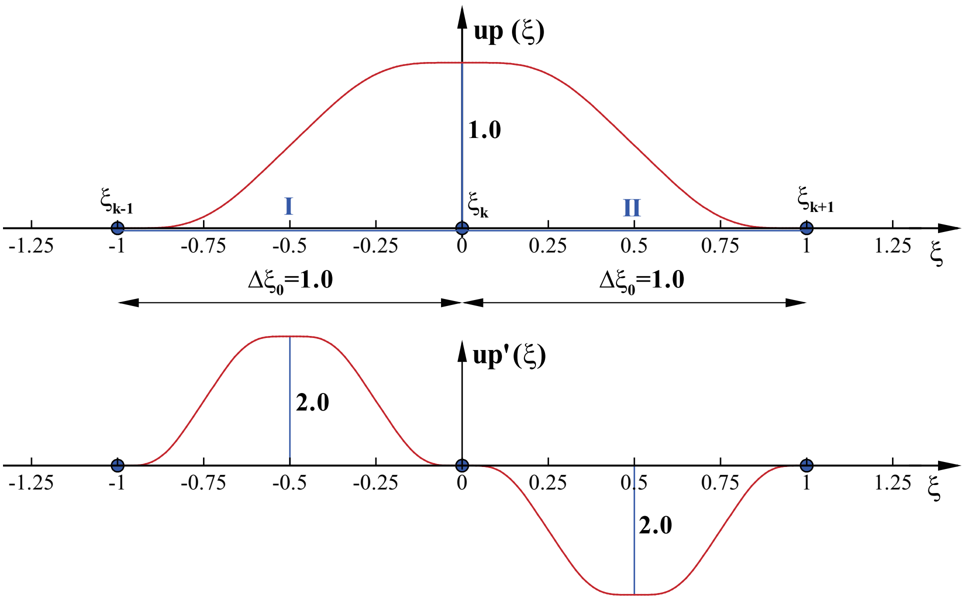



Figure 2: The function $up(\xi)$ and its first derivative

## 2.3 Derivatives of the function $up(\xi)$

Derivatives of the degree greater than $n$ of the $up(\xi)$ function have a zero value at binary rational points (5). This means that the Taylor series at these points is finite and that the function $up(\xi)$ for binary rational points coincides with a polynomial of $n$th degree, which is visible in Fig. 1.

The first derivative can be represented as a linear combination of shifted and compressed functions $up(\xi)$, as shown in equation (8). By differentiating the basic equation (8) and substituting the first derivative of the function $up(\xi)$ with the right side of the starting equation (8), it is shown that the second derivative can also be presented as a linear combination of shifted and compressed $up(\xi)$ functions:

$$up''(\xi) = 8up(4\xi+3) - 8up(4\xi+1) - 8up(4\xi-1) + 8up(4\xi-3)$$

By continuing the process of differentiation and replacing the first derivative from the basic equation, a general term for the derivative of the *m*-th degree is obtained:

$$up^{(m)}(\xi) = 2^{C_{m+1}^2} \sum_{k=1}^{2^m} \delta_k \cdot up\left(2^m \xi + 2^m + 1 - 2k\right), \quad m \in N$$

where $C_{m+1}^2 = m(m+1)/2$ is a binomial coefficient, and $\delta_k$ are coefficients that have a value of $\pm 1$ and determine the sign of the individual summand, which changes according to the following recursive formulas:

$$\delta_{2k-1} = \delta_k, \;\; \delta_{2k} = -\delta_k, \;\; k \in N, \;\; \delta_1 = 1$$

Figure 3 shows the function $up(\xi)$, its first four and seventh derivative. It can be seen that the derivatives are made up of functions $p(\xi)$, which are "compressed" on the interval with length $2^{-m+1}$ and which have ordinates multiplied by a factor $2^{C_{m+1}^2}$. A high degree derivative of the function $up(\xi)$ when $m \to \infty$ becomes a series whose individual member corresponds to Dirac's function.

### *2.4 The value of the function up(ξ) at an arbitrary point*

To calculate the values of the function $up(\xi)$ at points $\xi \in \left[-1+2^{-n}, -1+2^{-n+1}\right]$, it is necessary to know its values at points $\xi \in [-1, -1+2^{-n}]$. According to [Gotovac (1986); Gotovac and Kozulić (2002)]:

$$up(\xi) = \sum_{l=0}^{n} \frac{up^{(l)}\left(-1+2^{-n}\right)}{l!}\left(\xi + 1 - 2^{-n}\right)^l - up\left(\xi - 2^{-n}\right)$$

Given that the Taylor expansion of the function $up(\xi)$ at binary rational points (5) represents a polynomial of the *n*th degree, a special order for calculating the value of the function $up(\xi)$ at arbitrary point $\xi \in [0,1]$ was proposed in [Gotovac (1986)] in the form:

$$up(\xi) = 1 - up(\xi - 1) = 1 - \sum_{k=1}^{\infty} (-1)^{1+p_1+\cdots+p_k} p_k \sum_{j=0}^{k} C_{jk} \left(\xi - 0, p_1 \ldots p_k\right)^j \qquad (9)$$

where the coefficients $C_{jk}$ are rational numbers that are determined according to the following formula:

$$C_{jk} = \frac{2^{j(j+1)/2}}{j!} up(-1 + 2^{-(k-j)}); \;\; j = 0, 1, \ldots, k; \;\; k = 1, 2, \ldots, \infty$$

The values $up\left(-1+2^{-(k-j)}\right)$ are calculated as shown in [Gotovac (1986); Gotovac and Kozulić (2002)]. Expression $(\xi - 0, p_1 \ldots p_k)$ in (9) represents the difference

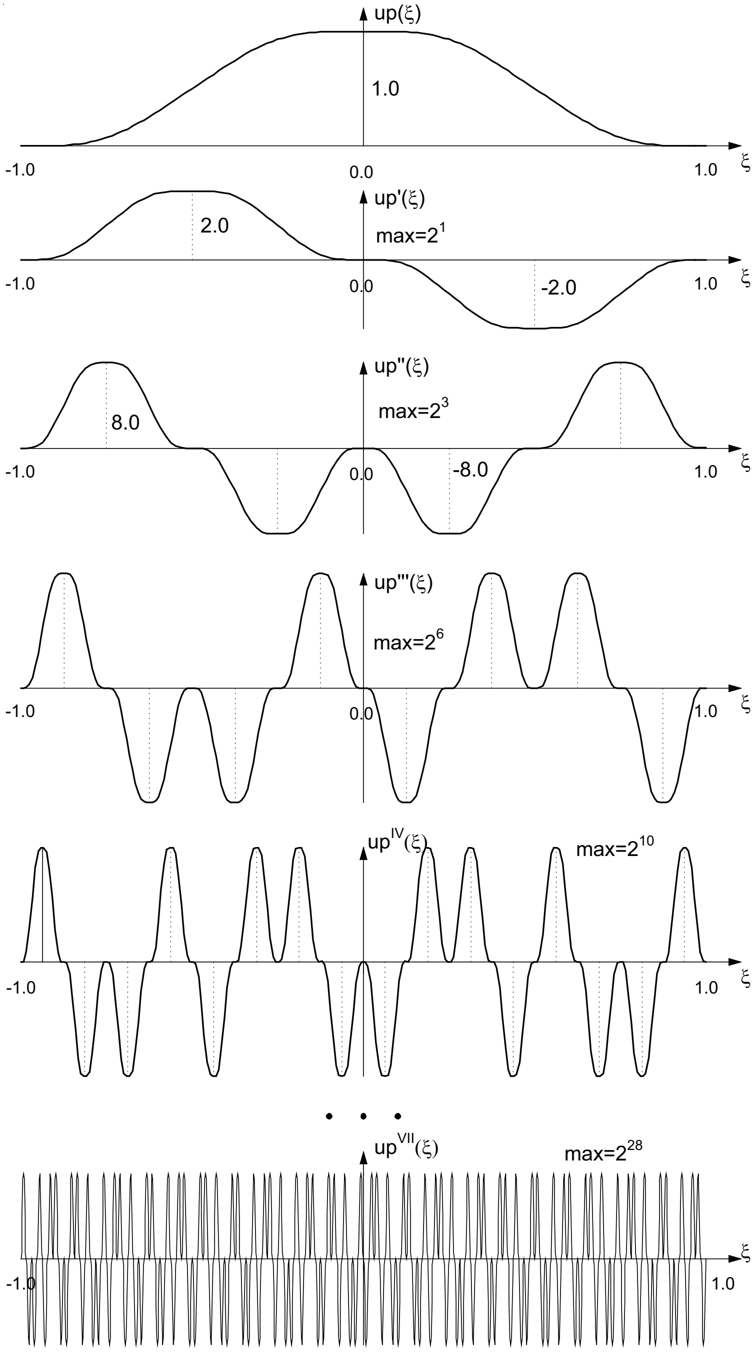



Figure 3: Function $up(\xi)$, the first four and the seventh derivative

between the real value of coordinate $\xi$ and its binary presentation with $k$ bits, where $p_1 \dots p_k$ are digits 0 or 1 of a binary development of the coordinate value $\xi$. Therefore, the accuracy of the coordinate $\xi$ and hence the accuracy of the function $up(\xi)$ in the arbitrary point, depends on the accuracy of an electronic computer. For some $n$, the error of the calculated value of the function $up(\xi)$ at arbitrary point $\xi$ and the rest of the series in (9), where $k = 1, \dots, n$, does not exceed the value of the function $up(-1+2^{-n})$.

### 2.5 Polynomial as a linear combination of shifted up($\xi$) functions

An arbitrary function, as a linear combination of the shifted $up(\xi)$ functions, can be written as:

$$\varphi(\xi) = \sum_{k=-\infty}^{\infty} C_k \cdot up(\xi - k \cdot \Delta\xi_n), \;\; k \in Z \tag{10}$$

where $\Delta\xi_n = 2^{-n}$ is a characteristic interval of the basis function. Particularly, if the coefficient $C_k$ is an algebraic polynomial of $m$th degree of the index $k$, i.e.,

$$\begin{aligned} &C_k \;\; \rightarrow \;\; (\Delta\xi_n^m \cdot \Delta\xi_n) \cdot C_n^{(m)}(k) = \Delta\xi_n^m \cdot \Delta\xi_n \cdot \sum_{i=0}^{m} A_{m-i}^{(m,n)} \cdot k^{m-i} \\ &m = 0, 1, \dots, n, \;\; n \in N \end{aligned} \tag{11}$$

then the function $\varphi(\xi)$ from (10) is the algebraic polynomial of $m$th degree, whereas $n$ denotes the greatest degree of the polynomial that is contained in a vector space $UP_n$.

The coefficients $A_{m-i}^{(m,n)}$ are calculated using the following formulas:

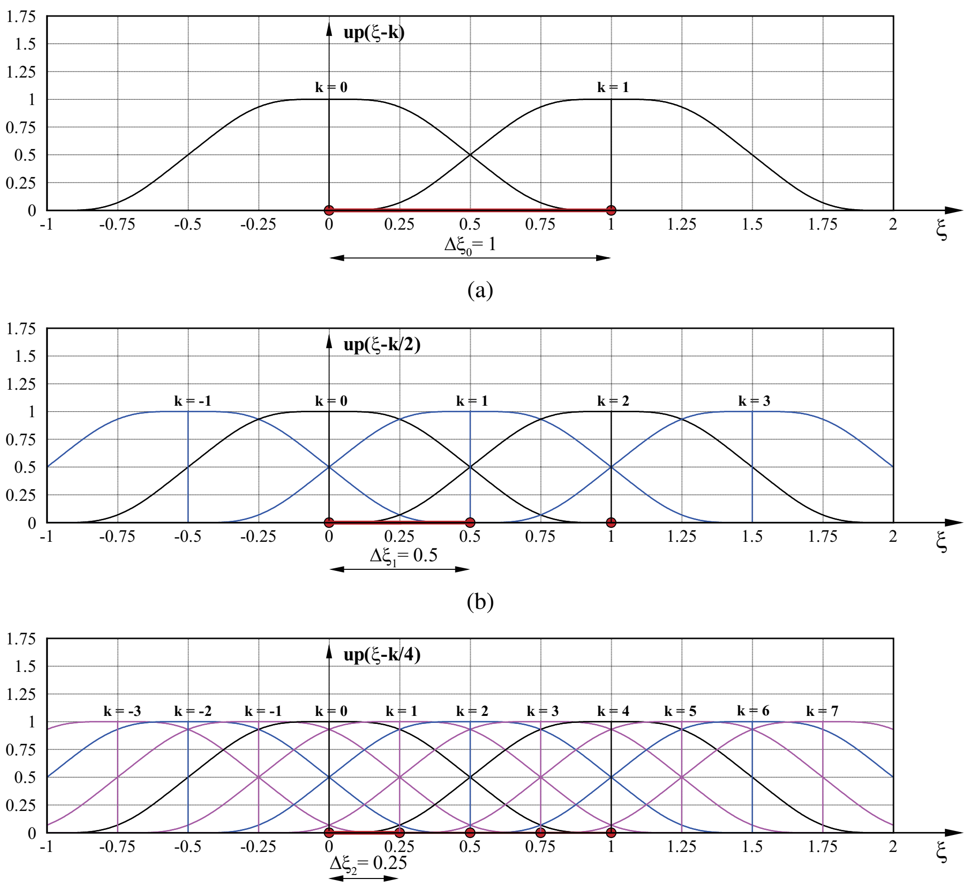
$$\begin{aligned} &A_m^{(m,n)} = 1 \\ &A_{m-2i+1}^{(m,n)} = 0, \;\; i = 1, 2, \dots, [m/2] \\ &A_{m-2i}^{(m,n)} = -\frac{2^{1-n}}{(m-2i)!} \sum_{l=0}^{i-1} A_{m-2l}^{(m,n)} \frac{(m-2l)!}{(2i-2l)!} \sum_{j=1}^{2^n-1} up\left(\frac{j}{2^n}\right) j^{2(i-l)} \end{aligned} \tag{12}$$

For example, $C_2^{(m)}(k)$ coefficients, for the monomials up to the second degree and the basis function distribution shown in Fig. 4c) (that is, for $n = 2$), according to (11), can be represented in a general form:

$$\begin{aligned} n = 2, \;\; &m = 0, \;\; \rightarrow \;\; C_2^{(0)}(k) = A_0^{(0,2)} \cdot k^0 \\ &m = 1, \;\; \rightarrow \;\; C_2^{(1)}(k) = A_1^{(1,2)} \cdot k^1 + A_0^{(1,2)} \cdot k^0 \\ &m = 2, \;\; \rightarrow \;\; C_2^{(2)}(k) = A_2^{(2,2)} \cdot k^2 + A_1^{(2,2)} \cdot k^1 + A_0^{(2,2)} \cdot k^0 \end{aligned} \tag{13}$$

and after calculating the coefficients $A_{m-i}^{(m,2)}, i =, 2, \ldots, [m/2]$ for the associated $m$ according to (12)

$$C_2^{(0)}(k) = 1, \;\; C_2^{(1)}(k) = k, \;\; C_2^{(2)}(k) = k^2 - \frac{16}{9} \tag{14}$$




(c)

Figure 4: The basis function distribution for an accurate representation of the polynomials of 0, 1 and 2 degrees

Using the coefficients (11) and (12) in the virtual domain and mapping the interval $\Delta\xi_n$ to the real length of the interval $\Delta x$, linear combination (10) in the real domain becomes

$$x^m = \Delta x^m \cdot \Delta\xi_n \cdot \sum_{k=-\infty}^{\infty} C_n^{(m)}(k) \cdot up\left(\frac{x}{2^n \Delta x} - \frac{k}{2^n}\right) \tag{15}$$

For example, algebraic polynomial $P_2(x) = a_0 + a_1x + a_2x^2$ in the real domain is obtained using expressions (14) and (15) in the following form:

$$P_2(x) = \Delta\xi_2 \cdot \sum_{k=-\infty}^{\infty} \left[ a_0 + a_1 \cdot \Delta x \cdot k + a_2 \cdot \Delta x^2 \cdot \left( k^2 - \frac{16}{9} \right) \right] \cdot up \left( \frac{x}{2^2 \Delta x} - \frac{k}{2^2} \right)$$

### *2.6 Vector space of functions $UP_n$*

Polynomials of the *n*th degree can be represented as a linear combination of basis functions obtained by moving the $up(\xi)$ function. The individual basis function $\varphi_k(\xi)$ is obtained by moving the $up(\xi)$ function on the abscissa axis for the value $k \cdot 2^{-n}$, so that:

$$\varphi_k(\xi) = up\left(\xi - k2^{-n}\right),\ k \in Z,\ n = 0, 1, \ldots$$

The exponent *n* determines the highest polynomial degree that can be accurately represented as a linear combination of basis functions $\varphi_k(\xi)$ according to (10). The coefficient *k* determines a displacement of the function $up(\xi)$ with respect to the origin of the global coordinate system with the length of a characteristic interval $\Delta\xi_n = 2^{-n}$ so that it becomes a basis function $\varphi_k(\xi)$ (Fig. 4.); thus, *k* has the role of a global index of the individual basis function.

As shown in Fig. 4, for an accurate representation of monomials $\xi^n$ on the interval of length $2^{-n}$, $2^{n+1}$ basis functions are required, so in this case, the dimension of vector space is $\dim(UP_n) = 2^{n+1}$.

For an accurate representation of monomials $\xi^{n+1}$, it is necessary to have $2^{n+2}$ basis functions, so vector space $UP_{n+1}$ has a dimension of $2^{n+2}$. Therefore, the linear vector space $UP_{n+1}$ contains a space $UP_n$ because it is obtained by extending $UP_n$ space with $2^{n+1}$ linearly independent vectors, i.e., displaced $up(\xi)$ functions. Accordingly, as distinct from the space built out of basis splines, the function space $UP_n$ is universal, i.e. :

$$UP_0 \subset UP_1 \subset \ldots \subset UP_n \subset UP_{n+1}$$

This fact makes it possible to form an iterative procedure in which the solution from the space $UP_n$ is used as a starting solution for searching the approximation in the space $UP_{n+1}$.

### *2.7 Approximation of the function that is an algebraic polynomial*

Let the given function be $f(x) = -1 + 4x - 2x^2, x \in [0, 1]$. We need to compute the approximation of the function using a linear combination of basis functions shown in Fig. 4a, 4b and 4c.

Using the collocation method, the system of equations and the corresponding approximation are obtained.

**a)** For two collocation points, the length of a characteristic interval is $\Delta x_a = 1$. The system of equations and the coefficients $C_i, i = 0,1$ are:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = \begin{bmatrix} f(0) \\ f(1) \end{bmatrix} \quad \to \quad \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = \begin{bmatrix} f(0) \\ f(1) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{16}$$

The approximation of the given function $f(x)$ has the form $\widetilde{f_a}(x) = C_0 \cdot up(x) + C_1 \cdot up(x-1)$ and is shown in Fig. 5.

**b)** For three collocation points, the distribution of basis functions and collocation points are shown in Fig. 4b. The length of a characteristic interval is $\Delta x_b = \Delta x_a/2$. There are five unknown coefficients and only three collocation points. When spline functions are used as an additional condition at the boundary, the first derivative is used so that there are double collocation points at the boundary. In the case of $up(\xi)$ basis functions, we obtain the following matrix $A_s$:

$$A_s = \begin{bmatrix} -2 & 0 & 2 & 0 & 0 \\ 1/2 & 1 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & -2 & 0 & 2 \end{bmatrix}, \quad A_R = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 1/2 & 1 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix} \tag{17}$$

A system $A_s$ in (17) is singular. In [Rvachev and Rvachev (1979)], the system is preconditioned and solved by an iterative procedure.

The polynomials of the $n$th degree can accurately be described using the basis functions of the vector space $UP_n$ as shown in the previous section. However, in addition to ABF $up(x)$, the vector space $UP_n$ contains the basis functions $Fup_n(x)$, [Gotovac (1986); Gotovac and Kozulić (2002); Rvachev and Rvachev (1979)], which can also exactly describe polynomials up to the $n$th degree. Basis function $Fup_n(x)$ is obtained as a result of a specific linear combination of $up(x)$ functions. Additionally, a smaller number of basis functions is needed when accurately describing the polynomials of the $n$th degree on the characteristic interval than when using the $up(x)$ functions.

Because all derivations over the order $n$ in all points must be equal to zero, additional equations, instead of known derivatives on the boundary, can be written with the condition that the $(n+1)$th derivation of linear combinations of ABFs $Fup_n(x)$ in the middle of the first and the last characteristic intervals are equal to zero.

Additional equations at the beginning and the end of the matrix formally coincide with the corresponding operators of the finite differences. For the first five derivations, the coefficients are as follows:

$$\begin{array}{llllllll} I. & 1 & -1 & & & & \\ II. & 1 & -2 & 1 & & & \\ III. & 1 & -3 & 3 & -1 & & \\ IV. & 1 & -4 & 6 & -4 & 1 & \\ V. & 1 & -5 & 10 & -10 & 5 & -1 \end{array} \tag{18}$$

For distribution of basis functions formation according to Fig. 4b ($n = 1$), the second derivative must be equal to zero, so using (18) and replacing the first and the last line in (17), matrix $A_s$ becomes a regular matrix $A_R$.

The right side vector is $B = [0, -1/8, 1/2, 7/8, 0]^T$, and the coefficients for linear combination are $C = [-3/2, -1/2, 1/2, 1/2, 1/2]^T$. The resulting approximation $\widetilde{f}_b(x)$ is shown in Fig. 5.

**c)** Using the linear combination of ABFs distributed according to Fig. 4c ($n = 2$), the algebraic monomials of the zero, first and second degree and the polynomial obtained by their combination can be accurately described. Using the formula $\Delta x_c = \Delta x_b/2 = \Delta x_a/4$, the collocation method and additional equations from (18) the values of required coefficients are obtained.

The matrix for calculating the coefficients, which – multiplied by the corresponding basis functions from Fig. 4c – accurately describe the given function, has the following form:

$$A = \begin{bmatrix} -1 & 3 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 5 & -10 & 10 & -5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5/72 & 1/2 & 67/72 & 1 & 67/72 & 1/2 & 5/72 & 0 & 0 & 0 & 0 \\ 0 & 5/72 & 1/2 & 67/72 & 1 & 67/72 & 1/2 & 5/72 & 0 & 0 & 0 \\ 0 & 0 & 5/72 & 1/2 & 67/72 & 1 & 67/72 & 1/2 & 5/72 & 0 & 0 \\ 0 & 0 & 0 & 5/72 & 1/2 & 67/72 & 1 & 67/72 & 1/2 & 5/72 & 0 \\ 0 & 0 & 0 & 0 & 5/72 & 1/2 & 67/72 & 1 & 67/72 & 1/2 & 5/72 \\ 0 & 0 & 0 & 0 & 0 & -1 & 5 & -10 & 10 & -5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -3 & 1 \end{bmatrix} \tag{19}$$

and the coefficients of the linear combination are

$$C = \frac{1}{288}[-353, -236, -137, -56, 7, 52, 79, 88, 79, 52, 7]^T$$

A linear combination of the basis functions shown in Fig. 4c exactly describes a given function $f(x)$ as shown in Fig. 5.

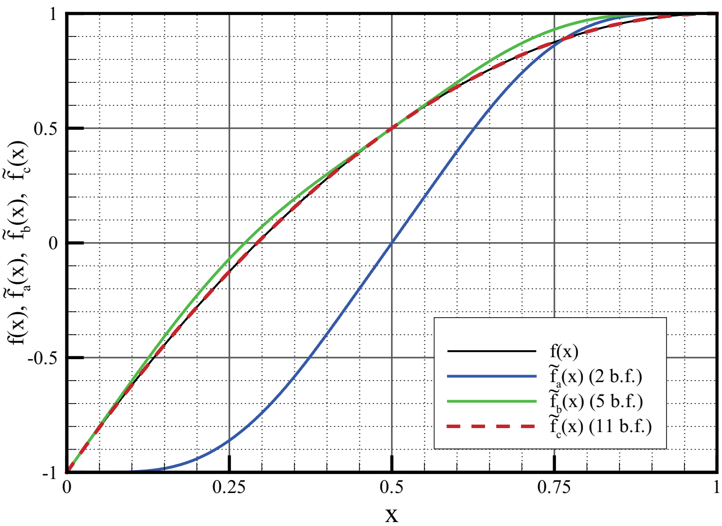



Figure 5: The function $f(x)$ and its approximations $\tilde{f}_a(x), \tilde{f}_b(x), \tilde{f}_c(x)$

## 3 "Mother" function $Eup(\xi,\omega)$ of the exponential ABFs

### *3.1 Generation of the Fourier transform of the function Eup$(\xi,\omega)$*

The FT of the $Eup(\xi,\omega)$ function is constructed by a similar procedure applied on the $up(\xi)$ function using the condition $\int_{-1}^{1} Eup(\xi,\omega)\,d\xi = 1$ on the corresponding compact support $supp\ Eup(\xi,\omega) = [-1,1]$.

Fig. 6 shows a graphical representation of the generation process for the function $Eup(\xi,\omega)$ using the convolution theorem. In this procedure, the support lengths of exponential splines of the zero degree $\varphi_j(\xi,\omega), j = 0,1,2,\ldots$ are reduced according to the law $h = 2^{-j}$:

$$Eup(\xi,\omega) = \underbrace{\varphi_0(\xi,\omega)}_{sup=[-1/2,1/2]} * \underbrace{\varphi_1(\xi,\omega)}_{sup=[-1/4,1/4]} * \ldots * \underbrace{\varphi_j(\xi,\omega)}_{\lim\limits_{j\to\infty}[-2^{-j-1},2^{-j-1}]=0} \tag{20}$$

where

$$\varphi_j(\xi,\omega) = \begin{cases} \dfrac{\omega \cdot e^{\omega/2^{j+1}}}{e^{\omega/2^j}-1} e^{\omega\xi} & \text{for } \xi \in \left[-2^{-(j+1)}, 2^{-(j+1)}\right] \\ 0 & \text{else} \end{cases} \tag{21}$$

FT of the function $Eup(\xi,\omega)$ from (20) corresponds to the product of an infinite

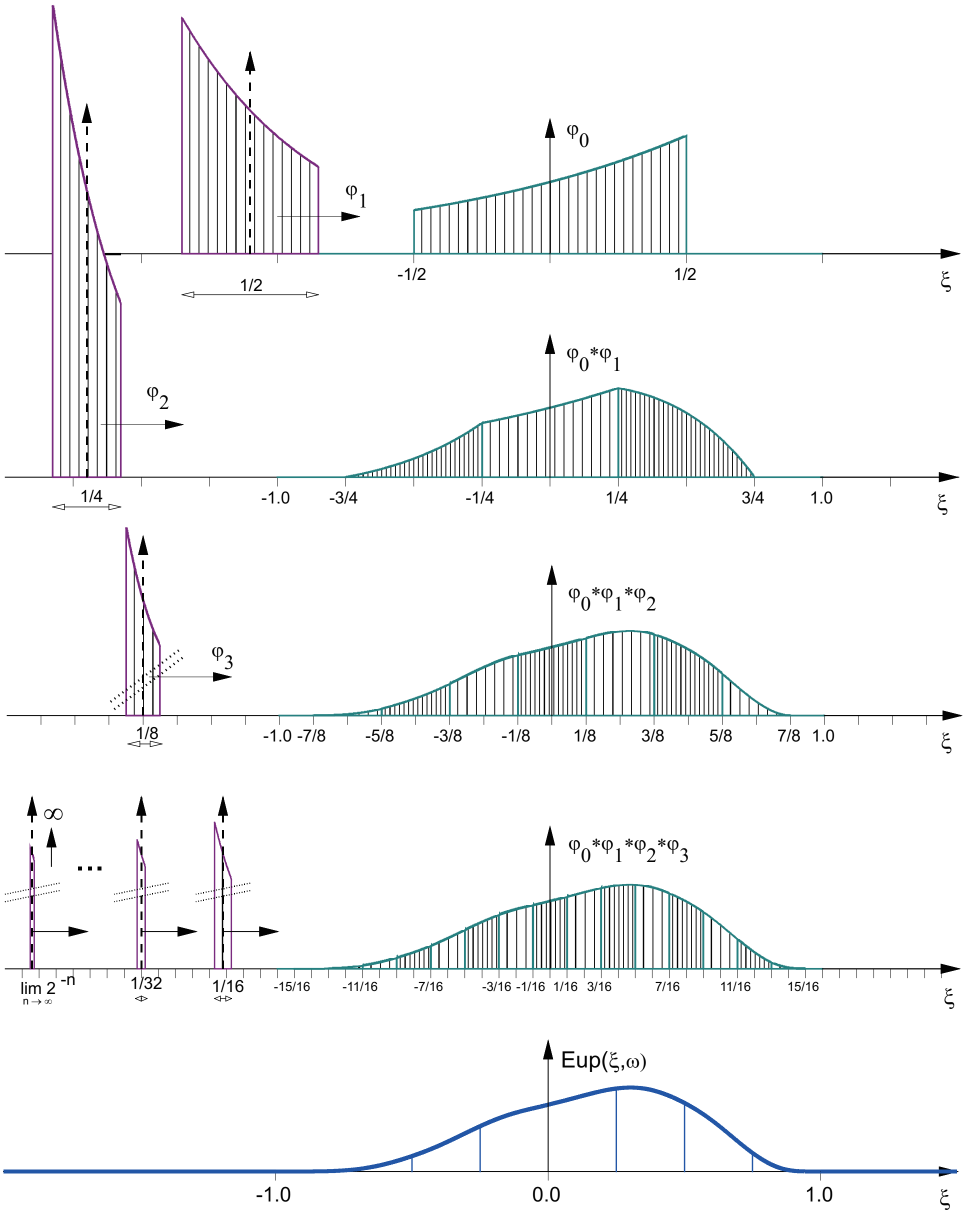


Figure 6: Exponential basis function $Eup(\xi, \omega)$ generation

number of Fourier transformations of compressed splines of zero degree (21):

$$F(t) = \prod_{j=1}^{\infty} \frac{\omega}{2sh(\omega/2)} \frac{\text{sh}\left(\omega/2 + i \cdot t/2^j\right)}{\omega/2 + i \cdot t/2^j} \tag{22}$$

When the parameter $\omega$ approaches zero, expression (22) becomes expression (4). Hence, exponential ABF $Eup(\xi,\omega)$ becomes algebraic ABF $up(\xi)$ when parameter $\omega$ becomes zero. The inverse FT or the function $Eup(\xi,\omega)$ itself, with the satisfaction of the normed condition, is:

$$Eup(\xi,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)\cdot e^{-it\xi}dt \tag{23}$$

By developing the right side of Eq. (23) in the Fourier series, the "original" of the function $Eup(\xi,\omega)$ can be determined.

The parameter $\omega$ has a role of frequency similar to trigonometric functions. Fig. 7 shows that the function $Eup(\xi,\omega)$ is inclined to the left for negative values of the frequency $\omega < 0$, whereas for positive values, it is inclined to the right. In the limiting case when $\omega \to 0$, the function $Eup(\xi,\omega)$ becomes $up(\xi)$. Thus, the vector space $EUP$ is denser than space $UP$ and $UP \subset EUP$.

### 3.2 Differential functional equation for the function Eup$(\xi,\omega)$

Differential functional equation for the function $Eup(\xi,\omega)$ is constructed from its known Fourier transform (22), which can be expressed in the following form:

$$F(t) = \frac{\omega}{2\cdot \mathrm{sh}(\omega/2)}\cdot\frac{\mathrm{sh}(\omega/2+it/2)}{\omega/2+it/2}\cdot F(t/2) \tag{24}$$

In the following text, the basis function $Eup(\xi,\omega)$ will also be denoted as $y(\xi,\omega)$ due to shortness of the writing.

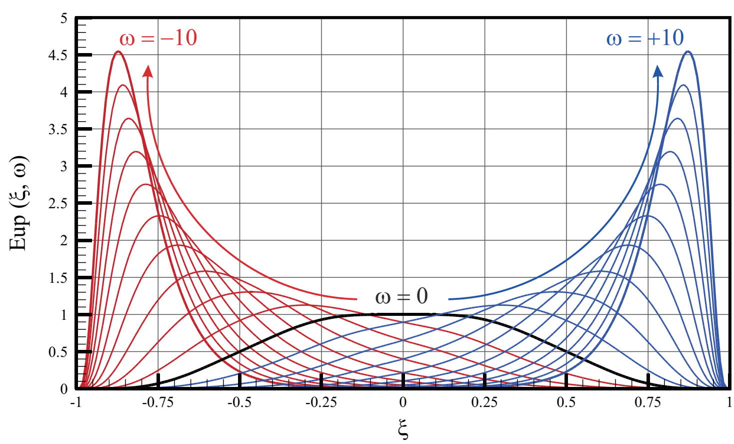



Figure 7: Function $Eup(\xi,\omega)$ for different values of parameter (frequency) $\omega$

Multiplying equation (24) with '$\omega + it$' and by short rearranging, the differential functional equation for the function $y(\xi,\omega)$ yields the final form

$$y'(\xi,\omega) - \omega \cdot y(\xi,\omega) = a \cdot y(2\xi+1,\omega) - b \cdot y(2\xi-1,\omega) \tag{25}$$

where the coefficients $a$ and $b$ are

$$a = \omega \cdot e^{-\omega/2}/\mathrm{sh}(\omega/2), \;\; b = a \cdot e^{\omega} \tag{26}$$

In particular, when the value of the parameter $\omega = 0$, coefficients $a = 2$, $b = 2$, so that in this case, equation (25) is equivalent to equation (8).

The expression for the first derivative of $y(\xi,\omega)$ directly follows from equation (25)

$$y'(\xi,\omega) = \omega \cdot y(\xi,\omega) + a \cdot y(2\xi+1,\omega) - b \cdot y(2\xi-1,\omega)$$

### 3.3 Derivatives of the function $Eup(\xi,\omega)$

Because on the left side of the differential functional equation (25) there is a member that contains a function $y(\xi,\omega)$, it is not possible, as it is in expression (8) for the $up(\xi)$ function, to express derivations directly through the compressed functions on the right side of the equation. However, it is still achieved by applying the appropriate differential operator, [Gotovac (1986); Gotovac and Kozulić (2002); Rvachev and Rvachev (1979)], which converts the function $y(\xi,\omega)$ into a combination of derivatives from the zero to the $m$-th order:

$$L_m = \prod_{j=0}^{m} \left(d/d\xi - 2^j\omega\right) \tag{27}$$

On the right side of the expression, similar to the ABF $up(\xi)$ (Fig. 3), only a linear combination of compressed basis functions appears (Fig. 8b). So, according to Eq. (27), the general form for an arbitrary derivative of the $y(\xi,\omega)$ function is determined by the following equation:

$$L_m y(\xi,\omega) = 2^{m(m+1)/2} \cdot \sum_{j=1}^{2^{m+1}} D_j^{(m)} y\left(2^{m+1}\xi - 2j + 2^{m+1} + 1, \omega\right) \tag{28}$$

Coefficients $D_j^{(m)}$ are calculated using the following recursive formula [Gotovac (1986)] for $m > 0$:

$$D_j^{(m)} = D_k^{(0)} \cdot D_l^{(m-1)} \quad j = 1,\ldots,2^{m+1};\; k = 1,2;\; l = 1,\ldots,2^m \tag{29}$$

where $D_k^{(0)}$ are defined by coefficients from equation (26), i.e.,

$$D_1^{(0)} = a; \; D_2^{(0)} = -b \tag{30}$$

From expressions (28)–(30), it is obvious that the first derivative of the function $y(\xi,\omega)$ is the sum of two components. The first component (Fig. 8a) is the zero derivative of the function $y(\xi,\omega)$ or the function itself multiplied by a coefficient, which in this case is just a parameter $\omega$. The second component (Fig. 8b) is a linear combination of compressed and displaced functions $y(\xi,\omega)$ similar to the first derivative of the function $up(\xi)$.

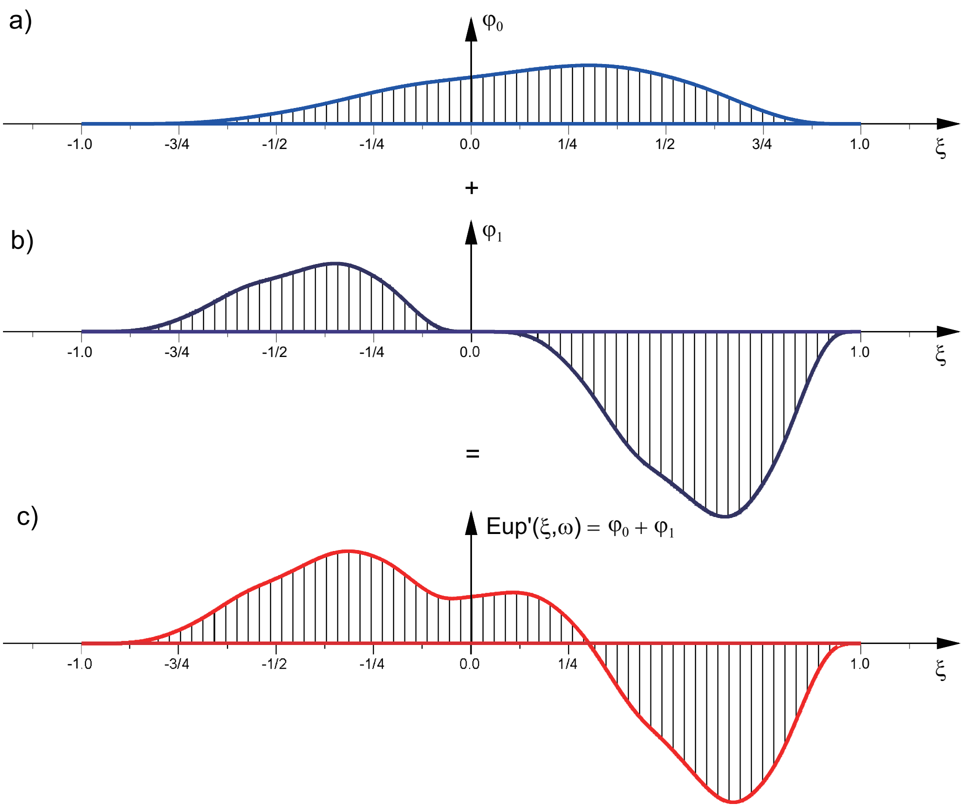



Figure 8: a) $\varphi_0 = \omega \cdot y(\xi,\omega)$, b) $\varphi_1 = a \cdot y(2\xi+1) - b \cdot y(2\xi-1)$, c) The first derivative of the function $Eup(\xi,\omega)$

By continuing the process of derivation, it follows that the derivative of the function $y(\xi,\omega)$ of the $m$th order is obtained as a linear combination of the function derivatives up to the order $(m-1)$ and the compressed and displaced function $y(\xi,\omega)$, similar to the $m$th derivative of the function $up(\xi)$.

### 3.4 Relationship between basis functions $Eup(\xi,\omega)$ and exponential polynomials $e^{2^m\omega\xi}$

By successive moving of just one finite basis function $Eup(\xi,\omega)$ along a real axis, the base of the vector space $EUP_n$ is formed. Arbitrary function $\varphi(\xi)$ can be represented as a linear combination of functions from the vector space $EUP_n$

$$\varphi(\xi)=\sum_{k=-\infty}^{\infty} C_n^{(m)}(k)\cdot Eup(\xi-k\cdot\Delta\xi_n,\omega),\; k\in Z,\; n\in N \tag{31}$$

where $C_n^{(m)}(k)$ are unknown coefficients of the linear combination, and $\Delta\xi_n=2^{-n}$ is a characteristic displacement of the basis function by the abscissa axis.

For a linear combination (31) to represent an exponential monomial $e^{2^m\omega\xi}$, it is necessary and sufficient that for the given $n\in N$ and by applying a differential operator (27) on the expression (31), the linear combination on the right side is annuled. Hence, according to Eq. (28), it follows that the coefficients $C_n^{(m)}(k)$ on the interval $[k\cdot 2^{-n},(k+1)\cdot 2^{-n}]$ must satisfy the following equation

$$\sum_{k=1}^{2^{n+1}} D_k^{(n)}C_n^{(m)}(k)=0,\; n\in N \tag{32}$$

where the coefficients $D_k^{(n)}$ are determined by expressions (29) and (30). Hence, $m=0,1,\ldots,n$ is the exponential polynomial degree, similar to the vector space $UP_n$, whereas $n$ is the highest degree of the exponential polynomial contained in the selected vector space $EUP_n$.

The coefficients of linear combination $C_n^{(m)}(k)$ from Eq. (31) are the roots of the characteristic equation of linear recursion (32).

For example, for $n=0$ ($m=0$) (Fig. 9a), according to expression (32), the following recursion is obtained

$$a\cdot C_0^{(0)}(k)-a\cdot e^{\omega}\cdot C_0^{(0)}(k-1)=0$$

Its solution is sought in the form of $C_0^{(0)}(k)=\lambda^k$. From the corresponding characteristic equation $\lambda^k-e^{\omega}\cdot\lambda^{k-1}=0$, $\lambda=e^{\omega}$ or $C_0^{(0)}(k)=e^{\omega k}/A_0^{(0)}$ is obtained.

So, the exponential monomial of zero degree $e^{2^0\omega\xi}$ has the final form

$$e^{2^0\omega\xi}=\frac{1}{A_0^{(0)}}\sum_{k=-\infty}^{\infty} e^{\omega k}\cdot Eup\left(\xi-k/2^0,\omega\right),\; k\in Z \tag{33}$$

where the coefficient $A_0^{(0)}$ is calculated from the expression (33) for $x=0$ and for $n=0$ is $A_0^{(0)}=Eup(0,\omega)=\lambda_0$.

Analogously, for $n=1$ ($m=0, m=1$), (Fig. 9b), a linear recursion is obtained whose roots are

$$\lambda_1 = e^{\omega/2};\;\; \lambda_2 = e^{\omega}$$

so the exponential monomials of the zero and the first degree can be expressed as a linear combination of the functions from the vector space $EUP_1$ in the following way

$$e^{2^0\omega\xi} = \sum_{k=-\infty}^{\infty} \frac{e^{2^0(\omega/2)k}}{A_1^{(0)}} \cdot Eup\left(\xi - k/2, \omega\right),\;\; k \in Z$$

$$e^{2^1\omega\xi} = \sum_{k=-\infty}^{\infty} \frac{e^{2^1(\omega/2)k}}{A_1^{(1)}} \cdot Eup\left(\xi - k/2, \omega\right),\;\; k \in Z$$

where

$$A_1^{(0)} = e^{-\omega/2} \cdot Eup\left(1/2, \omega\right) + Eup\left(0, \omega\right) + e^{\omega/2} \cdot Eup\left(-1/2, \omega\right)$$

$$A_1^{(1)} = e^{-\omega} \cdot Eup\left(1/2, \omega\right) + Eup\left(0, \omega\right) + e^{\omega} \cdot Eup\left(-1/2, \omega\right)$$

Generally, the exponential function $e^{2^m\omega\xi}, m = 0, 1, \ldots, n, n \in N$ on the interval $\Delta\xi_n = 2^{-n}$ can be accurately represented by a linear combination of $2^{n+1}$ basis functions $Eup\left(\xi, \omega\right)$ mutually displaced by $\Delta\xi_n$ in the form the $Eup\left(\xi - k2^{-n}, \omega\right)$ functions

$$e^{2^m\omega\xi} = \sum_{k=-\infty}^{\infty} \frac{e^{\omega\cdot k\cdot 2^{m-n}}}{A_n^{(m)}} \cdot Eup\left(\xi - \frac{k}{2^n}, \omega\right)$$

where

$$A_n^{(m)} = \sum_{i=-(2^n-1)}^{2^n-1} e^{i\cdot\omega\cdot 2^{m-n}} \cdot Eup\left(-\frac{i}{2^n}, \omega\right)$$

or in the real area of coordinate $x$:

$$e^{2^m\omega x} = \sum_{k=-\infty}^{\infty} \frac{e^{\omega\cdot\Delta x\cdot k\cdot 2^{m-n}}}{A_{n,x}^{(m)}} \cdot Eup\left(x - \frac{k}{2^n}, \omega\cdot\Delta x\right) \tag{34}$$

where

$$A_{n,x}^{(m)} = \sum_{i=-(2^n-1)}^{2^n-1} e^{i\cdot\omega\cdot\Delta x\cdot 2^{m-n}} \cdot Eup\left(-\frac{i}{2^n}, \omega\cdot\Delta x\right) \tag{35}$$

So, a binary increase in the number of basis functions in the linear combination on the interval of length $2^{-n}$ allows the development of an exponential function of degree $m = 0, 1, \ldots, n$.

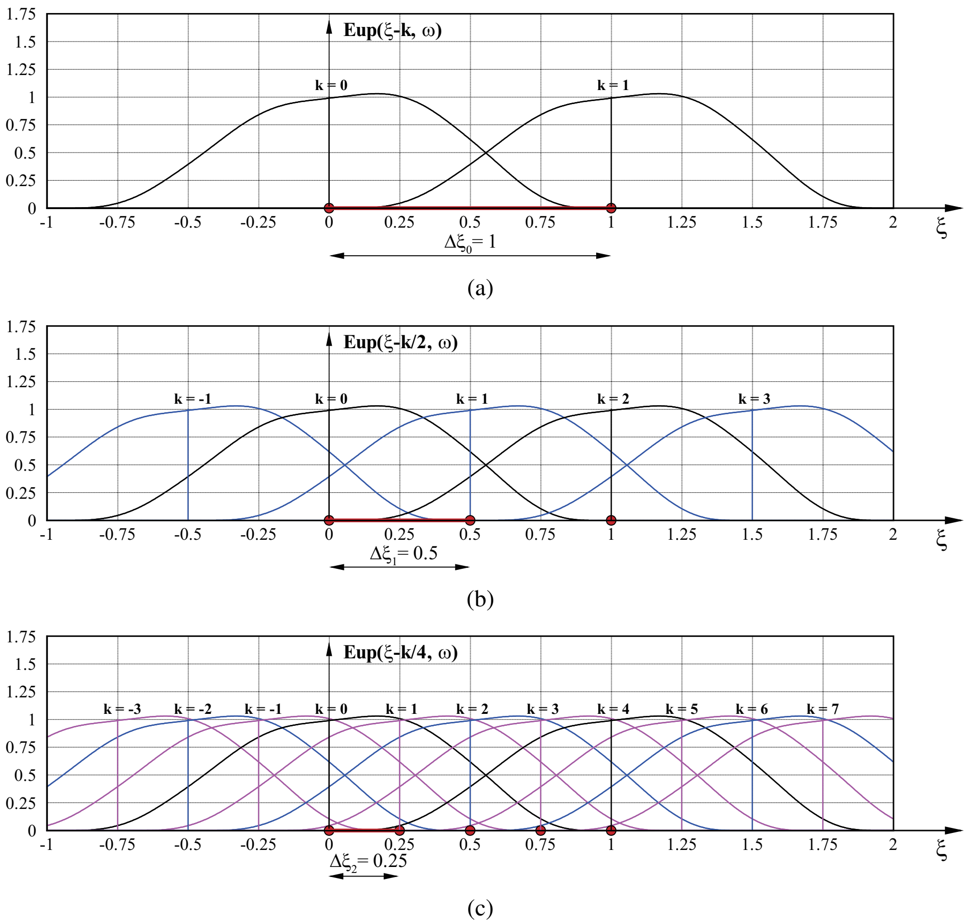



Figure 9: Development of exponential monomials of 0, 1 and 2 degrees by

### 3.5 *Example: Approximation of exponential polynomial function*

The given function is $f(x) = -230 \cdot e^{0.6 \cdot x} + 304 \cdot e^{1.2 \cdot x} - 65 \cdot e^{2.4 \cdot x}, x \in [0,1]$. Approximation of the function is sought in the form of a linear combination of basis functions shown in Fig. 9a), 9b) and 9c).

**a)** Using the collocation method and the associated frequency $\omega_a = 0.6$ for $\Delta x_a = 1$, the system of equations and the coefficients $C_i = 1, i = 0,1$ are obtained:

$$\begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{bmatrix} \cdot \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = \begin{bmatrix} f(0) \\ f(1) \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = \frac{1}{\lambda_0} \cdot \begin{bmatrix} f(0) \\ f(1) \end{bmatrix} = \frac{1}{\lambda_0} \begin{bmatrix} 9 \\ -126.278 \end{bmatrix} \tag{36}$$

Approximation of the function $f(x)$ in the form $\widetilde{f}_a(x) = C_0 \cdot Eup(x, \omega_a) + C_1 \cdot$

$Eup(x-1,\omega_a)$ is shown in Fig. 10.

**b)** For three collocation points, the distribution of basis functions is shown in Figure 9b). The length of a characteristic interval is $\Delta x_b = \Delta x_a/2$, and the associated frequency is $\omega_b = 2 \cdot \omega_a = 2 \cdot 0.6 = 1.2$. For three collocation points and two boundaries, conditional equations have a similar coefficient matrix as in Eq. (17). The obtained system is singular. The used vector space contains exponential polynomials up to the first degree. The first and the last equation should be replaced with the second derivative of $EFup_1(x,\omega)$, [Gotovac (1986); Gotovac and Kozulić (2002); Rvachev and Rvachev (1979)], which are also contained in a vector space formed by the basis functions $Eup(x,\omega)$. For ABF, $Eup(x,\omega)$ can be written as alternative equations, similar to the approach described in Section 2.7 for ABF $up(x)$

$$\begin{array}{lllllll} I. & e^{0\omega} & -e^{\omega} & & & & \\ II. & e^{0\omega} & -2e^{\omega} & e^{2\omega} & & & \\ III. & e^{0\omega} & -3e^{\omega} & 3e^{2\omega} & -e^{3\omega} & & \\ IV. & e^{0\omega} & -4e^{\omega} & 6e^{2\omega} & -4e^{3\omega} & e^{4\omega} & \\ V. & e^{0\omega} & -5e^{\omega} & 10e^{2\omega} & -10e^{3\omega} & 5e^{4\omega} & -5e^{5\omega} \end{array} \tag{37}$$

When $\omega \to 0$, the coefficients from (37) correspond to the coefficients in (18). The resulting approximation $\widetilde{f_b}(x)$ is shown in Fig. 10.

**c)** The linear combination of ABF of exponential type arranged according to Fig. 9c) accurately describes the exponential monomial of zero, first and second degree, or an exponential polynomial created by their combination. Using $\Delta x_c = \Delta x_b/2 = \Delta x_a/4$, the frequency $\omega_c = 2^1 \cdot \omega_b = 2^2 \cdot \omega_a$, the collocation method and additional equations from (37), the values of required coefficients are obtained.

The linear combination of basis functions shown in Fig. 9c) exactly describes the given function $f(x)$ as shown in Fig. 10.

The linear combination of basis functions arranged with displacement $\Delta x = h/4$ accurately approximates any algebraic polynomial up to the second degree as shown in Section 2.7 because of the universality of the vector space. The same is valid for the exponential ABF.

So, for $\Delta x = h/4 = 1/4$ and frequency $\omega = 2.4$, exponential monomials $e^{(2.4/4)\cdot x}$, $e^{(2.4/2)\cdot x}$, $e^{2.4\cdot x}$ and any of their linear combinations can be represented exactly. Thus, the given function $f(x)$ fully coincides with approximation $\widetilde{f_c}(x)$ as shown in Fig. 10.

### 3.6 The values of the function $Eup(\xi,\omega)$ at the origin $\xi = 0$ and $\xi_{br}$

By integrating the differential functional equation (25) in the range from $\xi = -1$ to $\xi = 0$, [Gotovac (1986); Rvachev and Rvachev (1979)], the formula for numerically

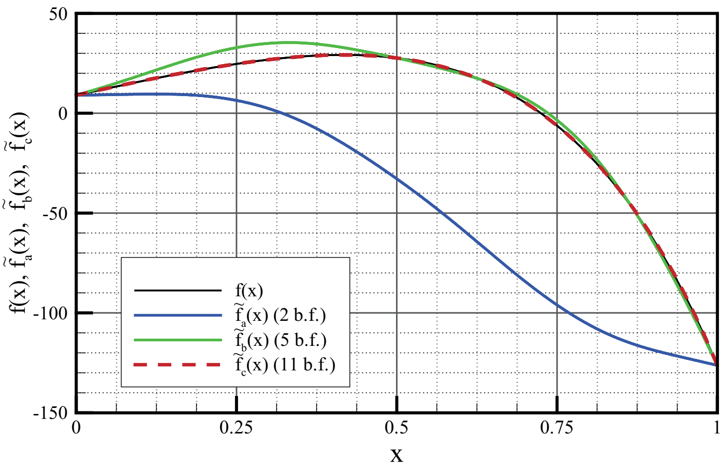



Figure 10: The function $f(x)$ and its approximations $\tilde{f}_a(x), \tilde{f}_b(x), \tilde{f}_c(x)$

calculating the value of the function $Eup(\xi,\omega)$ at the origin is obtained:

$$Eup(0,\omega) = \frac{\omega}{2\cdot \mathrm{sh}(\omega/2)} \cdot F(i\omega/2) = \lambda_0 \tag{38}$$

The expression for calculating the values of the function $Eup(\xi,\omega)$ in the binary rational points (5) is derived in Ref. [Rvachev and Rvachev (1979)]:

$$Eup\left(-1+k\cdot 2^{-m},\omega\right) = \sum_{p=0}^{m} \alpha_p \cdot F\left(\frac{i\cdot\omega}{2^{m+1-p}}\right)\cdot \sum_{r=1}^{k} D_r^{(m)} \cdot e^{\omega(2k-2r+1)/2^{m+1-p}}$$
$$m = 0,1,\ldots,N; p = 0,\ldots,m; k = 1,2,3\ldots,2^{m+1};\ r = 1,\ldots,k \tag{39}$$

Coefficients $D_r^{(m)}$ are determined by the expressions (29) and (30), and the integral operator $\alpha_p$ derived in [Gotovac (1986); Rvachev and Rvachev (1979)] is determined by the following formula:

$$\alpha_p = \frac{\omega^{-m}}{\prod_{p=0,p\neq j}^{m}(2^p-2^j)},\quad j=0,\ldots,m;\ \text{ for } m=0 \to \alpha_0 = 1$$

The member $F\left(\frac{i\cdot\omega}{2^{m+1-p}}\right)$ in (39) represents the Fourier transforms of the characteristic values of the variables:

$$F\left(\frac{i\omega}{2^{j+1}}\right) = \lambda_0 \cdot \frac{2\cdot sh(\omega/2)}{\omega} \cdot \prod_{k=1}^{j}\left(\frac{2\cdot sh(\omega/2)}{\omega}\cdot\frac{\omega/2+it/2^{k+1}}{sh(\omega/2+it/2^{k+1})}\right)$$

In this way, with (39), values of the function $Eup(\xi,\omega)$ in binary rational points $\xi_{br}$ can be determined in a way that they depend only on the value of the function $Eup(\xi,\omega)$ at the origin $\xi = 0$ from expression (38).

For example, the values of the function $Eup(\xi,\omega)$ at the binary rational points for $m = 1$ are:

$$Eup\left(-\frac{1}{2}\right) = \frac{e^{-\omega/2}}{e^{\omega/2}+1}\cdot\lambda_0, \quad Eup(0) = \lambda_0, \quad Eup\left(\frac{1}{2}\right) = \frac{e^{\omega}}{e^{\omega/2}+1}\cdot\lambda_0$$

### 3.7 Values of the function Eup$(\xi,\omega)$ and the nth order derivatives at an arbitrary point

The value of the function $Eup(\xi,\omega)$ at an arbitrary point is determined by the series of a special form that is constructed based on the fact that the development of the function $Eup(\xi,\omega)$ into a Taylor series at the binary rational points $\xi_{br}$ contains exponential polynomials analogous to the function $up(\xi)$, which contains algebraic polynomials:

$$y(\xi) = \sum_{k=0}^{\infty}(-1)^{p_0+p_1+\ldots+p_{k+1}}\cdot p_k\cdot e^{\omega\cdot(p_0+p_1+\ldots+p_{k+1})}\sum_{j=0}^{k}A_{jk}\left(\frac{a_\omega}{\omega}\right)^j\cdot y\left(-1+\frac{1}{2^{k-j}}\right) \tag{40}$$

The values for $y\left(-1+2^{-k+j}\right)$ are determined according to [Gotovac (1986); Rvachev and Rvachev (1979)], whereas coefficients $A_{jk}$ are rational expressions determined by:

$$A_{jk} = \frac{1}{\prod_{i=0}^{j-1}\left(2^{i+1}-1\right)}\sum_{m=0}^{j}(-1)^m\cdot 2^{m\cdot(m-1)/2}\cdot\prod_{i=0}^{m-1}\frac{2^{-i}\left(2^j-2^i\right)}{2^{i+1}-1}\cdot e^{\omega\cdot 2^{j-m}\cdot\left(\xi+\left(2^k-1\right)/2^k\right)}$$

If the arbitrary point is displayed in binary form $\xi = p_0, p_1, \ldots, p_k$, where $p_0, p_1, \ldots, p_k$ are the bits or digits 0 or 1 of the binary development of the coordinate $\xi$ value, then the accuracy of the function $Eup(\xi,\omega)$ at an arbitrary point depends on the accuracy of an electronic computer. For that level of accuracy, a very small number of members in Eq. (40) are required.

It is understood that only the value at the origin $\lambda_0 = Eup(0,\omega)$ must be calculated numerically according to (38).

Using expressions (28)–(30) to (40), the expression for the nth order derivative of the function $Eup(\xi,\omega)$ at an arbitrary point is derived:

$$y^{(n)}(\xi) = \sum_{i=0}^{n}\omega^{n-i}\cdot D_r^{(i-1)}\cdot 2^{i\cdot(i-1)/2}\cdot y\left(2^i\cdot\xi-2\cdot r+2^i+1\right)\cdot\prod_{k=0}^{i-1}\frac{\left(2^{n-k}-1\right)}{2^{k+1}-1}$$

where $n$ is the derivation order $D_r^{(-1)} = 1$ (by definition), whereas the coefficients $D_r^{(i)}$ are defined by expressions (29) and (30), and $r$ is an integer value defined by

$$r = \text{FLOOR}\left[(\xi+1)\cdot 2^{i-1}+1\right]$$

# 4 ABF Eup$(x,\omega)$ implementation

## *4.1 Basis functions distribution*

An ABF of exponential type in relation to the algebraic basis functions contains the parameter $\omega$ or, analogous to trigonometric functions, the frequency.

Vector space $EUP_n$ formed by ABF $Eup\,(x,\omega)$ with compact support $supp\ Eup(x,\omega) = [-\Delta x_0, \Delta x_0]$ has certain similarities to the vector space of trigonometric functions. Development of the unit value in both spaces is possible only in the case $\omega_0 = 0$, where

$$\omega \in \{\omega_k\},\ k \in Z.$$

Function $\sinh(\omega, x) = \left(e^{\omega x} - e^{-\omega x}\right)/2$ is composed of an exponential function with positive and negative frequencies. Thus, the vector space that contains the function $\sinh(\omega, x)$ should also contain the basis functions with positive and negative frequencies, as in Fig. 11b).

The odd indices are assigned to negative values of frequencies and even to positive ones. Approximation of a given function $f(x)$, $x \in [A,B]$, where the domain is divided into $n$ intervals $x_0$, can be described by a linear combination of basis functions $Eup\left(x/\left(2^k \Delta x_k\right) - i/2^k, \omega_k\right)$ mutually displaced per $\Delta x_k = \Delta x_0/2^k$ in the form:

$$f(x) = \sum_{i=1-2^k}^{n*2^{k+1}-1} C_i^{(k)} \cdot Eup\left(\frac{x}{2^k \Delta x_k} - \frac{i}{2^k}, \omega_k\right),\ \ k = 0,1,2,\ldots \tag{41}$$

If $k$ is finite, the function $f(x)$ is approximated so that from the selected initial value $x$, the length of interval $x_k$ and the associated frequency $\omega_k$ are determined.

Because the frequencies $\omega_k$, $k = 0,1,2,\ldots$ are unknown in advance, it is necessary to choose a criterion and to construct an algorithm for the best frequency selection.

The achieved accuracy of approximation $\varepsilon$ (the difference between the given function $f(x)$ and approximation $\tilde{f}(x)$) is compared with a given accuracy $\varepsilon^*$, according to the following expression:

$$\varepsilon = \|f(x) - \tilde{f}(x)\| \leq \varepsilon^*$$

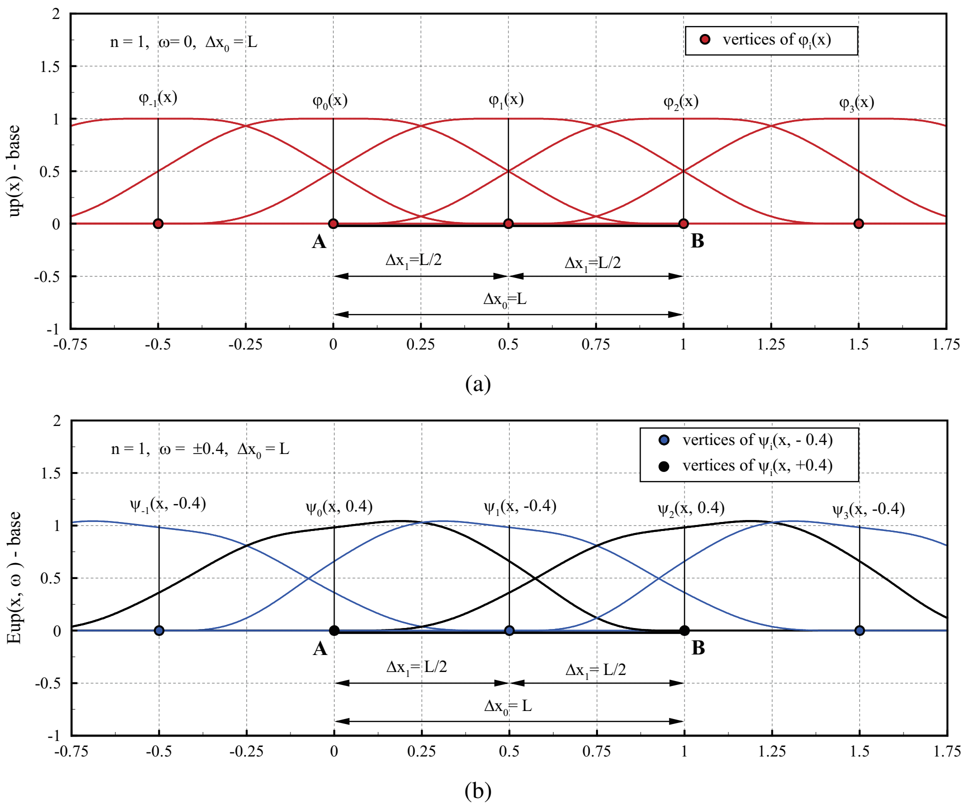



Figure 11: Distributions of the basis functions: a) $up(x)$ and b) $Eup\,(x,\omega_1)$

The numerical procedure can be realized in different ways. Regardless of the approach, a global system of equations according to (41) is formed. The approximation can be searched for by the chosen number of basis functions and the associated frequency. Another way is based on the residuua method, in which a partial contribution of a particular frequency to an entire approximation is solved. Hence, a successive approximation of the difference between the given function and the current sum of the approximations of the residuua functions is performed.

The distribution of basis functions $Eup\left(\frac{x}{2\cdot\Delta x}-\frac{i}{2},\pm\omega_1\right), i=-1,0,1,2,3$ for frequency $\omega_1=\mp 0.4$ and the length of a characteristic interval $\Delta x_1 = L/2$ is shown in Fig. 11b).

The basis functions are inclined to the right or to the left, depending on the sign of the frequency $\omega_1$. If the value of the frequency tends to zero, the basis functions become even and correspond to the algebraic ABF $up\left(\frac{x}{2\cdot\Delta x}-i/2\right), i=-1,0,1,2,3$ on Fig. 11a).

### 4.2 Approximation of the given exponential function $e^{\omega x}$

Let the function be given on segment $\overline{AB}$ in the form $f(x) = e^{10(x-1)}, x \in [0,1]$. The function approximations are searched using two characteristic intervals $x$ and the basis functions distribution according to Fig. 11b). The given function has an exponential character, and frequency $\omega$ has a value of 10 (only positive values of frequencies).

Using the collocation method, coefficients $C_i, i = -1,0,1,2,3$ of linear combinations of $up\left(\frac{x}{2\cdot\Delta x} - i/2\right)$ and $Eup\left(\frac{x}{2\cdot\Delta x} - \frac{i}{2}, \omega = 10\right)$ basis functions are calculated and presented in Table 1.

Coefficients $C_i$ with odd indices are equal to zero (see Table 1) because the corresponding basis functions do not contribute to the approximation. The given function has only positive frequency, so basis functions with a negative frequency must remain neutral so that the approximation is not spoiled. Approximation with the algebraic basis functions has a known oscillating character, whereas approximation obtained with exponential basis functions corresponds exactly to the given function.

Table 1: Coefficients of the linear combinations

| $C_i$ \ $i$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $C_i - Alg.$ | 0.2433 | 0.0 | $-0.2433$ | 0.5 | 1.2433 |
| $C_i - Exp.$ | **0.0** | 0.0291 | **0.0** | 640.7983 | **0.0** |

In general, the function to be approximated can include a frequency that is not known in advance by the sign or the value. Therefore, it is necessary to construct a method for determining the frequency of the basis functions $Eup(x,\omega)$ that gives the best approximation.

### 4.3 Determination of the best frequency

The approximation of the given function $f(x)$ is searched using the collocation method in the form of a linear combination of exponential basis functions that contain an unknown frequency $\omega$.

To determine the best frequency $\omega$, it is necessary to calculate the eigenvalues $\omega_i, i = 1,2,\ldots,n$ of the system matrix. For example, for a given function $f(x) = -tgh((x-3/4)/0.02), x \in [0,4]$ and selected interval $\Delta x$, five equations are obtained, and the eigenvalues of the corresponding coefficient matrix are illustrated on Fig. 13.

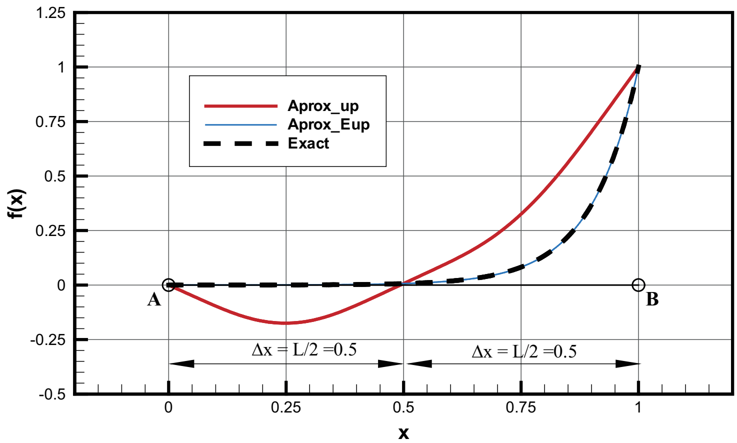



Figure 12: Comparison of approximations with the given function using algebraic and exponential basis functions with frequency $\omega = 10$

Hence, to determine the frequency $\omega$ that gives the best approximation, in a certain sense, appropriate additional criteria in the physical or some other sense, have to be chosen.

We selected the criterion of the least squares method for deviation between the given function and its approximation on each characteristic interval $\Delta x$:

$$\sum_{k=0}^{n-1} \int \left[f(x) - \tilde{f}(x)\right]^2 dx = \min \tag{42}$$

where $f(x)$ is the given function, $\tilde{f}(x)$ is an approximation of the function, and $n$ is the number of characteristic intervals $\Delta x$ in the domain $\overline{AB}$.

Using Eq. (42), the frequencies $\omega_i, i = 1, \ldots, 5$ are checked, and the one that gives the smallest square deviations is selected.

In this paper, a less economical though simpler method is selected. Generally, when the given function is not an algebraic polynomial or exponential function, the frequency step $\Delta\omega$ is selected, and starting from zero, the value of the last square deviation of the approximation with respect to the given function is determined according to (42).

The dependence between deviation and the frequency of the basis functions is shown in Fig. 14. The presented dependence is similar in the approximations of various functions. So, starting from zero, the deviation suddenly begins to rise and decline rapidly, thus achieving the local minima.

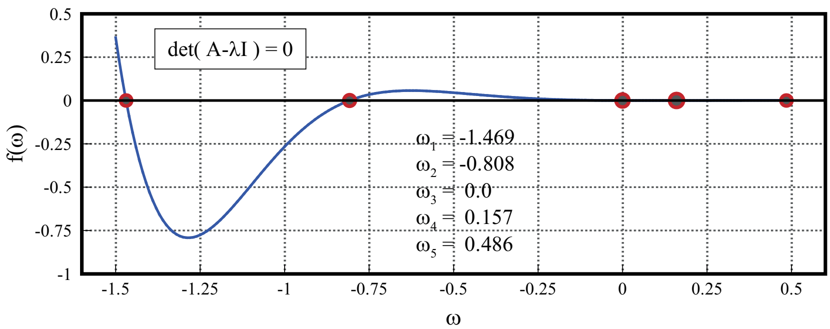



Figure 13: The roots of the frequency function $\omega_i, i = 1, \ldots, 5$

The absolute minimum, if not registered at $\omega = 0$, is in the area of a very slight change of the deviation square dependent on $\omega$, as shown in Fig. 13. The linear combination of basis functions $Eup\,(x, \omega)$ with a calculated frequency $\omega > 0$ generally gives a better approximation than the analogous algebraic atomic functions.

If the given function $f(x)$ is a polynomial $P_1(x)$, the algorithm for determining the best frequency finds the value $\omega = 0$ because the deviation square is then $LSS \equiv 0$, which corresponds to an absolute minimum.

Additionally, in the case of the exponential polynomials, e.g., $f(x) = e^{10(x-1)}$, the best frequency $\omega = 10$ is obtained by the proposed algorithm because $LSS \equiv 0$, which determines the absolute minimum of the deviation square of the approximation $\tilde{f}(x)$ from the given function $f(x)$.

### *4.4 Approximation on the uniform grid*

The function $f(x) = TANGH\,((x-4/3)/0.02)\,, x \in [0,4]$ is analysed. It is necessary to find an approximation by the linear combination of algebraic ABFs in the form of:

$$\tilde{f}(x)_{alg} = \sum_{i=-1}^{2n+1} C_i \cdot up\left(\frac{x}{2\cdot\Delta x} - \frac{i}{2}\right) \tag{43}$$

and exponential ABFs in the form:

$$\tilde{f}(x)_{exp} = \sum_{i=-1}^{2n+1} C_i \cdot Eup\left(\frac{x}{2\cdot\Delta x} - \frac{i}{2}, \omega_n\right) \tag{44}$$

where $n$ is the number of characteristic intervals on the length of the domain $L = 4.0$, and $i$ is the counter of the basis functions and collocation points except boundary points, which are double collocation points.

The numerical experiments were performed for the five different interval lengths $\Delta x = L/n$, where $n = 16, 32, 64, 128, 256$.

A comparison of the given function and its approximations according to Eq. (43) is shown in Fig. 14 in columns a1)–a5).

In the first four variants, the approximation oscillations are extremely expressive and are present even in the fifth variant for $\Delta x = L/256$. Oscillations occur when using any basis functions of the algebraic type.

An attempt is made to eliminate the negative effect of the algebraic functions using basis functions of exponential type. Columns b1)–b5) in Figure 14 show the approximations according to (44) for the same resolutions as in columns a1)–a5). An incomparably better approximation can be seen. However, for each variant, the most appropriate frequency $\omega$ should be previously determined. The frequency $\omega$ is the same for all basis functions in the linear combination for selected number of intervals $n$ according to (44).

According to the chosen criteria of least deviation squares between the approximation and the given function, the dependence between the deviation square and the frequency is shown in columns c1)–c5).

The algorithm for finding the best frequency is set only on the non-negative frequency values. If the frequency value is negative, it is controlled by the coefficients of the linear combination.

The effect of the exponential basis functions frequency impact on the approximation is visible already at $n = 8$ in Fig. 14 b1) with respect to a1). The frequency is directly related to the length of the interval $\Delta x$, and in all terms, the product $\Delta x \cdot \omega$ appears.

The value of this product remains constant in all variants c1)–c5) and is approximately 2.243. In other words, it is sufficient to accurately calculate the frequency for the largest $\Delta x$, and for the smaller values of $\Delta x$, the frequency value is $\omega_{k+1} = \frac{\Delta x_k}{\Delta x_{k+1}} \cdot \omega_k$, or in the case of diadic resolution increasing:

$$\omega_{k+1} = 2^k \cdot \omega_1, \;\; k = 0, 1, \ldots$$

### *4.5 Approximation by levels (Multilevel Approximation)*

Another approach to obtain approximations of the given function is a multilevel approach. We consider the function from the previous section.

In the zero step, the function approximation $\widetilde{f_0}(x)$ is subtracted from the given function $f(x) = f_0(x)$, and the new function $f_1(x)$ is obtained. Fig. 15 b) compares the function $f_1(x)$ with a prescribed accuracy, for example, $\varepsilon = \pm 0.02$. If accuracy is not satisfied, the approximation $\widetilde{f_1}(x)$ of the function $f_1(x)$ is searched (see Fig.

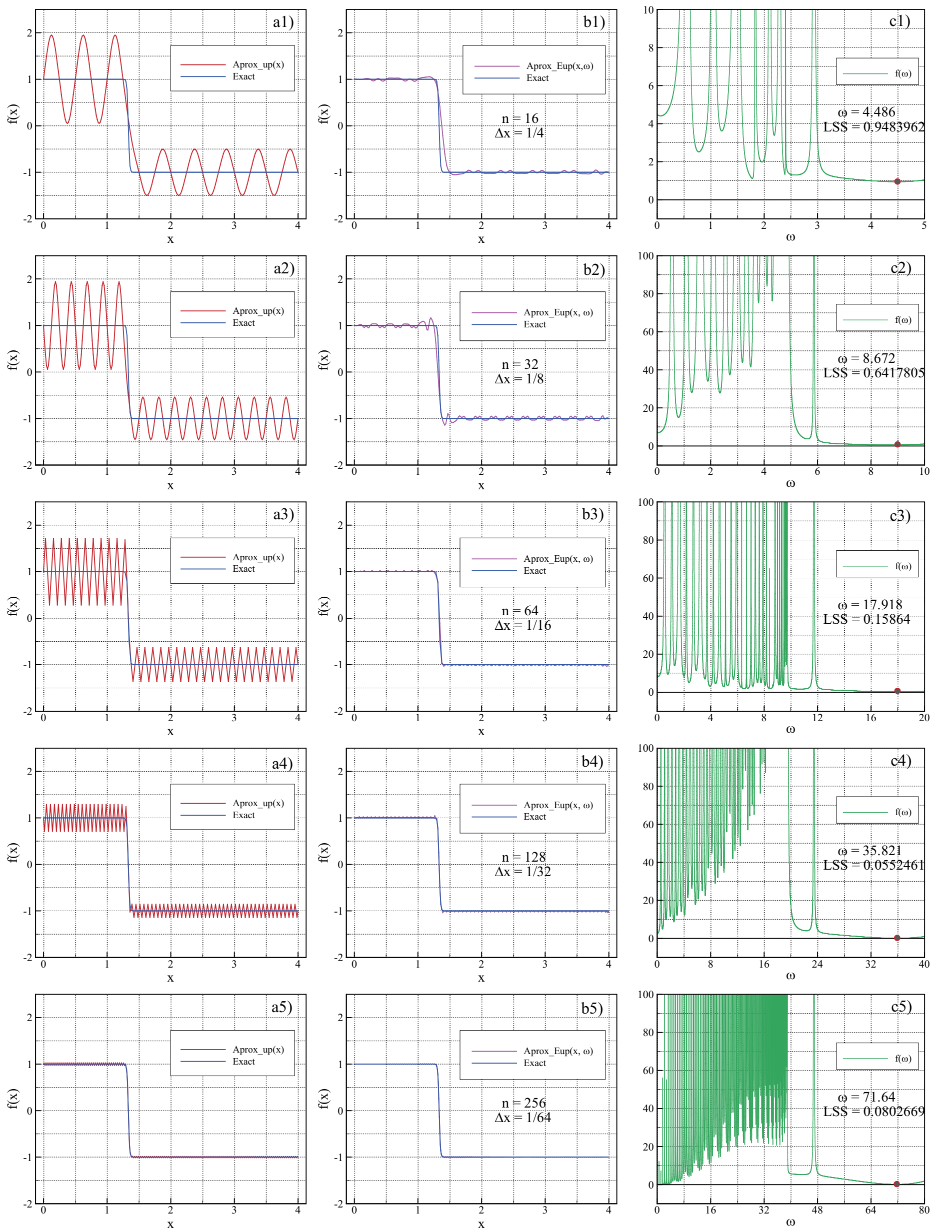



Figure 14: The approximations of the given function $f(x)$: a1)–a5) by algebraic ABFs; b1)–b5) by exponential ABFs; c1)–c5) finding the minimum of the deviations square depending on the frequency $\omega$

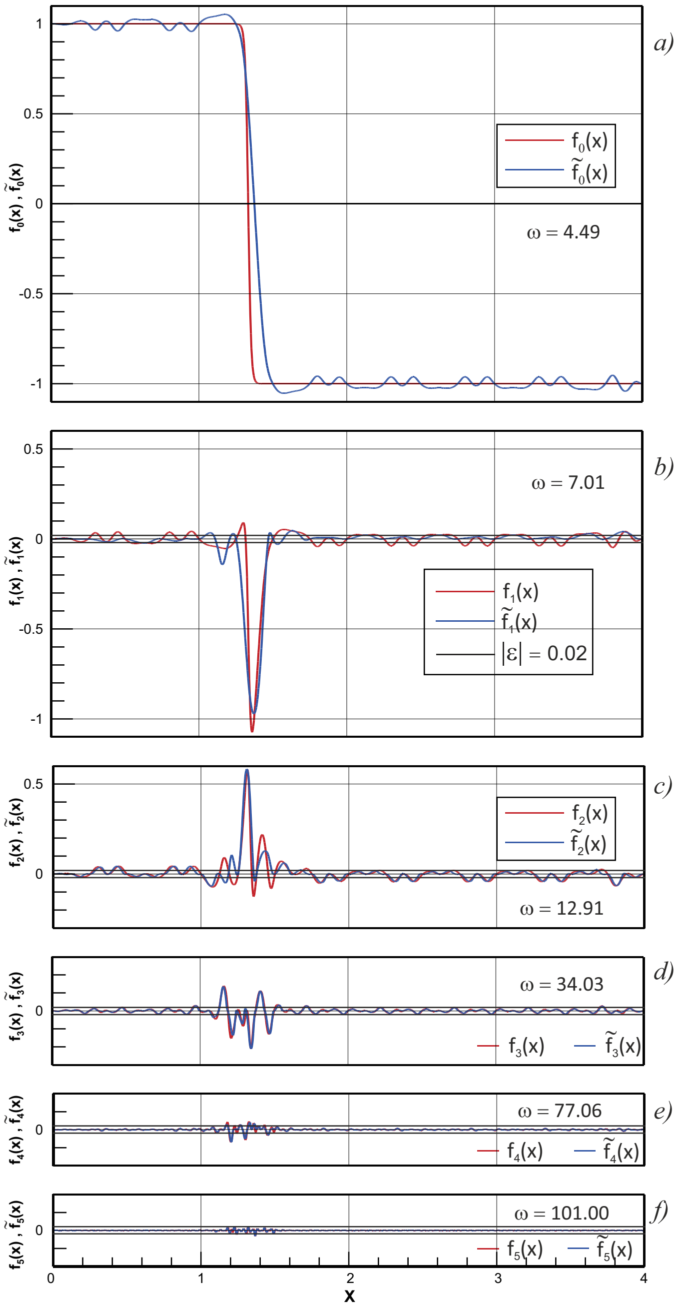



Figure 15: Given function $f(x)$, function approximation $\tilde{f}(x)$ and residuua $f(x) - \tilde{f}(x)$ by levels a)–f)

15 b)). Then, a comparison of the difference $f_1(x) - \widetilde{f}_1(x)$ and accuracy $\varepsilon$ follows, and the procedure is repeated until it reaches the requested accuracy, as shown in Fig. 15 f).

The final approximation of the function $f(x)$ is obtained as the sum of the individual approximations at every level

$$\tilde{f}(x) = \sum_{k=0}^{m} \widetilde{f}_k(x), \quad m \in N$$

where the individual approximation is in fact a linear combination of the basis functions with corresponding frequency $\omega_k$ determined according to the procedure described in Section 4.3:

$$\widetilde{f}_k(x) = \sum_{i=-1}^{2n_k+1} C_i \cdot Eup\left(\frac{x}{2 \cdot \Delta x} - \frac{i}{2}, \omega_k\right)$$

The starting grid $n_0$ is chosen arbitrarily, and the next is twice as dense, i.e., $n_1 = 2 \cdot n_0$, or generally

$$n_k = 2^k \cdot n_0; \quad k = 0, 1, 2, \ldots$$

For example, in Fig. 15, the initial grid $n_0 = 16$ is used, and the corresponding frequency $\omega_0 = 4.486$ is calculated.

In this multilevel approximation method, approximately twice as many basis functions are needed than in the procedure using uniform grid described in Section 4.4 for the same accuracy of approximation.

However, this multilevel procedure is analogous to the procedure that is used in the adaptive Fup collocation method [Gotovac, Andricevic and Gotovac (2007)]. In fact, at higher levels, only collocation points at which the residuum is higher than the prescribed accuracy are considered, whereas the other points do not have to be taken into consideration. Fig. 15 shows that such a criterion leads to an adaptive procedure, which saves CPU time and drastically reduces the number of collocation points at higher levels. We leave the development of an adaptive procedure for the presented *Eup* basis functions to future research in the development of new non-stationary algorithms. Thus, in this section, each level is observed with all collocation points as a non-adaptive algorithm.

## 5 Conclusion

In this paper, the current knowledge regarding mother ABF function $up(x)$ is once more synthesized. The approximation properties and expressions for the required mathematical operations are presented in a simpler, more understandable and user-friendly way. Using this approach, the basic atomic basis function of exponential type $Eup(x,\omega)$ is studied in detail [Gorškov, Kravčenko and Rvačev (1994)]. New expressions for calculating the values of the function, its derivatives and, of particular importance, the rules (elements) for its practical use are derived. The procedure for determining the frequency $\omega$ that gives the best approximation should especially be noted. The application of exponential ABF is shown in a few examples of the function approximations. The numerical results show excellent approximation properties of these basis functions, especially in the case of sharp gradient changes of the given function. Future research includes further development of the exponential ABF theory in the form of new $EFup$ basis functions that will significantly improve the approximation properties of the $Eup$ functions in the same way that algebraic $Fup$ basis functions do for $up$ functions. These more efficient exponential basis functions will be a basis for further development of the adaptive $EFup$ collocation method, which can be applied in non-stationary problem algorithms of, e.g., mass and heat conduction [Gotovac, Andricevic and Gotovac (2007)].

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