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Plane Vibrations in a Transversely Isotropic Infinite Hollow Cylinder Under Effect of the Rotation and Magnetic Field

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Abstract: The aim of this paper is to study the effects of rotation and magnetic field on the plane vibrations in a transversely isotropic material of an infinite hollow cylinder. The natural frequency of the plane vibrations in the case of harmonic vibrations has been obtained. The natural frequencies are calculated numerically and the effects of rotation and magnetic field are discussed. The numerical results obtained have been illustrated graphically to understand the behavior of frequency equation with different values of frequency ω under effects the rotation and magnetic field. Comparison was made with the results obtained in the presence and absence of the rotation and magnetic field. The results indicate that the effect of rotation and magnetic field are very pronounced.

Keywords: Plane transversely isotropic, rotating, magnetic field, homogeneous, transversely isotropic, Natural frequencies.

1 Introduction

The analysis of the dynamic problems of elastic bodies is an important and interesting research field for engineers and scientists. It is concerned with determining the strength and load carrying ability of engineering structures, including buildings, bridges, cars, planes, and thousands of machine parts that most of us never see. It is especially important in the fields of mechanical, civil, aeronautical and materials engineering. However, little attention has been given to the problem of the wave propagation in the isotropic circular cylinder. Boukhari et al [Boukhari et al. (2016)] studied an efficient shear deformation theory for wave propagation of functionally graded material plates. Tounsi et al [Tounsi et al. (2016)] investigated a new simple three-unknown sinusoidal shear deformation theory for functionally graded plates. Yahia et al [Yahia et al. (2015)] discussed the wave propagation in functionally graded plates with porosities using various higher-order shear deformation plate theories. Bellifa et al [Bellifa et al. (2016)] investigated the bending and free vibration analysis of functionally graded plates using a simple shear deformation theory and the concept the neutral surface position. Draiche et al [Draiche et al. (2016)] studied a refined theory with stretching effect for the flexure analysis of laminated composite plate. Mahmoud et al [Mahmoud et al. (2011)] investigated the effect of the rotation on the radial vibrations in a non-homogeneous

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orthotropic hollow cylinder. Abd-Alla et al. [Abd-Alla et al. (2008)] studied the effect of the non-homogenity on the composite infinite cylinder of isotropic material. Bourada et al. [Bourada et al. (2015)] studied a new simple shear and normal deformations theory for functionally graded beams. Gaoab et al [Gaoab et al. (2013)] investigated the wave propagation in poroelastic hollow cylinder immersed in fluid with seismoelectric effect Hebali et al. [Hebali et al. (2014)] investigated the a new quasi-3D hyperbolic shear deformation theory for the static and free vibration analysis of functionally graded plates. Hou et al. [Hou et al. (2006)] discussed the transient responses of a special non-homogeneous magneto-electro-elastic hollow cylinder for axisymmetric plane strain problem. Bennoun et al [Bennoun et al. (2016)] studied a novel five variable refined plate theory for vibration analysis of functionally graded sandwich plates. [Bounouara et al. [Bounouara et al. (2016)] studied a nonlocal zeroth-order shear deformation theory for free vibration of functionally graded nanoscale plates resting on elastic foundation. Meziane [Meziane (2014)] investigated an efficient and simple refined theory for buckling and free vibration of exponentially graded sandwich plates under various boundary conditions. Marin and Lupu [Marin and Lupu (1998)] investigated the harmonic Vibrations in Thermoelasticity of Micropolar Bodies. Marin [Marin (2010)] studied the domain of influence theorem for microstretch elastic materials. Marin [Marin (2010)] discussed the harmonic vibrations in thermoelasticity of microstretch materials. Marin [Marin (1997)] investigated the weak solutions in elasticity of dipolar bodies with voids. Hutchinson and El-Azhary [Hutchinson and El-Azhary (1986)] investigated the vibrations of free hollow circular cylinder. Abd-Alla and Farhan [Abd-Alla and Farhan (2008)] studied the effect of the non-homogeneous on the campsite infinite cylinder of isotropic material. Chen [Chen et al. (2005)] studied the free vibration of non-homogeneous transversely isotropic magneto-electro-elastic plates. Buchanan [Buchanan (2003)] discussed the free vibration of an infinite magneto-electro-elastic cylinder. Abd-Alla et al. [Abd-Alla et al. (2015)] investigated the wave propagation in fibre-reinforced anisotropic thermoelastic medium subjected to gravity field. The extensive literature on the topic is now available and we can only mention a few recent interesting investigations in Refs. [Abd-Alla and Mahmoud (2012), Abd-Alla, et al.((2013), (2017)), Bayones and Abd-Alla (2017)].

The main objective of the present research is to determine the eigenvalues of the natural frequency of the transversely isotropic infinite hollow cylinder for different boundary conditions in the cases of harmonic vibrations under effect the in rotation and magnetic field after determining the displacement components and stress components. The numerical results of the frequency equation are discussed in detail for homogeneous material and the effect of rotation and magnetic field for different cases by figures.

2 Formulation of the problem

Let us consider the electromagnetic field governs by Maxwell equations, under consideration that the medium is a perfect electric conductor taking into account absence of the displacement current (SI) in the from is in Abd-Alla and Mahmoud (2012).

$$j = curl h, \tag{1}$$

$$-\mu_e \frac{\partial \overrightarrow{h}}{\partial t} = curlE \ , \tag{2}$$

$$div\vec{h} = 0, \tag{3}$$

$$divE = 0 \tag{4}$$

$$\overline{E} = -\mu_e \left(\frac{\partial \overrightarrow{h}}{\partial t} \times \overrightarrow{H}\right) \tag{5}$$

where

$$\vec{h} = curl \quad \vec{u} \times \overrightarrow{H} \ , \ \overrightarrow{H} = H_0 + \vec{h} \tag{6}$$

where $\vec{h}$ the perturbed magnetic field is over the primary magnetic field $\vec{E}$ is the electric intensity, $\vec{J}$ is the electric current density, μ_e is the magnetic permeability, H is the constant primary magnetic field and $\vec{u}$ is there a displacement vector. Consider a homogeneous and isotropic elastic solid with a circular hollow cylinder of inner radius and outer radius b through a transversely isotropic material of Infinite extent. Taking the cylindrical polar coordinates such that the z-axis Pointing vertically upward along the axis of the cylinder. The stresses displacement relations for homogeneous cylindrical transversely isotropic materials in two dimensions are in the form

$$\sigma_{rr} = c_{11} \frac{\partial u_r}{\partial r} + c_{12} \frac{u_r}{r} + c_{13} \frac{\partial u_z}{\partial z} \tag{7}$$

$$\sigma_{\Theta\Theta} = c_{12} \frac{\partial u_r}{\partial r} + c_{11} \frac{u_r}{r} + c_{13} \frac{\partial u_z}{\partial z} \tag{8}$$

$$\sigma_{zz} = c_{13} \frac{\partial u_r}{\partial r} + c_{13} \frac{u_r}{r} + c_{33} \frac{\partial u_z}{\partial z} \tag{9}$$

$$\tau_{rz} = c_{44} \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) \tag{10}$$

$$\begin{aligned} \tau_{rr} &= \mu_e H_0^2 \left(\frac{\partial u_r}{\partial r} + \frac{1}{r} u_r + \frac{\partial u_z}{\partial z} \right), \\ \tau_{zz} &= \mu_e H_0^2 \left(\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} \right) \end{aligned} \tag{11}$$

where

$$c_{11} = c_{22}, c_{13} = c_{23}$$

The elasto-dynamic equations in rotating medium as:

$$\frac{\partial \sigma_{rr}}{\partial r}+\frac{\partial \tau_{rz}}{\partial z}+\frac{1}{r}(\sigma_{rr}-\sigma_{\Theta\Theta})+f_r=\rho\left[\frac{\partial^2 u_r}{\partial t^2}-\Omega^2 u_r\right] \tag{12}$$

$$\frac{\partial \tau_{rz}}{\partial r}+\frac{\partial \sigma_{zz}}{\partial z}+\frac{1}{r}\tau_{rz}+f_z=\rho\left[\frac{\partial^2 u_z}{\partial t^2}-\Omega^2 u_z\right] \tag{13}$$

where f_r and f_z are Lorenz's force-are defined by

$$f_r=\mu_e H_0^2\left[\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}+\frac{\partial^2 u_z}{\partial r\partial z}\right] \tag{14}$$

$$f_z=\mu_e H_0^2\left[\frac{\partial^2 u_r}{\partial r\partial z}+\frac{\partial^2 u_z}{\partial z^2}\right] \tag{15}$$

Substituting from Eqs.(1)-(10) into Eqs.(12) and (13), we

$$\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{1}{r^2}u_r+a_1\frac{\partial^2 u_z}{\partial r\partial z}+a_2\frac{\partial^2 u_r}{\partial z^2}=a_3\left(\frac{\partial^2 u_r}{\partial t^2}-\Omega^2 u_r\right), \tag{16}$$

$$\frac{\partial^2 u_r}{\partial r\partial z}+b_1\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+b_2\frac{\partial^2 u_r}{\partial r\partial z}+b_3\frac{1}{r}\frac{\partial u_r}{\partial z}=b_4\left(\frac{\partial^2 u_z}{\partial t^2}-\Omega^2 u_z\right), \tag{17}$$

where

$$a_1=\frac{(c_{13}+c_{44}+\mu_e H_0^2)}{(c_{11}+\mu_e H_0^2)},a_2=\frac{c_{44}}{(c_{11}+\mu_2 H_0^2)},a_3=\frac{\rho}{(c_{11}+\mu_e H_0^2)},$$

$$b_1=\frac{c_{44}}{(c_{44}+c_{13}+\mu_e H_0^2)},b_2=\frac{(c_{13}+c_{44})}{(c_{44}+c_{13}+\mu_e H_0^2)},b_3=\frac{(c_{33}+\mu_e H_0^2)}{(c_{44}+c_{13}+\mu_e H_0^2},\tag{18}$$

$$b_4=\frac{\rho}{(c_{44}+c_{13}+\mu_e H_0^2)}.$$

3 Solution of the problem

By [Morse and Feshbach (1953)], is the displacement vector u can be written

$$u=\underline{\nabla}\Phi+\underline{\nabla}\wedge\overrightarrow{\psi}, \tag{19}$$

where the two functions Φ and Ψ are known in the theory of elasticity, by Lame's potentials rotational and rotational parts of the displacement vector $\vec{u}$ respectively. The cylinder being bounded by the curved surface, therefore the stress distribution includes the effect of both Φ and Ψ. It is possible to take only one component of the vector $\vec{\psi}$ to be nonzero as

$$\vec{\psi}=(0,\psi,0) \tag{20}$$

From Eqs.(16) and (17), we obtain

$$u_r = \frac{\partial \Phi}{\partial r} - \frac{\partial \psi}{\partial z}, \tag{21}$$

$$u_z = \frac{\partial \Phi}{\partial z} + \frac{\partial \psi}{\partial r} + \frac{\psi}{r}. \tag{22}$$

Substituting from Eqs. (21) and (22) into Eq. (16), we get two independent equations for Φ and Ψ as follows:

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + (a_1 + a_2)\frac{\partial^2 \Phi}{\partial z^2} = a_3 (\frac{\partial^2 \Phi}{\partial t^2} - \Omega^2 \Phi) \tag{23}$$

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} - \frac{1}{r^2}\psi - \frac{a_2}{(a_1 - 1)}\frac{\partial^2 \psi}{\partial z^2} = -\frac{a_3}{(a_1 - 1)}(\frac{\partial^2 \Psi}{\partial t^2} - \Omega^2 \Psi) \tag{24}$$

To study the propagation of harmonic waves in the z-direction, we assume a solution in the form

$$[\Phi, \Psi](r, z, t) = [\Phi^*, \Psi^*](r) e^{i(\gamma z - \omega t)} \tag{25}$$

where γ is the wave number, ω is the angular frequency.
Substituting from Eq. (25) into Eqs. (23) and (24) we have

$$r^2 \frac{d^2 \Phi^*}{dr^2} + r\frac{d\Phi^*}{dr} + (\lambda^2 r^2)\Phi^* = 0, \tag{26}$$

and

$$r^2 \frac{d^2 \Psi^*}{dr^2} + r\frac{d\Psi^*}{dr} + (r^2 k^2 - 1)\ \psi^* = 0, \tag{27}$$

where

$$\begin{aligned} \lambda^2 &= \omega^2 a_3 + a_3 \Omega^2 - (a_1 + a_2)\ \gamma^2, \\ k^2 &= \frac{a_2 \gamma^2 - a_3 \omega^2 - a_3 \Omega^2}{(a_1 - 1)}. \end{aligned} \tag{28}$$

The solution of Eqs. (26) and (27) can be written in the following form:

$$\Phi^*(r) = A_1 J_0(\lambda r) + B_1 Y_0(\lambda r) \tag{29}$$

$$\psi^*(r) = A_2 J_1(kr) + B_2 Y_1(kr) \tag{30}$$

where A_1, A_2, B_1 and B_2 are arbitrary constants, J_0 and Y_0 are Bessel functions of the first and second kind of order zero, respectively, J_1 and Y_1 denote cylindrical Bessel's functions of the first and second kind of order one respectively. From Eqs. (29), (30) and (25) we get

$$\Phi(r, z, t) = e^{i(\gamma z - \omega t)} \{\ A_1 J_0(\lambda r) + B_1 Y_0(\lambda r)\ \}, \tag{31}$$

$$\psi(r,z,t)=e^{i(\gamma z-\omega t)}\left\{\ A_2 J_1(kr)+B_2 Y_1(kr)\ \right\}. \tag{32}$$

Substituting from Eqs.(31) and (32) into Eqs.(21) and (22) we obtain

$$u_r=e^{i(\gamma z-\omega t)}\left\{\ -\lambda[A_1 J_0(\lambda r)+B_1 Y_0(\lambda r)]-i\gamma[A_2 kJ_1(kr)+B_2 kY_1(kr)]\ \right\}, \tag{33}$$

$$u_z=e^{i(\gamma z-\omega t)}\left\{i\gamma[A_1 J_0(\lambda r)+B_1 Y_0(\lambda r)]+[A_2 kJ_0(kr)+B_2 kY_0(kr)]\ \right\}, \tag{34}$$

Substituting from Eqs.(33) and (34) into Eqs.(7) , (10) and (11), we get

$$\tau_{rz}=c_{44}e^{i(\gamma z-\omega t)}\left\{\begin{array}{l}-i\gamma\lambda A_1[J_0(\lambda r)+J_1(\lambda r)]-i\gamma\lambda B_1[Y_0(\lambda r)+Y_1(\lambda r)]+kA_2[\gamma^2\\-K]J_1(kr)+kB_2[\gamma^2-k]Y_1(kr)\end{array}\right\} \tag{35}$$

$$\tau_{rr}=\mu_e H_0^2 e^{i(\gamma z-\omega t)}\left(\begin{array}{l}[\lambda^2 J_1(\lambda r)-\frac{\lambda}{r}J_0(\lambda r)-\gamma^2 J_0(\lambda r)]A_1+[\lambda^2 Y_1(\lambda r)-\frac{\lambda}{r}Y_0(\lambda r)-\\-\gamma^2 Y_0(\lambda r)]B_1+[(-k^2+k)i\gamma J_0(kr)]A_2+[(-k^2+k)i\gamma Y_0(kr)]B_2\end{array}\right) \tag{36}$$

$$\sigma_{rr}=e^{i(\gamma z-\omega t)}\left\{\begin{array}{l}[c_{11}\lambda^2 J_1(\lambda r)-c_{12}\frac{\lambda}{r}J_0(\lambda r)-c_{13}\gamma^2 J_0(\lambda r)]A_1+[c_{11}\lambda^2 Y_1(\lambda r)-c_{12}\frac{\lambda}{r}Y_0(\lambda r)-\\-c_{13}\gamma^2 Y_0(\lambda r)]B_1+[(i\gamma\frac{k}{r}J_1(kr)-i\gamma k^2 J_0(kr))c_{11}-i\gamma c_{12}\frac{k}{r}J_1(kr)+\\+i\gamma kc_{13}J_0(kr)]A_2+[(i\gamma\frac{k}{r}Y_1(kr)-i\gamma k^2 Y_0(kr))c_{11}-i\gamma c_{12}\frac{k}{r}Y_1(kr)+\\+i\gamma kc_{13}Y_0(kr)]B_2\end{array}\right\} \tag{37}$$

In the following sections of the hollow circular cylinders with three different boundary conditions are performed.

4 Boundary conditions and frequency Equations

In this case, we are going to obtain the frequency equation for the boundary conditions. We consider the following transformations the boundary

$$c_1=\sqrt{\frac{c_{11}}{\rho}},\quad \gamma=\frac{\omega}{c_1},\quad h=\frac{a}{b},$$

$$\omega=\omega\omega_p,\ \ \omega_p=\frac{\pi\ \ c_1}{b(1-h)},\quad \Omega=\frac{\Omega\ \ b(1-h)}{\pi\ \ c_1}, \tag{38}$$

$$\lambda=\sqrt{a_3\left(\ \left(\frac{\omega\pi c_1}{b(1-h)}\right)^2+\left(\frac{\Omega b(1-h)}{\pi\ c_1}\right)^2\ \right)-\left(\frac{\omega\pi}{b(1-h)}\right)^2(a_1+a_2)}\ , \tag{39}$$

$$k=\sqrt{\frac{1}{a_1-1}\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2\left(a_2-c_1^2a_3\right)-a_3\left(\frac{\Omega b(1-h)}{\pi c_1}\right)^2\right]}. \tag{40}$$

To make all the quantities in (33)-(39), where λ, k do there frequency dimensions.

4.1 Plane vibrations cylindrical body-free surface traction:

In this case, we have

$$\tau_{rz}(a)=\tau_{rz}(b)=0, \tag{41}$$

$$\sigma_{rr}(a)+\tau_{rr}(a)=\sigma_{rr}(b)+\tau_{rr}(b)=0,$$

Which corresponds to the free inner and outer surfaces respectively. From Eqs.(35)- (41), we obtain four homogeneous linear equations in A_1, B_1, A_2 and B_2

$$\begin{gathered}-i\frac{\omega\pi}{b(1-h)}\lambda A_1[J_0(\lambda bh)+J_1(\lambda bh)]-i\frac{\omega\pi}{b(1-h)}\lambda B_1[Y_0(\lambda bh)+Y_1(\lambda bh)]+\\ kA_2\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2-K\right]J_1(kbh)+kB_2\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2-k\right]Y_1(kbh)=0,\end{gathered} \tag{42}$$

$$\begin{gathered}-i\frac{\omega\pi}{b(1-h)}\lambda A_1[J_0(\lambda bh)+J_1(\lambda bh)]-i\frac{\omega\pi}{b(1-h)}\lambda B_1[Y_0(\lambda bh)+Y_1(\lambda bh)]+\\ kA_2\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2-K\right]J_1(kbh)+kB_2\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2-k\right]Y_1(kbh)=0,\end{gathered} \tag{43}$$

$$\begin{gathered}-i\frac{\omega\pi}{b(1-h)}\lambda A_1[J_0(\lambda b)+J_1(\lambda b)]-i\frac{\omega\pi}{b(1-h)}\lambda B_1[Y_0(\lambda b)+Y_1(\lambda b)]+\\ kA_2\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2-K\right]J_1(kb)+kB_2\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2-k\right]Y_1(kb)=0,\end{gathered} \tag{44}$$

$$\begin{gathered}\left[\left(\left(\frac{\omega\pi}{b(1-h)}\right)^2(1+c_{13})+\frac{\lambda}{b}(1+c_{12})\right)J_0(\lambda b)-\lambda^2(1+c_{11})J_1(\lambda b)\right]A_1+\\ \left[\left(\left(\frac{\omega\pi}{b(1-h)}\right)^2(1+c_{13})+\frac{\lambda}{b}(1+c_{12})\right)Y_0(\lambda b)-\lambda^2(1+c_{11})Y_1(\lambda b)\right]B_1+\\ \frac{\omega\pi i}{b(1-h)}\left[k\left[k(1+c_{11})-(1+c_{13})\right]J_0(kb)-\frac{k}{b}(c_{11}-c_{12})J_1(kb)\right]A_2+\\ \frac{\omega\pi i}{b(1-h)}\left[k\left[k(1+c_{11})-(1+c_{13})\right]Y_0(kb)-\frac{k}{b}(c_{11}-c_{12})Y_1(kb)\right]B_2=0,\end{gathered} \tag{45}$$

These are a set of four homogeneous algebraic equations involving four unknown integration constants A_1, B_1, A_2 and B_2 .For a nontrivial solution of these equations, the determinant of the coefficient matrix must vanish. The zero determinant of the coefficient matrix will give the frequency equation for the surface waves. Thus, elimination of these unknowns would give us the frequency equations

$$\Delta = \begin{vmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{vmatrix} = 0, \tag{46}$$

where

$$D_{11} = -i\frac{\omega\pi}{b(1-h)}\lambda[J_0(\lambda bh) + J_1(\lambda bh)], \qquad D_{12} = -i\frac{\omega\pi}{b(1-h)}\lambda[Y_0(\lambda bh) + Y_1(\lambda bh)],$$

$$D_{13} = k\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2 - K\right]J_1(kbh), \qquad D_{14} = k\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2 - K\right]Y_1(kbh),$$

$$D_{21} = -i\frac{\omega\pi}{b(1-h)}\lambda[J_0(\lambda b) + J_1(\lambda b)], \qquad D_{22} = -i\frac{\omega\pi}{b(1-h)}\lambda[Y_0(\lambda b) + Y_1(\lambda b)],$$

$$D_{23} = k\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2 - K\right]J_1(kb), \qquad D_{24} = k\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2 - k\right]Y_1(kb),$$

$$D_{31} = \left[\left(\left(\frac{\omega\pi}{b(1-h)}\right)^2(1+c_{13}) + \frac{\lambda}{bh}(1+c_{12})\right)J_0(\lambda bh) - \lambda^2(1+c_{11})J_1(\lambda bh)\right],$$

$$D_{32} = \left[\left(\left(\frac{\omega\pi}{b(1-h)}\right)^2(1+c_{13}) + \frac{\lambda}{bh}(1+c_{12})\right)Y_0(\lambda bh) - \lambda^2(1+c_{11})Y_1(\lambda bh)\right],$$

$$D_{33} = \frac{\omega\pi i}{b(1-h)}\left[k\left[k(1+c_{11}) - (1+c_{13})\right]J_0(kbh) - \frac{k}{bh}(c_{11} - c_{12})J_1(kbh)\right],$$

$$D_{34} = \frac{\omega\pi i}{b(1-h)}\left[k\left[k(1+c_{11}) - (1+c_{13})\right]Y_0(kbh) - \frac{k}{bh}(c_{11} - c_{12})Y_1(kbh)\right],$$

$$D_{41} = \left[\left(\left(\frac{\omega\pi}{b(1-h)}\right)^2(1+c_{13}) + \frac{\lambda}{b}(1+c_{12})\right)J_0(\lambda b) - \lambda^2(1+c_{11})J_1(\lambda b)\right],$$

$$D_{42}=\left[\left(\left(\frac{\omega\pi}{b(1-h)}\right)^2(1+c_{13})+\frac{\lambda}{b}(1+c_{12})\right)Y_0(\lambda b)-\lambda^2(1+c_{11})Y_1(\lambda b)\right],$$

$$D_{43}=\frac{\omega\pi i}{b(1-h)}\left[k[k(1+c_{11})-(1+c_{13})]J_0(kb)-\frac{k}{b}(c_{11}-c_{12})J_1(kb)\right],$$

$$D_{44}=\frac{\omega\pi i}{b(1-h)}\left[k[k(1+c_{11})-(1+c_{13})]Y_0(kb)-\frac{k}{b}(c_{11}-c_{12})Y_1(kb)\right].$$

The roots of Eq. (46) gives the values of natural frequency for the free Δ surfaces of the cylinder.

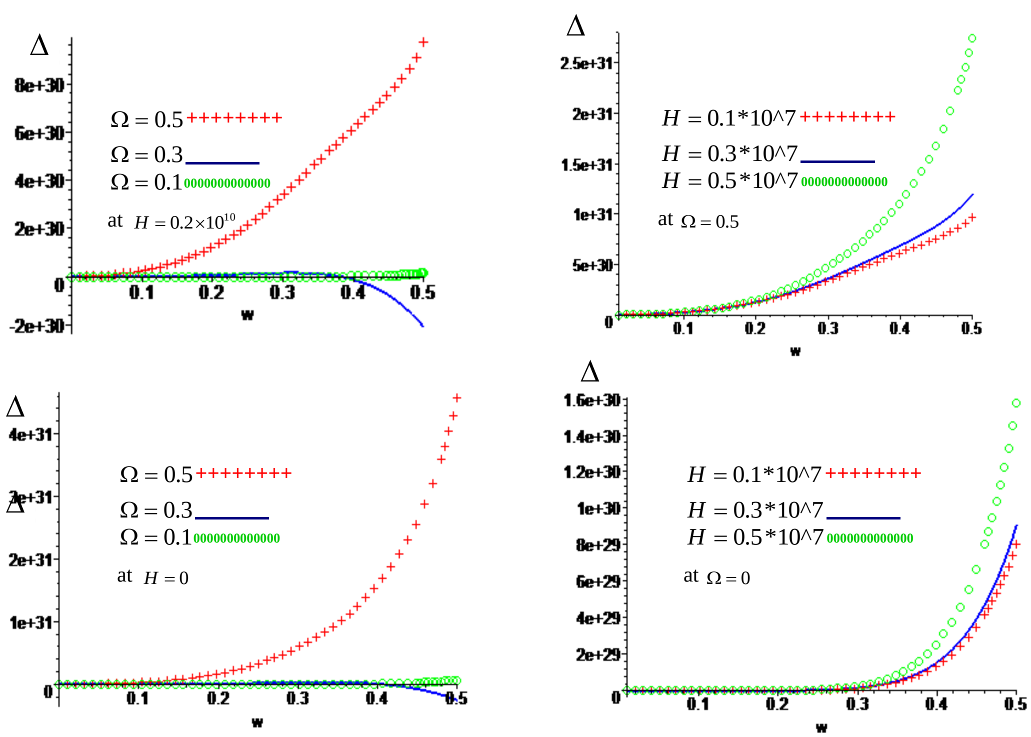

Figure 1: Variation of Δ with respect to frequency ω with effect and neglect respectively of rotation Ω and magnetic field H.

4.2 Plane vibrations of cylindrical body of fixed boundary conditions:

In this case, we have

$$u_r(a)=u_r(b)=0, \tag{47}$$

$$u_z(a)=u_z(b)=0, \tag{48}$$

which correspond to the fixed inner and outer surfaces respectively. From Eqs.(33), (34), (38), (39),(40), (47) and (48), we get

$$-\lambda[A_1J_0(\lambda bh)+B_1Y_0(\lambda bh)]-i\frac{\omega\pi}{b(1-h)}[A_2kJ_1(kbh)+B_2kY_1(kbh)]=0, \tag{49}$$

$$-\lambda[A_1J_0(\lambda b)+B_1Y_0(\lambda b)]-i\frac{\omega\pi}{b(1-h)}[A_2kJ_1(kb)+B_2kY_1(kb)]=0, \tag{50}$$

$$i\frac{\omega\pi}{b(1-h)}[A_1J_0(\lambda bh)+B_1Y_0(\lambda bh)]+[A_2kJ_0(kbh)+B_2kY_0(kbh)]=0, \tag{51}$$

$$i\frac{\omega\pi}{b(1-h)}[A_1J_0(\lambda b)+B_1Y_0(\lambda b)]+[A_2kJ_0(kb)+B_2kY_0(kb)]=0, \tag{52}$$

These are a set of four homogeneous algebraic equations involving four unknown integration constants A_1, B_1, A_2 and B_2. The condition for a nontrivial solution of these equations is that the determinant of the coefficients of these integration constants must vanish, which leads to the following frequency equation:

$$\Delta=\begin{vmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{vmatrix}=0 \tag{53}$$

where

$$d_{11}=-\lambda J_0(\lambda bh), \qquad d_{12}=-\lambda Y_0(\lambda bh),$$

$$d_{13}=-i\frac{\omega\pi}{b(1-h)}kJ_1(kbh), \qquad d_{14}=-i\frac{\omega\pi}{b(1-h)}kY_1(kbh),$$

$$d_{21}=-\lambda J_0(\lambda b), \qquad d_{22}=-\lambda Y_0(\lambda b),$$

$$d_{23}=-i\frac{\omega\pi}{b(1-h)}kJ_1(kb), \qquad d_{24}=-i\frac{\omega\pi}{b(1-h)}kY_1(kb),$$

$$d_{31}=i\frac{\omega\pi}{b(1-h)}J_0(\lambda bh), \qquad d_{32}=i\frac{\omega\pi}{b(1-h)}Y_0(\lambda bh),$$

$$d_{33}=kJ_0(kbh), \qquad d_{34}=kY_0(kbh),$$

$$d_{41}=i\frac{\omega\pi}{b(1-h)}J_0(\lambda b), \qquad d_{42}=i\frac{\omega\pi}{b(1-h)}Y_0(\lambda b),$$

$d_{43} = kJ_0(kb), \qquad d_{44} = kY_0(kb).$

The roots of Eq. (53) gives the values of natural frequency for the free surfaces of the cylinder.

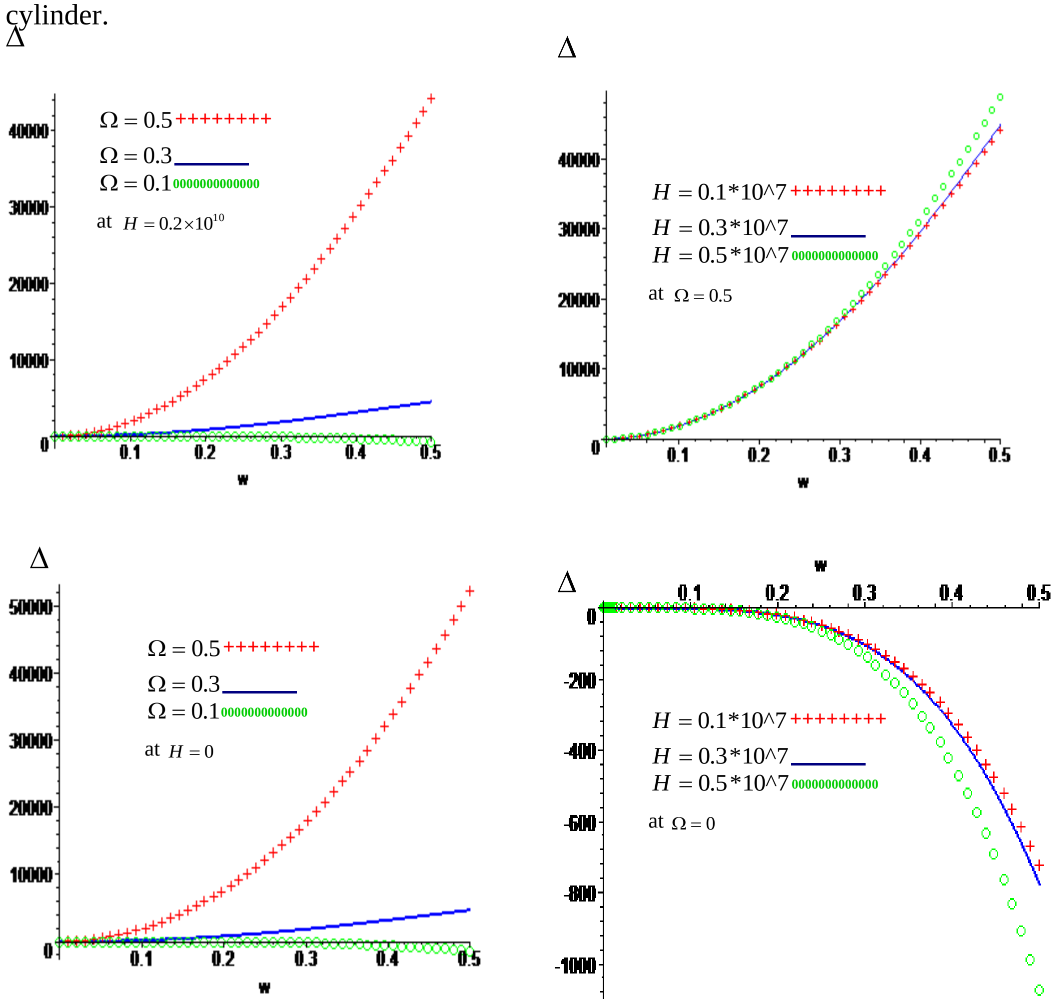

Figure 2: Variation of Δ with respect to frequenc ω with effect and neglect respectively of rotation Ω and magnetic field H.

4.3 Plane vibrations of cylindrical body with mixed boundary Conditions

In this application **,** we apply the mixed boundary conditions which consist of two kinds of boundary conditions, the first requires that the displacements vanish at the inner surface and the outer surface is traction-free i.e.,

$$\begin{aligned} &u_r(a) = u_z(a) = 0, \\ &\sigma_{rr}(b) + \tau_{rr}(b) = \tau_{rz}(b) = 0, \end{aligned} \tag{54}$$

while the second requires that the inner surface is traction-free and the displacements vanish at the outer surface, i.e.

$$\begin{aligned} &\sigma_{rr}(a)+\tau_{rr} = \tau_{rz}(a)=0, \\ &u_r(b) \;\; = u_z(b)=0, \end{aligned} \tag{55}$$

4.4 Free outer surface and fixed inner surface

In this case, from Eqs. (33)–(40) and (54), we get

$$-\lambda[A_1 J_0(\lambda bh)+B_1 Y_0(\lambda bh)] - i\frac{\omega\pi}{b(1-h)}[A_2 k J_1(kbh)+B_2 k Y_1(kbh)]=0, \tag{56}$$

$$i\frac{\omega\pi}{b(1-h)}[A_1 J_0(\lambda bh)+B_1 Y_0(\lambda bh)]+[A_2 k J_0(kbh)+B_2 k Y_0(kbh)]=0, \tag{57}$$

$$\begin{aligned} &\left[\left(\left(\frac{\omega\pi}{b(1-h)}\right)^2(1+c_{13})+\frac{\lambda}{b}(1+c_{12})\right)J_0(\lambda b)-\lambda^2(1+c_{11})J_1(\lambda b)\right]A_1 + \\ &\left[\left(\left(\frac{\omega\pi}{b(1-h)}\right)^2(1+c_{13})+\frac{\lambda}{b}(1+c_{12})\right)Y_0(\lambda b)-\lambda^2(1+c_{11})Y_1(\lambda b)\right]B_1 + \\ &\frac{\omega\pi i}{b(1-h)}\left[k\left[k(1+c_{11})-(1+c_{13})\right]J_0(kb)-\frac{k}{b}(c_{11}-c_{12})J_1(kb)\right]A_2 + \\ &\frac{\omega\pi i}{b(1-h)}\left[k\left[k(1+c_{11})-(1+c_{13})\right]Y_0(kb)-\frac{k}{b}(c_{11}-c_{12})Y_1(kb)\right]B_2 = 0, \end{aligned} \tag{58}$$

$$\begin{aligned} &-i\frac{\omega\pi}{b(1-h)}\lambda A_1[J_0(\lambda b)+J_1(\lambda b)]-i\frac{\omega\pi}{b(1-h)}\lambda B_1[Y_0(\lambda b)+Y_1(\lambda b)]+ \\ &kA_2\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2-K\right]J_1(kb)+kB_2\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2-k\right]Y_1(kb)=0. \end{aligned} \tag{59}$$

These are a set of four homogeneous algebraic equations involving four unknown integration constants A_1, B_1, A_2 and B_2 .The condition for a nontrivial solution of these equations is that the determinant of the coefficients of these integration constants must vanish, which leads to the following frequency equation:

$$\Delta = \begin{vmatrix} \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} \\ \mu_{21} & \mu_{22} & \mu_{23} & \mu_{24} \\ \mu_{31} & \mu_{32} & \mu_{33} & \mu_{34} \\ \mu_{41} & \mu_{42} & \mu_{43} & \mu_{44} \end{vmatrix} = 0, \tag{60}$$

where

$$\mu_{11} = -\lambda J_0(\lambda bh), \qquad \mu_{12} = -\lambda Y_0(\lambda bh),$$

$$\mu_{13} = -i\frac{\omega\pi}{b(1-h)}kJ_1(kbh), \qquad \mu_{14} = -i\frac{\omega\pi}{b(1-h)}kY_1(kbh),$$

$$\mu_{21} = i\frac{\omega\pi}{b(1-h)}J_0(\lambda bh), \qquad \mu_{22} = i\frac{\omega\pi}{b(1-h)}Y_0(\lambda bh),$$

$$\mu_{23} = kJ_0(kbh), \qquad \mu_{24} = kY_0(kbh),$$

$$\mu_{31} = \left[\left(\left(\frac{\omega\pi}{b(1-h)}\right)^2(1+c_{13}) + \frac{\lambda}{b}(1+c_{12})\right)J_0(\lambda b) - \lambda^2(1+c_{11})J_1(\lambda b)\right],$$

$$\mu_{32} = \left[\left(\left(\frac{\omega\pi}{b(1-h)}\right)^2(1+c_{13}) + \frac{\lambda}{b}(1+c_{12})\right)Y_0(\lambda b) - \lambda^2(1+c_{11})Y_1(\lambda b)\right],$$

$$\mu_{33} = \frac{\omega\pi i}{b(1-h)}\left[k\left[k(1+c_{11}) - (1+c_{13})\right]J_0(kb) - \frac{k}{b}(c_{11}-c_{12})J_1(kb)\right],$$

$$\mu_{34} = \frac{\omega\pi i}{b(1-h)}\left[k\left[k(1+c_{11}) - (1+c_{13})\right]Y_0(kb) - \frac{k}{b}(c_{11}-c_{12})Y_1(kb)\right],$$

$$\mu_{41} = -i\frac{\omega\pi}{b(1-h)}\lambda[J_0(\lambda b) + J_1(\lambda b)], \quad \mu_{42} = -i\frac{\omega\pi}{b(1-h)}\lambda[Y_0(\lambda b) + Y_1(\lambda b)],$$

$$\mu_{43} = k\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2 - K\right]J_1(kb), \qquad \mu_{44} = k\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2 - k\right]Y_1(kb),$$

The roots of Eq. (60) gives the values of natural frequency for the free outer surface and fixed inner surface of the cylinder.

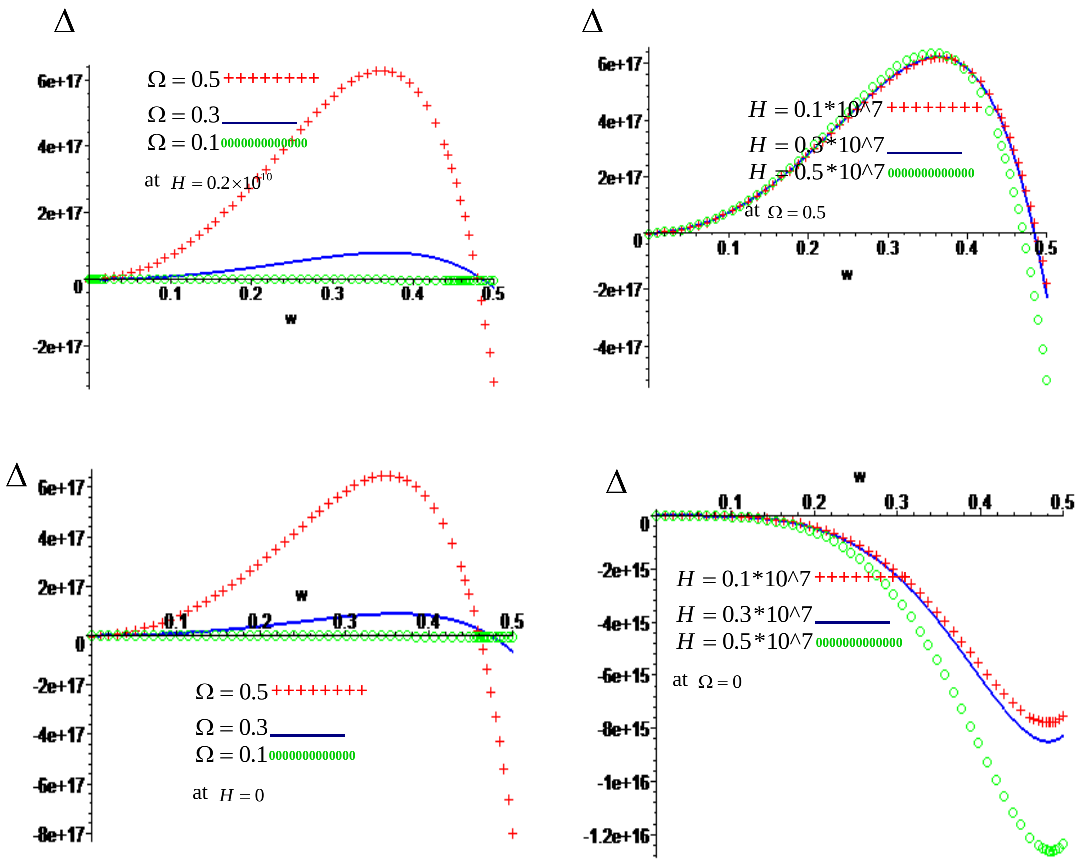

Figure 3: Variation of Δ with respect to frequen ω with effect and neglect respectively of rotation Ω and magnetic field H.

5 Free inner surface and fixed outer surface

In this case, from Eqs. (33)-(40) and (55), we get

$$\left[\left(\left(\frac{\omega\pi}{b(1-h)}\right)^2(1+c_{13})+\frac{\lambda}{bh}(1+c_{12})\right)J_0(\lambda bh)-\lambda^2(1+c_{11})J_1(\lambda bh)\right]A_1+$$
$$\left[\left(\left(\frac{\omega\pi}{b(1-h)}\right)^2(1+c_{13})+\frac{\lambda}{bh}(1+c_{12})\right)Y_0(\lambda bh)-\lambda^2(1+c_{11})Y_1(\lambda bh)\right]B_1+ \tag{61}$$
$$\frac{\omega\pi i}{b(1-h)}\left[k\left[k(1+c_{11})-(1+c_{13})\right]J_0(kbh)-\frac{k}{bh}(c_{11}-c_{12})J_1(kbh)\right]A_2+$$
$$\frac{\omega\pi i}{b(1-h)}\left[k\left[k(1+c_{11})-(1+c_{13})\right]Y_0(kbh)-\frac{k}{bh}(c_{11}-c_{12})Y_1(kbh)\right]B_2=0,$$

$$-i\frac{\omega\pi}{b(1-h)}\lambda A_1[J_0(\lambda bh)+J_1(\lambda bh)]-i\frac{\omega\pi}{b(1-h)}\lambda B_1[Y_0(\lambda bh)+Y_1(\lambda bh)]+$$
$$kA_2\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2-K\right]J_1(kbh)+kB_2\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2-k\right]Y_1(kbh)=0, \tag{62}$$

$$-\lambda[A_1J_0(\lambda b)+B_1Y_0(\lambda b)]-i\frac{\omega\pi}{b(1-h)}[A_2kJ_1(kb)+B_2kY_1(kb)]=0, \tag{63}$$

$$i\frac{\omega\pi}{b(1-h)}[A_1J_0(\lambda b)+B_1Y_0(\lambda b)]+[A_2kJ_0(kb)+B_2kY_0(kb)]=0. \tag{64}$$

These are a set of four homogeneous algebraic equations involving four unknown integration constants A_1, B_1, A_2 and B_2.The condition for a nontrivial solution of these equations is that the determinant of the coefficients of these integration constants must vanish ,which leads to the following frequency equation:

$$\Delta=\begin{vmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{vmatrix}=0, \tag{65}$$

where

$$m_{11}=\left[\left(\left(\frac{\omega\pi}{b(1-h)}\right)^2(1+c_{13})+\frac{\lambda}{bh}(1+c_{12})\right)J_0(\lambda bh)-\lambda^2(1+c_{11})J_1(\lambda bh)\right],$$

$$m_{12}=\left[\left(\left(\frac{\omega\pi}{b(1-h)}\right)^2(1+c_{13})+\frac{\lambda}{bh}(1+c_{12})\right)Y_0(\lambda bh)-\lambda^2(1+c_{11})Y_1(\lambda bh)\right],$$

$$m_{13}=\frac{\omega\pi i}{b(1-h)}\left[k\left[k(1+c_{11})-(1+c_{13})\right]J_0(kbh)-\frac{k}{bh}(c_{11}-c_{12})J_1(kbh)\right],$$

$$m_{14}=\frac{\omega\pi i}{b(1-h)}\left[k\left[k(1+c_{11})-(1+c_{13})\right]Y_0(kbh)-\frac{k}{bh}(c_{11}-c_{12})Y_1(kbh)\right],$$

$$m_{21}=-i\frac{\omega\pi}{b(1-h)}\lambda[J_0(\lambda bh)+J_1(\lambda bh)],$$

$$m_{22}=-i\frac{\omega\pi}{b(1-h)}\lambda[Y_0(\lambda bh)+Y_1(\lambda bh)],$$

$$m_{23}=k\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2-K\right]J_1(kbh), \qquad m_{24}=k\left[\left(\frac{\omega\pi}{b(1-h)}\right)^2-k\right]Y_1(kbh),$$

$$m_{31} = -\lambda J_0(\lambda b), \qquad m_{32} = -\lambda Y_0(\lambda b),$$

$$m_{33} = -i\frac{\omega\pi}{b(1-h)}kJ_1(kb), \qquad m_{34} = -i\frac{\omega\pi}{b(1-h)}kY_1(kb),$$

$$m_{41} = i\frac{\omega\pi}{b(1-h)}J_0(\lambda b), \qquad m_{42} = i\frac{\omega\pi}{b(1-h)}Y_0(\lambda b),$$

$$m_{43} = kJ_0(kb), \qquad m_{44} = kY_0(kb),$$

The roots of Eq. (65) gives the values of natural frequency for the free surfaces of the cylinder.

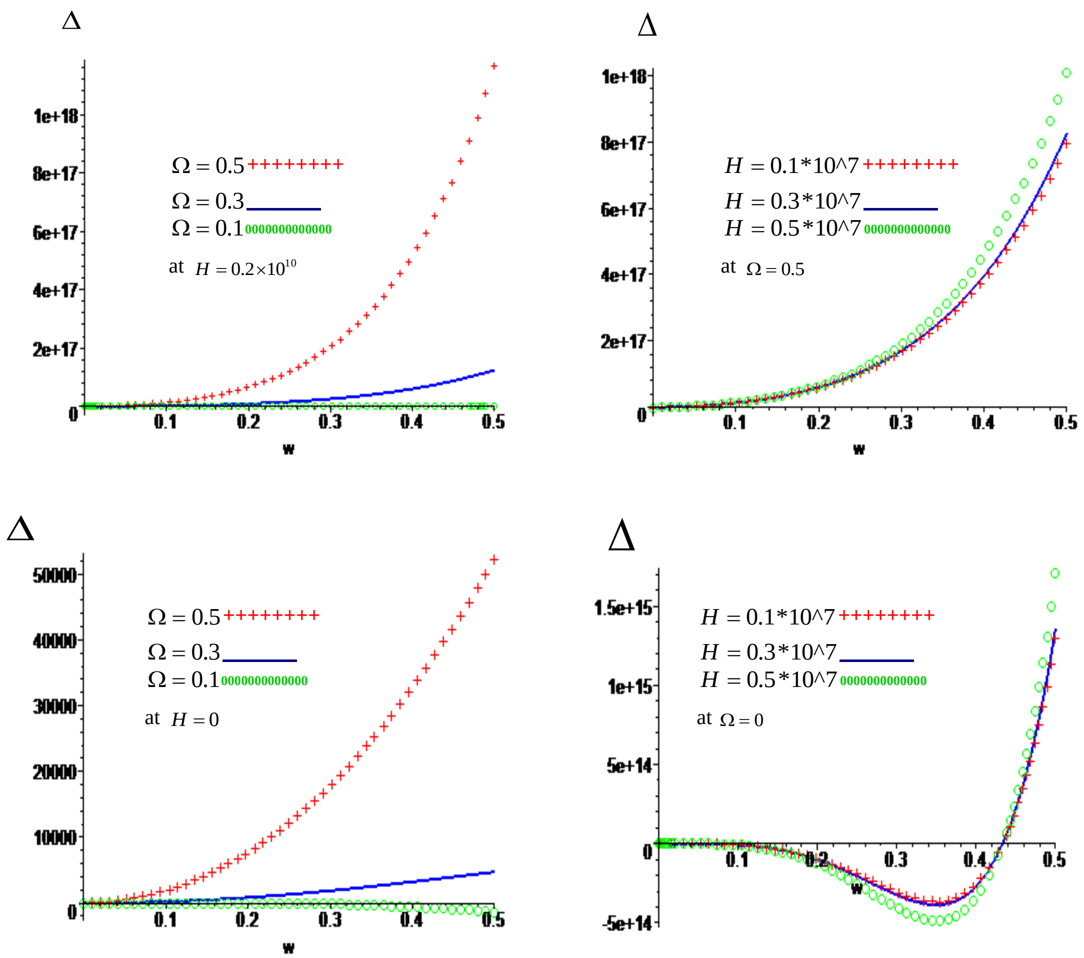

Figure 4: Variation of Δ with respect to frequency ω with effect and neglect respectively of rotation Ω and magnetic field H .

6 Numerical results and discussion

Here, we shall investigate the frequency equations given by Eqs. (46), (53), (60) and (65) numerically for a particular model. Since these equations are an implicit function relation of natural frequency ω, therefore one can proceed to find the variation of natural frequency (the eigenvalues) with rotation Ω and magnetic field H, the cylinder has the following geometric and material constants given by Chen et al [Chen et al. (2005)].

$$\rho = 3.986\times10^{3}\,kg/m^{3},\ c_{11} = 4.968\times10^{11},\ c_{33} = 4.981\times10^{11},\ c_{44} = 1.474\times10^{11}$$

$$c_{13} = 1.109\times10^{11},\ c_{12} = 0.93\times10^{11},\ a = 3,\ b = 4.$$

The variations are shown in Figures. (1)-(4), respectively.

Figure 1: show that the variation of the magnitude of the frequency equation Δ with respect to frequency ω for different values of rotation Ω and magnetic field H at free surface traction. The magnitude of the frequency equation increases with increasing of rotation and frequency, while it dispersion at $\Omega = 0.3$ in the presence and absence of magnetic field, while it increases with increasing of magnetic field in the presence and absence of the rotation.

Figure 2: show that the variation of the magnitude of the frequency equation Δ with respect to frequency ω for different values of rotation Ω and magnetic field H at a fixed surface. The magnitude of the frequency equation increases with increasing of rotation and frequency in the presence and absence of magnetic field, while it increases with increasing of magnetic field and frequency in the presence of the rotation, while it decreases with increasing of magnetic field and frequency in the absence of rotation.

Figure 3: show that the variation of the magnitude of the frequency equation Δ with respect to frequency ω for different values of rotation Ω and magnetic field H at Free outer surface and fixed inner surface. The magnitude of the frequency equation increases with increasing of rotation in the presence and absence of magnetic field, while it increases and decreases with increasing of frequency in the presence and absence of the magnetic field, as well as it increases with increasing of magnetic field in the interval $[0, 0.3]$, while it decreases with increasing of magnetic field in the interval $[0.3, 0.5]$ and it increases and decreases with increasing of frequency in the presence of rotation, while it decrease with increasing of magnetic field and frequency in the absence of rotation.

Figure 4: show that the variation of the magnitude of the frequency equation Δ with respect to frequency ω for different values of rotation Ω and magnetic field H at Free inner surface and fixed outer surface. The magnitude of the frequency equation increases with increasing of rotation and frequency in the presence and absence of magnetic field, while it dispersion at $\Omega = 0.1$, as well as it increases with increasing of magnetic field and frequency in the presence of the rotation, while it increases and decreases with increasing of magnetic field and frequency in the absence of rotation.

7 Conclusions

1. Harmonic vibrations of an elastic cylinder have been studied using a half-interval method. The governing equations in cylindrical coordinates are recorded for future reference. The magnitude of the frequency equations has been obtained under the effect of rotating Ω and magnetic field H. The numerical results of the natural frequency are obtained and represented graphically in detail for different cases.
2. All the physical quantities satisfy the boundary conditions.
3. The magnetic field and rotation play a significant role in the distribution of all the physical quantities.
4. The results presented in this paper will be very helpful for researchers in structures and material science, designers of new materials and the study of the phenomenon of rotation and magnetic field is also used to improve the conditions of oil extractions. Finally, if the rotation and magnetic field are neglected, the relevant results obtained are deduced to the results obtained by Abd-Alla et al [Abd-Alla et al. (2013)].

References

Abd-Alla, A. M.; Abo-Dahab, S. M., Bayones, F. S. (2015): Wave propagation in fibre-reinforced anisotropic thermoelastic medium subjected to gravity field, *Structural Engineering and Mechanics*, vol.53, pp.76-90.

Abd-Alla, A.M.; Farhan, A.M. (2008): Effect of the non-homogeneous on the campsite infinite cylinder of orthotropic material, *J. Physics Letters A*, vol.372, pp.756-760.

Abd-Alla, A.M.; Nofal, T.A.; Farhan A.M. (2008): Effect of the non-homogenity on the composite infinite cylinder of isotropic material, *Physics Letters A*, vol.372, pp.4861-4864.

Abd-Alla, A.M.; Abo-Dahab, S.M.; Khan, A. (2017): Rotational effect on thermoelastic Stoneley, Love and Rayleigh waves in fibre-reinforced anisotropic general viscoelastic media of higher order, *Structures Engineering and Mechanics*, vol.61, pp.221-230.

Abd-Alla, A. M.; Mahmoud, S. R. (2012): Analytical solution of wave propagation in a non-homogeneous orthotropic rotating elastic media, *Journal of Mechanical Science and Technology*, vol. 26, pp.917-926.

Abd-Alla, A.M.; Yahya, G.A.; Mahmoud, S.R. (2013): Effect of magnetic field and non-homogeneity on the radial vibrations in hollow rotating elastic cylinder, *Meccanica*, vol.47, pp.76-88.

Buchanan, G.R. (2003) Free vibration of an infinite magneto-electro-elastic cylinder, *J. Sound and Vibration*, vol. 268, pp.413-426.

Bayones, F.S.; Abd-Alla, A.M. (2017): Eigenvalue approach to two dimensional coupled magneto-thermoelasticity in a rotating isotropic medium, *Results in Physics*, vol.7, pp.2941-2949.

Bounouara, F.; Benrahou, K.H.; Belkorissat, I.; Tounsi, A. (2016): A nonlocal zeroth-order shear deformation theory for free vibration of functionally graded nanoscale plates resting on elastic foundation, *Steel and Composite Structures*, vol. 20, no.2, pp.227-249.

Bellifa, H.; Benrahou, K. H.; Hadji, L.; Houari,M. S. A. ; Tounsi, A. (2016): Bending and free vibration analysis of functionally graded plates using a simple shear deformation theory and the concept the neutral surface position, *J. Braz. Soc. Mech. Sci. Eng.*, vol. 38, no.1, pp.265-275.

Bennoun, M.; Houari, M. S. A. ; Tounsi, A. (2016): A novel five variable refined plate theory for vibration analysis of functionally graded sandwich plates, *Mechanics of Advanced Materials and Structures*, vol. 23, no.4, pp.423-431.

Boukhari, A.; Hassen, A.A.; Tounsi, A.; Mahmoud, S.R. (2016): An efficient shear deformation theory for wave propagation of functionally graded material plates, *Structural Engineering and Mechanics*, vol. 57, no.5, PP.837-859.

Bourada, M. ; Kaci, A. ; Ahmed, H. M. S. ; Tounsi, A. (2015): A new simple shear and normal deformations theory for functionally graded beams, *Steel and Composite Structures*, vol.18, no.2, pp.409-423.

Chen, W. Q.; Lee, K. Y.; Ding, H. J. (2005): On free vibration of non-homogeneous transversely isotropic magneto-electro-elastic plates, *J. Sound and Vibration*, vol.279, pp.237-251.

Draiche, K.; Tounsi, A.; Mahmoud, S.R. (2016): A refined theory with stretching effect for the flexure analysis of laminated composite plates, *Geomechanics and Engineering*, vol. 11, no.5, pp.671-690.

Gaoab ,W.; Cuia, Z.; Wanga, K. (2013): Wave propagation in poroelastic hollow cylinder immersed in fluid with seismoelectric effect, *Journal of Sound and Vibration*, vol. 332, 20, pp.5014-5028.

Hebali, H.; Tounsi, A.; Houari, M.S.A. ; Bessaim, A. (2014): A new quasi-3D hyperbolic shear deformation theory for the static and free vibration analysis of functionally graded plates, *ASCE J. Engineering Mechanics*, vol.140, pp.374-383.

Hou, P.F.; Ding, H.J.; Leung, A.Y.T. (2006): The transient responses of a special non-homogeneous magneto-electro-elastic hollow cylinder for axisymmetric plane strain problem, *Journal of Sound and Vibration*, vol. 291, no.1, pp.19-47.

Hutchinson, J.R.; El-Azhary, S.A. (1986): Vibrations of free hollow circular cylinder, *J. Applied Mechanics*, vol. 53, pp. 641-646.

Mahmoud, S.R.; Abd-Alla, A.M. and AL-Shehri, N.A. (2011): Effect of the rotation on the radial vibrations in a non-homogeneous orthotropic hollow cylinder, *Int. J. of Modern Physics B*, vol.25, pp. 3513-3528.

Marin, M. and Lupu, M. (1998): On Harmonic Vibrations in Thermoelasticity of Micropolar Bodies, *Journal of Vibration and Control*, vol.4, pp. 507-518.

Marin, M. (2010): A domain of influence theorem for microstretch elastic materials, *Nonlinear, Analysis: Rea World Applications*, vol.11, pp. 3446-3452.

Marin, M. (2010): Harmonic vibrations in thermoelasticity of microstretch materials, *J. Vibr. Acoust., ASME*, vol. 132, no.4, pp.1-6.

Marin, M. (1997): On weak solutions in elasticity of dipolar bodies with voids, *J. Comp. Appl. Math.*, vol.82, no.1-2, 291-297, 1997

Meziane, M. A.A.; Abdelaziz, H.H.; Tounsi, A. (2014): An efficient and simple refined theory for buckling and free vibration of exponentially graded sandwich plates under various boundary conditions, *J. Sandw. Struct. Mater*, vol.16, no.3, pp.293-318.

Morse, P. M.; Feshbach, H.(1953): *Methods of theoretical physics, Part I,* McGraw-Hill, New York.

Tounsi, A.; Ahmed, H.; Bessaim, A. (2016): A new simple three-unknown sinusoidal shear deformation theory for functionally graded plates, *Steel and Composite Structures*, vol.22, no. 2, pp.257-276.

Yahia, S. A.; Hassen, A.A.; Ahmed, H. M.S.; Tounsi, A. (2015): Wave propagation in functionally graded plates with porosities using various higher-order shear deformation plate theories, *Structural Engineering and Mechanics*, vol. 53, no.6, pp.1143-1165.