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Subdivision of Uniform ωB-Spline Curves and Two Proofs of Its C^{k-2}-Continuity

Jing Tang[1], Mei-e Fang[1, *] and Guozhao Wang[2]

Abstract: ωB-splines have many optimal properties and can reproduce plentiful commonly-used analytical curves. In this paper, we further propose a non-stationary subdivision method of hierarchically and efficiently generating ωB-spline curves of arbitrary order of ωB-spline curves and prove its C^{k-2}-continuity by two kinds of methods. The first method directly prove that the sequence of control polygons of subdivision of order k converges to a C^{k-2}-continuous ω B-spline curve of order k. The second one is based on the theories upon subdivision masks and asymptotic equivalence etc., which is more convenient to be further extended to the case of surface subdivision. And the problem of approximation order of this non-stationary subdivision scheme is also discussed. Then a uniform ωB-spline curve has both perfect mathematical representation and efficient generation method, which will benefit the application of ωB-splines.

Keywords: ωB-spline, subdivision, C^{k-2}-continuity, asymptotic equivalence, approximation order.

1 Introduction

Polynomial B-splines and NURBS are important modeling tools in CAD/CAM. But polynomial B-splines are not able to exactly represent often-used conics (except for parabola), trigonometric functions and hyperbolic functions etc. NURBS can represent conics, but its ration form results in complicated computations about differential and integral. Then all kinds of B-like splines are proposed [Fang and Wang (2008); Zhang (1996); Vasov and Sattayatham (1999); Mainar and Pe˜na (2002)]. In paper [Wang, Chen and Zhou (2004)], we further unified these B-like splines into ωB-splines, which are constructed over $\left\{\cos\omega t, \sin\omega t, 1, t, \ldots, t^n, \ldots\right\}$. ω can be non-negative real number and pure imaginary number. If taking the value of ω as a constant 0,1 or i, we will get usual polynomial B-splines, trigonometric polynomial B-splines and hyperbolic polynomial B-splines respectively. ωB-splines inherit most of optimal properties from polynomial B-splines, including the subdivision property. Due to optimal properties of these B-like splines, many applications are studied in recent years [Mannia, Pelosi and Speleers (2012); Xu, Sun, Xu et al. (2017)]. In this paper, we perfect the subdivision

[1] School of Computer Sciences and Educational Softwares, Guangzhou University, Guangzhou, China.

[2] Department of Mathematics, Zhejiang University, Hangzhou, China.

* Corresponding Author: Mei-e Fang. Email: fme@gzhu.edu.cn.

method and theory of ωB-splines in order to apply them better in the future.

Subdivision is a standard technique of recursively generating smooth curves/surfaces from an initial polygon/mesh. Please see paper [Chaikin (1974); Doo and Sabin (1978); Catmull and Clark (1978); Dyn (1992); Stam (2001); Jena, Shunmugaraj and Das(2002); Jena, Shunmugaraj and Das (2003); Andersson, Lars-Erik, Stewart et al. (2010); Conti and Romani (2011); Conti, Cotronei and Sauer (2017)] for more details. This kind of modeling method is popularly applied in geometric modelling and 3D animation because of its numerical stability, simple implement and suitability for arbitrary topology. But most of subdivision curves and surfaces lack exactly mathematical representations, which are the fundamental of all kinds of differential/integral computations. So subdivision methods which have spline backgrounds are very interesting. Subdivision models with spline backgrounds include all merits mentioned above. For example, Doo-Sabin method [Doo and Sabin (1978)], Catmull-Clark method [Catmull and Clark (1978)], and the subdivision method proposed in paper [Stam (2001)] respectively have their spline backgrounds of B-splines of degree 2, cubic B-splines, polynomial B-splines of arbitrary order. These subdivision methods are all stationary, i.e, their subdivision rules persist unchanged in each level of subdivision. While stationary subdivision can not generate ωB-spline curves with frequency parameters.

In this paper, we introduce a parameter relative to the frequency parameter to build a non-stationary subdivision method with the background of ωB-splines. Then this kind of modeling method has the merits of both subdivision and ωB-splines. Concretely, we consider the subdivision of uniform ωB-splines with uniform knot intervals and ω taking a certain constant. At first, we derive the definition of uniform ωB-spline bases and curves according to the corresponding definitions in paper [Wang, Chen and Zhou (2004)].

Definition 1.1 (uniform ωB-spline bases) Let T be a given uniform knot sequence $\{t_i = i\ell\}_{i\ell=-\infty}^{+\infty}$, ℓ be the length of uniform knot intervals, k refers to the order of splines .ω be a given frequency parameter, where ω can take value as a non-negative real number ($\omega\ell \in [0, \pi)$ in this case) or a pure imaginary number whose imaginary part is positive. $N_{i,k}(t)$ constructed by the following formula are called uniform ωB-spline bases in the span of $\{\cos\omega t, \sin\omega t, 1, t, \ldots, t^{k-3}\}$ for $t \in [i\ell, (i+k)\ell)$. We first define uniform ωB-spline basic functions of order k=2 as follows.

$$N_{0,2}(t) = \begin{cases} \frac{\sin\omega(t-i\ell)}{\sin\omega\ell}, & i\ell \le t < (i+1)\ell, \\ \frac{\sin\omega((i+2)\ell-t)}{\sin\omega\ell}, & (i+1)\ell \le \mathrm{t} < (i+2)\ell, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

and $N_{i,2}(t) = N_{0,2}(t - i\ell)$.

In formula (1.1), when ω = 0, we compute it by the L'Hospital rule about ω.

For $k \geq 3$, $N_{i,k}(t)$ are defined recursively by

$$N_{i,k}(t) = \frac{1}{\ell}\int_{t-\ell}^{t} N_{i,k-1}(s)\,ds \tag{2}$$

Definition 1.2 (uniform ωB-spline curves) Let $\{P_i\}_{i=1}^{n} \in \Re^3$, $N_{i,k}(t)$ be uniform ωB-spline bases of order *k* corresponding to the partition $T = \{t_i = i\ell\}_{i=-\infty}^{+\infty}$ of the parameter axis t.

Then $P(t) = \sum_{i=1}^{n} N_{i,k}(t)P_i, \left((k-1)\ell \leq t \leq (n+1)\ell, n \geq k-1\right)$ is called an uniform ωB-spline curve of order *k* corresponding to the knot vector *T*. $P_i\ (i = 1, \ldots, n)$ are control points.

ωB-spline curves can reproduce conics, trigonometric and hyperbolic curves. They also have many useful properties for geometry modelling, including those inherited from common B-spline curves and some special merits. Please refer to paper [Fang and Wang (2008)] for details. But we can see that the basic functions need to be recursively computed by integration from their definition, which results in low efficiency of evaluation. In this paper, we devote to build a high-efficiency subdivision method of generating ωB-spline curves.

The rest of this paper is organized as follows. In Section 2, we derive the relation formula of control points between two representations of the same uniform ωB-spline curve of order *k* respectively with the original knot intervals and their bisections. Then the explicit subdivision rule is constructed based on this. By this kind of subdivision rule of order *k*, a sequence of control polygons generates from the original control polygon of an uniform ωB-spline curve of order *k*. We directly prove that the limit of this sequence converges to the C^{k-2}-continuous uniform ωB-spline curve in Section 3. But this kind of proof method is hard to be applied in the corresponding proof of the continuity for the case of surface subdivision. So in Section 4, we reconsider the proof from the aspect of subdivision masks and provide a more general proof of the continuity of subdivision which will be easier to be extended to the case of surface subdivision. Because our proposed surface scheme is non-stationary, we use the theories of asymptotic equivalence between non-stationary subdivision and the corresponding stationary subdivision with the rule in limit status to complete the proof. The approximation order of the proposed subdivision scheme is also discussed. Section 5 makes a conclusion.

2 The subdivision method of uniform ωB-spline curves

According to Definition 1.1 and Definition 1.2, we find that an uniform ωB-spline curve can also be equivalently represented by another uniform ωB-spline curve with knot intervals after bisection.

Theorem 2.1 Let $N_{i,k}\left(\ell, t\right)$ and $N_{i,k}\left(\ell/2\,, t\right)$ represent bases with knots $t_i = i\ell\left(i = 0, \pm1, \pm2, \ldots\right)$ and $t'_i = i\ell/2\left(i = 0, \pm1, \pm2, \ldots\right)$ respectively. Then an uniform ωB-spline curve of order $k\left(k \geq 3\right)$ defined by

$$p_k\left(t\right) = \sum_{i=1}^{n} P_i N_{i,k}\left(\ell, t\right), t \in \left[k\ell, \left(n+1\right)\ell\right],$$

and also be defined by

$$p_k\left(t\right) = \sum_{i=1}^{2n-k+1} P_i^k N_{i,k}\left(\ell/2\,, t\right), t \in \left[k\ell, \left(n+1\right)\ell\right].$$

where

$$P_i^k = \left(P_i^{k-1} + P_{i+1}^{k-1}\right)/2, \qquad \text{k>3},$$

$$p_i^3 = \begin{cases} \dfrac{\left(4\cos\left(\omega\ell/2\right)-1\right)\cdot P_{(i+1)/2} + P_{(i+3)/2}}{4\cos\left(\omega\ell/2\right)}, & \text{i is odd;} \\ \dfrac{P_{i/2} + \left(4\cos\left(\omega\ell/2\right)-1\right)\cdot P\left(i+2\right)/2}{4\cos\left(\omega\ell/2\right)}, & \text{i is even.} \end{cases} \tag{3}$$

Proof. According to the meanings of $N_{i,2}\left(\ell, t\right)$ and $N_{i,2}\left(\ell/2\,, t\right)$, it is easy to obtain their representation formula by formula (1). And the relation formula between two bases can be deduced as below:

$$N_{i,2}\left(\ell, t\right) = \frac{1}{2\cos\left(\omega\ell/2\right)} \cdot N_{2i-3,2}\left(\ell/2, t\right) + N_{2i-2,2}\left(\ell/2, t\right) + \frac{1}{2\cos\left(\omega\ell/2\right)} \cdot N_{2i-1,2}\left(\ell/2, t\right)$$

Furthermore by the above formula, formula (1.1) and the recursive formula (2), we get

$$
\begin{aligned}
N_{i,3}(\ell,t) &= \frac{1}{\ell}\int_{t-\ell}^{t} N_{i,2}(\ell,\mathrm{s})\,ds \\
&= \frac{1}{2}\cdot\frac{1}{\ell/2}\cdot\left(\int_{t-\ell/2}^{t} + \int_{t-\ell}^{t-\ell/2}\right)\left(N_{i,2}(\ell,s)\right)ds \\
&= \frac{1}{4\cos(\omega\ell/2)}\cdot N_{2i-3,3}(\ell/2\,,t) + \frac{4\cos(\ell/2)-1}{4\cos(\omega\ell/2)}\cdot N_{2i-2,3}(\ell/2,t) \\
&\quad + \frac{4\cos(\omega\ell/2)-1}{4\cos(\omega\ell/2)}\cdot N_{2i-1,3}(\ell/2,t) + \frac{1}{4\cos(\omega\ell/2)}\cdot N_{2i-1,3}(\ell/2,t).
\end{aligned}
$$

In the following, we prove the conclusion by induction.

When $k = 3, t \in \left[3\ell, (n+1)\ell\right]$, we know $N_{i,3}(\ell/2,t)$, $i = -1, 0, 2n-1, 2n$.

Then

$$
\begin{aligned}
\sum_{i=1}^{n} P_i N_{i,3}(\ell,t) &= \sum_{i=1}^{n-1} \Bigg(\frac{(4\cos(\omega\ell/2)-1)P_i + P_{i+1}}{4\cos(\omega\ell/2)}\cdot N_{2i-1,3}(\ell/2,t) \\
&\quad + \frac{P_i + \left(4\cos(\omega\ell/2)-1\right)P_{i+1}}{4\cos(\omega\ell/2)}\cdot N_{2i,3}(\ell/2,t) \\
&= \sum_{i=1}^{2n-2} P_i^3 N_{i,3}(\ell/2,t),
\end{aligned}
$$

$$
P_i^3 = \begin{cases}
\frac{4\cos(\omega\ell/2)-1}{4\cos(\omega\ell/2)}\cdot P_{(i+1)/2} + \frac{1}{4\cos(\omega\ell/2)}\cdot P_{(i+3)/2}, & \text{i is odd,} \\
\frac{1}{4\cos(\omega\ell/2)}\cdot P_{i/2} + \frac{4\cos(\omega\ell/2)-1}{4\cos(\omega\ell/2)}\cdot P_{i/2+1}, & \text{i is even.}
\end{cases}
$$

When k=4,

$$
\begin{aligned}
\sum_{i=1}^{n} P_i N_{i,4}\left(\ell, t\right) &= \frac{1}{\ell}\sum_{i=1}^{n} P_i \int_{t-\ell}^{t} N_{i,3}\left(\ell, s\right) ds \\
&= \frac{1}{\ell}\int_{t-\ell}^{t} \sum_{i=1}^{n} P_i N_{i,3}\left(\ell, s\right) ds \\
&= \frac{1}{\ell}\int_{t-\ell}^{t} \sum_{i=1}^{2n-2} P_i^3 N_{i,3}\left(\ell/2\,, s\right) ds \\
&= \frac{1}{2}\sum_{i=1}^{2n-2} P_i^3 \cdot \left(\frac{1}{\ell/2}\int_{t-\ell}^{t} N_{i,3}\left(\ell/2, s\right) ds\right) \\
&= \frac{1}{2}\sum_{i=1}^{2n-2} P_i^3 \cdot \left(N_{i,4}\left(\ell/2, t\right) + N_{i-1,4}\left(\ell/2, t\right)\right)
\end{aligned}
$$

We know $N_{i,4}\left(\ell/2, t\right) = 0,\ i = 0, 2n-2,$ then we get

$$
\begin{aligned}
\sum_{i=1}^{n} P_i N_{i,4}\left(\ell, t\right) &= \frac{1}{2}\left(\sum_{i=2}^{2n-2} P_i^3 N_{i-1,4}\left(\ell/2, t\right) + \sum_{i=1}^{2n-3} P_i^3 N_{i,4}\left(\ell/2, t\right)\right) \\
&= \sum_{i=1}^{2n-3}\left(\frac{1}{2}\left(P_i^3 + P_{i+1}^3\right) \cdot N_{i,4}\left(\ell/2, t\right)\right)
\end{aligned}
$$

That is $P_i^4 = \left(P_i^3 + P_{i+1}^3\right)/2$.

Now assume that the conclusion holds for k ($k \geq 4$), i.e.

$$
\sum_{i=1}^{n} P_i N_{i,k}\left(\ell, t\right) = \sum_{i=1}^{2n-k+1}\left(\frac{P_i^{k-1} + P_{i+1}^{k-1}}{2}\right) \cdot N_{i,k}\left(\ell/2, t\right)
$$

$$
P_i^k = \frac{P_i^{k-1} + P_{i+1}^{k-1}}{2} .
$$

Then for k+1, we have

$$\begin{aligned}\sum_{i=1}^{n} P_i N_{i,k+1}(\ell, t) &= \sum_{i=1}^{n} P_i \left(\frac{1}{\ell}\int_{t-\ell}^{t} N_{i,k}(\ell, s)\right) ds \\ &= \frac{1}{\ell}\int_{t-\ell}^{t} \sum_{i=1}^{n} P_i N_{i,k}(\ell, s) ds \\ &= \frac{1}{\ell}\int_{t-\ell}^{t} \sum_{i=1}^{2n-k+1} \left(\frac{P_i^{k-1} + P_{i+1}^{k-1}}{2}\right) N_{i,k}(\ell/2, s) ds \\ &= \frac{1}{\ell}\sum_{i=1}^{2n-k+1} \left(\int_{t-\ell}^{t} N_{i,k}(\ell/2, s) ds\right) \\ &= \sum_{i=1}^{2n-(k+1)+1} \left(\frac{P_i^{k} + P_{i+1}^{k}}{2}\right) \cdot N_{i,k+1}(\ell/2, t) \\ &= \sum_{i=1}^{2n-(k+1)+1} P_i^{k+1} N_{i,k+1}(\ell/2, t).\end{aligned}$$

So the conclusion holds for $k + 1$.

Based on this, an uniform ωB-spline curve can be generated by continuously using formula (4) from its initial control polygon. Let $u = \cos(\omega\ell/2)$, we get the following definition of generating uniform ωB-spline curves by subdivision (ωBS for short).

Definition 2.1 (ωBS scheme) Let $P = \{P_1, P_2, \ldots P_n\}$ be the initial control polygon and u be the tension parameter. The subdivision rule of ωBS curves of order $k\ (k \geq 3)$ S_k is defined as:

$$\begin{aligned} &S_k : P \to P^k = \{P_1^k, P_2^k, \ldots P_{2n-k+1}^k\}, \\ &P_i^k = (P_i^{k-1} + P_{i+1}^{k-1})/2, \quad k>3, \end{aligned} \tag{4}$$

$$P_i^3 = \begin{cases} \dfrac{(4u-1)P_{(i+1)/2} + P_{(i+3)/2}}{4u}, & \text{i is odd ;} \\ \dfrac{P_{i/2} + (4u-1)P_{(i+2)/2}}{4u}, & \text{i is even ;} \end{cases} \tag{5}$$

Using the subdivision rule S_k, the iterative process of ωBS is described as below.

Table 1: The time report of generating ωBS curves and ωB-spline curves from the same control polygon

the type of curve	k=3	k=4	k=5	k=10
ωBS	0.007531 s	0.011816 s	0.013624 s	0.013835s
ωB-spline	0.014133 s	0.076115 s	2.027394 s	35.855619 s

1. Give an initial control polyline P, a tension parameter u ($u \geq 0$) and the subdivision order k;
2. Determine a subdivision times N_0 in advance;
3. $N = 0$;
4. $S_k : P' = S_k(P)$, $N = N + 1$;
5. If $N = N_0$

    go to step 6;

    else

    $u = \sqrt{\frac{1+u}{2}}$;

    $P = P'$;

    go to step 3;

    end
6. Output P′ ;

From the above definition, we can see that the parameter u updates in each level of subdivision. So ωBS method is a non-stationary subdivision method. The updating formula $u := \sqrt{\frac{1+u}{2}}$ is derived from the half-angle cosine formula because $u = \cos(\omega\ell/2)$.

Fig. 1 illustrates the proposed subdivision rules and an example.

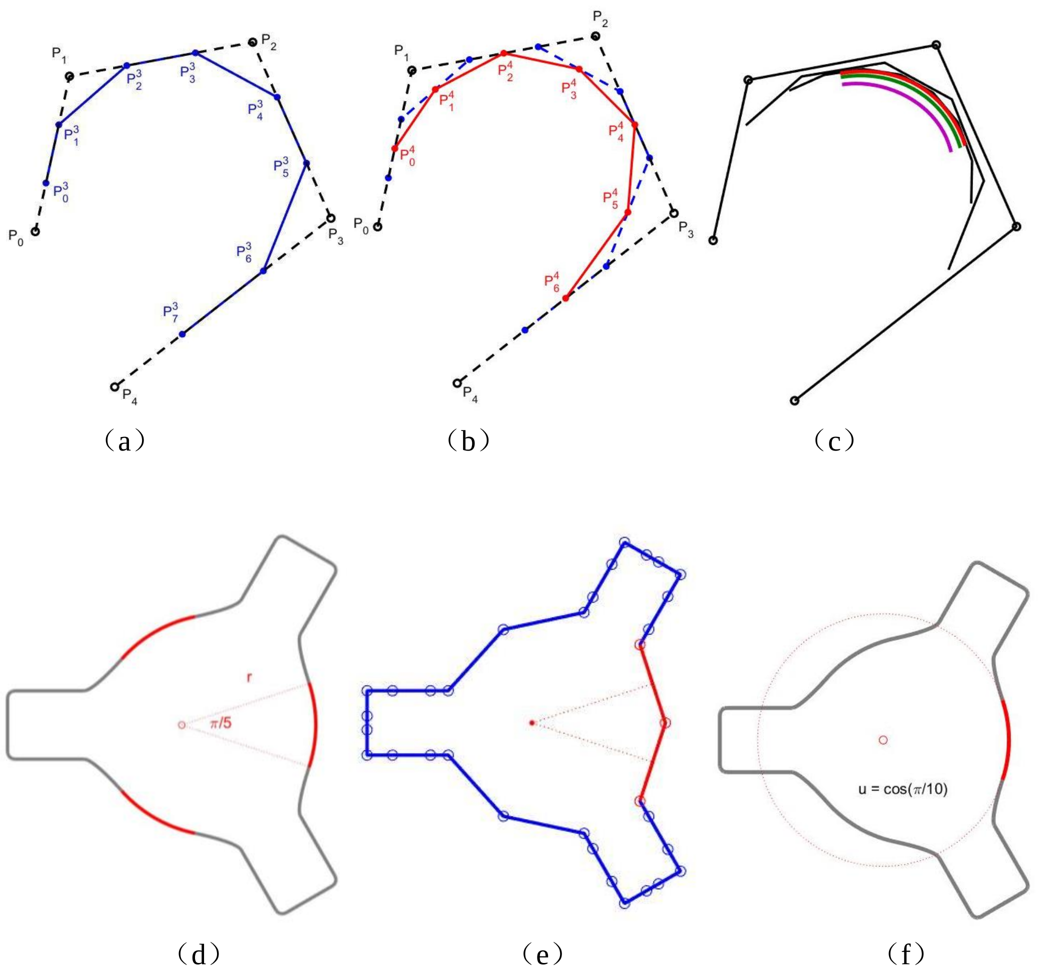

Figure 1: The subdivision rules (a)(b)(c) and an example (d)(e)(f).

In Fig. 1 (a), $\left\{P_i^3\right\}_{i=0}^{7}$ is computed by formula (5) from the initial control polyline $\left\{P_i\right\}_{i=0}^{4}$. In Fig. 1(b), $\left\{P_i^4\right\}_{i=0}^{6}$ is computed by formula (5) from $\left\{P_i^3\right\}_{i=0}^{7}$. Similarly, $\left\{P_i^k\right\}$ can be computed by formula (5) from $\left\{P_i^{k-1}\right\}$ when $k > 4$. In Fig. 1(c), the black poly lines are respectively the results after one level and two levels of subdivision from the initial control poly line when *k*=5, *u*=3. The red curve is the results after six levels of subdivision which can be seen as the approximation of the limit curve. The green and purple curves respectively correspond to the cases of *k*=5, *u*=1 and *k*=5, *u*=0.5. In Fig. 1(d), the profile of an industrial model which consists of three pieces of circular arcs (red), some line segments and some cushioning curves. In Fig. 1(e), the control polygon of the profile is computed according to the ωB-spline representation proposed in paper [Fang and Wang (2008)]. In Fig. 1(f), the profile is reproduced by subdividing the control polygon according to the proposed method in this paper, with $k = 3, u = \cos(\pi/10)$.

Comparing Definition 2.2 with Definition 1.1 and 1.2, we can see that ωBS curves only include linear computations, which is much simpler and more efficient than those recursive integral computations included in the definition of uniform ωB-spline bases. This is very important for real-time rendering and hierarchically displaying curves and surfaces. Taking the control polygon illustrated in Fig. 1(e) with 33 control points as an example, Tab. 1 shows the comparison of the efficiency of both methods to render the curve jointed with the same number (about 300) of points. Apparently, the efficiency of rendering ωBS curves is much faster than rendering ωB-spline curves. And with the increase of order, the difference between them becomes bigger and bigger.

From Theorem 2.1, we know ωBS curve is derived by the knot interpolation method of uniform ωB-spline curves. The sequence of control polygons formed by continuous bisections of knot intervals will converge to smooth ωB-spline curves, which are C^{k-2} -continuous. That is to say, ωB-spline curve is the limit curve of ωBS curve with the same control polygon when the subdivision level tends to infinity. In the next two sections, we prove that ωBS curves are also C^{k-2} continuous using two proving methods.

3 One proof of C^{k-2}-continuity of ωBS curves

Theorem 2.1 shows how the new control polygon can be obtained from the old control polygon after a round of subdivision. We have the following theorem.

Theorem 3.1 Let $B_k\left(P^0;\ell\right)(t) = p_k(t) = \sum_{i=1}^{n} P_i^0 N_{i,k}(t)$ be an uniform ωB-spline curve of order *k* whose control polygon is $P^0 = S_k^{(0)}\left[P^0\right] = \left[P_1^0, P_2^0, \ldots, P_n^0\right]$ and the length of knot interval is ℓ.

Let $S_k^{(N)}\left[P^0\right] = S_k\left[S_k^{(N-1)}\left[P^0\right]\right] = \left[P_1^{k,N}, P_2^{k,N}, \ldots, P_{2^N(n-k+1)+k-1}^{k,N}\right]$, where $S_k^1\left[P^0\right] = \left[P_1^{k,1}, P_2^{k,1}, \ldots, P_{2n-k+1}^{k,1}\right]$. Then $\lim_{N\to\infty} S_k^{(N)}\left[P^0\right] = B_k\left(P^0;\ell\right)(t)$.

Proof. By Theorem 2.1 and simple induction on *N*, we have

$$B_k\left(S_k^{(N)}\left[P^0\right];\frac{\ell}{2^N}\right)(\mathrm{t}) = B_k\left(P^0;\ell\right)(t).$$

Let $M = \max_i \left|P_{i+1}^0 - P_i^0\right|$, then

$$\left|P_{i+1}^{k,1} - P_i^{k,1}\right| \le \left(1 - \frac{1}{2\cos\frac{\omega\ell}{2}}\right) M .$$

Furthermore, we get

$$\left|P_{i+1}^{k,N} - P_i^{k,N}\right| \le \left(1 - \frac{1}{2\cos\frac{\omega\ell}{2}}\right)\cdots\left(1 - \frac{1}{2\cos\frac{\omega\ell}{2^N}}\right)M.$$

By Definition 1.1, we know $0 \le \frac{\omega\ell}{2} < \frac{\pi}{2}$ when ω is a real number. And clearly $\cos\frac{\omega\ell}{2^i} = \cosh\frac{|\omega|\ell}{2^i}\left(i = 1, 2, \ldots, N\right)$ when ω is a pure imaginary number. Then we have

$$\left|P_{i+1}^{k,N} - P_i^{k,N}\right| \le \begin{cases} \left(1-\frac{1}{2\cos\frac{\omega\ell}{2^N}}\right)^N M, & \omega \text{ is real and } 0\le\frac{\omega\ell}{2}<\frac{\pi}{2}; \\ \left(1-\frac{1}{2\cosh\frac{|\omega|\ell}{2}}\right)^N M, & \omega \text{ is imaginary and } 0\le\frac{|\omega|\ell}{2}<\frac{\pi}{2}. \end{cases}$$

Because $\lim\limits_{N\to\infty}\frac{1}{2\cos\frac{\omega\ell}{2^N}} = 1$ in the first case, and $\cosh\frac{|\omega|\ell}{2} \ge 1$ in the second case, we have $\lim\limits_{N\to\infty}\left|P_{i+1}^{k,N} - P_i^{k,N}\right| = 0$ in both cases. Therefore

$$\lim_{N\to\infty}\left|P_{i+j}^{k,N} - P_i^{k,N}\right| = 0 \tag{6}$$

for any
$i \in \left\{1, \ldots, 2^N\left(n-k+1\right)+k-1\right\}, 1, \ldots, k, i+j \le 2^N\left(n-k+1\right)+k-1.$

From the convex hull property, for any $t_0 \in \left[k|\omega|\ell, \left(n+1\right)|\omega|\ell\right]$, we know that $B_k\left(P^0;\ell\right)\left(t_0\right)$ lies within the convex hull of $P_i^{k,N}, P_{i+1}^{k,N}, \ldots, P_{i+k}^{k,N}$ for some *i*. Together with (6), we conclude that

$$\lim_{N\to\infty} S_k^{(N)}\left[P^0\right] = B_k\left(P^0;\ell\right)\left(t\right).$$

Theorem 3.2 ωBS curves of order k (k≥3) converge to C^{k-2}-continuous uniform ωB-spline curves.

Proof. Based on Definition 2.1 and Theorem 3.1, we can conclude that ωBS curves of order k (k≥3) converge to uniform ωB-spline curves of order k whose C^{k-2}-continuity are obvious according to the definition of ωB-spline basis functions and paper [Wang, Chen and Zhou (2004)]. So the conclusion holds.

4 Another proof of C^{k-2}-continuity of ωBS curves

The proof in Section 3 is simple. But this proof method is difficult to be extended to the case of surface subdivision, especially non-tensor product surface subdivision. So we provide another proof method for C^{k-2}-continuity of ωBS curves based on those theories upon subdivision masks, which will be advantageous to be applied in the proof of our further surface subdivision.

From the steps of ωBS described in Definition 2.2, we can see that the tension parameter is changing with the subdivision level, so ωBS is a non-stationary subdivision scheme. For convenience of proving its continuity, we introduce the corresponding notions of the mask of ωBS at first.

Given a set of control points $P^0 = \left\{P_j^0 \in \mathbb{R}^3 : j \in \mathbb{Z}\right\}$, a non-stationary subdivision scheme for curves defines recursively new sets of control points $P^N = \left\{P_j^N \in \mathbb{R}^3 : j \in \mathbb{Z}\right\}$ formally by

$$P^{N+1} = S^{(N)}P^N, \quad N=0, 1, \ldots$$

Each point in P^{N+1} is defined by a linear combination of points in P^N. The rule defining the new points in P^{N+1} is denoted by $m^{|N|} := \left\{m_i^{|N|} : i \in \mathbb{Z}\right\}$ such that

$$P_j^{N+1} = \sum_{i \in \mathbb{Z}} m_{j-2i}^{|N|} P_i^N. \tag{7}$$

The sequence $m^{[N]}$ of coefficients is called the subdivision mask at level N. If $m^{[N]}$ is independent of N, i.e. $m^{[N]}$ = m, the corresponding scheme is said to be stationary. It is assumed that only a finite number of coefficients $m_i^{[k]}$ are non-zero so that changes in a control point affect only its local neighborhood.

The specific rules of ωBS are given as follows. First consider the case k = 3. For a given tension parameter $u_0 > 0$, the ωBS scheme of order 3 generates a new set of the control points by the rule $S_3^{(N)}$:

$$\left(S_3^{(N)}P^N\right)_{2i} = \frac{4u_N - 1}{4u_N}P_i^N + \frac{1}{4u_N}P_{i+1}^N \ ,$$
$$\left(S_3^{(N)}P^N\right)_{2i+1} = \frac{1}{4u_N}P_i^N + \frac{4u_N - 1}{4u_N}P_{i+1}^N \ . \tag{8}$$

At each subdivision level *N*, the parameters u_N are also updated as

$$u_{N+1} = \sqrt{\frac{1 + u_N}{2}} \ . \tag{9}$$

Since $u_0 > 0$,we see that the sequence $\{u_N\}_{N=0}^{\infty}$ is monotone and bounded such that it converges to 1 as $N \to \infty$.When $u_0 = 1\left(i.e., u_N = 1 \ for \ \text{all N} \in \mathbb{N}\right)$, the scheme becomes stationary and it is the well-known Chaikin's algorithm [Chaikin (1974)]. In view of (4.1), the non-stationary mask $m^{[N]}$ for P^{N+1} can be written as

$$m^{[N]} = \left(\frac{1}{4u_N}, \frac{4u_N - 1}{4u_N}, \frac{4u_N - 1}{4u_N}, \frac{1}{4u_N}\right). \tag{10}$$

Further, based on the rule in (8), the scheme $S_k^{(N)}$ for *k* > 3 is defined recursively by

$$\left(S_{k+1}^{(N)}P^N\right)_j = \frac{1}{2}\left(\left(S_k^{(N)}P^N\right)_j + \left(S_k^{(N)}P^N\right)_{j+1}\right) \tag{11}$$

It can be easily checked that the support of the mask $m^{[N]}$ is indeed the same as the one of the classical B-spline of order *k* [Stam (2001)].

It's difficult to directly prove the continuity of a kind of non-stationary subdivision scheme. So we prove C^{k-2} -continuity of ωBS curves according to the theorems including asymptotic equivalence proposed in paper [Dyn and Levin (1995)]. Here we cite the notion of asymptotic equivalence between two schemes defined in paper [Dyn and Levin (1995)].

Definition 4.1 A non-stationary scheme with the masks $\left\{m^{[N]}\right\}$ is said to be asymptotically equivalent to a stationary scheme with the mask m if

$$\sum_{N \in \mathbb{Z}_+} \left\|m^{[N]} - m\right\|_{\infty} < \infty \tag{12}$$

Lemma 4.1 Let $\left\{S^{(N)}\right\}$ be a non-stationary subdivision scheme with the mask $\left\{m^{[N]}\right\}$.Assume that $\left\{S^{(N)}\right\}$ is asymptotically equivalent to a stationary scheme S with the mask m. If S is in C^q with $q \in \mathbb{N}$ and

$$\sum_{N=0}^{\infty} 2^{qN} \left\| m^{[N]} - m \right\| < \infty,$$

then the non-stationary scheme $\left\{ S^{(N)} \right\}$ is also C^q.

Theorem 4.1 The ωBS scheme of order 3 with the mask (10) generates C^1-continuous limit curves.

Proof. For a given parameter $u_0 \geq 0$, we can induce from (9) that $\left\{ u_N \right\}_{N=1}^{\infty}$ is a monotone and bounded sequence. To be more precise, if $0 \leq u_0 \leq 1$, the sequence $\left\{ u_N \right\}_{N=1}^{\infty}$ is monotonically increasing to 1. Otherwise, it is monotonically decreasing to 1. Hence, it is immediate that $m^{[N]}$ converges to

$$m_3 := \left(\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4} \right).$$

This is the mask of the Chaikin's corner cutting algorithm and it generates C^1 limit curve (see Chaikin [Chaikin (1974); Dyn and Levin (1995)]). Now, to estimate the C^1-smoothness of the proposed scheme of order 3, it is necessary to estimate the difference between $m^{[N]}$ and m_3. From (9), we see that

$$\left\| m^{[N]} - m_3 \right\|_\infty = \max \left(\left| \frac{1}{4u_N} - \frac{1}{4} \right|, \left| \frac{4u_N - 1}{4u_N} - \frac{3}{4} \right| \right).$$

If $u_0 = 1$, then $u_N = 1$, $i.e.$, $m^{[N]} = m_3$. Thus, in what follows, we assume that $u_0 \neq 1$. In fact, some elementary calculation easily reveals that

$$\left\| m^{[N]} - m_3 \right\|_\infty = \frac{1}{4} \left| \frac{1 - u_N}{u_N} \right| \tag{13}$$

Then, following Lemma 4.1, we only need to show that

$$\sum_{N=0}^{\infty} 2^N \left\| m^{[N]} - m_3 \right\|_\infty < \infty \tag{14}$$

for the proof of the C^1-smoothness of the proposed scheme of order 3. To this end, for simple notation, we use the abbreviation

$$U_N := 2^N \left\| m^{[N]} - m_3 \right\|_\infty = 2^{N-2} \left| \frac{1 - u_N}{u_N} \right|$$

Here, by (9), it is clear that $u_N = 2u_{N+1}^2 - 1$. Hence, $U_N = \frac{2\left(1-u_{N+1}^2\right)}{2u_{N+1}^2-1}$. Thus, we have

$$\begin{aligned}\frac{U_{N+1}}{U_N} &= 2\left|\frac{1-u_{N+1}}{u_{N+1}}\right| \cdot \left|\frac{u_N}{1-u_N}\right| \\ &= 2\left|\frac{1-u_{N+1}}{u_{N+1}}\right| \cdot \left|\frac{2u_{N+1}^2-1}{2-2u_{N+1}^2}\right| \\ &= \left|\frac{2u_{N+1}^2-1}{u_{N+1}\left(1+u_{N+1}\right)}\right|\end{aligned}$$

Consequently, since u_N converges to 1 as $N \to \infty$, it follows that

$$\lim_{N\to\infty} \frac{U_{N+1}}{U_N} = \frac{1}{2} < 1$$

Following the D'Alembert criteria for convergence of positive series and in view of (13), the claim (14) is proved.

We are now ready to prove the smoothness of the proposed scheme of order *k*>3. In the following analysis, we will see that it is convenient to represent a subdivision rule with the mask $m^{[N]}$ in terms of the symbol

$$m^{[N]}(z) := \sum_{j\in\mathbb{Z}} m_j^{[N]} z^j$$

Since $m^{[N]}$ is finitely supported, the symbol $m^{[N]}(z)$ is in fact a Laurent polynomial. The following lemma is obtained from Dyn et al. [Dyn and Levin (1995)].

Lemma 4.2 Let $\left\{S_{a^{[N]}}\right\}$ be a non-stationary subdivision scheme associated with the symbol of the form

$$a^{[N]}(z) = \frac{1}{2}(1+z)\, b^{|N|}(z)$$

and the non-stationary scheme corresponding to $\left\{S_{b^{[N]}}\right\}$ is C^q with $q \in \mathbb{Z}_+$.Then the scheme $\left\{S_{a^{[N]}}\right\}$ is C^{q+1}.

Theorem 4.2 The ωBS scheme of order $k \geq 3$ generate C^{k-2}-continuous limit curves.

Proof. We prove this theorem by mathematical induction for *k*. The case *k*=3 indeed holds immediately by Theorem 4.2. For the case *k*>3, we use the notation

$m^{[k,N]} = \left\{ m_j^{[k,N]} : j \in \mathbb{Z} \right\}$ for the mask of the ωBS scheme of order k at level *N*. By construction, the mask $m^{[k,N]}$ can be iteratively obtained by using the equation

$$m_j^{[k,N]} = \left(m_j^{[k-1,N]} + m_{j+1}^{[k-1,N]} \right) \Big/ 2 \tag{15}$$

which is an immediate consequence of the relation (11). Accordingly, the Laurent polynomial associated to the mask $m^{[k,N]}$ can be written as

$$m_k^{[N]}(z) := \left(\frac{z+1}{2} \right)^{k-3} m_3^{[N]}(z), \quad k>3 \ .$$

where $m_3^{[N]}(z)$ is the symbol of the ωBS scheme of order 3 with the mask $m^{[3,N]}$ in (10). By Theorem 4.1, the scheme associated to the Laurent polynomial $m_3^{[N]}(z)$ is C^1. Hence, applying Lemma 4.2 inductively, we can conclude that the proposed scheme of order k is C^{k-2}.

The approximation order of the proposed non-stationary subdivision is also important. In the following, we discuss this problem. Theorem 4.3 shows that it is of approximation order *k*-1, where *k* refers to the order of the corresponding ωB-splines.

Theorem 4.3 For the ωBS scheme $S^{[N]}$ of order $k \geqslant 3$, the approximation order of this non-stationary subdivision is *k*-1.

Proof.Based on Lemma 4.1, the proposed non-stationary subdivision scheme is asymptotically equivalent to a stationary scheme S. S converges to ωB-splines of order *k*, with a constant frequency sequence, which can reproduce polynomials of order *k*-1. According to the results concluded in paper [Conti, Dyn, Manni et al. (2015); Conti, Romani and Yoon (2016)], we know a non-stationary subdivision implies approximation order k-1 (k-1 refers to the degree of ωB-splines) asymptotic similarity to stationary scheme is assumed. So based on the above, the conclusion of Theorem 4.3 is proved.

5 Conclusion

In this paper, we proposed the subdivision scheme for uniform ωB-spline curves. Then a uniform ωB-spline curve has both perfect mathematical representation and efficient generation method. We also provide two proofs of C^{k-2}-continuity ωBS curves of *k*-order in two different aspects and discuss its approximation order. The first method is direct and simple. The second kind of proof is based on subdivision masks and some corresponding theories, which will be advantageous to prove the corresponding conclusions of surface subdivision. In the future, we will extend the subdivision scheme to the case of surfaces with tensor product form and further arbitrary topology as well. In addition, we will apply ωB-splines and especially the subdivision scheme in the all kinds of applications relative to finite element method (FEM) and isogeometric analysis (IGA) to improve the accuracy during modeling and analysis [Wang, Shen, Zou et al. (2018);

Guo and Nairn (2017); Xu, Sun, Xu et al. (2017)].

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