 CMES, vol.115, no.3, pp.313-326, 2018

Extrapolation Method for Cauchy Principal Value Integral with Classical Rectangle Rule on Interval

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Abstract: In this paper, the classical composite middle rectangle rule for the computation of Cauchy principal value integral (the singular kernel $1/(x\text{-}s)$) is discussed. With the density function approximated only while the singular kernel is calculated analysis, then the error functional of asymptotic expansion is obtained. We construct a series to approach the singular point. An extrapolation algorithm is presented and the convergence rate of extrapolation algorithm is proved. At last, some numerical results are presented to confirm the theoretical results and show the efficiency of the algorithms.

Keywords: Cauchy principal value integral, Extrapolation method, Composite rectangle rule, Superconvergence, Error expansion.

1 Introduction

In recent years, much attention has been paid to the singular integral of the form

$$I(f,s) = \fint_a^b \frac{f(t)}{t-s} dt, s \in (a,b), \tag{1}$$

where $\fint_a^b$ denotes a Cauchy principal value integral and s is the singular point.

There are many definition of the Cauchy principal value integral, in the following we adopt the definition as below:

$$\fint_a^b \frac{f(t)}{t-s} dt = \lim_{\varepsilon \to 0} \left\{ \int_a^{s-\varepsilon} \frac{f(t)}{t-s} dt + \int_{s+\varepsilon}^b \frac{f(t)}{t-s} dt \right\}, s \in (a,b). \tag{2}$$

Lots of numerical methods for such singular integrals have been studied previously by many authors [Choi, Kim and Yun (2004); Ioakimidis (1985); Kim and Jin (2003); Li, Yang and Yu (2014); Yu (1992)]. The classical extrapolation method based on polynomial and rational function has been widely studied. The extrapolation methods as an accelerating convergence technique has been applied to many fields in computational mathematics [Liem, Lü and Shih (1995)]. One of such extrapolation methods is Richardson extrapolation with the error functional as

$$T(h) - a_0 = a_1 h^2 + a_2 h^4 + a_3 h^6 + \cdots,$$

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here $T(0) = a_0$, a_j is constant independent of h.

In reference Choi et al. [Choi, Kim and Yun (2004)], the asymptotic error analysis of the Euler-Maclaurin formula is obtained by using the parametric sigmoidal transformation, with traditional sigmoidal transformations, a distinct improvement on its predecessors is p-resented. Then in the reference Elliott et al. [Elliott and Venturino (1997)], sigmoidal transformations to obtain better approximation to Cauchy principal value integrals is employed, which is also extended the Euler-Maclaurin formula to Hadamard finite-part integrals. In the reference Sidi [Sidi (2003)] and Zeng et al. [Zeng, Lei and Huang (2014)] presented high-accuracy numerical quadrature methods for integrals of singular periodic functions which are based on the appropriate Euler-Maclaurin expansions of trapezoidal rule approximations and their extrapolations. In recent reference, the classical Euler-Maclaurin summation formula [Sidi (2003)] expresses the difference between a definite integral over $[0,1]$ and its approximation using the trapezoidal rule with step length $h = 1/m$ as an asymptotic expansion in powers of h together with a remainder term.

The extrapolation method for the computation of Hadamard finite-part integrals on the interval and in a circle are studied in Li et al. [Li, Wu and Yu (2009)] and Li et al. [Li, Zhang and Yu (2013)] respectively which focus on the asymptotic expansion of error function. Based on the asymptotic expansion of the error functional, algorithm with theoretical analysis of the generalized extrapolation are given. In reference Zeng et al. [Zeng, Lei and Huang (2014)], quadrature formulae for hypersingular integrals and their asymptotic error expansions and the extrapolation methods for hypersingular integrals with either periodic integrand or non-periodic integrand are presented.

In this paper, we firstly obtain the error expansion of the classical rectangle rule. Then with certain special function, we present the explicit part for the first part of the error expansion. Based on this asymptotic expansion, we suggest an extrapolation algorithm. A series of s_j is selected to approximate the singular point s accompanied by the refinement of the meshes. Moreover, by means of the extrapolation technique, we not only obtain an approximation with higher order accuracy but also get a posteriori estimate of the error functional.

The rest of this paper is organized as follows. In Section 2, after introducing some basic formulas of the classical rectangle rule, we present the asymptotic error expansion of classical rectangle rule for Cauchy principal value integrals. In Section 3, we finish the proof of the main theorem. In Section 4, extrapolation algorithm and a posteriori asymptotic error estimation to compute Cauchy principal value integrals are obtained. Finally, several numerical examples are provided to validate our analysis.

2 Main result

Before we give our main results, we firstly let $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ be a uniform partition of the interval $[a,b]$ with mesh size $h = (b-a)/n$.

Theorem 1 *Let $f(t) \in C^{\infty}[a,b]$ and let $\theta \in [0,1]$be fixed. Set $h = (b-a)/n$ for integer n and define*

$$E_n(f,h) = h\sum_{i=0}^{n-1} f(a+ih+\theta h), \quad \theta \in [0,1] \tag{3}$$

and

$$I = \int_a^b f(t)\,dt, \tag{4}$$

then we have

$$E_n(f,h) = I + \sum_{k=1}^{\infty} \frac{B_k(\theta)}{k!}[f^{(k-1)}(b) - f^{(k-1)}(a)]h^k. \tag{5}$$

where $B_k(\theta)$ is the Bernoulli number.

Define $f_C(t)$ as the constant interpolant for $f(t)$

$$f_C(t) = f(t_{j-1}),\ t \in [t_{j-1}, t_j]. \tag{6}$$

and also define a linear transformation

$$t = \hat{t}_j(\tau) := (\tau+1)(t_j - t_{j-1})/2 + t_{j-1},\ \ \tau \in [-1,1], \tag{7}$$

from the standard reference element $[-1,1]$ to the subinterval $[t_{j-1}, t_j]$. Replacing $f(t)$ in (1) with $f_C(t)$ gives the composite middle rectangle rule:

$$I_n(f;s) := \fint_a^b \frac{f_C(t)}{t_{j-1} - s} dt = \sum_{j=1}^{n} \omega_j f(t_{j-1}) = \fint_a^b \frac{f(t)}{t-s} dt - E_n(f,s), \tag{8}$$

where ω_j denotes the Cote coefficient given by

$$\omega_j = \frac{h}{t_{j-1} - s}. \tag{9}$$

We also define

$$F^j(t) = t - t_{j-1}. \tag{10}$$

By linear transformation (7), we have

$$F^j(t) = \frac{h}{2}(\tau+1) = \frac{h}{2}F(\tau), \tag{11}$$

where

$$F(\tau) = \tau + 1 \tag{12}$$

and $\phi_0(t)$, defined by

$$\phi_0(t) = \begin{cases} -\dfrac{1}{2}\displaystyle\fint_{-1}^{1} \frac{F(\tau)}{(\tau - t)(t+1)} d\tau, & |t| < 1, \\ -\dfrac{1}{2}\displaystyle\int_{-1}^{1} \frac{F(\tau)}{(\tau - t)(t+1)} d\tau, & |t| > 1. \end{cases} \tag{13}$$

If $F(\tau)$ is the Legendre polynomial of first kind, $\phi_0(t)$ defines the Legendre function of the second kind [Andrews (2002)].

Now we present the main results below.

Theorem 2 *Assume $f(t) \in C^{\infty}[a,b]$. For the middle rectangle rule $I_n(f;s)$ defined in (8), there exist certain constant c_i, independent of h, such that*

$$E_n(f,s) = f(s)\pi \tan\frac{\pi\tau}{2} + \sum_{i=1}^{\infty} c_i h^i \tag{14}$$

where $s = t_{m-1} + (1+\tau)h/2, \ \ m = 1,2,\cdots,n$.

Based on the theorem 2, we present the modify middle rectangle rule

$$\tilde{I}_n(f;s) = I_n(f;s) - f(s)\pi \tan\frac{\pi\tau}{2}, \tag{15}$$

and

$$\tilde{E}_n(f;s) = \int_a^b \frac{f(x)}{x-s}dx - \tilde{I}_n(f;s). \tag{16}$$

Then we have the corollary

Corollary 1 *Under the same assumption of theorem 2, for the modify middle rectangle rule (15), we have*

$$|\tilde{E}_n(f;s)| \leq Ch. \tag{17}$$

3 Proof of the Theorem 2

Lemma 1 *Assume that $s \in (t_{j-1}, t_j)$ for some m and let $c_j = 2(s-t_{j-1})/h - 1, 1 \leq j \leq n$. Then, we have*

$$\phi_0(c_j) = \begin{cases} -2 \displaystyle\int_{t_{m-1}}^{t_m} \frac{t_{m-1}-t}{(t-s)(t_{m-1}-s)}\, dt, & j = m, \\ -2 \displaystyle\int_{t_{j-1}}^{t_j} \frac{t_{j-1}-t}{(t-s)(t_{j-1}-s)}\, dt, & j \neq m. \end{cases} \tag{18}$$

Proof: By following the definition of (1) and the linear transformation (7), we have

$$\begin{aligned} &\int_{t_{m-1}}^{t_m} \frac{t_{m-1}-t}{(t-s)(t_{m-1}-s)}\, dt \\ &= \lim_{\varepsilon \to 0} \left\{ \int_{t_{m-1}}^{s-\varepsilon} \frac{t_{m-1}-t}{(t-s)(t_{m-1}-s)}\, dt + \int_{s+\varepsilon}^{t_m} \frac{t_{m-1}-t}{(t-s)(t_{m-1}-s)}\, dt \right\} \\ &= -\frac{1}{2} \int_{-1}^{1} \frac{\tau+1}{(\tau - c_m)(c_m+1)}\, d\tau \\ &= -\frac{1}{2}\phi_0(c_m). \end{aligned} \tag{19}$$

The case $j \neq m$ can be proved by applying the same approach to the correspondent Riemann integral.

Lemma 2 *Under the same assumptions of theorem 2, it holds that*

$$\frac{f(t)}{t-s} - \frac{f_C(t)}{t_{j-1}-s} = \left[\frac{1}{t-s} - \frac{1}{t_{j-1}-s}\right] f(s) \tag{20}$$

$$+ \sum_{k=1}^{\infty} \frac{f^{(k)}(s)}{k!}[(t-s)^{k-1} - (t_{j-1}-s)^{k-1}] \tag{21}$$

Proof: By performing Taylor expansion of $f_C(t), f(t)$ at the point s, we have

$$f_C(t) = f(s) + \sum_{k=1}^{\infty} \frac{f^{(k)}(s)}{k!}(t_{j-1}-s)^k \tag{22}$$

and

$$f(t) = f(s) + \sum_{k=1}^{\infty} \frac{f^{(k)}(s)}{k!}(t-s)^k \tag{23}$$

Combining (22) and (23) together we get the results (20).

Lemma 3 *Under the same assumptions of theorem 2, there holds*

$$S_0(\phi_0, \tau) = \sum_{j=1}^{n} \phi_0(c_j) = \pi \tan \frac{\pi(\tau)}{2}$$

Proof: By straightly calculation of $\phi_0(x)$, we easily get

$$\begin{aligned} \phi_0(x) &= \frac{2}{x-1} + \ln\left|\frac{1-x}{1+x}\right| \\ &= 2\mathbf{Q}_0(x) + 2Q_0(x) \end{aligned} \tag{24}$$

where

$$\mathbf{Q}_0(x) = \frac{1}{x-1}$$

and

$$Q_0(x) = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|, \quad Q_1(x) = xQ_0(x) - 1. \tag{25}$$

Here $Q_n(x)$ be the function of the second kind associated with the Legendre polynomial $P_n(x)$, defined by Andrews [Andrews (2002)]. Then, we have,

$$\begin{aligned} \lim_{n\to\infty} \sum_{j=1}^{n} Q_0(2j+\tau) &= \lim_{m\to\infty} \left(\sum_{j=0}^{n-m-1} Q_0(2j+\tau) + \sum_{j=1}^{m} Q_0(-2j+\tau) \right) \\ &= \lim_{m\to\infty} \frac{1}{2} \ln \frac{2(n-m)-1+\tau}{2m+1-\tau} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\sum_{j=1}^{n}\mathbf{Q}_0(2j+\tau) &= \lim_{m\to\infty}\left(\sum_{j=0}^{n-m-1}\mathbf{Q}_0(2j+\tau)+\sum_{j=1}^{m}\mathbf{Q}_0(-2j+\tau)\right)\\ &= \lim_{m\to\infty}\left(\sum_{j=0}^{n-m-1}\frac{2}{2j+\tau-1}+\sum_{j=1}^{m}\frac{2}{-2j+\tau-1}\right)\\ &= \lim_{m\to\infty}\sum_{j=-m}^{m}\frac{1}{j+\frac{1}{2}-\frac{\tau}{2}}=\pi\tan\frac{\pi\tau}{2},\end{aligned}$$

where we have use the identity [Andrews (2002)], it follows that

$$S_0(\phi_0,\tau)=\pi\tan\frac{\pi\tau}{2}. \tag{26}$$

3.1 Proof of Theorem 1

Proof: By Lemma 2, we have

$$\begin{aligned}&\left(\int_a^{t_{m-1}}+\int_{t_m}^{b}\right)\frac{f(t)}{t-s}dt-\sum_{j=1,j\neq m}^{n}\frac{hf_C(t)}{t_{j-1}-s}\\ &=\sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}\left[\frac{f(t)}{t-s}-\frac{f_C(t)}{t_{j-1}-s}\right]dt\\ &=\sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}\left[\frac{1}{t-s}-\frac{1}{t_{j-1}-s}\right]f(s)dt\\ &+\sum_{i=1}^{\infty}\frac{f^{(i)}(s)}{i!}\sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}[(t-s)^{i-1}-(t_{j-1}-s)^{i-1}]\,dt.\end{aligned} \tag{27}$$

For $i=m$, we have

$$\begin{aligned}\int_{t_{m-1}}^{t_m}\frac{f(t)}{t-s}dt-\frac{hf_C(t)}{t_{m-1}-s} &= \int_{t_{m-1}}^{t_m}\left[\frac{f(t)}{t-s}-\frac{f_C(t)}{t_{m-1}-s}\right]dt\\ &= \int_{t_{m-1}}^{t_m}\left[\frac{1}{t-s}-\frac{1}{t_{m-1}-s}\right]f(s)dt\\ &+\sum_{i=1}^{\infty}\frac{f^{(i)}(s)}{i!}\int_{t_{j-1}}^{t_j}[(t-s)^{i-1}-(t_{m-1}-s)^{i-1}]\,dt.\end{aligned} \tag{28}$$

Putting (27) and (28) together yields

$$\begin{aligned}
&\int_a^b \frac{f(t)}{t-s}dt - \sum_{j=1}^{n} \frac{hf_C(t)}{t_{j-1}-s} \\
&= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left[\frac{f(t)}{t-s} - \frac{f_C(t)}{t_{j-1}-s}\right] dt \\
&= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left[\frac{1}{t-s} - \frac{1}{t_{j-1}-s}\right] f(s)dt \\
&+ \sum_{i=1}^{\infty} \frac{f^{(i)}(s)}{i!} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} [(t-s)^{i-1} - (t_{j-1}-s)^{i-1}]\, dt \\
&= S_0(\phi_0,\tau)f(s) + \sum_{i=1}^{\infty} \frac{f^{(i)}(s)}{i!} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} [(t-s)^{i-1} - (t_{j-1}-s)^{i-1}]\, dt.
\end{aligned} \tag{29}$$

Here

$$S_0(\phi_0,\tau) = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left[\frac{1}{t-s} - \frac{1}{t_{j-1}-s}\right] dt$$

with the linear transformation from $[t_{j-1},t_j]$ to the identity interval $[-1,1]$. As for the last part of

$$\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} [(t-s)^{i-1} - (t_{j-1}-s)^{i-1}]\, dt$$

which can be considered as the error estimate of middle rectangle rule for the definite integral$\int_a^b (t-s)^{i-1}\, dt, i \geq 2$. Obviously,by the Theorem 1, it can be expanded by the Euler-Maclaurin expansions and we have

$$E_n^i(f,h) = \int_a^b (t-s)^{i-1}\, dt + \sum_{k=1}^{\infty} \frac{B_k(\theta)}{k!}[(b-s)^{(k-1)} - (a-s)^{(k-1)}]h^k, \;\; k \leq i-1. \tag{30}$$

It is easy to see that there are not relation with the singular point s which can be written as

$$E_n^i(f,h) = \int_a^b (t-s)^{i-1}\, dt + \sum_{k=1}^{\infty} c_k h^k, \;\; k \leq i-1. \tag{31}$$

The proof is completed.

We actually obtain the error expansion of the middle rectangle rule and moreover, get the explicit expression of the first order t erm. So it is easy for us to get the superconvergence point with $S_0(\phi_0,\tau) = 0$, which means that $\tau = 0$ is the superconvergence point in subinterval not near the end of the interval.

4 Extrapolation method

In the above sections, we have proved that the error functional of the middle rectangle rule have the following asymptotic expansion

$$E_n(f,s) = f(s)\pi \tan\frac{\pi\tau}{2} + \sum_{k=1}^{\infty} c_k h^k \tag{32}$$

It is easily to see that the error functional depended on the value of $c_i(\tau)$. In order to present our extrapolation algorithm, we give the Lemma below

Lemma 4 *Assume $f(t) \in C^{\infty}[a,b]$. For $I(f;s)$ defined in (1), there holds that $I(f;s) \in C^{\infty}[a,b]$.*

The proof is similarly to the Lemma 2 in Du [Du (2001)], here we omit it.

For the given s, now we present algorithm. There exists positive integer n_0 such that

$$m_0 := \frac{n_0(s-a)}{b-a}$$

is a positive number. Firstly, we partition $[a,b]$ into n_0 equal subinterval and get a mesh denoted by Π_1 with mesh size $h_1 = (b-a)/n_0$ as the starting meshes. Then we refine the starting meshes Π_1 to get mesh Π_2 with mesh size $h_2 = h_1/2$. In this way, a series of meshes $\{\Pi_j\}(j=1,2,\cdots)$ is obtained in which Π_j is refined from Π_{j-1} with mesh size denoted by h_j. Then we get extrapolation scheme in Tab. 1.

Table 1: Extrapolation scheme of $T_i^{(j)}$

$T(h_1) = T_1^{(1)}$				
$T(h_2) = T_1^{(2)}$	$T_2^{(1)}$			
$T(h_3) = T_1^{(3)}$	$T_2^{(2)}$	$T_3^{(1)}$		
$T(h_4) = T_1^{(4)}$	$T_2^{(3)}$	$T_3^{(2)}$	$T_4^{(1)}$	
$T(h_5) = T_1^{(5)}$	$T_2^{(4)}$	$T_3^{(3)}$	$T_4^{(2)}$	$T_5^{(1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

For a coordinate $\tau \in (-1,1)$ is given, and define

$$s_j = s + \frac{\tau+1}{2}h_j, \quad j=1,2,\cdots. \tag{33}$$

and

$$T(h_j) = I_{2^{j-1}n_0}(f,s_j). \tag{34}$$

the following extrapolation algorithm is presented:

Step one:

$$Compute \;\; T_1^{(j)} = T(h_j), \quad j=1,\cdots,m.$$

Step two:

$$Compute \;\; T_i^{(j)} = T_{i-1}^{(j+1)} + \frac{T_{i-1}^{(j+1)} - T_{i-1}^{(j)}}{2^{i-1}-1}, \quad i=2,\cdots,m \quad j=1,\cdots,m-i.$$

Theorem 3 *Under the asymptotic expansion of theorem 2, for $\tau = 0$ and the series of meshes defined by (33), we have*

$$|I(f,s) - T_i^{(j)}| \le Ch^i \tag{35}$$

and a posteriori asymptotic error estimate is given by

$$\left| \frac{T_{i-1}^{(j+1)} - T_{i-1}^{(j)}}{2^{i-1} - 1} \right| \le Ch^{i-1}, \quad i \ge 2.$$

Proof: For a given τ, by the asymptotic expansion of (32) we have

$$\begin{aligned} I(f,s) - T(h_j) &= I(f,s) - I(f,s_j) + I(f,s_j) - T(h_j) \\ &= I(f,s) - I(f,s_j) + f(s_j)\pi \tan\frac{\pi\tau}{2} + \sum_{i=1}^{\infty} c_i h^i \end{aligned} \tag{36}$$

By the definition of Cauchy principal value integrals and (33), for the first two part of (36), by Taylor expansion for $I(f;s_j)$ at the singular point s, we have

$$\begin{aligned} I(f;s_j) &= I(f;s) + I'(f;s)\frac{\tau+1}{2}h_j + \frac{I''(f;s)}{2!}(\frac{\tau+1}{2}h_j)^2 \\ &+ \cdots + \frac{I^{(l)}(f;s)}{l!}(\frac{\tau+1}{2}h_j)^l + \cdots, \end{aligned} \tag{37}$$

Putting (37) and (36) together, yields

$$I(f,s) - T(h_j) = f(s_j)\pi \tan\frac{\pi\tau}{2} + \sum_{i=1}^{\infty} b_i(s,\tau)h_j^i, \tag{38}$$

where

$$b_i(s,\tau) = c_i - \frac{(\tau+1)^i}{2^i i!} I^{(i)}(f,s), \tag{39}$$

for a given τ, $b_i(s,\tau)$ is a constant. By (38), we also have

$$I(f,s) - T(h_{j+1}) = f(s_{j+1})\pi \tan\frac{\pi\tau}{2} + \sum_{i=1}^{\infty} b_i(s,\tau)h_{j+1}^i. \tag{40}$$

By (38) and (40), with $h_j = 2h_{j+1}$ we also have

$$\begin{aligned} I(f,s) &= 2T(h_{j+1}) - T(h_j) + \sum_{i=1}^{\infty} b_i(s,\tau)\left(\frac{1}{2^{i-1}} - 1\right)h_j^i \\ &= T_2^{(j)} + \sum_{i=1}^{\infty} b_i(s,\tau)\left(\frac{1}{2^{i-1}} - 1\right)h_j^i, \end{aligned} \tag{41}$$

which implies

$$I(f,s) - T_2^{(j)} = \sum_{i=1}^{\infty} b_i(s,\tau)\left(\frac{1}{2^{i-1}} - 1\right)h_j^i \tag{42}$$

and

$$T_2^{(j)} = 2T(h_{j+1}) - T(h_j). \tag{43}$$

Continuing to use extrapolation process again, we can obtain accuracy $O(h^3)$. Similarly we can get the accuracy $O(h^4)$. In this way, we continue extrapolation process and finish the proof.

5 Numerical example

In this section, computational results are reported to confirm our theoretical analysis.

Example 1 *We consider the Cauchy principal value integrals with* $f(t) = t^3$, $a = 0, b = 1$. *Obviously the integrand function* $f(t)$ *is smooth enough and by (1), we examine the dynamic point* $s = t_{[n/4]} + (\tau + 1)h/2$ *with* $\tau = 0, \pm 2/3, 1/2$.

Table 2: The error of the rectangle rule to $s = t_{[n/4]} + (\tau + 1)h/2$

	0	$-2/3$	$2/3$	$1/2$
32	2.1095e-02	1.1125e-01	-9.3054e-02	-4.2904e-02
64	1.0481e-02	9.8114e-02	-8.8455e-02	-4.5782e-02
128	5.2243e-03	9.1563e-02	-8.6596e-02	-4.7382e-02
256	2.6081e-03	8.8291e-02	-8.5774e-02	-4.8222e-02
512	1.3031e-03	8.6656e-02	-8.5389e-02	-4.8651e-02
1024	6.5129e-04	8.5839e-02	-8.5203e-02	-4.8869e-02
h^α	1.0035	–	–	–

Table 3: The error modify of the rectangle rule to $s = t_{[n/4]} + (\tau + 1)h/2$

	0	$-2/3$	$2/3$	$1/2$
32	2.1095e-02	2.0798e-02	2.1401e-02	2.1324e-02
64	1.0481e-02	1.0408e-02	1.0555e-02	1.0537e-02
128	5.2243e-03	5.2061e-03	5.2426e-03	5.2380e-03
256	2.6081e-03	2.6036e-03	2.6127e-03	2.6116e-03
512	1.3031e-03	1.3019e-03	1.3042e-03	1.3039e-03
1024	6.5129e-04	6.5101e-04	6.5157e-04	6.5150e-04
h^α	1.0035	0.9995	1.0075	1.0065

From Tab. 2, we know that the convergence rate is $O(h)$ with the coordinate location of singular point equal zero, while for the local coordinate of singular point do not equal zero, it is not convergence in general which coincide with our analysis.

For the modify classical rectangle rule, from Tab. 3, we can see that the convergence rate can reach $O(h)$ for the local coordinate of singular point equal zero or not, which is also coincide with our corollary.

Example 2 *We consider the Cauchy principal value integrals with* $f(t) = t^3$ $a = 0, b = 1$. *Obviously the integrand function* $f(t)$ *is smooth enough and by (1),with* $s = 0.25$ *and the exact value is 5.379991503437726e-01 , we use* $s = t_{[n/4]} + (\tau + 1)h/2$ *with* $\tau = 0$, *to approximation* $s = 0.25$.

From Tab. 4, we know that the convergence rate of error estimate of the classical rectangle rule is $O(h)$ for the first column, and the convergence rate of second column, third column and fourth column are $O(h^2)$, $O(h^3)$ and $O(h^4)$ respectively. From Tab. 5, we know that the convergence rate of posteriori estimate of the classical rectangle rule rule is the same as the the convergence rate of error estimate of the classical rectangle rule which agree with our theorem.

Table 4: Error estimate of the classical rectangle rule $s_j = s + (\tau+1)h_j/2$

	0	h^2-extra	h^3-extra	h^4-extra
32	3.3328e-03			
64	1.6541e-03	-2.4542e-05		
128	8.2476e-04	-4.5983e-06	2.0496e-06	
256	4.1190e-04	-9.6219e-07	2.4985e-07	-7.2613e-09
512	2.0584e-04	-2.1742e-07	3.0838e-08	-4.4913e-10
1024	1.0289e-04	-5.1482e-08	3.8303e-09	-2.7926e-11

Table 5: A posteriori estimate of the classical rectangle rule rule $s_j = s + (\tau+1)h_j/2$

	0	h^2-extra	h^3-extra	h^4-extra
32				
64	1.6787e-03			
128	8.2936e-04	-6.6479e-06		
256	4.1286e-04	-1.2120e-06	2.5711e-07	
512	2.0606e-04	-2.4826e-07	3.1287e-08	-4.5415e-10
1024	1.0295e-04	-5.5312e-08	3.8582e-09	-2.8080e-11

For the case the singular point is near the end of the interval with $s = 1/1024$,and the exact value is 3.338225747121760e-01. We choose the starting meshes $n_0 = 1024$,the convergence rate is also $O(h)$ for the first column, and the convergence rate of second column, third column and fourth column are $O(h^2)$, $O(h^3)$ and $O(h^4)$ respectively in Tab. 6 and Tab. 7 which agree with our theorem.

For the case the singular point is not located as the mesh-point, we can not find the proper starting meshes. We have lots of methods to solve the problem, we adopt the methods by moving the starting meshes a little to make the singular point be located at the mesh point. In fact, it is not difficult to extend our methods to the quasi-uniform meshes and the proof is similarly to Theorem 2.

Example 3 *Let $f(t) = t^3,\ \ a = 0, b = 1$ for the case of quasi-uniform meshes, we consider the case of $s = 1/\sqrt{2}$ and make sure s is located at the meshes point by moving the starting meshes a little and refine the meshes each time.*

Table 6: Error estimate of the classical rectangle rule $s = 1/1024, s_j = s + (\tau + 1)h_j/2$

	0	h^2−extra	h^3−extra	h^4−extra
1024	2.4349e-04			
2048	1.2179e-04	8.1779e-08		
4096	6.0904e-05	2.0399e-08	-6.0707e-11	
8192	3.0454e-05	5.0939e-09	-7.7871e-12	-2.2704e-13
16384	1.5228e-05	1.2727e-09	-9.8932e-13	-1.8208e-14
32768	7.6141e-06	3.1809e-10	-1.2346e-13	2.2204e-16

Table 7: A posteriori estimate of the classical rectangle rule $s_j = s + (\tau + 1)h_j/2$

	0	h^2−extra	h^3−extra	h^4−extra
1024				
2048	1.2170e-04			
4096	6.0883e-05	2.0460e-08		
4096	3.0449e-05	5.1017e-09	-7.5600e-12	
16384	1.5227e-05	1.2737e-09	-9.7111e-13	-1.3922e-14
32768	7.6137e-06	3.1822e-10	-1.2369e-13	-1.2286e-15

From Tab. 8, for the singular point $s = 1/\sqrt{2}$, we know that the convergence rate of error estimate of the classical rectangle rule is $O(h)$ for the first column, and the convergence rate of second column, third column and fourth column are $O(h^2)$, $O(h^3)$ and $O(h^4)$ respectively. From Tab. 9, we know that the convergence rate of posteriori estimate of the classical rectangle rule is the same as the the convergence rate of error estimate of the classical rectangle rule which agree with our theorem.

Table 8: Error estimate of the classical rectangle rule with $s = 1/\sqrt{2}$

	0	h^2−extra	h^3−extra	h^4−extra
32	7.5514e-02			
64	3.6402e-02	-2.7097e-03		
128	1.7875e-02	-6.5214e-04	3.3719e-05	
256	8.8575e-03	-1.6001e-04	4.0393e-06	-2.0067e-07
512	4.4089e-03	-3.9631e-05	4.9451e-07	-1.1884e-08
1024	2.1995e-03	-9.8618e-06	6.1181e-08	-7.2315e-10

Table 9: A posteriori estimate of the classical rectangle rule with $s = 1/\sqrt{2}$

	0	h^2-extra	h^3-extra	h^4-extra
32				
64	3.9112e-02			
128	1.8527e-02	-6.8586e-04		
256	9.0175e-03	-1.6405e-04	4.2400e-06	
512	4.4486e-03	-4.0125e-05	5.0640e-07	-1.2586e-08
1024	2.2094e-03	-9.9230e-06	6.1904e-08	-7.4403e-10

Acknowledgment: The work of Jin Li was supported by National Natural Science Foundation of China (Grant No. 11471195), China Postdoctoral Science Foundation (Grant No. 2015T80703), Shan-dong Provincial Natural Science Foundation of China (Grant No. ZR2016JL006) and Na-tional Natural Science Foundation of China (Grant No. 11771398).

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