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Structural Design Optimization Using Isogeometric Analysis: A Comprehensive Review

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Abstract: Isogeometric analysis (IGA), an approach that integrates CAE into conventional CAD design tools, has been used in structural optimization for 10 years, with plenty of excellent research results. This paper provides a comprehensive review on isogeometric shape and topology optimization, with a brief coverage of size optimization. For isogeometric shape optimization, attention is focused on the parametrization methods, mesh updating schemes and shape sensitivity analyses. Some interesting observations, e.g. the popularity of using direct (differential) method for shape sensitivity analysis and the possibility of developing a large scale, seamlessly integrated analysis-design platform, are discussed in the framework of isogeometric shape optimization. For isogeometric topology optimization (ITO), we discuss different types of ITOs, e.g. ITO using SIMP (Solid Isotropic Material with Penalization) method, ITO using level set method, ITO using moving morphable com-ponents (MMC), ITO with phase field model, etc., their technical details and applications such as the spline filter, multi-resolution approach, multi-material problems and stress con-strained problems. In addition to the review in the last 10 years, the current developmental trend of isogeometric structural optimization is discussed.

Keywords: Isogeometric analysis, structural optimization, shape optimization, topology optimization, sensitivity analysis.

1 Introduction

Structural optimization, as an important tool in the product design process, deals with the optimal design of load-carrying structures. In general, structural optimization methods can be divided into three categories: Size, shape and topology optimization methods [Bendsoe and Sigmund (2003); Huang and Xie (2010); Mortazavi and Toğan (2016); Tejani, Savsani, Patel et al. (2018)]. An illustration of these three types of structural optimization is shown

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in Fig. 1. Size optimization deals with the optimum size parameters such as cross-section dimensions and thickness, while fixing the shape and topology of the structure. Shape optimization works by modifying the outer boundary of the structure to obtain the optimal shape. Topology optimization optimizes the material layout within a given design domain, which is capable to find the optimal topology from different topologies rather than a particular topology. Of the three categories, size optimization is a relatively simple parameter optimization. We will thus focus on the other two types of structural optimization in this review.

In shape optimization, the geometric parameterization, an adequate boundary description and the design variables are essential for a successful optimization. To be consistent with the analysis model, the coordinates of the boundary nodes were used as design variables in the early days of shape optimization [Francavilla, Ramakrishnan and Zienkiewicz (1975)]. However, utilizing the nodal coordinates as design variables may result in unrealistic designs with irregular boundaries [Haftka and Grandhi (1986); Hsu (1994)]. To avoid such irregular optimized results, researchers tried to use geometric boundary representations in shape optimization, such as polynomials, B-spline and NURBS [Ding (1986); Samareh (2001a)]. Although the usage of spline representations improves the smoothness of boundaries, the design model based on splines is separated from the analysis model based on finite element, resulting in a disconnection between the design and analysis models. To update the shape geometry during the iterative process of shape optimization, a relation between the design variables and the analysis model is required, e.g. the design element [Imam (1982)] and the natural design variables [Belegundu and Rajan (1988)]. Besides, some strategies such as adaptive meshing [Bennett and Botkin (1985)] and remeshing [Wilke, Kok and Groenwold (2006)] are required to maintain the accuracy of shape optimization, especially for the shape optimization with large deformations.

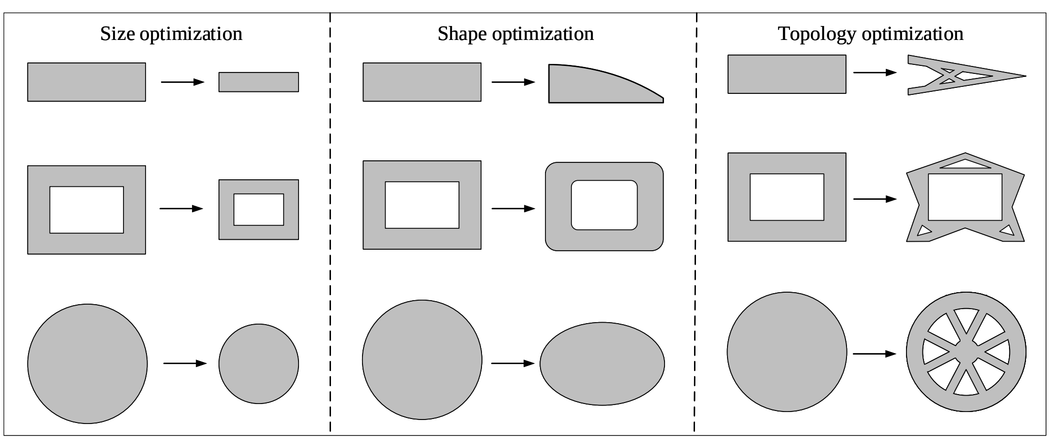


Figure 1: An illustration of structural optimization

Shape optimization cannot change the topology of the structure so it is usually used to optimize the detailed features of the product after the basis geometrical shape has been obtained. In contrast, topology optimization aims to find the optimal material distribution in a design domain, such that the structural topologies can change during the optimization process. The first attempt was made by Cheng et al. [Cheng and Olhoff (1981)] on optimal thickness distribution of solid elastic plates, and the pioneering work of Bendsøe and Kikuchi [Bendsøe and Kikuchi (1988)] that used a homogenization method to generate the optimal topologies was a milestone in the topology optimization research. Since then, other important methods have been proposed to solve topology optimization problems, e.g. solid isotropic material with penalization (SIMP) [Bendsøe (1989); Sigmund (2001); Andreassen, Clausen, Schevenels et al. (2011)], evolutionary structural optimization [Xie and Steven (1993); Querin, Steven and Xie (1998); Huang and Xie (2009)] and level-set based topology optimization [Wang, Wang and Guo (2003); Allaire, Jouve and Toader (2004); Luo, Tong, Wang et al. (2007); Xia, Shi and Xia (2019)]. Topology optimization is becoming an efficient tool in product designs, and has been applied to a wide range of engineering domains such as aerospace [Zhu, Zhang and Xia (2016)], automobile [Yang and Chahande (1995)], additive manufacturing [Mass and Amir (2017)] and photonics [Borel, Harpøth, Frandsen et al. (2004)].

The abovementioned shape and topology optimization methods are based on the finite element method (FEM), some limitations of the FE approach include the disconnection between analysis and geometric models, low continuity between the elements and low efficiency of high-order elements. These limitations are successfully addressed by a new method called isogeometric analysis (IGA) [Hughes, Cottrell and Bazilevs (2005)], which directly uses the basis functions of a computer aided design (CAD) model as the shape functions of analysis model, IGA has since been used in a variety of domains [Benson, Bazilevs, Hsu et al. (2011); Hsu and Bazilevs (2012); Wang and Benson (2015); Wang, Benson and Nagy (2015); Yan, Deng, Korobenko et al. (2017); Deng, Wu, Yang et al. (2018)]. In the last decade, IGA has been adopted for both shape and topology optimizations, which provides several advantages over the established methods. For example, the exact geometric representation of IGA merges analysis and design model in shape and topology optimizations [Wall, Frenzel and Cyron (2008); Seo, Kim and Youn (2010b,a)], and the spline functions of IGA can avoid the checkerboards in the topology optimization without additional filters [Hassani, Khanzadi and Tavakkoli (2012); Dedè, Borden and Hughes (2012); Lieu and Lee (2017a)]. Although the research in isogeometric structural optimization has been growing for the past decade (See Fig. 2), to the best of our knowledge, no comprehensive review has been published yet to help the researchers working in this domain to systematically understand isogeometric structural optimization.

In this paper, we intend to provide a comprehensive review on the isogeomtric structural optimization from the first work of Wall et al. [Wall, Frenzel and Cyron (2008)] proposed in 2008. An outline of the remainder of this paper is as follows: Section 2 introduces the fundamentals of isogeometric structural optimization. Section 3 gives a comprehensive review for isogeometric shape optimization including shape sensitivity analysis, regulation method, special application etc. Section 4 discusses different types of isogeometric topology optimization (ITO) including but not limited to density-based and level-set-based ITOs. The conclusions and research trends are provided in Section 5.

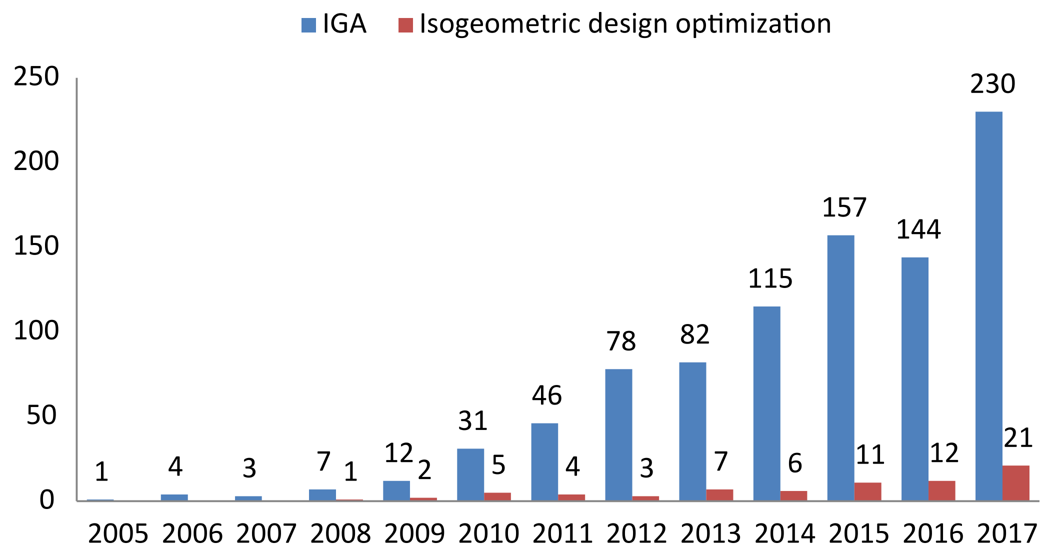


Figure 2: Numbers of publications on IGA and isogeometric design optimization per year (data from Scopus search engine)

2 Fundamntals of isogeometric structural optimization

Isogeometric analysis (IGA), proposed by Hughes et al. [Hughes, Cottrell and Bazilevs (2005)], is an efficient alternative to conventional FEM, which represents the analysis model with CAD mathematical primitives such as B-spline and NURBS. Over ten years of development, IGA is now the most active research topic in both computational mechanics and computer aided geometric design, and has been applied to solve many problems in-cluding structural optimizations. In this section, we will summarize the fundamentals of isogeometric structural optimization consisting of IGA and structural optimization.

2.1 Summary of isogeometric analysis

2.1.1 Numerical geometry description tools

A number of new spline technologies have been used in IGA, including T-splines [Bazilevs, Calo, Cottrell et al. (2010); Scott, Li, Sederberg et al. (2012)], PHT-splines [Nguyen-Thanh, Kiendl, Nguyen-Xuan et al. (2011)] hierarchical B-splines or NURBS [Vuong, Giannelli, Jüttler et al. (2011); Wu, Huang, Liu et al. (2015)] and Powell-Sabin splines [Speleer-s, Manni, Pelosi et al. (2012)]. Presently B-splines and nonuniform rational B-splines (NURBS) [Piegl and Tiller (1997)] are commonly used in CAD systems, and are still the most prevalent spline technology in IGA. In B-splines, the a knot vector $\Xi=\{\xi_1, \xi_2, ..., \xi_{n+p+1}\}$ is a sequence of non-decreasing real numbers in the parametric space, where n is the num-ber of control points and p is the order of the spline curve. The interval $[\xi_1,\, \xi_{n+p+1})$ is called a patch and the knot interval $[\xi_i,\, \xi_{i+1})$ is called a span. The B-spline basis functions can be recursively defined by the Cox-de Boor formula [Boor (1972)]:

$$
\begin{aligned}
B_{i,0}(\xi) &= \begin{cases} 1 & \text{if } \xi_i \le \xi < \xi_{i+1}, \\ 0 & \text{otherwise}, \end{cases} \\
B_{i,p}(\xi) &= \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} B_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} B_{i+1,p-1}(\xi), \quad (p \neq 0).
\end{aligned} \tag{1}
$$

The continuity of the B-spline basis functions between knot spans is C^{p-k} with k as the multiplicity of the knots. NURBS basis functions can be defined based on B-spline basis function by assigning a positive weight w_i to each basis function or control variables:

$$
N_{i,p}(\xi) = \frac{B_{i,p}(\xi) w_i}{\sum_{j=1}^{n} B_{j,p}(\xi) w_j}. \tag{2}
$$

Note that B-splines are a special case of NURBS where the weights are uniform.

A point location vector $\boldsymbol{x}(\boldsymbol{\chi})$ in a geometry patch described using NURBS can be generally expressed as

$$
\boldsymbol{x}(\boldsymbol{\chi}) = \sum R_I(\boldsymbol{\chi}) \boldsymbol{x}_I, \tag{3}
$$

where $\boldsymbol{\chi}$ is the location vector in the parametric space:

$$
\boldsymbol{\chi} = \begin{cases} (\xi) & \text{for 1D parametric space} \\ (\xi, \eta) & \text{for 2D parametric space} \\ (\xi, \eta, \zeta) & \text{for 3D parametric space} \end{cases}
$$

I is a index function with a form of $I(i)$ for 1D problems, $I(i,j)$ for 2D problems and $I(i,j,k)$ for 3D problems; the weighted basis function R_I is obtained from the tensor products of the 1D case shown in Eq. (2) along different directions and correspond to the Ith control point $\boldsymbol{x}_I$.

Taking two dimensional B-spline for example, the basis functions are constructed as tensor products,

$$
B_{i,p}^{j,q}(\xi, \eta) = B_{i,p}(\xi) B_{j,q}(\eta). \tag{4}
$$

$B_{i,p}(\xi)$ and $B_{j,q}(\eta)$ are univariate B-spline basis functions of order p and q, corresponding to knot vectors $\Xi = \{\xi_1, \xi_2, ..., \xi_{n+p+1}\}$ and $\mathcal{H} = \{\eta_1, \eta_2, ..., \eta_{m+q+1}\}$. According to the tensor product formulation, two-dimensional NURBS basis functions with order p in ξ direction and order q in η direction are constructed as

$$
N_{i,p}^{j,q}(\xi, \eta) = N_{i,p}(\xi) N_{j,q}(\eta). \tag{5}
$$

The four important properties of NURBS basis functions (likewise for B-spines) are briefly listed as: (1) Nonnegativity: $N_{i,j}(\xi, \eta) \ge 0$; (2) Partition of unity: $\sum_{i=1}^{n} \sum_{j=1}^{m} N_{i,j}(\xi, \eta) =$

1; (3) Local support: $N_{i,j}(\xi, \eta) = 0$ if (ξ, η) is outside the span domain of $[\xi_i, \xi_{i+p+1}) \times [\eta_j, \eta_{j+q+1})$; and (4) Continuity: the interior of knot span domain is continuous up to C^∞, and the continuity between knot spans are C^{p-k} and C^{q-l} where k and l are the multiplicity, respectively.

2.1.2 NURBS-based IGA for solving boundary value problems

A typical boundary value problem formulation defined over domain Ω with boundary Γ can be interpreted as

$$\begin{cases} \boldsymbol{c}(\boldsymbol{u}) := \operatorname{div} \mathbb{C}\nabla \boldsymbol{u} + \boldsymbol{f} = \boldsymbol{0} & \text{in } \Omega \\ (\mathbb{C}\nabla \boldsymbol{u})\, \boldsymbol{n} - \hat{\boldsymbol{t}} = \boldsymbol{0} & \text{on } \Gamma_t \\ \boldsymbol{u} - \hat{\boldsymbol{u}} = \boldsymbol{0} & \text{on } \Gamma_u \end{cases} \tag{6}$$

where $\boldsymbol{c}(\boldsymbol{u})$ represents the equilibrium equation, $\boldsymbol{f}$ is body force, $\mathbb{C}$ is a forth-order tensor related to material properties, $\boldsymbol{n}$ is the outward normal vector, div is the divergence operator, ∇ is the gradient operator, $\hat{\boldsymbol{t}}$ and $\hat{\boldsymbol{u}}$ are the prescribed Neumann and Dirichlet boundary conditions defined on Γ_t and Γ_u, respectively.

Introducing a test function field $\bar{\boldsymbol{u}}$, the weak formulation of the strong form defined in Eq. (6) can be written as:

$$\begin{aligned} \langle \boldsymbol{c}(\boldsymbol{u}), \bar{\boldsymbol{u}} \rangle_\Omega = & - \int_\Omega \mathbb{C}\nabla \boldsymbol{u} \cdot \nabla \bar{\boldsymbol{u}} \, \mathrm{d}\Omega + \int_\Omega \boldsymbol{f} \cdot \bar{\boldsymbol{u}} \, \mathrm{d}\Omega \\ & + \int_{\Gamma_\mathrm{t}} \hat{\boldsymbol{t}} \cdot \bar{\boldsymbol{u}} \, \mathrm{d}\Gamma + \int_{\Gamma_\mathrm{u}} \boldsymbol{t} \cdot \bar{\boldsymbol{u}} \, \mathrm{d}\Gamma = 0. \end{aligned} \tag{7}$$

This BVP can be solved using finite element method. Using the Bubnov-Galerkin method, the state variable $\boldsymbol{u}$ and the force field are discretized and subsequently substituted into Eq. (7) to obtain a system of linear equations such that the problem can be solved:

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{F} \tag{8}$$

where $\boldsymbol{K}$ is the stiffness matrix, $\boldsymbol{U}$ is the unknown discrete variables and $\boldsymbol{F}$ is a force vector. Note that the terminology used here originated in mechanical problems for the development of FEM and can be used to non-mechanical problems that may have different terminologies.

The NURBS-based IGA follows a similar approach to obtain Eq. (8), but the shape functions used for discretization are the same as the basis functions used for geometry description, i.e. the the state variables and the force field are discretized using the basis functions in Eq. (3). In the framework of IGA, the state variable $\boldsymbol{u}$ is interpolated as

$$\boldsymbol{u} = \sum R_I \boldsymbol{u}_I, \tag{9}$$

where $\boldsymbol{u}_I$ is the state variable vector of the Ith control point $\boldsymbol{x}_I$. Using this interpolation scheme, the element stiffness matrix $\boldsymbol{K_e}$ that can be written as

Table 1: Comparison Between IGA and FEM

	IGA	FEM
Analysis model	spline element	finite element
Basis function	spline basis function	lagrange basis function
Degree of freedom	control point	node
Element refinement	h, p, k-refinement	h-refinement
Element continuity	up to C^{p-1}	C^0

$$\begin{aligned} \boldsymbol{K_e} &= \int_{\Omega_e} \boldsymbol{B^T D B} d\Omega \\ &= \int_{\widehat{\Omega}_e} \boldsymbol{B^T D B} |\boldsymbol{J}_1| d\widehat{\Omega} \qquad , \\ &= \int_{\overline{\Omega}_e} \boldsymbol{B^T D B} |\boldsymbol{J}_1||\boldsymbol{J}_2| d\overline{\Omega} \end{aligned} \tag{10}$$

where $\boldsymbol{B}$ represents the strain-displacement matrix, $\boldsymbol{D}$ denotes the stress-strain matrix, Ω_e represents the element domain, $\widehat{\Omega}_e$ is the mapping domain in the NURBS parametric space, and $\overline{\Omega}_e$ is integration domain in the integration parametric space $\{\overline{\xi}, \overline{\eta}\}$. Jacobian $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ indicate the transformation relation from the NURBS parametric space to the physical space and the integration parametric space to the NURBS parametric space, respectively. More details can be found in Wang et al. [Wang, Xu and Pasini (2017)].

Compared with conventional FEM, IGA has a series of advantages such as nonnegativity and high continuity for high-order basis functions. A comparison between IGA and FEM is shown in Tab. 1. It is noted that the geometry remains constant when elements are refined in IGA, since IGA uses the accurate geometric model. This is not true with FEM, which uses finite elements to approximately discretize the geometric models. Besides, the accuracy per degree of freedom (DOF) can be improved by the smoothness of splines in IGA.

2.2 Basis statement of structural optimization

A typical structural optimization problem involves a set of design variables $\boldsymbol{d} = (d_1,\ d_2,\ \cdots)$, an objective function $\Psi\big(\boldsymbol{u}(\boldsymbol{x}); \boldsymbol{d}\big)$, a system of equality constraints charactered by the boundary value problem (BVP) defined in Eq. (6), and some other constraints, which can be expressed as:

$$\begin{aligned} \text{Min}: &\ \Psi\big(\boldsymbol{u}; \boldsymbol{d}\big) \\ \text{Subject to}: &\ \langle \boldsymbol{c}(\boldsymbol{u}, \bar{\boldsymbol{u}} \rangle_\Omega = 0 \\ &\ h_i(\boldsymbol{u}; \boldsymbol{d}\big) = 0,\ \ i = 1, 2, ..., L \\ &\ g_i(\boldsymbol{u}; \boldsymbol{d}\big) \leq 0,\ \ i = 1, 2, ..., M \end{aligned} \tag{11}$$

where the constraint defined by the BVP is nested in a weak form $\langle \boldsymbol{c}(\boldsymbol{u}), \bar{\boldsymbol{u}} \rangle_\Omega = 0$; $\boldsymbol{x}$ is location vector; h_i are equality constraints; and g_i are the inequality constraints.

The above formulation of a structural optimization problem is based on continuum variables. An alternative interpretation is to define the problem based on discrete formulations:

$$\begin{aligned} \text{Min}: & \ \Psi\big(\boldsymbol{U};\boldsymbol{d}\big) \\ \text{Subject to}: & \ \boldsymbol{K}\boldsymbol{U} = \boldsymbol{F} \\ & \ h_i\big(\boldsymbol{U};\boldsymbol{d}\big) = 0, \ \ i = 1, 2, ..., L \\ & \ g_i\big(\boldsymbol{U};\boldsymbol{d}\big) \leq 0, \ \ i = 1, 2, ..., M \end{aligned} \tag{12}$$

where the variables have been discretized.

In general, the gradient of the objective and constraint with respect to the design variables needs to be calculated so that a gradient-based method such as steepest descent method [Boyd and Vandenberghe (2009)], sequential quadratic programming (SQP) [Boggs and Tolle (1995)], the optimality criteria (OC) method [Hassani and Hinton (1998)], convex linearization method (CONLIN) [Fleury (1989)], the method of moving asymptotes (M-MA) [Svanberg (1987)] and Globally convergent version of MMA (GCMMA) [Svanberg (2002)] can be used to solve the optimization problem. There are also non-gradient-based methods such as genetic algorithm (GA) [Adeli and Kumar (1995)] and Particle swarm optimization (PSO) [Eberhart and Kennedy (1995)] to solve the optimization problem with absence of the gradient, but the efficiency of such non-gradient-based methods is much lower compared to the gradient method.

3 Isogeometric shape optimization

Structural shape optimization is a type of design methodology that optimizes structures by changing the geometry shape such that a cost function is minimized. It has been a research interest since 1970s [Haftka and Grandhi (1986)]. As the computational mechanics was restricted by the computational resources at that time, the effectiveness of the size and topology optimization was limited. Meanwhile, shape optimization was able to provide optimal designs suitable for practical applications. Hence it was considered as a more effective design method. However, compared to size and topology optimization where the finite element mesh remains same and the perturbations of the design variables only affect the material property tensor, shape optimization is relatively more challenging because (i) it often requires the finite element mesh to be updated during an iterative optimization process and (ii) the perturbation of the design variables directly affects the geometry boundary and the integration domain. This poses three critical problems: the parameterization method, shape updating scheme and the sensitivity analysis.

With IGA, a seamless integration between analysis and design is achieved, and hence automatically averts typical parameterization problems in traditional FEM-based shape optimizations. Compared to FEM-based shape optimizations, there are three additional advantages of isogeometric shape optimization [Wall, Frenzel and Cyron (2008); Cho and Ha (2009)]: (i) its ability of preserving the curved geometry features for analysis, which makes it ideal to design structures or domains with curved boundary features; (ii) its higher accuracy of evaluating structural responses, especially for boundary variables and dynamic responses, which is desirable for sensitivity analysis involving boundaries and interfaces; and (iii) the integrated design-analysis-optimization model, which can significantly reduce the transition effort between CAD and analysis models. With these two advantages, isogeometric shape optimization method has demonstrated its capability in designing curved domains, e.g. Nagy et al. [Nagy, Abdalla and Gürdal (2010a); Hosseini, Moetakef-Imani, Hadidi-Moud et al. (2018); Choi and Cho (2018a); Weeger, Narayanan and Dunn (2018); Liu, Yang, Wang et al. (2018)] about curved beams; Nagy et al. [Nagy, IJsselmuiden and Abdalla (2013); Kiendl, Schmidt, WWüchner et al. (2014); Kang and Youn (2016); Bandara and Cirak (2018); Hirschler, Bouclier, Duval et al. (2018)] about shells; Gillebaart et al. [Gillebaart and De Breuker (2016); Herath, Natarajan, Prusty et al. (2015); Kostas, Ginnis, Politis et al. (2015, 2017)] about fluid-interacted structural geometries; Nagy et al. [Nagy, Abdalla and Gürdal (2011); Manh, Evgrafov, Gersborg et al. (2011); Taheri and Hassani (2014)] about dynamic problems; Wang et al. [Wang, Poh, Dirrenberger et al. (2017); Wang and Poh (2018); Choi and Cho (2018a)] about smoothed curved auxetics; Hao et al. [Hao, Wang, Ma et al. (2018)] with reliability-based design optimization; and Nguyen et al. [Nguyen, Evgrafov and Gravesen (2012); Pels, Bontinck, Corno et al. (2015)] about magnetic-related design domains. The integrated analysis-design framework enables to build a platform where the structural responses of designs with different parameters can be observed easily, and hence a sequence of the results can be used as samples for a surrogate model to quickly explore the design space [Benzaken, Herrema, Hsu et al. (2017)].

In this section, the issues of parameterization methods, shape updating schemes and sensitivity analysis in the context of isogeometric shape optimization is comprehensively reviewed with comparisons to traditional-FEM based shape optimizations in Sections 3.1 and 3.2. Studies based on gradient-free optimization methods are discussed in Section 3.3. Some remarks and discussions about isogeometric shape optimization are briefly summarized in Section 3.4.

3.1 Design parameterization, analysis techniques and updating schemes

3.1.1 Parameterization methods in traditional FEM-based shape optimization

A major limitation with the traditional FEM-based shape optimization lies with the transition between a geometry model described in a CAD system and the analysis model discretized with finite element mesh. Depending on the design parametrizations, different approaches have been developed.

An intuitive way is to use the finite element mesh directly, with the nodal locations as design variables (e.g. [Zienkiewicz (1973)]). This is also termed as a "parameter-free" approach (e.g. [Le, Bruns and Tortorelli (2011); Riehl, Friederich, Scherer et al. (2014)]). This approach naturally avoids a CAD description and hence the corresponding transition difficulties. However, in addition to introduce a large number of design variables, it also leads to non-optimal jagged boundaries and irregular meshes, as illustrated in Fig. 3, and the geometry features such as curvature can only be approximated [Schmitt and Steinmann (2017)].

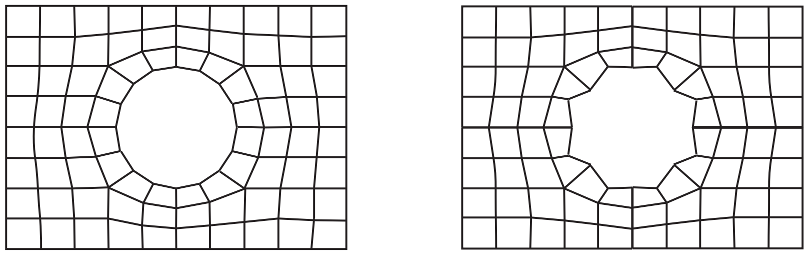
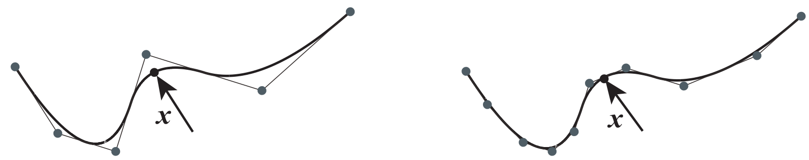


Figure 3: Schematic of jagged boundaries occurring during a shape optimization for a FEM-based shape optimization

An alternative approach is to use parametric curves or surfaces to describe the design objects. These parametric schemes include polynomials (e.g. [Imam (1982)]), Bezier and B-Splines (e.g. [Braibant and Fleury (1984, 1985)]), NURBS (e.g. [Schramm and Pilkey (1993)]), mesh mapping/morphing (e.g. [Wang, Zhang, Wang et al. (2010); Wang, Wang, Zhu et al. (2011)]) and bi-space parametrization (e.g. [Wang and Zhang (2012); Wang, Zhang and Zhu (2016)]). More details of the parameterization can be found in the surveys of Haftka et al. [Haftka and Grandhi (1986); Samareh (2001b); Daxini and Prajapati (2017)]. The two major advantages of using a parametric method for shape optimization lie in two major aspects: (i) to reduce the number of design variables; (ii) to improve the accuracy of shape sensitivity analysis. However, the disadvantage of using a separate description for the geometry is that the mesh updating process requires the transition between parametric and analysis models. One tricky way of avoiding the mesh updating problem is to embed the structural domain on a background mesh where only the domain covered regions are assumed to have solid material, with the remaining domain composing of voids (e.g. [Bendsøe (1989); Norato, Haber, Tortorelli et al. (2004); Wang and Zhang (2013)]). Generally, the level-set-based topology optimization is a variant of this method, which will be addressed later in section Section 4.2.

3.1.2 Isogeometric shape optimization with different parameterization methods and analysis techniques

In the framework of isogeometric analysis, the transition between geometry description and analysis model is seamless. Hence, using IGA for shape optimization automatically avoids the problems induced by the parameterization approaches, and naturally leads to a simple integrated design framework. This was first addressed in the pioneering works [Wall, Frenzel and Cyron (2008); Cho and Ha (2009)]. Since the NURBS refinement can be easily



(a) Design discretization space (b) Analysis discretization space

Figure 4: Double levels of discretization scheme: (a) A coarse discretization for design parameterization to reduce design variables and (b) A refined discretization for analysis to guarantee accurate structural responses [Wang, Turteltaub and Abdalla (2017); Wang, Abdalla and Turteltaub (2017)]. Note that an interesting observation is found in Lieu et al. [Lieu and Lee (2017a,b)] for isogeometric topology optimization, an contrary strategy is used such that a coarse mesh is used for analysis to improve the computational efficiency, yet a refined mesh is used for design to achieve topology designs with a high resolution

achieved using standard algorithms to insert knots into the knot vectors, the isogeometric shape optimization framework also provides a platform to use multi-levels discretization for design and analysis [Nagy, Abdalla and Gürdal (2010a)], e.g. a coarse discretization for design variables to reduce the number of design variables and a refined discretization for analysis to ensure the accuracy, as illustrated in Fig. 4. It should be noted that in [Lieu and Lee (2017a,b)] for isogeometric topology optimization, an contrary strategy is used such that a coarse mesh is used for analysis to improve the computational efficiency, yet a refined mesh is used for design to achieve topology designs with a high resolution. Although NURBS is a powerful tool to describe complicated geometries, the CAD models are not always analysis-suitable [Schmidt, Kiendl, Bletzinger et al. (2010); Scott, Li, Seder-berg et al. (2012); da Veiga, Buffa, Cho et al. (2012)], especially for complex geometries such as structures with cut-out openings using Boolean operations. This strongly restricts the development of isogeometric shape optimization. This problem can be solved using two approaches: one is to propose new parameterization methods that are analysis-suitable; the other one is to develop new analysis techniques to conform to existing CAD modeling methods. For such purposes, various parameterization methods and analysis techniques have been developed to further promote the design optimization. Representative parameterization methods include analysis suitable T-splines in Scott et al. [Scott, Li, Sederberg et al. (2012); da Veiga, Buffa, Cho et al. (2012)], an analysis-suitable parameterization framework using harmonic method in Xu et al. [Xu, Mourrain, Duvigneau et al. (2013)], a method of using mapped B-spline basis functions in [Yuan and Ma (2014)] and an optimized trivariate B-spline solids parameterization approach in Wang et al. [Wang and Qian (2014)]. Representative analysis techniques for shape design optimization problems in-clude the T-Spline based IGA in Ha et al. [Ha, Choi and Cho (2010)], boundary element method (BEM) based IGA in Li et al. [Li and Qian (2011); Lian, Kerfriden and Bordas (2016); Yoon and Cho (2016)], a combination of T-Spline and BEM solver in Schillinger et al. [Schillinger, Dede, Scott et al. (2012); Kostas, Ginnis, Politis et al. (2015); Yoon, Choi and Cho (2015); Lian, Kerfriden and

Bordas (2017); Kostas, Fyrillas, Politis et al. (2018)], Powell-Sabin splines based IGA in Speleers et al. [Speleers and Manni (2015)], isogeometric B-Rep analysis for trimmed surfaces in Philipp et al. [Philipp, Breitenberger, D'Auria et al. (2016)], an immersed method termed immersogeometric methods in Wu et al. [Wu, Kamensky, Wang et al. (2017)], an combination of immersed method and bound-ary method in Marco et al. [Marco, Ródenas, Fuenmayor et al. (2018)], and triangulations based IGA in Wang et al. [Wang, Xia, Wang et al. (2018)]. It should be noted that among these methods, the boundary element method is popular for its certain advantages over the domain-based methods, e.g. the unnecessary of generate domain mesh such that the interior mesh irregularity can be avoided during an optimization process. With the development of IGA, analysis-suitable parameterization techniques can provide a truly seamless analysis-design environment that is capable of dealing with complex geometries for multi-physics problems, e.g. the works in Herrema et al. [Herrema, Wiese, Darling et al. (2017)].

3.1.3 Shape and mesh updating schemes

For the traditional FEM-based shape optimization, it is important to use proper mesh updating techniques to deal with the jagged boundary and irregular mesh. These techniques include the traction method that takes the shape derivatives on the boundary as a type of Neumann boundary condition in Inzarulfaisham et al. [Inzarulfaisham and Azegami (2004); Azegami and Takeuchi (2006); Shimoda, Tsuji and Azegami (2007); Riehl, Friederich, Scherer et al. (2014)], an automatic regridding method that takes the shape derivatives on the boundary as a type of Dirichlet boundary condition in Choi et al. [Choi and Kim (2005a)] and filtering method in Bletzinger et al. [Bletzinger, Firl, Linhard et al. (2010); Le, Bruns and Tortorelli (2011); Firl, Wüchner and Bletzinger (2013)].

In the framework of isogeometric shape optimization, although it is mentioned in Braibant et al. [Braibant and Fleury (1984, 1985)] that a B-Spline parameterization automatically accounts for boundary irregularities, proper boundary and domain mesh regularizations are still required. This mainly because that (i) the boundary shape change may cause irregular mesh if the interior control points are not moved properly [Yuan and Ma (2015); Choi and Cho (2015)] and (ii) the non-uniform local supports of the control points can lead to unbalanced step-sizes for different design control points to give an irregular geometry [Nagy (2011); Wang, Abdalla and Turteltaub (2017)], as illustrated in Fig. 5. To address these problems, several methods have been developed. For simple problems, it is possible to use a linear interpolation scheme for relocating the interior control points, e.g. the works in Wang et al. [Wang, Abdalla and Turteltaub (2017)]. In Nagy et al. [Nagy, Abdalla and Gürdal (2010a)], a Sobolev semi-norm, referred to as "shape change norm", is proposed. In Azegami et al. [Azegami, Fukumoto and Aoyama (2013)], the H1 gradient method, which is essentially similar to the traction method, is utilized. Both the shape change norm and the H1 gradient methods require to solve a system of equations. In Yuan et al. [Yuan and Ma (2015)], a parametric mesh regularization based on a method of mapped basis functions is demonstrated with a iterative procedure. In Choi et al. [Choi and Cho (2015)], a mesh regularization scheme is proposed by minimizing the Dirichlet energy functional and a dimensionless functional such that the uniform parametrization and the mesh orthogonality can be obtained. In Kiendl et al. [Kiendl, Schmidt, WWüchner et al. (2014)], a sensitivity weighting method is presented to normalize the step-sizes of the movements of the control points. This approach is further studied systemically in Wang et al. [Wang, Abdalla and Turteltaub (2017)], where additional normalization approaches are proposed and the underlying reasons of using the normalization approaches analysed.

3.2 Shape sensitivity analysis methods

Shape sensitivity analysis plays a critical rule in a gradient-based optimization process. In this work, we distinguish between the various sensitivity analysis methods based on how the derivatives of the state variable field are treated, to give the: (i) Direct method (DM);(ii) Overall finite d ifferences (OFD); (iii) S emi-analytical (SA) m ethod; a nd (iv) adjoint method. An alternative classification method used in the surveys of [Van Keulen, Haftka and Kim (2005); Newman III, Taylor III, Barnwell et al. (1999)] is based on whether the derivation is derived from a continuum or discrete forms. This classification is also used here to have a comprehensive discussion for each type of sensitivity analysis method, e.g. for the adjoint method, it is further distinguished into two types: (i) Discrete adjoint method and (ii) Continuum adjoint method, also known as continuous or variational adjoint method (see e.g. [Newman III, Taylor III, Barnwell et al. (1999); Choi and Kim (2005a,b)]). The fundamental aspects of these sensitivity methods were developed in the 1980s, and become increasingly applicable for shape optimization problems with the development of software and hardware platforms for numerical analysis. Since surveys and reviews have been avail-able for these informations in the works of Adelman et al. [Adelman and Haftka (1986); Haftka and Adelman (1989); Tortorelli and Michaleris (1994); Hsu (1994); Newman III, Taylor III, Barnwell et al. (1999); Van Keulen, Haftka and Kim (2005)], we will focus on shape sensitivity analyses in the context of isogeometric shape optimization.

3.2.1 Preliminaries: Material derivatives and transport relations

For simplicity, the problem here assumes that the load is design-independent. Henceforth, to elaborate on the shape sensitivity analysis methods for the continuum-based formulations shown in Eq. (11), we define the objective function in a generic manner as

$$\Psi(\boldsymbol{u};\ s) := \int_\Omega \psi_\omega\big(\boldsymbol{u}(\boldsymbol{x};s)\big)\,\mathrm{d}\Omega + \int_\Gamma \psi_\gamma\big(\boldsymbol{u}(\boldsymbol{x};s)\big)\,\mathrm{d}\Gamma, \tag{13}$$

where s is a time-like parameter, analogous to the time parameter in continuum mechanics. In analogy to the configuration change with time in continuum mechanics, the *material* derivative, also termed material design derivative in Wang et al. [Wang and Turteltaub (2015)] to distinguish with the material time derivatives (e.g. in Jao et al. [Jao and Arora (1992); Wang and Kumar (2017)]) of time-dependent problems, of a generic function $f(\boldsymbol{x};s)$ is defined as

$$\dot{f}(\boldsymbol{x},t;s) := \frac{\mathrm{D}f}{\mathrm{D}s} = \frac{\partial f}{\partial s} + \nabla f \boldsymbol{\nu} = f' + \nabla f \boldsymbol{\nu}, \tag{14}$$

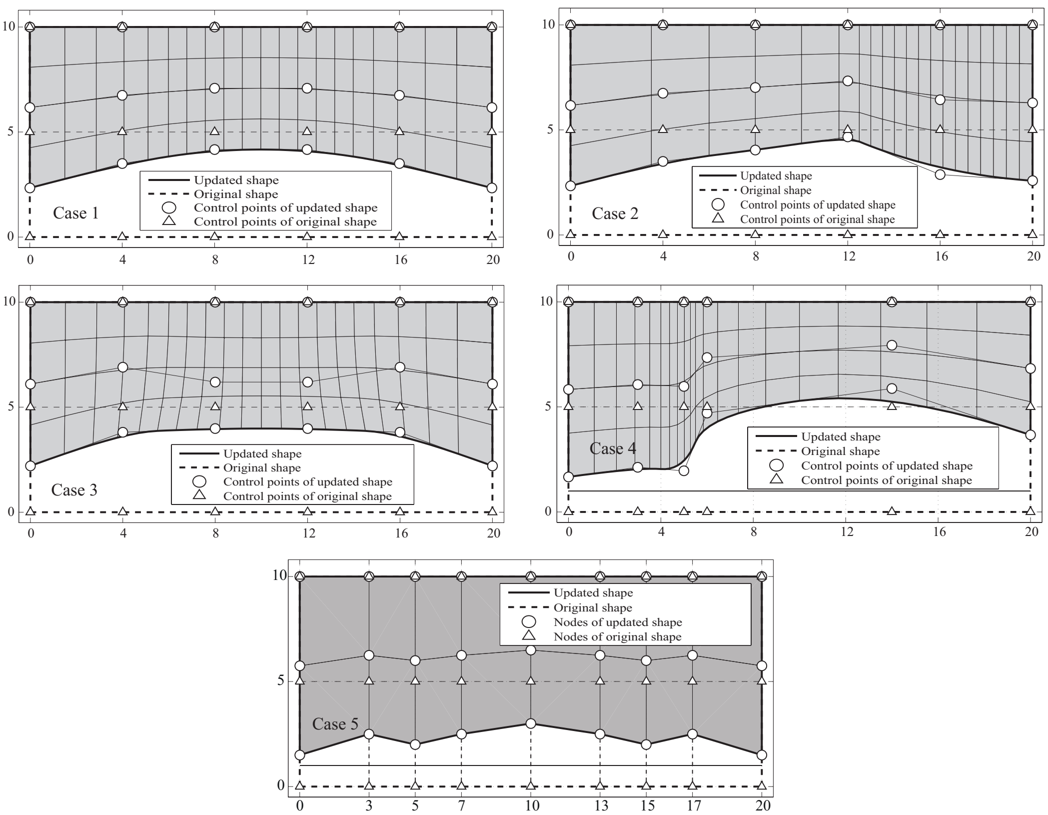


Figure 5: The design parameterization dependency of NURBS-based shape updating scheme [Wang, Abdalla and Turteltaub (2017)]: Case 1 corresponds to a parameterization with uniformly distributed control points, uniform weights $\boldsymbol{w}_{|y=0,5,10} = \{1, 1, 1, 1, 1, 1\}$ and uniform knot spans $\boldsymbol{\xi} = \{0\ 0\ 0\ 0.25\ 0.5\ 0.75\ 1\ 1\ 1\ \}$; Case 2 corresponds to a parameterization with uniformly distributed control points, uniform weights $\boldsymbol{w}_{|y=0,5,10} = \{1, 1, 1, 1, 1, 1\}$ and non-uniform knot spans $\boldsymbol{\xi} = \{0\ 0\ 0\ 0.1\ 0.2\ 0.3\ 1\ 1\ 1\ \}$; Case 3 corresponds to a parameterization with uniformly distributed control points, non-uniform weights $\boldsymbol{w}_{|y=0} = \{1, 1, 0.6, 0.6, 1, 1\}$ and uniform knot spans $\boldsymbol{\xi} = \{0\ 0\ 0\ 0.25\ 0.5\ 0.75\ 1\ 1\ 1\ \}$; Case 4 corresponds to a parameterization with non-uniformly distributed control points, uniform weights $\boldsymbol{w}_{|y=0,5,10} = \{1, 1, 1, 1, 1, 1\}$ and uniform knot spans $\boldsymbol{\xi} = \{0\ 0\ 0\ 0.25\ 0.5\ 0.75\ 1\ 1\ 1\ \}$; Case 5 corresponds to a parameterization with linear basis functions and non-uniformly distributed control points

where $\frac{\mathrm{D}}{\mathrm{D}}(\cdot) = \dot{(\cdot)}$ denotes the material or total derivatives of a quantity, $\frac{\partial f}{\partial s} = f'$ is the *spatial* (design) or partial derivatives and $\boldsymbol{\nu}$ is the design velocity field:

$$\boldsymbol{\nu} := \dot{\boldsymbol{x}} = \sum R_I \frac{\mathrm{D}\boldsymbol{x}_I}{\mathrm{D}d_i}\frac{\mathrm{D}d_i}{\mathrm{D}s} = \boldsymbol{V}\frac{\mathrm{D}d_i}{\mathrm{D}s}. \tag{15}$$

The symbol $\boldsymbol{V}$ in Eq. (15) denotes the perturbation of point $\boldsymbol{x}$ with respect to design variable d_i. Note that in many references, e.g. [Cho and Ha (2009)], for simplicity, the definition of the time-like parameter is absence, and the quantity $\boldsymbol{V}$ defined here is used as 'design velocity'. Essentially, there is no difference between these two terminologies.

The transport relations for domain and boundary integrals are expressed, respectively, as

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_\Omega f \mathrm{d}\Omega = \int_\Omega f' \mathrm{d}\Omega + \int_\Gamma f\boldsymbol{\nu}\cdot\boldsymbol{n}\,\mathrm{d}\Gamma \tag{16}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_\Gamma f \mathrm{d}\Gamma = \int_\Gamma \big(f' + (\nabla f\boldsymbol{v})\boldsymbol{n} - \kappa f\boldsymbol{\nu}\cdot\boldsymbol{n}\big)\;\mathrm{d}\Gamma \tag{17}$$

where κ is the mean curvature and $\boldsymbol{n}$ is the unit outward normal vector.

3.2.2 Direct method

The direct method (DM), also called direct differentiation method (e.g. [Arora (1993); Tortorelli and Michaleris (1994); Cho and Ha (2009)]), is to take the derivatives of the objective function Ψ with respect to the design variables d_i using the chain rule. For both continuum and discrete approaches, the DM requires to solve the system of equations in Eq. (8) to obtain the discrete state variable field $\boldsymbol{U}$, such that the continuum state variable $\boldsymbol{u}$ can be evaluated at any given location.

Using the transport relation defined in Eqs. (16) and (17), the continuum-based derivatives of the objective function reads

$$\dot{\Psi}(\boldsymbol{u};\ s) = \int_\Omega \psi_{\omega,\boldsymbol{u}}\boldsymbol{u}'\,\mathrm{d}\Omega + \int_\Gamma \Big(\psi_\omega\nu_n + \psi_{\gamma,\boldsymbol{u}}\boldsymbol{u}' + (\nabla\psi_\gamma\boldsymbol{\nu})\boldsymbol{n} - \kappa\psi_\gamma\nu_n\Big)\Gamma \tag{18}$$

where $\nu_n = \boldsymbol{\nu}\cdot\boldsymbol{n}$. Then, the transport relations are applied to the weak formulation in Eq. (7) to obtain its derivatives. Note that the spatial derivatives of the test function $\bar{\boldsymbol{u}}$ vanishes using $\langle \boldsymbol{c}(\boldsymbol{u}), \bar{\boldsymbol{u}}'\rangle_\Omega = 0$. Following this, discretization is applied to obtain a discrete system:

$$\boldsymbol{K}\boldsymbol{U}' = \boldsymbol{T} \tag{19}$$

where $\boldsymbol{U}'$ is the vector of the spatial derivative of the discrete state variables and $\boldsymbol{T}$ is a dummy force vector that is a function of $\boldsymbol{u}$, $\nabla\boldsymbol{u}$, $\boldsymbol{\kappa}$, $\boldsymbol{n}$ and $\boldsymbol{V}$. Eventually, solving Eq. (19) and use the result to obtain $\boldsymbol{u}'$ in Eq. (18), the shape sensitivity can be evaluated. It should be noted that Eq. (19) is derived to compute the spatial derivatives $\boldsymbol{u}'$, which is easy for computing the sensitivity of objective function with stress/strain terms. This is because the spatial derivatives $\boldsymbol{u}'$ with respect to design variables, and the gradient $\nabla\boldsymbol{u}$, with respect to spatial locations, commute (e.g., see [Dems and Haftka (1988); Arora (1993); Choi and Kim (2005a); Wang and Poh (2018)]), i.e.

$$\boldsymbol{\epsilon}'(\boldsymbol{u}) = (\nabla \boldsymbol{u})' = \nabla(\boldsymbol{u}') = \boldsymbol{\epsilon}(\boldsymbol{u}'); \tag{20}$$

while the material derivative $\dot{\boldsymbol{u}}$ and the spatial gradient do not commute, i.e.

$$\dot{\boldsymbol{\epsilon}}(\boldsymbol{u}) = \dot{\overline{\nabla \boldsymbol{u}}} = (\nabla \boldsymbol{u})' + \nabla(\nabla \boldsymbol{u})\boldsymbol{v} = \nabla \dot{\boldsymbol{u}} - (\nabla \boldsymbol{u})(\nabla \boldsymbol{v}) = \boldsymbol{\epsilon}(\dot{\boldsymbol{u}}) - (\nabla \boldsymbol{u})(\nabla \boldsymbol{v}). \tag{21}$$

It is possible to derive Eq. (19) in terms of material derivatives, paying attention to cases involving stress/strain terms. The derivation of this approach in the FEM-based regime can be found, e.g. in the works of Arora et al. [Arora (1993); Tortorelli and Michaleris (1994); Korycki (2001); Kuci, Henrotte, Duysinx et al. (2017)].

For the discrete derivations, the derivative of objective $\Psi(\boldsymbol{U})$ with respect to design variable d_i can be written as

$$\dot{\Psi}(\boldsymbol{U}) = \frac{\mathrm{D}\Psi(U)}{\mathrm{D}d_i} = \frac{\mathrm{D}\boldsymbol{\Psi}}{\mathrm{D}\boldsymbol{U}}\frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}d_i} = \Psi_{,\boldsymbol{U}}\dot{\boldsymbol{U}} \tag{22}$$

where $\dot{\boldsymbol{U}}$ can be computed from the derivatives of Eq. (8):

$$\boldsymbol{K}\dot{\boldsymbol{U}} = \dot{\boldsymbol{F}} - \dot{\boldsymbol{K}}\boldsymbol{U}. \tag{23}$$

Note that term $\dot{\boldsymbol{F}}$ vanishes for cases where the force vector is design-independent. Solving this equation for $\dot{\boldsymbol{U}}$ and substitute it into Eq. (22), the shape sensitivity can be obtained. The derivation of this approach can be found in many works, e.g. [Braibant and Fleury (1984, 1985); Hsieh and Arora (1985); Tortorelli (1992); Tortorelli and Michaleris (1994)].

Compared to the continuum-based direct method, the discrete formulations are simpler to derive. However, the discrete version requires the computation of derivative term $\dot{\boldsymbol{K}} = \frac{\mathrm{D}\boldsymbol{K}}{\mathrm{D}d_i}$, which can be problematic for FEM-based shape optimization problems with linear shape functions. Although it is possible to use high order shape functions in traditional FEM, the C^0 continuity between each element may still lead to an inaccurate strain or stress field and hence cannot guarantee the accuracy of the sensitivity analysis. Hence, except for the case where the objective or constraint is defined independently of the BVP in Eq. (6), it is tricky to implement the direct method. Moreover, the direct method requires to solve an extra system of equations, either Eq. (19) or Eq. (23), for each design variables. Apart from simple problems where the inverse of matrix $\boldsymbol{K}$ can be obtained and stored, the overall cost of using direct method can be significantly big. This severely restrict the DSA method to be used in typical FEM-based shape optimization problems, especially in the early years when the computational hardware was much less powerful. Eventually, although the derivations can be found in above-mentioned works, the actual implementations are limited to simple examples, such as Hsieh et al. [Hsieh and Arora (1985); Tortorelli (1992); Tortorelli and Michaleris (1994)].

The IGA utilizes shape functions with a high order of continuity and is able to increase the order using standard p-refinement algorithms, which makes it easy to obtain the second order derivatives of the state variables $\boldsymbol{u}$, and the first order derivative of stiffness $\boldsymbol{K}$.

In addition, IGA is capable of reaching a certain level of accuracy with less degrees of freedoms, hence reducing the dimensions of matrix $\boldsymbol{K}$. This makes it possible to store the inverse of matrix $\boldsymbol{K}$ to significantly reduced the computational costs [Tortorelli and Michaleris (1994)]. Eventually, unlike the limitations of using direct method in a FEM-bashed shape optimization, in Cho et al. the context of IGA, shape sensitivity analysis using direct method is relatively popular. In Cho et al. [Cho and Ha (2009); Liu, Chen, Zhao et al. (2017)], the direct method based on continuum derivation using isogeometric discretization is presented. In Qian et al. [Qian (2010)], a full analytical direct method based on the discrete formulations for isogeometric shape sensitivity analysis is developed by considering both the locations and the weights of control points as design variables. Isogeometric shape optimization using direct method with discrete derivation for sensitivity analysis can also be found in other works such as Li et al. [Li and Qian (2011); Manh, Evgrafov, Gersborg et al. (2011); Qian and Sigmund (2011); Nørtoft and Gravesen (2013); Taheri and Hassani (2014); Lian, Kerfriden and Bordas (2016); Kang and Youn (2016); Ding, Cui and Li (2016); Lian, Kerfriden and Bordas (2017); Lei, Gillot and Jezequel (2018b)]. For the isogeometric topological shape optimization, the shape sensitivity analysis is relatively simple because a background mesh is used, which makes it desirable to use direct differential method, e.g. in the works of Seo et al. [Seo, Kim and Youn (2010a); Cai, Zhang, Zhu et al. (2014); Zhang, Zhao, Gao et al. (2017); Ahn, Koo, Kim et al. (2018)].

3.2.3 Finite difference (FD) approach

The finite difference (FD) or overall finite difference (OFD) approach is to compute the shape sensitivity using the difference quotients:

$$\frac{\mathrm{D}\Psi}{\mathrm{D}d_i} = \frac{\Psi(d_i+\Delta)-\Psi(d_i)}{\Delta} \quad \text{and} \quad \frac{\mathrm{D}\Phi}{\mathrm{D}d_i} = \frac{\Phi(d_i+\Delta)-\Phi(d_i)}{\Delta} \tag{24}$$

where Δ is the perturbation of the design variable d_i. Among all of the shape sensitivity analysis methods, the FD approach is the easiest to implement. However, the FD approach is not efficient for problems with large number of design variables. For a problem with n design variables, the BVP problem has to be solved for $n+1$ times. The computational cost of the FD approach can be significantly high for non-linear and time-dependent large scale problems. Hence, the FD approach is often used to verify the results of other sensitivity analysis methods, e.g. in the works of Cho et al. [Cho and Ha (2009); Wang and Kumar (2017)], or to deal with special cases where other sensitivity analyses are not easy to implement, e.g. in Park et al. [Park, Seo, Sigmund et al. (2013)], where it is used to compute the shape sensitivity of intersecting knots of trimmed surfaces.

It should be noted that the accuracy of FD approach depends on the perturbation size. Thus, a proper convergence analysis needs to be performed with respect to the perturbation size for each design variable.

3.2.4 Semi-analytical (SA) approaches

An alternative simple and efficient approach is the semi-analytical (SA) method. For each design variable, a perturbation Δ is applied to obtain the the perturbations of the loading vector $\boldsymbol{F}$ and stiffness matrix $\boldsymbol{K}$, such that by solving $\boldsymbol{K}^{-1}$ just once, the full derivatives of $\boldsymbol{U}$ in Eq. (22) can be obtained:

$$\dot{\boldsymbol{U}} = \boldsymbol{K}^{-1}\Big(\frac{\Delta \boldsymbol{F}}{\Delta d_i} - \frac{\Delta \boldsymbol{K}}{\Delta d_i}\boldsymbol{U}\Big). \tag{25}$$

Then, solving Eq. (25) and substituting it into Eq. (18) with proper derivation using Eqs. 20 and 21, e.g. in Wang et al. [Wang and Poh (2018)], or into Eq. (22) directly, e.g. in Haftka et al. [Haftka (1981)], the sensitivity analysis can be computed. Isogeometric shape optimization using SA method can be found in the works of Seo et al. [Seo, Kim and Youn (2010a); Park, Seo, Sigmund et al. (2013); Kiendl, Schmidt, WWüchner et al. (2014); Wang and Poh (2018); Hosseini, Moetakef-Imani, Hadidi-Moud et al. (2018)].

It should be noted that for the cases where the objective function involves stress/strain variables, the problem can be slightly more complicated. First, Eq. (22) needs to be modified into $\frac{\mathrm{D}\Psi}{\mathrm{D}x_i^j} = \Psi_{,\boldsymbol{U}}\dot{\boldsymbol{U}} + \Psi_{,\boldsymbol{\epsilon}}\dot{\boldsymbol{\epsilon}}$. Then, the material derivative of $\boldsymbol{\epsilon}$ can be obtained from $\dot{\boldsymbol{u}}$ using Eq. (21):

$$\dot{\boldsymbol{\epsilon}}(\boldsymbol{u}) = \boldsymbol{\epsilon}(\dot{\boldsymbol{u}}) - (\nabla \boldsymbol{u})(\nabla \boldsymbol{v}), \tag{26}$$

where $\dot{\boldsymbol{u}}$ can be computed from the discrete vector $\dot{\boldsymbol{U}}$, and the gradient of design velocity $\nabla \boldsymbol{v}$ can be computed either analytically or based on perturbations of the design variables. This is addressed in the work of Wang et al. [Wang and Poh (2018)].

Despite the fact that SA approach is computationally efficient and popular, the accuracy of the sensitivity analysis can be relatively unsatisfactory for some special cases [Barthelemy and Haftka (1990)].

3.2.5 Discrete adjoint method

The shape sensitivity analysis using adjoint method is performed by introducing an adjoint variable field to replace the derivatives of the implicitly-dependent terms. For problems with n objectives and constraints that are dependent on structural responses, it requires $n+1$ times of analysis. Hence, this approach is efficient for problems with large number of design variables and small number of constraints [Haftka (1981)].

The derivations based on the discrete fields $\boldsymbol{K}$, $\boldsymbol{U}$ and $\boldsymbol{F}$ using the form of $\Psi(\boldsymbol{U})$ for the objective are called discrete adjoint method, i.e the derivations are done after the system has been discretized. Firstly, an augmented function is introduced based on the discrete fields:

$$\tilde{\Psi} = \Psi = \Psi + \boldsymbol{U}^{*\mathrm{T}}(-\boldsymbol{K}\boldsymbol{U} + \boldsymbol{F}) \tag{27}$$

where $\boldsymbol{U}^*$ is the so-called adjoint variable field. Note that $\dot{\boldsymbol{U}}^*(\boldsymbol{K}\boldsymbol{U} - \boldsymbol{F}) = 0$, the full derivative of above augmented function can be derived as

$$\dot{\tilde{\Psi}} = \Psi_{,\boldsymbol{U}}\dot{\boldsymbol{U}} + \boldsymbol{U}^{*\mathrm{T}}(-\dot{\boldsymbol{K}}\boldsymbol{U} - \boldsymbol{K}\dot{\boldsymbol{U}} + \dot{\boldsymbol{F}}) = (\Psi_{,\boldsymbol{U}} - \boldsymbol{U}^{*\mathrm{T}}\boldsymbol{K})\dot{\boldsymbol{U}} + \dot{\boldsymbol{F}} - \boldsymbol{U}^{*\mathrm{T}}\dot{\boldsymbol{K}}\boldsymbol{U} \tag{28}$$

Introducing an adjoint problem $\boldsymbol{K}\boldsymbol{U}^* = \Psi_{,\boldsymbol{U}}$ and solve it, the shape sensitivity can be computed using

$$\dot{\tilde{\Psi}} = \dot{\Psi} = \dot{\boldsymbol{F}} - \boldsymbol{U}^{*\mathrm{T}}\dot{\boldsymbol{K}}\boldsymbol{U}. \tag{29}$$

Despite the computing effect to obtain the full derivative of the stiffness matrix $\dot{\boldsymbol{K}}$, the discrete adjoint method is relatively popular because the derivations are simple and straightforward. In the pioneering work of Wall et al. [Wall, Frenzel and Cyron (2008)], the derivations and some numerical aspects of the discrete adjoint method are presented in the framework of IGA. Following this, this method is also used in the context of isogeometric design optimization to derive the shape and size sensitivity for curved beams to minimize the structure compliance in Nagy et al. [Nagy, Abdalla and Gürdal (2010a)] and maximize the fundamental frequency in Nagy et al. [Nagy, Abdalla and Gürdal (2011)], for topological shape optimization in Seo et al. [Seo, Kim and Youn (2010b)], for composite shells to consider the buckling load and structural stiffness in Nagy et al. [Nagy, IJsselmuiden and Abdalla (2013)], for geometrically exact Timoshenko beams in Ferede et al. [Ferede, Abdalla and van Bussel (2017)], for shells with multiresolution subdivision surfaces in Cirak et al. [Cirak and Bandara (2015); Bandara and Cirak (2018)] and for airfoil design in Wang et al. [Wang, Yu, Wang et al. (2018)]. Details with systematic explanations for implementations can be found in Nagy et al. [Nagy (2011)].

3.2.6 Continuum adjoint method

Another type of adjoint method, termed continuum adjoint method, e.g. in Choi et al. [Choi and Kim (2005a)] or continuous adjoint method, e.g. in Papadimitriou et al. [Papadimitriou and Giannakoglou (2007)], derives the sensitivity based on the continuum formulations. The derivations of the adjoint problem are based on the continuum variable fields using the BVP equations in Eq. (6) and $\Psi(\boldsymbol{u})$ to form an augmented function:

$$\tilde{\Psi} := \Psi + \langle \boldsymbol{c}(\boldsymbol{u}), \boldsymbol{u}^* \rangle_\Omega \tag{30}$$

where $\langle \boldsymbol{c}(\boldsymbol{u}), \boldsymbol{u}^* \rangle_\Omega$ nests the BVP problem defined in Eq. (6) and $\boldsymbol{u}^*$ is the so-called adjoint state variables.

The derivatives of the augmented function with respect to the time-like parameter s can be derived using the transport relations defined in Eqs. (16) and (17). To eliminate the implicitly dependent terms such as $\boldsymbol{u}'$ and $\boldsymbol{t}'$, an adjoint BVP is introduced as

$$\begin{aligned} \mathbb{C}\nabla^2\boldsymbol{u}^* + \boldsymbol{f}^* = 0 \quad &\text{with} \quad \boldsymbol{f}^* = \psi_{\omega,\boldsymbol{u}} \qquad \text{in} \quad \Omega; \\ \boldsymbol{u}^* = \hat{\boldsymbol{u}}^* \quad &\text{with} \quad \hat{\boldsymbol{u}}^* = -\psi_{\gamma,\boldsymbol{t}} \qquad \text{on} \quad \Gamma_u; \\ \boldsymbol{t}^* = (\mathbb{C}\nabla\boldsymbol{u}^*)^{\mathrm{T}}\boldsymbol{n} = \hat{\boldsymbol{t}}^* \quad &\text{with} \quad \hat{\boldsymbol{t}}^* = \psi_{\gamma,\boldsymbol{u}} \qquad \text{on} \quad \Gamma_t \end{aligned} \tag{31}$$

where variable fields with a star symbol are variables for the adjoint BVP. Eventually, the shape sensitivity can be obtained using terms without involving implicitly dependent derivatives. For problems with design-independent boundary conditions, it is possible to express the shape sensitivity as

$$\dot{\Psi} = \int_{\Gamma} \boldsymbol{g} \cdot \boldsymbol{\nu} \mathrm{d}\Gamma, \tag{32}$$

where $\boldsymbol{g}$ is local shape derivative that includes the terms with the primary and adjoint variables (see e.g. [Wang and Turteltaub (2015)]). Using the chain rule to get $\frac{\mathrm{D}\Psi}{\mathrm{D}s} = \frac{\mathrm{D}\Psi}{\mathrm{D}d_i}\frac{\mathrm{D}d_i}{\mathrm{D}s}$ and $\boldsymbol{\nu} = \boldsymbol{V}\frac{\mathrm{D}d_i}{\mathrm{D}s}$, the sensitivity with respect to each design variable $\frac{\mathrm{D}\Psi}{\mathrm{D}d_i}$ can be obtained. This framework can be easily applied to multiple levels discretization for design and analysis, where the terms in $\boldsymbol{g}$ are computed in the analysis discretization space with refined mesh for obtaining accurate responses, while the design velocity is implemented in the design discretization space with coarse mesh for reducing the number of design variables.

Compared to the discrete approach, the continuum adjoint method involves a relatively more complicated derivation, leading to a tedious numerical implementation. Against its complexity, various interpretations and derivative approaches have been proposed in literature, e.g. derivations based on a mutual Hu-Washizu energy principle in Haber et al. [Haber (1987)], Lagrange multipliers interpretations in Belegundu et al. [Belegundu (1985); Tortorelli, Haber and Lu (1989)], energy bilinear forms in Komkov et al. [Komkov, Choi and Haug (1986); Choi and Haug (1983); Choi (1987)], reciprocity theorems in Tortorelli et al. [Tortorelli, Subramani, Lu et al. (1991); Dems and Mroz (1984)], a mixed variational formulation in [Rodrigues (1988)] and a mixed Lagrange multiplier approach in Wang et al. [Wang and Turteltaub (2015)].

In terms of numerical implementation, the continuum adjoint method is tricky for FEM-based shape optimization. Apart from a significant account of boundary integrations where the structural response may not be accurately computed, the boundary geometric parameters, such as the normal vector and the curvature, cannot be accurately preserved. In the context of isogeometric shape optimization, the exact geometric representation enables the computation of the boundary geometric parameters easily and seamlessly, hence, providing a natural platform to implement the continuum adjoint method. This has been intensively studied by Seonho Cho and his co-authors in Cho et al. [Cho and Ha (2009)] for general discussions, in Ha et al. [Ha, Choi and Cho (2010)] using T-spline based isogeometric method, in Ahn et al. [Ahn and Cho (2010)] for level-set-based topology optimization of heat conduction problems, in Koo et al. [Koo, Yoon and Cho (2013)] coping with Kronecker delta property, in Yoon et al. [Yoon, Ha and Cho (2013); Yoon, Choi and Cho (2015)] for shape design optimization of heat conduction problems, in Choi et al. [Choi and Cho (2014)] for stress intensity factors of curved crack problems, in Lee et al. [Lee and Cho (2015)] for optimizing built-up structures, in Lee et al.[Lee, Lee and Cho (2016)] for ferromagnetic materials in magnetic actuators, in Yoon et al. [Yoon and Cho (2016)] for boundary integral equations, in Choi et al. [Choi, Yoon and Cho (2016)] for designing curved Kirchhoff beams with finite deformations, in Lee et al. [Lee, Yoon and Cho (2017)] for topological shape optimization using dual evolution with boundary integral equation and level sets, in Choi et al. [Choi and Cho (2018a)] for designing lattice structures embedded on curve surfaces, in Choi et al. [Choi and Cho (2018b)] for using curved beams to design auxetic structures and in Ahn et al. [Ahn, Choi and Cho (2018); Ahn, Koo, Kim et al. (2018)] for designing nanoscale structures. Wang and his co-authors implemented the continuum adjoint method in Wang et al. [Wang and Turteltaub (2015)] for quasi-static problems considering discontinuities in the objective functions, in Wang et al. [Wang, Turteltaub and Abdalla (2017); Wang and Kumar (2017)] for transient heat conduction problems and in Wang et al. [Wang, Poh, Dirrenberger et al. (2017)] to design auxetic structures using numerical homogenization method. The continuum adjoint method is also used in Azegami et al. [Azegami, Fukumoto and Aoyama (2013)] enforced with $H1$ gradient method, Ha [Ha (2015)] for curvilinear coordinate systems, and Cirak et al. [Cirak and Bandara (2015)] for structural compliance and boundary flux optimizations. The equivalence of the discrete and continuum adjoint methods is discussed with a simple selft-adjoint example under the framework of IGA in Fusseder et al. [Fußeder, Simeon and Vuong (2015)].

3.3 Gradient-free optimization methods

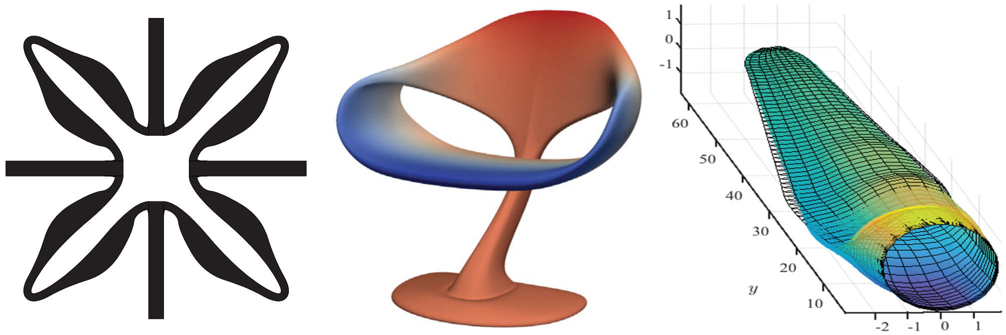
The gradient-based optimization methods are efficient to find optimal solutions, which may, however, lead to local optimal solutions that depends on start point. Moreover, despite the great improvements of the computation techniques in the past decades, sensitivity analysis for complex problems is still remarkably challenging, e.g. designing complex geometry generated with Boolean operations in Wang et al. [Wang, Zhang, Yang et al. (2012)], problems with large elasto-plastic deformations in Zhu et al. [Zhu, Wang and Poh (2018)], nonlinear problems in Meng et al. [Meng, Breitkopf, Raghavan et al. (2015); Hou, Sapanathan, Du-mon et al. (2018)], complicated reliability-based design problems in [Dubourg, Sudret and Bourinet (2011)] and uncertainty in crashworthiness optimization in Qiu et al. [Qiu, Gao, Fang et al. (2018)]. Eventually, gradient-free optimization methods become essentially important to diminish these obstacles [Queipo, Haftka, Shyy et al. (2005); Forrester and Keane (2009)]. Particularly, the gradient-free optimization methods are easy and straightforward to im-plement, which is attractive for engineering applications. The IGA-based gradient-free optimization methods can also benefit from IGA with its seamless geometry parameteri-zation, possibly higher accuracy of structure responses with the same degrees of freedom, and more efficient computation for the same level of accuracy, especially for the structures with complicated curved features. Representative works of such studies can be found in Benzaken et al. [Benzaken, Herrema, Hsu et al. (2017)] and Herrema et al. [Herrema, Wiese, Darling et al. (2017)], where a surrogate model technique and a generalized pattern search algorithm, respectively, are adopted to optimize the wind turbine blades (see Fig. 6(c)) with complex geometries, Kostas et al. [Kostas, Ginnis, Politis et al. (2015)] where evolutionary algorithms are utilized to optimize ship-hull shape, Herath et al. [Herath, Natarajan, Prusty et al. (2015); Kostas, Ginnis, Politis et al. (2017); Kostas, Fyrillas, Politis et al. (2018)] with genetic algorithms, Lieu et al. [Lieu, Lee, Lee et al. (2018); Lieu and Lee (2019)] with an adaptive hybrid evolutionary firefly algorithm, Wang et al. [Wang, Yu, Shao et al.(2018)] with a chaotic particle swarm optimization method, and Zhang et al. [Zhang, Li, Shen et al. (2019)] with multi-island genetic algorithm and adaptive simulated annealing methods.

3.4 Some remarks and discussions in isogeometric shape optimization

The concept of isogeometric analysis raised renewed interests in developing structural shape optimization.Together with the advancement of computational technologies, the increasing interest in isogeometric shape optimization has sparked the development of some fundamental theories. To deal with stress constraint problems in isogeometric design framework, Nagy et al. [Nagy, Abdalla and Gürdal (2010b)] presented a novel variational formulation using isogeometric basis functions to represent Lagrange multipliers. For geometry parameterization, the NURBS-based CAD geometry description in the past often develops diversely aiming simply to improve the geometry modeling. The development of isogeometric design and analysis have directed the attention towards an CAD parameterization concept that is analysis-suitable, which further enhances the design ability, as discussed in Section 3.1. With these efforts, highly integrated platforms with design and analysis for large scale and geometrically complicated structures (see the concept presented in e.g. [Herrema, Wiese, Darling et al. (2017)]) is becoming more practical and achievable. For structural shape sensitivity analysis, IGA has significantly reduced the complexity of implementing the direct differentiation method, leading it to become a popular method, as presented in Section 3.2.2. Meanwhile, in the context of isogeometric shape optimization, implementing continuum adjoint method is easier than traditional FEM-based shape optimization, which makes it possible to adopt it for more practical structure designs, e.g. in Wang et al. [Wang, Poh, Dirrenberger et al. (2017); Choi and Cho (2018a)] for auxetic structures with complicated curved features. In Wang et al. [Wang and Turteltaub (2015)], the shape sensitivity analysis considering discontinuities in the objective functions is derived using continuum adjoint method. In Choi et al. [Choi, Yoon and Cho (2016)], the continuum adjoint method is derived for finite deformations of curved beams. It is also notable that with the development of isogeometric analysis, shape optimization has been applied to structures with increasingly complicated geometries, e.g. the structures depicted in Fig. 6, which can be difficult for a FEM-based shape optimization.

4 Isogeometric topology optimization

Isogeometric topology optimization (ITO) using IGA to replace FEM in the topology optimization was first proposed by Seo et al. [Seo, Kim and Youn (2010a)], where trimmed surfaces were used to accurately represent the topology of the optimized structure, since the design variables are the coordinates of control points. However, this method needs to deal with inner front creation and inner front merging, which is complicated and thus increases the computational time significantly when the number of trim curves is large. Later, researchers begin to devote their time to the ITO research and a series of results are obtained based on different types of topology optimization framework. In this section, we will review the ITO research in the past ten years according to the types of ITO design variables.



(a) (b) (c)

Figure 6: Isogeometric shape optimization for geometrically complicated structures: (a) petal-shaped auxetic structure design in Wang et al. [Wang, Poh, Dirrenberger et al. (2017)]; (b) Curved 3D chair shape design in Lian et al. [Lian, Kerfriden and Bordas (2017)]; and (c) Curved wind turbine blade shape design in Benzaken et al. [Benzaken, Herrema, Hsu et al. (2017)]

4.1 Density-based isogeometric topology optimization

Density-based topology optimization is a category of topology optimizations whose design variables are comprised of a certain kind of density such as element density and nodal density.

For a classical topology optimization problem, i.e. the minimum compliance problem, the mathematical formulation of the optimization problem can be written as

$$\begin{aligned} \text{Min} : & \; c(\boldsymbol{d}) = \boldsymbol{U}^T \boldsymbol{K} \boldsymbol{U}, \\ \text{Subject to} : & \; \boldsymbol{K}\boldsymbol{U} = \boldsymbol{F}, \\ & \; V(\boldsymbol{d}) \leq V_m, \\ & \; \boldsymbol{d}_{\min} \leq \boldsymbol{d} \leq \boldsymbol{d}_{\max} \end{aligned} \tag{33}$$

where c is the compliance, $\boldsymbol{K}$ is the global stiffness matrix, $\boldsymbol{U}$ and $\boldsymbol{F}$ are the displacement and force vectors. $V(\boldsymbol{d})$ and V_m are the material volume and volume constraint, and vector $\boldsymbol{d}$ consists of design variables, e.g. element densities and control point densities.

A well-known density-based topology optimization scheme is the modified SIMP scheme [Andreassen, Clausen, Schevenels et al. (2011)], where for each element e, a density x_e is assigned such that the Young's modulus E_e can be identified proportionally

$$E_e(x_e) = E_{\min} + x_e^p(E_0 - E_{\min}), \quad x_e \in [0, 1], \tag{34}$$

where E_0 is the Young's modulus of solid material, $E_{\min}$ is a small value to prevent the stiffness matrix from becoming singular, and p is a penalization factor (typically $p = 3$) used to ensure black-and-white solutions.

To avoid a checkerboard pattern, filtering techniques such as the sensitivity filter and the density filter are used in topology optimizations [Andreassen, Clausen, Schevenels et al. (2011)]. The sensitivity filter modifies the sensitivities $\partial c/\partial x_e$ as

$$\widetilde{\frac{\partial c}{\partial x_e}} = \frac{1}{\max(\gamma, x_e)\sum_{i\in N_e} H_{ei}} \sum_{i\in N_e} H_{ei} x_i \frac{\partial c}{\partial x_i}, \tag{35}$$

where $\gamma(= 10^{-3})$ is a small positive number introduced to avoid division by zero, N_e is the set of elements i whose distance $\triangle(e, i)$ to element e is smaller than the filter radius $r_{\min}$ and H_{ei} is a weight defined as

$$H_{ei} = \max(0, r_{\min} - \triangle(e, i)), \tag{36}$$

and the density filter is

$$\widetilde{x_e} = \frac{1}{\sum_{i\in N_e} H_{ei}} \sum_{i\in N_e} H_{ei} x_i. \tag{37}$$

In IGA, a variable value can be represented by the control point values as Eq. (3), and thereby the density functions associated to the control points can be utilized as design parameters. Kumar et al. [Kumar and Parthasarathy (2011)] presented a topology optimization framework using B-spline finite elements where control point densities were used as design variables, and the B-spline basis functions had a smoothing effect similar to density filtering schemes for eliminating mesh dependence. Hassani et al. [Hassani, Khanzadi and Tavakkoli (2012)] presented a similar NURBS-based ITO and pointed out that the ITO was able to obtain the correct topology without checkerboard even with a coarse net of control points. However, in such schemes, the filtering region is dependent on the element order since the higher order B-spline basis function has a larger support region, e.g. the 2D support region of a B-spline control point with order $p = q = 3$ is larger that with order $p = q = 2$ as shown in Fig. 7. Therefore, when a given design domain is refined with different element sizes, the filtering region changes, which will result in mesh dependence as the MBB example in Fig. 8 [Qian (2013)]. To solve the mesh dependence problem, a weighted penalty on the square of the density gradient was added to the objective function as follows [Kumar and Parthasarathy (2011)]

$$\min : \prod = c(\phi) + w \int_V (\bigtriangledown \phi)^2 dV, \tag{38}$$

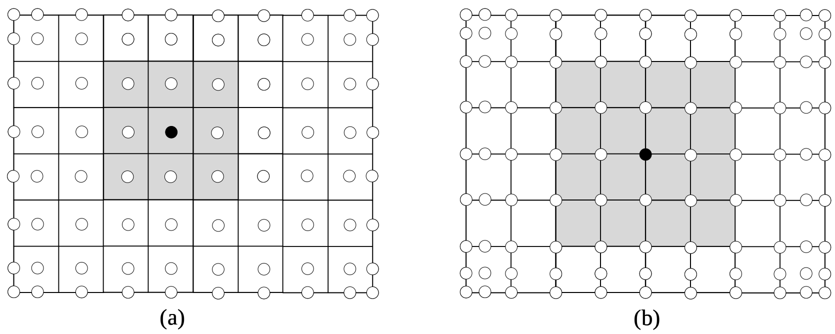


Figure 7: An illustration of the B-spline basis support region corresponding to the black solid control point: (a) $p = q = 2$ and (b) $p = q = 3$

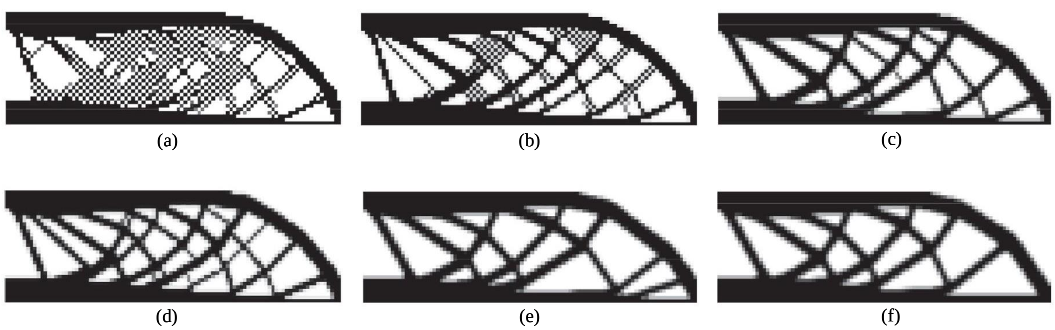


Figure 8: Mesh dependence caused by the B-spine orders: (a) $p = q = 2$, (b) $p = q = 4$, (c) $p = q = 5$, (d) $p = q = 6$, (e) $p = q = 7$, (f) $p = q = 9$ [Qian (2013)]

where ϕ is the control point density, and ω is the weighting factor. However, the weight ω which has an important influence on the topology of optimized result, needs to be manually defined. In Qian [Qian (2013)], the design domain was embedded into a domain with density field presented by tensor-product B-splines, and used different B-spline orders to change the filtering region. However, the analysis was implemented by FEM, and the topology optimization therewith cannot be regarded as an ITO.

Note that the conventional filtering techniques as Eqs. (35) and (37) can be used in the ITO by setting a filter radius. Although the basis functions are able to avoid checkerboards,

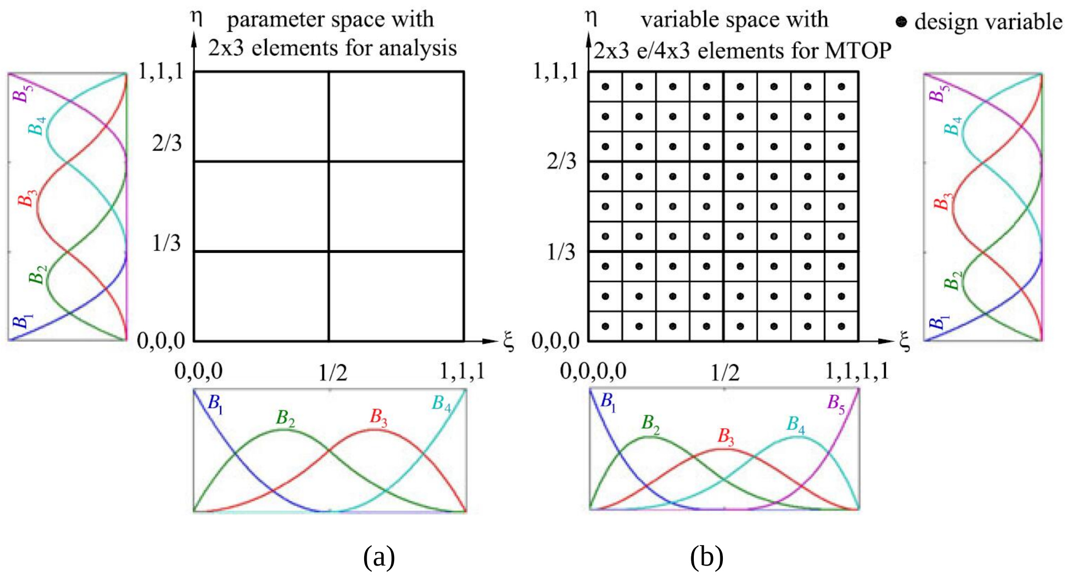


Figure 9: The parameter space and variable space of MTOP: (a) Discretization in parameter space and the basis functions for analysis, and (b) Discretization in variable space and the basis functions for MTOP [Lieu and Lee (2017b)]

the filtering techniques are still recommended to avoid the mesh dependence. In Lieu et al. [Lieu and Lee (2017b)], Lieu and Lee proposed a multiresolution topology optimization (MTOP) using IGA, where the variable space is separated from the parameter space, as depicted in Fig. 9. The density value at the center of each subelement (i.e. the element of variable space) is considered as a design variable by assuming uniformity within each subelement. In the MTOP using IGA, due to the high accuracy of IGA, the analysis space can be refined with fewer elements to reduce the computational cost, and the design space is refined with more elements ensuring a sufficient number of design variables to generate a correct structural topology with a high resolution. Note that, interestingly, this strategy is contrary to the one used in isogeometric shape optimization where coarse mesh is used to reduce the design variables while refined mesh is used to guarantee the accuracy of the analysis (e.g. in Nagy et al. [Nagy, Abdalla and Gürdal (2010a); Wang and Turteltaub (2015)]). In addition, the sensitivity filter (Eq. (35)) was used to prevent the mesh dependence, which demonstrated that the filtering techniques are also effective for the ITO.

Advancing for the single-material topology optimization, the IGA has been also used in multi-material topology optimization. Lin et al. [Lin, Rayasam and Subbarayan (2015)] proposed a strategy to simultaneously insert inclusions or holes as well as a redistribution of material to obtain the optimal topology, where NURBS is utilized to interpolate the geometry, material and physical field. Taheri et al. [Taheri and Suresh (2017)] proposed an ITO approach for multi-material and functionally graded structures. For the multi-material problem, the elasticity matrix $\boldsymbol{D}$ is represented as [Stegmann and Lund (2005)]

$$\boldsymbol{D} = x_0^p \sum_{i=1}^{m} \left[\prod_{j=1}^{i-1} [1 - (x_{i \neq m})^p] x_j^p \right] \boldsymbol{D}_i, \tag{39}$$

where $x \in [0, 1]$ is the design variable, p is the penalization factor which is used to impose the intermediate density approaching 0 (void) or 1 (solid), m is the number of materials, and $\boldsymbol{D}_i$ represents the elasticity matrix of material i. For the functionally graded structure, the average of the lower and upper Hashin-Shtrikman bounds [Hashin and Shtrikman (1962)] is used as the material model

$$\boldsymbol{D} = x_0^p \left[\frac{1}{2} (\boldsymbol{D}_{HS}^{-} + \boldsymbol{D}_{HS}^{+}) \right], \tag{40}$$

where $\boldsymbol{D}_{HS}^{-}$ and $\boldsymbol{D}_{HS}^{+}$ are the lower and upper bounds on the elasticity matrix, which are obtained based on lower and upper bounds of Young's modulus

$$E_{HS}^{-} = \frac{(2 + x)E_1 + (1 - x)E_2}{2(1 - x)E_1 + (1 + 2x)E_2} E_2, \tag{41}$$

$$E_{HS}^{+} = \frac{xE_1 + (3 - x)E_2}{(3 - 2x)E_1 + 2xE_2} E_1, \tag{42}$$

where E_1 is Young's modulus of harder material.

Lieu et al. [Lieu and Lee (2017a)] extended the multi-resolution approach (MTOP) [Lieu and Lee (2017b)] to the multi-material topology optimization, in which the multi-material model is written as

$$E(x_i) = \sum_{i=1}^{m} (x_i)^p E_i, \tag{43}$$

where E_i is the Young's modulus of the ith material and the penalization factor p is usually set to 3. A bride structure model is shown in Fig. 9, and the optimized topologies shows that the MTOP within the IGA framework can achieve higher resolution designs with a lower computational cost.

Recently, Wang et al. [Wang, Xu and Pasini (2017)] proposed an ITO method for graded lattice materials. In this work, the mechanical properties were expressed as a function of relative density of the unit cell, which avoided the iterative calculations during ITO. The optimization results of the cantilever beam are shown in Fig. 11, where we can find that topologies that differ between lattice cells influence the material distribution and the optimal solutions. Moreover, using fitting functions to evaluate the derivatives of the objective provides a higher accuracy than the SIMP approach.

As mentioned in Section 2.1, one of the most important advantages of IGA is the high continuity between elements. Therefore, the IGA is especially suitable for topology optimization with stress constraint, since the stress is discontinuous between elements in FEM. Liu et al. [Liu, Yang, Hao et al. (2018)] presented a stress-constrained ITO of thin bending plates, where two stability transformation methods (STMs) were proposed to achieve the stable iterations. Due to the high continuity of IGA, the ITO can meet the requirement of

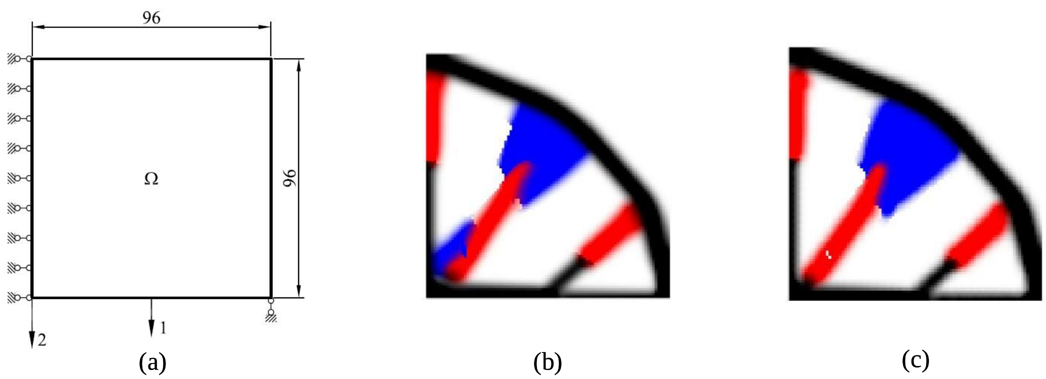


Figure 10: Optimized topologies of the bride structure: (a) Design domain and boundary condition, (b) 96×96 elements without subelements, $c = 49.5033$, and (c) 32×32 elements with 3×3 subelements in each element, $c = 47.8111$ [Lieu and Lee (2017b)]

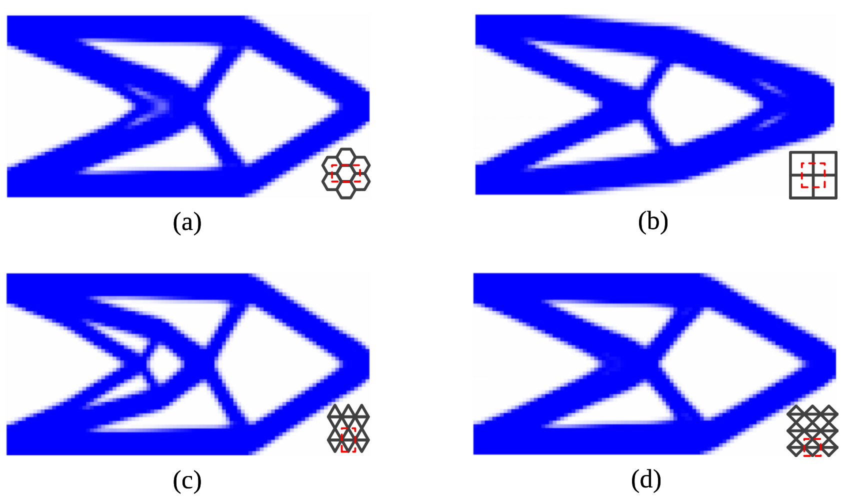


Figure 11: The optimization results of the cantilever beam: (a) Hexagon lattice, (b) Square lattice, (c) Acute triangle lattice and (d) Obtuse triangle lattice [Wang, Xu and Pasini (2017)]

C1 continuity for the Kirchhoff plate formulations, and the example results indicate that the ITO shows superior performance for both accuracy and efficiency.

4.2 Level-set-based isogeometric topology optimization

The level set method (LSM) was first proposed by Osher et al. [Osher and Sethian (1988)] to track the evolution of free surfaces in computational fluid dynamics, and has proven to be effective in representing complicated boundaries in a wide variety of applications. In the LSM, the structural boundary $\partial\Omega$ is implicitly embedded as the zero level set of a one-dimensional-higher level set function (LSF) $\Phi(\boldsymbol{x},\, t)$, where $\boldsymbol{x}$ is the coordinate vector of a point and t is a pseudo time. The level-set-based topology optimization handles the complex topological changes by the motions of implicit boundaries represented by the LSF $\Phi(\boldsymbol{x},\, t)$ which is defined over a reference domain $D \subset R^d$ $(d = 2\ or\ 3)$. The level set representation of the structure is defined as

$$\begin{cases} \Phi(\boldsymbol{x},t) > 0 & \forall \boldsymbol{x} \in \Omega \backslash \partial\Omega & \text{(inside)}, \\ \Phi(\boldsymbol{x},t) = 0 & \forall \boldsymbol{x} \in \partial\Omega \bigcap D & \text{(boundary)}, \\ \Phi(\boldsymbol{x},t) < 0 & \forall \boldsymbol{x} \in D \backslash \Omega & \text{(outside)} \end{cases} \tag{44}$$

Differentiating the LSF $\Phi(\boldsymbol{x},t)$ with respect to the pseudo-time t, the Hamilton-Jacobi equation is obtained as [Wang, Wang and Guo (2003)]

$$\frac{\partial\Phi(\boldsymbol{x},t)}{\partial t} - v_n|\bigtriangledown\Phi| = 0,\ \ \Phi(\boldsymbol{x},0) = \ \Phi_0(\boldsymbol{x}), \tag{45}$$

where the normal velocity v_n is

$$v_n = -\frac{d\boldsymbol{x}}{dt}\cdot\frac{\nabla\Phi}{|\nabla\Phi|}. \tag{46}$$

The Hamilton-Jacobi equation Eq. (45) defines an initial value problem for the time dependent function Φ. In the optimization process, v_n is the movement of a point on a surface driven by the objective function of the optimization, and the optimal structural boundary is obtained by solving Eq. (45).

A level-set-based topology optimization for the minimum compliance design problem may be defined mathematically as [Wang and Benson (2016a)]

$$\begin{aligned} \text{Min} : c(\Phi) &= \int_\Omega \boldsymbol{\varepsilon}^T \boldsymbol{E}\boldsymbol{\varepsilon} H(\Phi) d\Omega, \\ \text{Subject to} : V(\Omega) &= \int_\Omega H(\Phi) d\Omega \le V_{\max}, \end{aligned} \tag{47}$$

where $H(\Phi)$ is the Heaviside function defined as

$$H(\Phi) = \begin{cases} 1, & \text{if } \Phi \ge 0, \\ 0, & \text{if } \Phi < 0, \end{cases} \tag{48}$$

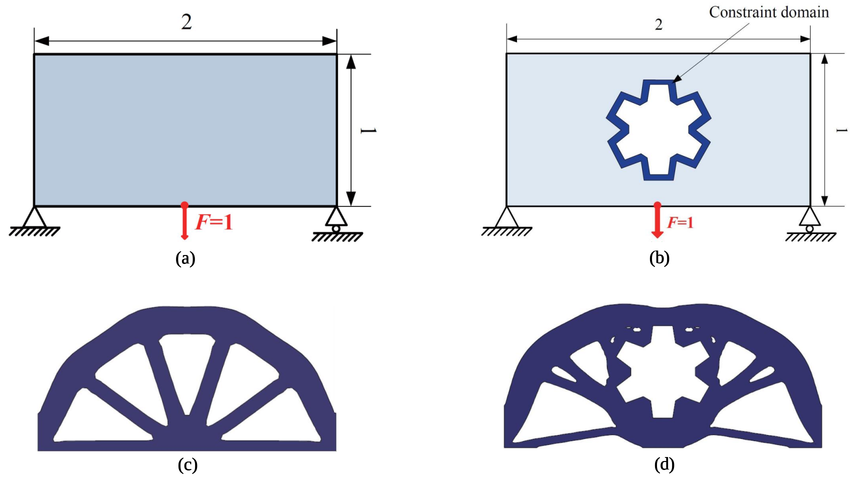


Figure 12: The level-set-based ITO of the Michell type structures: (a) Design domain without geometric constraint, (b) Design domain with geometric constraint, (c) Optimization result without geometric constraint and (d) Optimization result with geometric constraint [Wang and Benson (2016b,a)]

c is the compliance, $\boldsymbol{E}$ is the elasticity matrix and $\boldsymbol{\varepsilon}$ is the strain. The inequality $V(\Omega) \leq V_{\max}$ defines the volume constraint.

To avoid solving the Hamilton-Jacobi equations, Wang et al. [Wang and Benson (2016b)] used the NURBS basis functions to represent the LSF in a parameterized mode as

$$\Phi(\boldsymbol{x}, t) = \boldsymbol{N}(\boldsymbol{x})\boldsymbol{\phi}(t) = \sum_i N_i(\boldsymbol{x})\phi_i(t), \tag{49}$$

where $\phi_i(t)$ is the ith expansion coefficient and $N_i(\boldsymbol{x})$ is the corresponding basis function. After this parameterization, the LSF associated with both space and time is divided into the spatial terms $N_i(\boldsymbol{x})$ and the time dependent terms $\phi_i(t)$, and only the latter are updated during the optimization procedure.

The same NURBS basis functions were also used to evaluate the objective function in the level-set-based isogeometric topology optimization (ITO) proposed by Wang and Benson [Wang and Benson (2016b)]. They later extended the level-set-based ITO to solve geometrically constrained problems using trimmed elements [Wang and Benson (2016a)]. According to the work of Wang and Benson, using IGA to replace FEM can accelerate the optimization more than 3 times when quadratic elements are utilized and make the optimization iterations easier to converge. Fig. 12 shows the optimization results for the geometrically constrained and the classical Michell type problems.

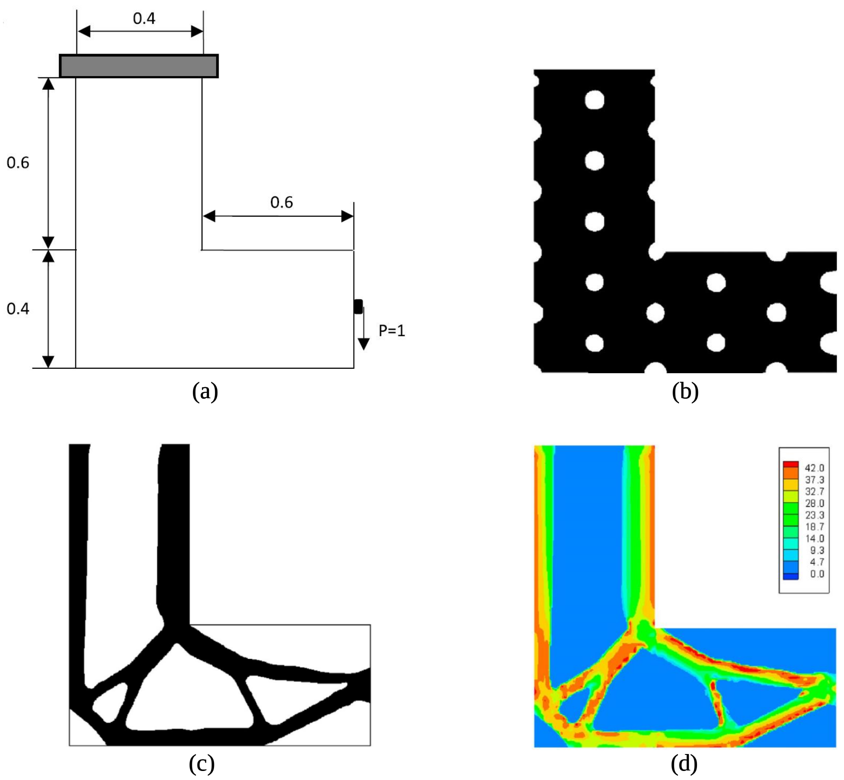


Figure 13: The level-set-based ITO of the L-shape structure: (a) Design domain and boundary conditions, (b) Initial topology, (c) Optimization result and (d) Von Misses stress contour [Ghasemi, Park and Rabczuk (2017)]

Jahangiry et al. [Ghasemi, Park and Rabczuk (2017)] proposed a similar level-set-based ITO where the LSF is parameterized using NURBS basis functions, and applied the level-set-based ITO to a series of problems including minimizing the mean compliance with a material volume constraint, minimizing weight with consideration of reducing local stress concentration, and simultaneously minimizing the weight and strain energy considering local stress constraints. An L-shape structure with stress constraint that is optimized by the proposed method is given in Fig. 13.

With the help of the high-continuity of IGA, the ITO is able to solve some problems that may be very difficult for the conventional topology optimization using FEM. For example, Ghasemi et al. [Ghasemi, Park and Rabczuk (2017)] presented a level-set-based ITO of flexoeletric materials, which requires at least C^1 continuous approximations because of the fourth order partial differential equations (PDEs). The proposed ITO is also able to noticeably increase the electromechanical coupling coefficient, with substantial enhancements observed for higher aspect ratios.

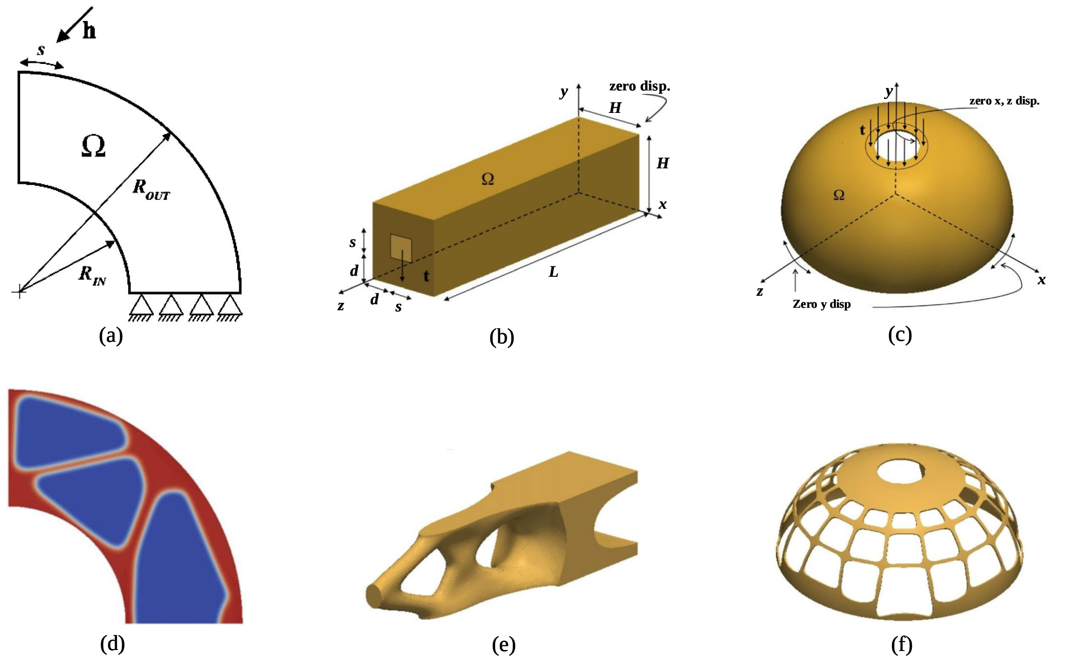


Figure 14: Numerical examples of the ITO using the phase field m odel: (a) Design domain of the 2D example, (b) Design domain of the 3D example, (c) Design domain of the thin shell example, (d) Optimization result of the 2D example, (e) Optimization result of the 3D example, and (f) Optimization result of the thin shell example [Dedè, Borden and Hughes (2012)]

4.3 Other types of isogeometric topology optimization

Besides density-based and level-set-based topology optimizations, the IGA can also be integrated with other types of topology optimization methods. Dedè et al. [Dedè, Borden and Hughes (2012)] used IGA for topology optimization with a phase field model in which the optimal topology is obtained as the steady state of the phase transition described by the generalized Cahn-Hilliard equation, and the IGA for the spatial approximation encapsulating the exactness of the representation of the design domain and is particularly suitable for the analysis of phase filed problems. The proposed method has solved both two and three-dimensional topology optimization problems and some of the results are shown in Fig. 14. Recently, Guo et al. [Guo, Zhang and Zhong (2014); Zhang, Yuan, Zhang et al. (2016)] developed an explicit topology design optimization approach using the concept of moving morphable components (MMC). The basic idea of this method is based on that arbitrary complicated topology can be decomposed into a finite number of components and

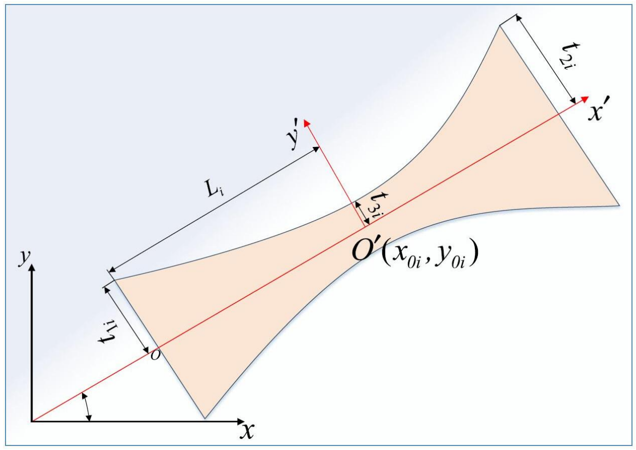


Figure 15: Geometry description of the ith component with straight skeleton and quadratically variable thicknesses [Xie, Wang, Xu et al. (2018)]

the desired topology structure can be obtained by optimizing the central position, length, inclined angles, and some other geometrical features of components. In the MMC topology optimization, a component with straight skeleton and quadratically variable thicknesses (i.e. the ith component) can be described by a set of geometric parameters consisting of coordinates of center, half length, variable thicknesses and inclined angle as $\boldsymbol{D}_i = (x_{0i}, y_{0i}, L_i, t_{1i}, t_{2i}, t_{3i}, \theta_i)$, and the detail meaning is shown in Fig. 15. Based on these geometric parameters, the structural topology can be explicitly and uniquely described by a vector of design variables $\boldsymbol{D} = (\boldsymbol{D}^T, \boldsymbol{D}_2^T, ..., \boldsymbol{D}_n^T)^T$.

Hou et al. [Hou, Gai, Zhu et al. (2017)] first proposed an explicit ITO using IGA to perform the MMC-based topology optimization, which not only explicitly preserve the geometric and mechanical information in topology optimization, but also provides a more flexible process for analysis and optimization. However, there is an imperfection of both conventional MMC-based topology optimization and MMC-based ITO that the overlapped region of components is dealt with by the max function with only C^0 continuity, which results in the objective function becoming nondifferentiable and may cause a low convergence rate of optimization. To overcome the C^1 discontinuity problem, Xie et al. [Xie, Wang, Xu et al. (2018)] used the differentiable R-function [Shapiro (1991)] to replace the max function and proposed a new MMC-based ITO. Besides, they also presented three new ersatz material models based on uniform, Gauss and Greville abscissae collocation schemes to represent both the Young's modulus of material and the density field based on the Heaviside values of collocation points, and successfully solve both 2D and 3D optimization problems (see Fig. 16).

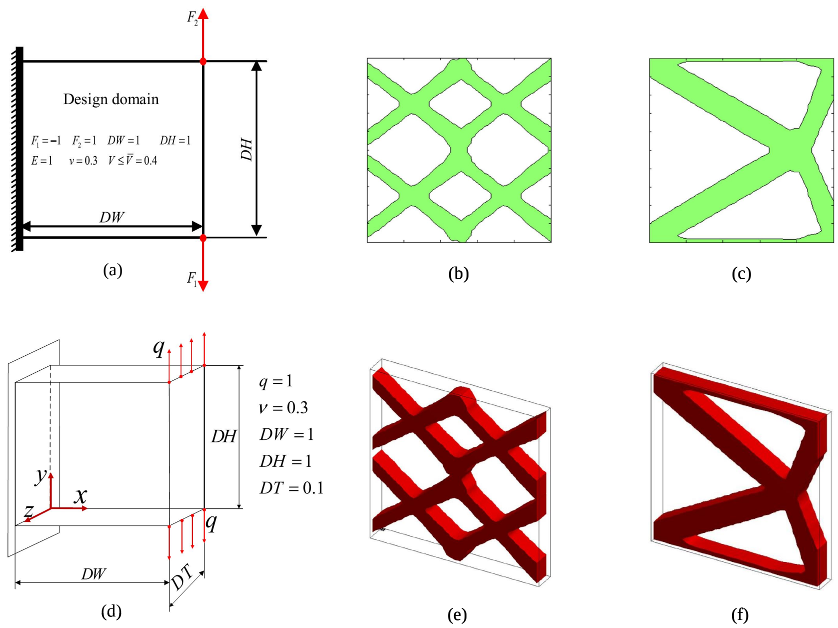


Figure 16: Numerical examples of the MMC-based ITO: (a) Design domain of the 2D example, (b) Initial design of the 2D example, (c) Optimization result of the 2D example, (d) Design domain of the 3D example, (e) Initial design of the 3D example and (f) Optimization result of the 3D example [Xie, Wang, Xu et al. (2018)]

5 Conclusions and research trend

In this work, the development of a decade of isogeometric design optimization is comprehensively reviewed with proper comparisons to traditional FEM-based design optimization. Attention is focused on shape and topology optimizations, with size optimization briefly covered in between.

For isogeometric shape optimization, the typical parameterization problems in FEM-based shape optimization can be avoided due to the seamless CAD-CAE integration framework, however, analysis-suitable parameterization methods for designing complicated geometries become the new challenges, which has been intensively studied in the past few years (see Section 3.1). Although the exact geometry representation of isogeometric framework averts the problem of jagged boundary in FEM-based shape optimization, proper boundary and domain mesh regulation methods are still necessary to ensure the stability of the optimization process (see Section 3.1.3). Isogeometric shape optimization also reduces the difficulties of computing derivative terms with respect to design parameters, leading the direct differential method for shape sensitivity analysis to be popularly used (see Section 3.2).

The seamless CAD-CAE integration framework also promote the development of structural designs with increasingly complicated geometries. Novel software platforms with highly integrated design-analysis engines are becoming more practical and achievable, which will be a great revolution for computer added design technologies.

Currently, shape design for curved structures with special applications using IGA is of great interest, e.g. using isogeometric shape optimization to design auxetic structures with smoothed features in Wang et al. [Wang, Poh, Dirrenberger et al. (2017); Choi and Cho (2018b); Weeger, Narayanan and Dunn (2018)], which focuses on the design of lattice structures with enhanced properties by exploring possible curved unit structures. Combining the finite cell method (FCM) [Guo and Ruess (2015)] with isogeometric analysis for designing curved structures with openings (e.g. in He et al. [He, Pan, Zou et al. (2014)]) or further apply it to topology optimization (e.g. in Cai et al. [Cai, Zhang, Zhu et al. (2014); Zhang, Zhao, Gao et al. (2017)]) is also interesting as it promotes the isogeometric design optimization to be applied to geometrically complicated structures. With the development of analysis-suitable parameterization methods, proper design platforms integrating these new methods for industrial applications would also be interesting. As the IGA has such unique advantages in both analysis [Hughes, Cottrell and Bazilevs (2005)] and design, it should be used to promote the development of challenging problems, e.g. design optimizations for time-dependent, nonlinear solid and fluid mechanical problems. The convenience and efficiency of both geometry and analysis modeling can help to improve global optimization algorithms, which are becoming increasingly practical and favorable (see Haftka et al. [Haftka (2016); Haftka, Villanueva and Chaudhuri (2016)]).

The integrated design-analysis framework in IGA averts the manual transition effort from a CAD model to a FE model, which makes it possible to perform large amount of analysis automatically by varying the geometry design parameters. This is relatively advantageous in the age of big data. With massive results, data-driven design for structural optimization can become increasingly applicable and useful [Queipo, Haftka, Shyy et al. (2005); Forrester and Keane (2009); Xu, You and Du (2015)]. Hence, developing proper methods integrating surrogate modeling, e.g. in Benzaken et al. [Benzaken, Herrema, Hsu et al. (2017)], and artificial intelligence using IGA should also be encouraged to benefit the engineering industry and human society.

For isogeometric topology optimization (ITO), the advantages of IGA has been successfully integrated into topology optimizations. For example, the local support property can free the checkerboards, better efficiency for high order elements greatly increase the efficiency of topology optimization with high-order elements, and the high continuity property can easily solve topology optimizations requiring high continuity, such as the Kirchhoff plate optimization and flexoeletric material optimization. In the future, the IGA will be extended to solve more topology optimization problems, especially for the problems which are impossible or very difficult to complete by conventional topology optimizations. Some of the ITO research trends are summarized hereinafter.

The IGA with high continuity and accuracy is suitable for shell topology optimization, especially for the shell with curved boundary and the curved shell in 3D space [Benson, Bazilevs, Hsu et al. (2010, 2011); Lei, Gillot and Jezequel (2018a)]. Since the shell structures are widely used in the engineering domains such as aerospace, vehicle and ship, the ITO for shell structure optimization has a bright prospect for solving engineering problems. Moreover, the IGA has the local refinement ability [Vuong, Giannelli, Jüttler et al. (2011); Scott, Li, Sederberg et al. (2012); Wu, Huang, Liu et al.(2015)], which provides an opportunity to easily implement the topology optimization with adaptive mesh [Lambe and Czekanski (2018); Wang, Kang and He (2013)]. Definitely, the ITO can be also combined with other approaches used in conventional topology optimizations, e.g. Shepard interpolation [Xia and Shi (2018)], size control approach [Zhou, Lazarov, Wang et al. (2015)], etc., and extend its application for more physical domains such as thermoelasticity [Xia, Xia and Shi (2018)], fluid-structure interaction [Jenkins and Maute (2015)] and thermal-fluid flow [Yaji, Yamada, Yoshino et al. (2016)].

Besides, Topology optimization has demonstrated its great potential in designing architectured materials with tunable properties since 1980s [Sigmund (1994, 1995)]. The effort of designing such materials in a concurrent multiscale levels is gaining renewed interests, e.g. in Abdalla et al. [Abdalla, Setoodeh and Gürdal (2007); Wang, Abdalla and Zhang (2017, 2018); Wang, Abdalla, Wang et al. (2018)] for designing composite laminate with tailored local properties; in Xia et al. [Xia and Breitkopf (2014a,b); Da, Cui, Long et al. (2017); Da, Yvonnet, Xia et al. (2018); Wang, Arabnejad, Tanzer et al. (2018); Wu, Xia, Wang et al. (2019)] for general lattice structures designs. Such work can been extended in the context of isogeometric analysis and will become a new research trend.

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