# Solving the Nonlinear Variable Order Fractional Differential Equations by Using Euler Wavelets

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**Abstract:** An Euler wavelets method is proposed to solve a class of nonlinear variable order fractional differential equations in this paper. The properties of Euler wavelets and their operational matrix together with a family of piecewise functions are first presented. Then they are utilized to reduce the problem to the solution of a nonlinear system of algebraic equations. And the convergence of the Euler wavelets basis is given. The method is computationally attractive and some numerical examples are provided to illustrate its high accuracy.

**Keywords:** Euler wavelets, variable order fractional differential equations, caputo fractional derivatives, operational matrix, convergence analysis.

## 1 Introduction

Many phenomena in fluid mechanics, chemistry, physics, finance and other sciences can be described successfully by models using mathematical tools from fractional calculus, i.e. the theory of derivatives and integrals of fractional order [Miller and Ross (1993)]. Due to the fractional order exponents in differential operators, analytical solutions of fractional equations are usually difficult to obtain. Consequently, different methods have been developed to give numerical solutions for fractional equations, including fractional differential transform method [Wei and Chen (2014)], Adomian decomposition method [Song and Wang (2013)], Chebyshev pseudo-spectral method [Khader and Sweilam (2013)], Homotopy perturbation method [Abdulaziz, Hashim and Momani (2008)], Homotopy analysis method [Dehghan, Manafian and Saadatmandi (2010)], and wavelet method [Li and Zhao (2010); Wang and Fan (2012); Wang and Zhu (2016)].

Recently, the concepts of fractional derivatives of variable order have been introduced and some research works of the relative practical applications have arisen [Sun, Chen and Chen (2009); Samko (2013)]. Several numerical approximation methods are proposed to solve the variable order fractional differential equation [Chen, Liu, Li et al. (2014); Chen, Liu, Turner et al. (2013); Lin, Liu, Anh et al. (2009); Chen, Wei, Liu et al. (2015); Zhao, Sun

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and Karniadakis (2015); Zayernouri and Karniadakis (2015); Li and Wu (2017); Jia, Xu and Lin (2017); Doha, Abdelkawy, Amin et al. (2018)].

Recently, a Euler wavelet numerical method has been presented to solve the nonlinear Volterra integro-differential equations [Wang and Zhu (2017)]. The Euler wavelets are constructed by Euler polynomials. It is well known that Euler polynomials have many advantages over Legendre polynomials in approximating arbitrary functions. Fristly, the Euler polynomials have less terms than Legendre polynomials. Hence, for approximating an arbitrary function, we use less CPU time by applying Euler polynomials as compared to Legendre polynomials. Secondly, the computational errors are less by using Euler polynomials in function approximation. Therefore, we have reason to believe that Euler wavelets inherit these advantages of Euler polynomials. So the main purpose in this paper is to introduce the Euler wavelets operational matrix method to solve the nonlinear variable order fractional differential equation. The Euler wavelets method is based on reducing the equation to a system of algebraic equations by expanding the solution as Euler wavelets with unknown coefficients. The convergence analysis of the Euler wavelets basis is given. Also the characteristic of the operational matrix method is to transform the differential equations into the algebraic one. It not only simplifies the problem, but also speeds up the computation.

The outline of this paper is as follows: In Section 2, some necessary mathematical preliminaries and notations of variable order fractional derivatives are given. Section 3 is devoted to the basic formulation of wavelets and the Euler wavelets. In Section 4, we derive the Euler wavelets operational matrices. Section 5 is devoted to the numerical method for solving the variable order fractional differential equations, and in Section 6 we report our numerical findings and demonstrate the accuracy of the proposed numerical scheme by considering two numerical examples. Finally, conclusions are drawn in last Section.

## 2 Basic definitions and properties of the variable order fractional integrals and derivatives

There are several definitions for variable order fractional derivatives, such as the one in Riemann-Liouville's sense and the one in Caputo's sense [Samko (2013)].

**Definition 1.** Riemann-Liouville fractional integral of the first kind with order $\alpha(t)$ is defined as

$$I_{a+}^{\alpha(t)}u(t) = \frac{1}{\Gamma(\alpha(t))}\int_{a+}^{t}(t-\tau)^{\alpha(t)-1}u(\tau)\mathrm{d}\tau, \ \ t>0, \ \ [Re(\alpha(t))>0]. \tag{1}$$

**Definition 2.** Riemann-Liouville fractional derivate of the first kind with order $\alpha(t)$ is defined as

$$D_{a+}^{\alpha(t)}u(t) = \frac{1}{\Gamma(m-\alpha(t))}\frac{d^m}{dt^m}\int_{a+}^{t}\frac{u(\tau)}{(t-\tau)^{\alpha(t)-m+1}}\mathrm{d}\tau, \ \ m-1\leq\alpha(t))<m, \tag{2}$$

but $D_{a+}^{\alpha(t)}I_{a+}^{\alpha(t)}u \neq u$.

**Definition 3.** The Caputo definition of fractional derivate with order $\alpha(t)$

$$D^{\alpha(t)}u(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_{0+}^{t}\frac{u'(\tau)}{(t-\tau)^{\alpha(t)}}d\tau + \frac{(u(0+) - u(0-))t^{-\alpha(t)}}{\Gamma(1-\alpha(t))}, \tag{3}$$

where $0 < \alpha(t) \leq 1$. If we assume the starting time in a perfect situation, we can get the definition as follows:

$$D^{\alpha(t)}u(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_{0+}^{t}\frac{u'(\tau)}{(t-\tau)^{\alpha(t)}}d\tau, \quad 0 < \alpha(t) < 1, \tag{4}$$

with the definition above, we can get the following formula ($0 < \alpha(t) \leq 1$)

$$D^{\alpha(t)}t^{\beta} = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha(t))}t^{\beta-\alpha(t)} & \beta = 1, 2, 3 \ldots, \\ 0, & \beta = 0. \end{cases} \tag{5}$$

## 3 Euler wavelets and function approximation

### 3.1 Euler wavelets

Euler wavelets $\psi_{nm}(t) = \psi(k, n, m, t)$ involve four arguments, $n = 1, ..., 2^{k-1}$, $k$ is assumed any positive integer, $m$ is the degree of the Euler polynomials and $t$ is the normalized time. They are defined on the interval $[0, 1)$ as [Wang and Zhu (2017)]

$$\psi_{nm}(t) = \begin{cases} 2^{\frac{k-1}{2}}\tilde{E}_m(2^{k-1}t - n + 1), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

with

$$\tilde{E}_m(t) = \begin{cases} 1, & m = 0, \\ \frac{1}{\sqrt{\frac{2(-1)^{m-1}(m!)^2}{(2m)!}E_{2m+1}(0)}}E_m(t), & m > 0, \end{cases} \tag{7}$$

where $m = 0, 1, \ldots, M - 1$ and $n = 1, ..., 2^{k-1}$. The coefficient $\frac{1}{\sqrt{\frac{2(-1)^{m-1}(m!)^2}{(2m)!}E_{2m+1}(0)}}$

is for normality, the dilation parameter is $a = 2^{-(k-1)}$ and the translation parameter $b = (n-1)2^{-(k-1)}$. Here, $E_m(t)$ are the well-known Euler polynomials of order $m$ which can be defined by means of the following generating functions [He (2015)]

$$\frac{2e^{ts}}{e^s + 1} = \sum_{m=0}^{\infty}E_m(t)\frac{s^m}{m!} \quad (|s| < \pi). \tag{8}$$

In particular, the rational numbers $E_m = 2^m E_m(1/2)$ are called the classical Euler numbers. Also, the Euler polynomials of the first kind for $m = 0, \cdots, N$ can be constructed from the following relation

$$\sum_{k=0}^{m}\binom{m}{k}E_k(t) + E_m(t) = 2t^m,$$

where $\binom{m}{k}$ is a binomial coefficient. Explicitly, the first basic polynomials are expressed by

$$E_0(t) = 1, \quad E_1(t) = t - \frac{1}{2}, \quad E_2(t) = t^2 - t, \quad E_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{4}, \cdots.$$

These polynomials satisfy the following formula

$$\int_0^1 E_m(t)E_n(t)\mathrm{d}t = (-1)^{n-1}\frac{m!(n+1)!}{(m+n+1)!}E_{m+n+1}(0), \quad m, n \geq 1. \tag{9}$$

Euler polynomials form a complete basis over the interval $[0, 1]$. Furthermore, when $t = 0$, we have

$$E_0(0) = 1, \quad E_1(0) = -\frac{1}{2}, \quad E_3(0) = \frac{1}{4}, \quad E_5(0) = -\frac{1}{2}, \cdots.$$

### 3.2 Function approximation

A function $f(t)$, square integrable in [0,1], may be expressed in terms of the Euler wavelets as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} c_{nm}\psi_{nm}(t),$$

and we can approximate the function $f(t)$ by the truncated series

$$f(t) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^{\mathrm{T}}\Psi(t), \tag{10}$$

where the coefficient vector $C$ and Euler function vector $\Psi(t)$ are given by

$$C = [c_{10}, c_{11}, \ldots, c_{1(M-1)}, c_{20}, \ldots, c_{2(M-1)}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}(M-1)}]^{\mathrm{T}} \tag{11}$$

$$\Psi(t) = [\Psi_1(t), \Psi_2(t), \ldots, \Psi_{2^{k-1}M}(t)]^{\mathrm{T}} = [\psi_{10}(t), \psi_{11}(t), \ldots, \psi_{2^{k-1}(M-1)}(t)]^{\mathrm{T}}. \tag{12}$$

To evaluate $C$, we let $a_{ij} = \int_0^1 \psi_{ij}(t)f(t)\,\mathrm{d}t$. Using Eq. (10) we obtain

$$a_{ij} = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\int_0^1 \psi_{nm}(t)\psi_{ij}(t)f(t)\,\mathrm{d}t = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}d_{nm}^{ij},$$

where $d_{nm}^{ij} = \int_0^1 \psi_{nm}(t)\psi_{ij}(t)f(t)\,\mathrm{d}t$ and $i = 1, 2, \cdots, 2^{k-1}$, $j = 0, 1, \cdots, M-1$. Therefore,

$$A^{\mathrm{T}} = C^{\mathrm{T}}D,$$

with

$$A = [a_{10}, a_{11}, \ldots, a_{1(M-1)}, a_{20}, \ldots, a_{2(M-1)}, \ldots, a_{2^{k-1}0}, \ldots, a_{2^{k-1}(M-1)}]^{\mathrm{T}}$$

and

$$D = [d_{nm}^{ij}],$$

where $D$ is a matrix of order $2^{k-1}M \times 2^{k-1}M$ and is given by

$$D = \int_0^1 \Psi(t)\Psi^T(t)\,\mathrm{d}t. \tag{13}$$

The matrix $D$ in Eq. (13) can be calculated by using Eq. (9) in each interval $n = 1, 2, \cdots, 2^{k-1}$.

Hence, $C^T$ in Eq. (10) is given by

$$C^T = A^T D^{-1}. \tag{14}$$

We investigate the convergence of the Euler wavelets expansion in the following Theorem.

**Theorem 4**. Suppose that the function $f : [0, 1] \to \mathrm{R}$ is $m + 1$ times continuously differentiable and $f \in C^{m+1}[0, 1]$. Then $\tilde{f}(t) = C^T \Psi(t)$ approximate $f(t)$ with mean error bounded as follows

$$\|f(t) - \tilde{f}(t)\|_2 \leq \frac{\sqrt{2}\tilde{M}}{2^{(k-1)(m+1)}(m+1)!\sqrt{2m+3}},$$

where $\tilde{M} = \max_{t \in [0,1]} |f^{(m+1)}(t)|$.

*Proof:* The proof is similar to that of the Theorem 1 in Wang et al. [Wang and Zhu (2017)], so it is omitted here.

## 4 Operational matrix of Euler wavelets

### 4.1 A new family of functions

Since Euler wavelets functions are formulated based on polynomials, a family of functions defined on [0, 1] is constructed as follows

$$\varphi_{nm}(t) = \begin{cases} t^m, & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{15}$$

for $n = 1, 2, \ldots, 2^{k-1}, m = 0, 1, \ldots, M - 1$. Unlike Euler wavelets functions, this family of functions is not normalized. The connection between these functions and Euler wavelets can be demonstrated as follows:

$$\Phi_i(t) = \sum_{j=1}^{2^{k-1}M} p_{ij}\Psi_j(t), \quad i = 1, 2, \cdots, 2^{k-1}M, \tag{16}$$

where

$$\Phi(t) = [\Phi_1(t), \Phi_2(t), \ldots, \Phi_{2^{k-1}M}(t)]^{\mathrm{T}} = [\varphi_{10}(t), \varphi_{11}(t), \ldots, \varphi_{2^{k-1}(M-1)}(t)]^{\mathrm{T}}.$$

Then, the following result is obtained:

$$\Phi(t) = P\Psi(t), \tag{17}$$

where $P = [p_{ij}]$ is calculated by the means of $C$ in Eq. (14).

### 4.2 Transformation of differential operators

In order to numerically solve the variable order fractional differential equations, we propose to transform both integer and fractional order differential operators into matrix forms.

First, the following equation can be easily obtained for the first order differential operator:

$$\frac{d}{dt}\Phi(t) = S\Phi(t) = \begin{bmatrix} H & 0 & 0 & \cdots & 0 \\ 0 & H & 0 & \cdots & 0 \\ 0 & 0 & H & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & H \end{bmatrix}\Phi(t). \tag{18}$$

where

$$H = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & m-1 & 0 \end{bmatrix}.$$

and the number of $H$ in matrix $S$ is $n$. Using (17)-(18), the following equation is derived

$$\frac{d}{dt}\Psi(t) = R\Psi(t) = P^{-1}SP\Psi(t). \tag{19}$$

Second, using Eq. (5) the following equation can be obtained for the variable order fractional differential operator:

$$D^{\alpha(t)}\Phi(t) = T\Phi(t) = \begin{bmatrix} G & 0 & 0 & \cdots & 0 \\ 0 & G & 0 & \cdots & 0 \\ 0 & 0 & G & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & G \end{bmatrix}\Phi(t). \tag{20}$$

where

$$G = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha(t))}t^{-\alpha(t)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\alpha(t))}t^{-\alpha(t)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha(t))}t^{-\alpha(t)} \end{bmatrix}. \tag{21}$$

The number of $G$ in matrix $T$ is $n$. Using (17) and (20), the following equation is derived:

$$D^{\alpha(t)}\Psi(t) = Q\Psi(t) = P^{-1}TP\Psi(t). \tag{22}$$

where $Q$ is the variable order fractional differential operational matrix of Euler wavelets.

## 5 Method of numerical solution

Consider the nonlinear variable order fractional differential equation

$$D^{\alpha(t)}u(t) + \lambda_1 u'(t) + \lambda_2 u(t) + \lambda_3 u''(t)u(t) = f(t), \tag{23}$$

subject to initial conditions $u(0) = u_0$, where $f(t) \in L^2[0,1]$ is known, $u(t) \in L^2[0,1]$ is the unknown function which we want to approximate, $\lambda_1, \lambda_2, \lambda_3$ and $u_0$ are all constants. If $\lambda_3$ is equal to 0, the equation reduces to linear case.

Now we approximate $u(t)$ in terms of Euler wavelets as follows

$$u(t) \approx C^{\mathrm{T}}\Psi(T), \tag{24}$$

Using Eq. (19), we have

$$\frac{d}{dt}u(t) \approx \frac{d}{dt}(C^T\Psi(t)) = C^T R\Psi(t) = C^T(P^{-1}SP)\Psi(t). \tag{25}$$

By combining (10) with (22), the following result is obtained:

$$D^{\alpha(t)}u(t) \approx D^{\alpha(t)}(C^T\Psi(t)) = C^T Q\Psi(t) = C^T(P^{-1}TP)\Psi(t). \tag{26}$$

Finally, Eq. (23) can be rewritten into the following matrix form:

$$C^T Q\Psi(t) + \lambda_1 C^T R\Psi(t) + \lambda_2 C^T\Psi(t) + \lambda_3 C^T R^2\Psi(t)C^T\Psi(t) = f(t). \tag{27}$$

Then, by taking $t_i = \frac{2i-1}{2^{k-1}M}$, for $i = 1, 2, ..., 2^{k-1}M$ and solving the equations, we can obtain the unknown $C$ by solving a system of algebraic equations.

## 6 Numerical examples

In this section, two examples are given to demonstrate the applicability and accuracy of our method.

**Example 1.** Consider the following linear variable order fractional differential equation:

$$D^{\alpha(t)}u(t) - 10u'(t) + u(t) = f(t), \quad 0 \le t < 1, \tag{28}$$

where $\alpha(t) = \frac{t+2e^t}{7}$, $f(t) = 10(\frac{t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} + \frac{t^{1-\alpha(t)}}{\Gamma(2-\alpha(t))}) + 5t^2 - 90t - 95$, and the initial condition is equal to $u(0) = 5$. The exact solution of this equation is given by $u(t) = 5(1+t)^2$. By applying the proposed method with $k = 1, 2, M = 2$, the numerical solutions can be obtained. Tab. 1 shows the numerical errors obtained by Euler wavelets (Euler), Legendre wavelets (Leg) and finite difference scheme (FDS) [Chen, Wei, Liu et al. (2015)] respectively. In fact, for Legendre wavelets, the basis functions are not wavelets functions, but normalized Legendre polynomial when $k = 0$. Similarly, for Euler wavelets, they are no longer the wavelets functions but the Euler polynomials when $k = 1$. From Tab. 1 we can see, the polynomials functions and the wavelets functions have the same error order with the same $M$, but the coefficient matrix obtained by wavelet decomposition is sparse, which can greatly reduce the calculation amount and reduce the storage space. However, the polynomials basis functions have not such advantages. Meanwhile, The comparisons between approximate and exact solutions for various $k$ are shown in Fig. 1. The numerical results for $\alpha_1(t) = \frac{t+2e^t}{3}, \alpha_2(t) = \frac{t+2e^t}{5}, \alpha(t) = \frac{t+2e^t}{7}, \alpha_3(t) = \frac{t+2e^t}{9}$ are shown in Tab.
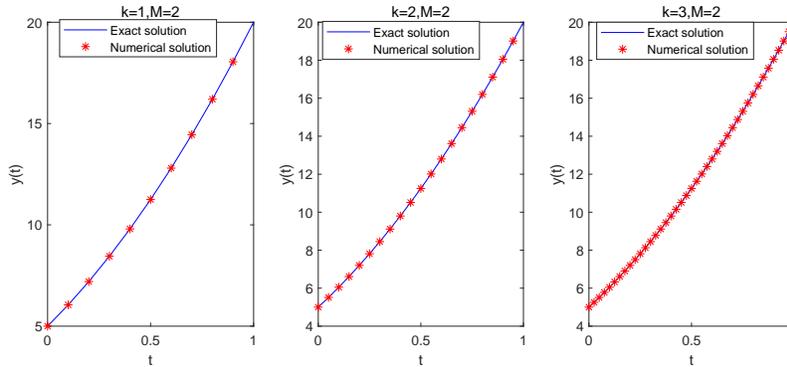
**Figure 1:** The approximate solution of Example 1 for different $k$

2. The absolute errors in Tab. 2 corresponding to different $\alpha(t)$ indicate that the numerical solutions are in good agreement with the exact values.

**Table 1:** Numerical errors for Example 1

| t | Exact solution | FDS (N = 20) | Absolute errors | | | |
|---|---|---|---|---|---|---|
| | | | Leg (k=0,M=4) | Leg (k=2,M=4) | Euler (k=1,M=2) | Euler (k=2,M=2) |
| 0.2 | 7.200000 | 7.247370 | 4.245503e-11 | 8.091305e-12 | 1.443737e-12 | 1.993516e-011 |
| 0.4 | 9.800000 | 9.877180 | 4.334849e-11 | 2.024535e-09 | 1.473957e-12 | 2.035882e-011 |
| 0.6 | 12.800000 | 12.878905 | 4.428849e-11 | 9.564669e-10 | 1.505782e-12 | 8.341771e-012 |
| 0.8 | 16.200000 | 16.248209 | 4.527084e-11 | 1.696030e-10 | 1.539072e-12 | 8.530065e-012 |
| 1.0 | 20.000000 | 19.990749 | 4.628863e-11 | 1.734222e-10 | 1.573855e-12 | 8.622436e-012 |

**Table 2:** The absolute errors of different $\alpha(t)$ for Example 1

| t | Absolute errors | | | |
|---|---|---|---|---|
| | $\alpha_3(t)$ | $\alpha_2(t)$ | $\alpha(t)$ | $\alpha_1(t)$ |
| 0.2 | 7.636113e-13 | 1.436628e-13 | 2.524647e-13 | 1.993516e-011 |
| 0.4 | 7.769340e-13 | 1.467714e-13 | 2.577937e-13 | 2.035882e-011 |
| 0.6 | 6.705747e-14 | 9.983125e-13 | 4.085176e-12 | 8.341771e-012 |
| 0.8 | 6.927791e-14 | 1.019850e-12 | 4.176659e-12 | 8.530065e-012 |
| 1.0 | 7.105427e-14 | 1.040945e-12 | 4.270361e-12 | 8.622436e-012 |

**Example 2.** Consider the following nonlinear variable order fractional differential equation:

$$D^{\alpha(t)}u(t) - 7u'(t) + 5u(t) - 6u''(t)u(t) = f(t), \quad 0 \leq t < 1, \tag{29}$$

where $\alpha(t) = \frac{3(\cos(t)+\sin(t))}{5}$, and $f(t) = 5\left(\frac{2t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} + \frac{3t^{1-\alpha(t)}}{\Gamma(2-\alpha(t))}\right) - 275t^2 - 895t - 105$. The exact solution of this equation is $u(t) = 5(3t + t^2)$. Then, by applying the proposed
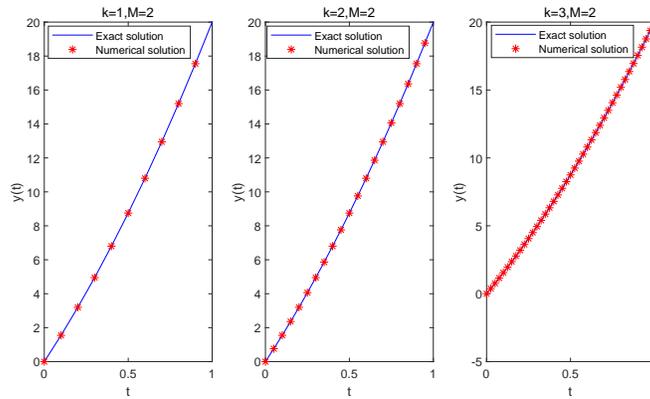
**Figure 2:** The approximate solution of Example 2 for different $k$

method with $k = 1, 2$ and $M = 2$, the numerical solutions can be obtained. Tab. 3 shows the numerical errors obtained by Euler wavelets(Euler), Legendre wavelets (Leg) and finite difference scheme (FDS) [Chen, Wei, Liu et al. (2015)] respectively. The comparisons between approximate and exact solutions for various $k$ are shown in Fig. 2. The numerical results for $\alpha_1(t) = \frac{3(\cos(t)+\sin(t))}{3}, \alpha_2(t) = \frac{3(\cos(t)+\sin(t))}{5}, \alpha(t) = \frac{3(\cos(t)+\sin(t))}{7}, \alpha_3(t) = \frac{3(\cos(t)+\sin(t))}{9}$ are shown in Tab 4.

**Table 3:** Numerical errors for Example 2

| | | | Absolute errors | | | |
|---|---|---|---|---|---|---|
| t | Exact solution | FDS (N = 10) | Leg (k=0,M=4) | Leg (k=2,M=4) | Euler (k=1,M=2) | Euler (k=2,M=2) |
| 0.2 | 3.200000 | 3.199530 | 1.989522e-13 | 1.199041e-14 | 1.09690035e-13 | 4.440892e-015 |
| 0.4 | 6.800000 | 6.805765 | 8.526513e-14 | 1.421085e-14 | 4.24105195e-14 | 1.598721e-014 |
| 0.6 | 10.800000 | 10.812471 | 3.410605e-13 | 2.842171e-14 | 1.86239912e-14 | 2.842170e-014 |
| 0.8 | 15.200000 | 15.218643 | 5.719869e-13 | 1.669775e-13 | 7.36632977e-14 | 3.907985e-014 |
| 1.0 | 20.000000 | 20.024094 | 7.958079e-13 | 2.273737e-13 | 1.23234756e-13 | 4.973799e-014 |

**Table 4:** The absolute errors of different $\alpha(t)$ for Example 2

| | | Absolute errors | | |
|---|---|---|---|---|
| t | $\alpha_1(t)$ | $\alpha(t)$ | $\alpha_2(t)$ | $\alpha_3(t)$ |
| 0.2 | 2.584044e-14 | 4.440892e-015 | 0 | 0 |
| 0.4 | 1.814104e-13 | 1.598721e-014 | 0 | 0 |
| 0.6 | 2.625431e-13 | 2.842170e-014 | 8.093525e-14 | 2.220446e-16 |
| 0.8 | 2.346275e-13 | 3.907985e-014 | 1.251776e-14 | 5.551115e-17 |
| 1.0 | 4.764582e-14 | 4.973799e-014 | 5.101474e-14 | -5.551115e-17 |

According to the numerical examples, the numerical solutions obtained via Euler wavelets method have better accuracies than by Legendre wavelets and finite difference methods. Since Euler wavelets functions and the new functions are both constructed based on

polynomials of commensurate degrees and also have a similarity on their forms, the derived operational matrices are quite accurate. Therefore, due to the uniform convergence of Euler wavelets functions, satisfactory approximation of exact solution could be found by appropriately choosing $k$ and $M$.

## 7 Conclusion

A general formulation for the Euler wavelets operational matrix of variable order fractional order differential has been derived. This matrix is used to approximate numerical solutions of variable order fractional differential equations. Numerical examples are given to illustrate the powerfulness of the proposed method. Moreover, the achieved results are compared with the exact solution and the solutions derived by other approaches presented in open literatures, which further demonstrate the effectiveness of the presented method.

## References

**Abdulaziz, O.; Hashim, I.; Momani, S.** (2008): Solving systems of fractional differential equations by homotopy-perturbation method. *Physics Letters A*, vol. 372, no. 4, pp. 451-459.

**Chen, C. M.; Liu, F.; Turner, I.; Anh, V.; Chen, Y.** (2013): Numerical approximation for a variable-order nonlinear reaction-subdiffusion equation. *Numerical Algorithms*, vol. 63, no. 2, pp. 265-290.

**Chen, Y.; Liu, L.; Li, B.; Sun, Y.** (2014): Numerical solution for the variable order linear cable equation with bernstein polynomials. *Applied Mathematics and Computation*, vol. 238, pp. 329-341.

**Chen, Y. M.; Wei, Y. Q.; Liu, D. Y.; Yu, H.** (2015): Numerical solution for a class of nonlinear variable order fractional differential equations with legendre wavelets. *Applied Mathematics Letters*, vol. 46, pp. 83-88.

**Dehghan, M.; Manafian, J.; Saadatmandi, A.** (2010): Solving nonlinear fractional partial differential equations using the homotopy analysis method. *Numerical Methods for Partial Differential Equations: An International Journal*, vol. 26, no. 2, pp. 448-479.

**Doha, E.; Abdelkawy, M.; Amin, A.; Baleanu, D.** (2018): Spectral technique for solving variable-order fractional volterra integro-differential equations. *Numerical Methods for Partial Differential Equations*, vol. 34, no. 5, pp. 1659-1677.

**He, Y.** (2015): Some new results on products of apostol-bernoulli and apostol-euler polynomials. *Journal of Mathematical Analysis and Applications*, vol. 431, no. 1, pp. 34-46.

**Jia, Y. T.; Xu, M. Q.; Lin, Y. Z.** (2017): A numerical solution for variable order fractional functional differential equation. *Applied Mathematics Letters*, vol. 64, pp. 125-130.

**Khader, M.; Sweilam, N.** (2013): On the approximate solutions for system of fractional integro-differential equations using chebyshev pseudo-spectral method. *Applied Mathematical Modelling*, vol. 37, no. 24, pp. 9819-9828.

**Li, X.; Wu, B.** (2017): A new reproducing kernel method for variable order fractional boundary value problems for functional differential equations. *Journal of Computational and Applied Mathematics*, vol. 311, pp. 387-393.

**Li, Y.; Zhao, W.** (2010): Haar wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations. *Applied Mathematics and Computation*, vol. 216, no. 8, pp. 2276-2285.

**Lin, R.; Liu, F.; Anh, V.; Turner, I.** (2009): Stability and convergence of a new explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation. *Applied Mathematics and Computation*, vol. 212, no. 2, pp. 435-445.

**Miller, K. S.; Ross, B.** (1993): An introduction to the fractional calculus and fractional differential equations.

**Samko, S.** (2013): Fractional integration and differentiation of variable order: an overview. *Nonlinear Dynamics*, vol. 71, no. 4, pp. 653-662.

**Song, L.; Wang, W.** (2013): A new improved adomian decomposition method and its application to fractional differential equations. *Applied Mathematical Modelling*, vol. 37, no. 3, pp. 1590-1598.

**Sun, H.; Chen, W.; Chen, Y.** (2009): Variable-order fractional differential operators in anomalous diffusion modeling. *Physica A: Statistical Mechanics and its Applications*, vol. 388, no. 21, pp. 4586-4592.

**Wang, Y.; Fan, Q.** (2012): The second kind chebyshev wavelet method for solving fractional differential equations. *Applied Mathematics and Computation*, vol. 218, no. 17, pp. 8592-8601.

**Wang, Y.; Zhu, L.** (2016): Scw method for solving the fractional integro-differential equations with a weakly singular kernel. *Applied Mathematics and Computation*, vol. 275, pp. 72-80.

**Wang, Y.; Zhu, L.** (2017): Solving nonlinear volterra integro-differential equations of fractional order by using euler wavelet method. *Advances in Difference Equations*, vol. 2017, no. 1, pp. 27.

**Wei, Y.; Chen, Y.** (2014): Legendre spectral collocation method for neutral and high-order volterra integro-differential equation. *Applied Numerical Mathematics*, vol. 81, pp. 15-29.

**Zayernouri, M.; Karniadakis, G. E.** (2015): Fractional spectral collocation methods for linear and nonlinear variable order fpdes. *Journal of Computational Physics*, vol. 293, pp. 312-338.

**Zhao, X.; Sun, Z. Z.; Karniadakis, G. E.** (2015): Second-order approximations for variable order fractional derivatives: algorithms and applications. *Journal of Computational Physics*, vol. 293, pp. 184-200.