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Metric Identification of Vertices in Polygonal Cacti

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ABSTRACT

The distance between two vertices u and v in a connected graph G is the number of edges lying in a shortest path (geodesic) between them. A vertex x of G performs the metric identification for a pair $(u,\ v)$ of vertices in G if and only if the equality between the distances of u and v with x implies that $u = v$ (That is, the distance between u and x is different from the distance between v and x). The minimum number of vertices performing the metric identification for every pair of vertices in G defines the metric dimension of G. In this paper, we perform the metric identification of vertices in two types of polygonal cacti: chain polygonal cactus and star polygonal cactus.

KEYWORDS

Metric; metric identification; metric generator; metric dimension; cactus graph

1 Introduction

The metric dimension is used in a variety of scientific disciplines, including chemical graph theory and computer networking. A technique for finding a vertex's precise location or position in a network is called localization. In a work environment, localization is used when a computer sends a printing command to help locate nearby printers, broken equipment, network intruders, illegal or unauthorised connections, and wandering robots. The localization of a network is a difficult, expensive, time-consuming, and arduous procedure. The minimum number of vertices (the metric dimension of a graph representing the network) is picked in such a way that, with the aid of selected vertices, the location of the required vertex may be identified by its distinctive representation.

In robotic engineering, there is no concept of direction and no visibility on a polygonal type planar surface/mesh. So handling the navigation of a robot (a navigation agent) from point to point is a crucial task, which can be done quickly with the help of distinctively labelled landmarks. These landmarks help the robot locate itself on the surface (or graph). The visual detection of a landmark sends information to the robot about its direction, allowing it to determine its position. In this way, the robot's position on the graph can be determined by its distances to the elements of the set of distinctively labelled landmarks. The problem of locating the fewest number of landmarks to determine the robot's position is equivalent to finding a minimum metric generator of the graph on which the robot's navigation is required [1].

Consider a connected graph $G = (V(G), E(G))$, where $V(G)$ and $E(G)$ represent vertex set and edge set of G, respectively. The distance, $d(v, w)$ between vertices $v, w \in V(G)$ is the length of a shortest path between v and w. We use the notation $u \sim v$ to indicate that the vertices u and v are adjacent in G.

A vertex x identifies two distinct vertices $v, w \in V(G)$ if $d(x, w) \neq d(x, w)$. The metric vector of a vertex $v \in V(G)$ with respect to an ordered set $W = \{w_1, w_2, \ldots, w_\kappa\} \subseteq V(G)$, is the κ-ordered tuple

$m_W(v) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_\kappa))$.

The set W performs the metric identification of vertices x and y in G if and only if $m_W(x) = m_W(y)$ implies $x = y$. A set of vertices, performing the metric identification of G, is called a metric generator for G. The minimum cardinality of a metric generator for G is called the metric dimension of G, symbolized by $dim(G)$ [2,3].

Slater introduced the concept of metric identification by using the concept of metric generator with name reference (locating) set [3]. Since that, this concept was studied independently, by Melter and Harary where they used the terminology of resolving set for metric generator [2]. While working on the idea of navigating long range aids, Slater examined the usage of the concept of metric identification in 1975 [3]. Moreover, it has been described in [1,4] that how the navigation of robots and likely objects can be performed with this concept. The following short survey will develop the interest of relevant researchers working with the problem of metric identification for various graph families:

- Some fundamental problems related to the metric identification in tree graphs and graphs having minimum and maximum metric dimension have been addressed in [4].
- Using an algorithmic technique with mathematical induction, the problem of metric identification has been solved for a family of 3-regular circulant graphs by Salman et al. [5], and for two 4-regular families of circulant graphs by Khalid et al. [6].
- For three families, $P(2n, n-1)$, $P(n, 4)$ and $P(n, 3)$, of well-known generalized Petersen graphs, the metric identification problem has been solved in [7–9], respectively.
- The study of metric identification has also been taken into account for various chemical networks such as for chordal ring networks in [10] (the authors used the algorithmic technique), for silicate networks in [11], for torus networks in [12] and for two hexa chemical networks in [13].
- Various graph products have also been considered in the context of metric identification problem such as the lexicographic product in [14], the cartesian product in [15,16], and the corona product in [17].

The following theorem provides the minimum metric dimension for a connected graph:

Theorem 1.1. [1,4] Let G be a connected graph, then $dim(G) = 1$ if and only if G is a path graph.

A connected graph in which no edge is a part of more than one cycle is called a cactus graph, see Fig. 1. A cycle C_κ of length κ is called a κ-polygon. If each edge in a cactus graph is a part of a κ-polygon, then the cactus is called a κ-polygonal (or simply polygonal) cactus. If C_κ contains precisely one cut-vertex, then C_κ is called a pendent polygon. Otherwise, C_κ is called a non-pendent polygon [18]. An induced subgraph of a jth κ-cycle C_κ in a polygonal cactus obtained by deleting its cut (vertex) vertices will be called a jth block B_j in the cactus. Two distinct vertices x and y in a polygonal cactus are said to be block-wise distance similar (in short BDS) if the distance of x and y is same with all the vertices of at least one block of the cactus. We label the vertices of a polygonal cactus as follows.

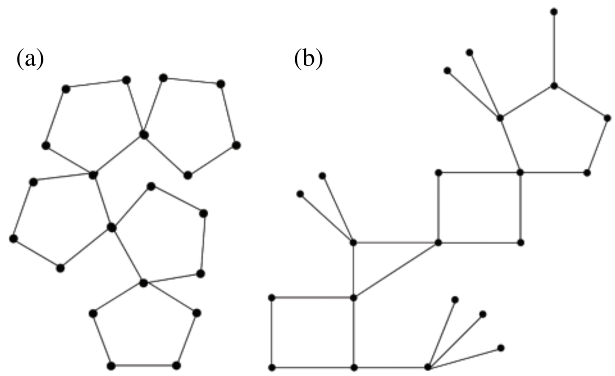

Figure 1: Cactus graphs: (a) is polygonal and (b) is non-polygonal

Let $V_j = \{v_i^j : 1 \le i \le \kappa\}$ be the set of vertices in jth κ-cycle of a polygonal cactus for $1 \le j \le n$. Then the vertex set of the cactus is $\bigcup_{j=1}^{n} V_j$. The aim of this paper is to explore the metric identification of vertices in polygonal cacti. We investigate the minimum number of vertices which perform the metric identification in chain and star polygonal cacti. It is worth noticing that the metric identification of certain graphs have been studied [19,20]. However, this notion has not been explored for the chain and star polygonal cacti which makes the paper different from the available literature.

2 Chain Polygonal Cactus

A chain polygonal cactus, denoted by $T_{n,\kappa}$, is a class of polygonal cactus in which each polygon has at most two cut vertices, where n is the number of κ-polygons, known as the length of $T_{n,\kappa}$.

Lemma 2.1. For $\kappa \ge 3$ with $n \ge 2$, if S is a metric generator for $T_{n,\kappa}$, then S must contain at least one vertex from both the end blocks of $T_{n,\kappa}$.

Proof. Without loss of generality, suppose that S does not contain any vertex from the first block of $T_{n,\kappa}$. Then for two vertices x, y such that $x \sim v_1$ and $y \sim v_1$ (v_1 is the cut vertex between first and second polygons of $T_{n,\kappa}$), we have $m_S(x) = m_S(y)$, a contradiction.

According to the definition, cactus chain $T_{n,\kappa}$ has exactly $n - 2$ non-pendent polygons and two pendent polygons. For $n \ge 3$, $T_{n,3}$ is unique. However, $T_{n,\kappa}$ is not unique for $\kappa \ge 4$ and $n \ge 3$. Hereafter, we define two special classes of cactus chains for $\kappa \ge 4$ and $n \ge 3$.

2.1 $T_{n,\kappa}$ with Adjacent Cut Vertices

In $T_{n,\kappa}$, if we let $v_\kappa^j = v_1^{j+1} = v_j$ (a joint/cut vertex between jth and $(j+1)$th polygons/cycles) for $1 \le j \le n-1$, then cut vertices in $T_{n,\kappa}$ are adjacent, and this type of chain polygonal cactus is denoted by $H_{n,\kappa}$. In fact, cut-vertices in $H_{n,\kappa}$, lying in the same non-pendent polygon, are adjacent. See Fig. 2.

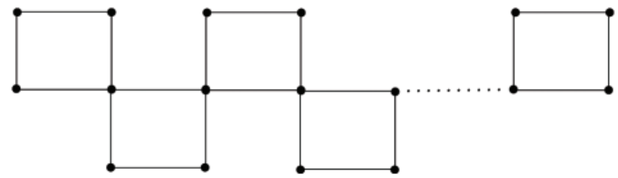

Figure 2: A 4-polygonal chain cactus with adjacent cut vertices

Lemma 2.2. For $\kappa, n > 3$, it is not possible that two consecutive blocks do not contribute to form a metric generator for $H_{n,\kappa}$.

Proof. Contrarily, suppose that two consecutive blocks B_i and B_{i+1} do not contribute to form a metric generator S for $H_{n,\kappa}$. Then, there are BDS x in B_i and y in B_{i+1} and both x and y are neighbors of the joint v_i, such that no vertex $s \in S$ identified x and y. So, S is not a metric generator for $H_{n,\kappa}$, a contradiction.

Theorem 2.1. For $n \geq 3$, $dim(H_{n,3}) = 2$.

Proof. By Theorem 1.1, only path graph has metric dimension equal to 1, so $dim(H_{n,3}) \geq 2$. Let $W = \{v_1^1, v_3^n\}$ be a set of vertices of $H_{n,3}$, then metric vectors of all the vertices in $H_{n,3}$ with respect to W are:

$m_W(v_1^1) = (0, n-1), m_W(v_3^n) = (n-1, 0),$

$m_W(v_j) = (j, n-j)$ for $1 \leq j \leq n-1,$

$m_W(v_2^{j+1}) = (1+j, n-j)$ for $0 \leq j \leq n-1.$

We can see that all the metric vectors are distinct. So, W is a metric generator for $H_{n,3}$, and therefore $dim(H_{n,3}) = 2$.

Theorem 2.2. For odd $\kappa \geq 5$, $dim(H_{3,\kappa}) = 2$.

Proof. By Theorem 1.1, only path graph has the metric dimension equals to 1. So, $dim(H_{3,\kappa}) \geq 2$. Further, consider $W = \left\{v^1_{\frac{\kappa-1}{2}}, v^3_{\frac{\kappa+1}{2}}\right\}$, and accordingly metric vectors of the vertices are:

$$m_W\left(v_i^1\right) = \begin{cases} \left(\frac{\kappa-1}{2} - i, \frac{\kappa+1}{2} + i\right), & 1 \leq i \leq \frac{\kappa-3}{2}, \\ \left(i + \frac{1-\kappa}{2}, \frac{3\kappa+1}{2} - i\right), & \frac{\kappa-1}{2} < i \leq \kappa - 1, \\ \left(\frac{\kappa-1}{2}, \frac{\kappa+1}{2}\right), & i = \kappa. \end{cases}$$

$$m_W\left(v_i^2\right) = \begin{cases} \left(\frac{\kappa-3}{2} + i, \frac{\kappa-1}{2} + i\right), & 1 \leq i \leq \frac{\kappa-1}{2}, \\ \left(i + \frac{\kappa-3}{2}, \frac{3\kappa-1}{2} - i\right), & i = \frac{\kappa+1}{2}, \\ \left(\frac{3\kappa+1}{2} - i, \frac{3\kappa-1}{2} - i\right), & \frac{\kappa+1}{2} < i \leq \kappa. \end{cases}$$

$$m_W\left(v_i^3\right)=\begin{cases}\left(\dfrac{\kappa-1}{2}+i,\dfrac{\kappa+1}{2}-i\right), & 1\le i\le\dfrac{\kappa+1}{2},\\ \left(\dfrac{3\kappa+3}{2}-i,i-\dfrac{\kappa+1}{2}\right), & \dfrac{\kappa+1}{2}<i\le\kappa.\end{cases}$$

Obviously for every two vertices x, y of $H_{3,\kappa}$ with $x \neq y$, $m_W(x) \neq m_W(y)$. Thus, W is a metric generator for $H_{3,\kappa}$ and $dim(H_{3,\kappa}) \geq 2$.

Lemma 2.3. For even $\kappa \geq 4$, if S is a minimum metric generator for $H_{4,\kappa}$, then $|S| \geq 4$.

Proof. We contrarily assume that $|S| = 3$. By Lemma 2.1, S must contain one vertex from each end block. Let a vertex u be taken from the block B_1 and a vertex w be taken from the block B_4. Then S does not contain any vertex from one of the remaining two blocks. Without loss of generality, we suppose that S does not contain any vertex from block B_3, then we have two possibilities:

1. Whenever $d(u, v_1) \neq \frac{\kappa}{2} \neq d(w, v_3)$, then there are two vertices x and y, both are the neighbors of the joint v_3, such that they are BDS in $H_{4,\kappa}$ and $m_S(x) = m_S(y)$, a contradiction.
2. Whenever $d(u, v_1) = \frac{\kappa}{2}$ $\left(\text{or } d(w, v_3) = \frac{\kappa}{2}\right)$, then there are two vertices x and y both are lying in the block B_1 (or B_3) and the neighbors of the joint v_1(or v_3) such that they are BDS in $H_{4,\kappa}$ and $m_S(x) = m_S(y)$, a contradiction.

Hence, our supposition is wrong and $|S| \geq 4$.

Theorem 2.3. For even $\kappa \geq 4$, $dim(H_{4,\kappa}) = 4$.

Proof. Lemma 2.3 provides the lower bound for $dim(H_{4,\kappa})$.

Now, we prove that $dim(H_{4,\kappa}) \leq 4$. For this, let $S = \left\{v^1_{\frac{\kappa}{2}+1}, v^2_{\frac{\kappa}{2}+1}, v^3_{\frac{\kappa}{2}+1}, v^4_{\kappa}\right\}$, and we have to show that for any pair (x, y) of vertices in $H_{4,\kappa}$, there is always a vertex in S which identifies the pair (x, y). For this, we consider three cases:

Case-I Whenever x, y belong to the same block B_i of $H_{4,\kappa}$, then there are two possibilities:

1. If x and y are BDS, then a vertex in S, chosen from the block B_i, will identifies the pair (x, y).
2. If x and y are not BDS, then $d(x, v) \neq d(y, v)$, where v is the cut vertex of a cycle C_i. Thus, $d(x, s) \neq d(y, s)$ for at least one vertex s of S lying in the block B_{i+1} (or in the block B_{i-1}).

Case-II If x, y do not belong to the same block B_i, then there are two possibilities:

1. When x belongs to the block B_i and y belongs to the block B_{i+1} (or B_{i-1}). If x and y are BDS, then there is a vertex of S lying either in the block B_i or B_{i-1} or B_{i+1}, which identifies the pair (x, y). Otherwise, there is always a vertex s in S lying in the block containing x or y such that $d(x, s) \neq d(y, s)$.
2. If x and y do not belong to the two adjacent blocks, i.e., $x \in B_i$ and $y \in B_j$ for $j \neq i + 1$ and $i - 1$, then we always find a vertex w of S lying in B_i (or B_j) such that $d(x, w) \neq d(w, y)$.

Case-III Whenever x or y or both x and y is (are) a joint (s), then there are two possibilities:

1. If x and y are adjacent, then there is a vertex $u \in S$, such that $d(x, u) = 1 + d(y, u)$ where u and y lie on a same cycle, or $d(y, u) = d(x, u) + 1$ where u and x lie on a same cycle. Accordingly, u identifies the pair (x, y).

2. If x and y are not adjacent, then there are s_1, s_2 in S such that s_1, x lie on the same cycle C_i (say) and s_2, y lie on the a same cycle C_j, where $j \neq i,\ i+1,\ i-1$. In this case, both s_1 and s_2 identify the pair *(x, y)*, because

$$d(x, s_2) = d(y, x) + d(y, s_2) \text{ and } d(y, s_1) = d(x, y) + d(x, s_1).$$

According to all these cases, it can be concluded that S is a metric generator of $H_{4,\kappa}$.

Lemma 2.4. For even $\kappa \geq 4$ and any $n \geq 3$ with $n \neq 4$, if S is a minimum metric generator for $H_{n,\kappa}$, then $|S| \geq [\frac{n}{2}] + 2$.

Proof. Let $[\frac{n}{2}] + 2 = m$ and S has two vertices x and y (say) from end blocks B_1 and B_n respectively, by Lemma 2.1. Next, we show that S must contain at least $m - 2$ more vertices from $H_{n,\kappa}$. Contrarily, assume that S contains $m - 3$ more vertices. There are two claims to discuss:

Claim-1 Whenever $d\,(x, v_1) = \dfrac{\kappa}{2}$ $\left(or\, d\,(y, v_{n-1}) = \dfrac{\kappa}{2}\right)$, then S must contain one more vertex from B_1 *(or* B_n*)*.

Neighbors u and v of x (or y) satisfy $m_S(u) = m_S(v)$, so S is not a metric generator. In this way, we get two consecutive blocks among them no vertex will contribute in S, because S contains $m - 3$ more vertices from $(n - 2)$ blocks. It yields a contradiction of Lemma 2.2.

Claim-II Whenever $d\,(x, v_1) \neq \dfrac{\kappa}{2} \neq d\,(x, v_{n-1})$, then S must have at least one vertex from both the blocks B_2 and B_{n-1}.

We suppose that S does not contain a vertex from the block B_2 (say). Then there are two vertices, u_1 in the block B_1 and w_1 in the block B_2, such that $m_S(u_1) = m_S(w_1)$, where $u_1 \sim v_1 \sim w_1$. So, S is not a metric generator. Thus our claim is true. Now, S must have at least one vertex from both the block B_2 and B_{n-1}, and S must contain $m-3$ vertices from $(n-2)$ blocks. So, there always exist two consecutive blocks from each and among them no vertex will contribute to form the set S, which is contradiction of Lemma 2.2.

Both the claims provide that our assumption is wrong. Hence S must contain at least $m-2$ more vertices other than x and y, which implies that $|S| \geq m$.

Theorem 2.4. For even $\kappa \geq 4$ and any $n \geq 3$ with $n \neq 4$, $dim\left(H_{n,\kappa}\right) = \left[\frac{n}{2}\right] + 2$.

Proof. An establishment of upper and lower bounds for $dim(H_{n,\kappa})$ will complete the proof.

Lower bound: Lemma 2.4 provides the minimum metric generator for $H_{n,\kappa}$ of cardinality $\left[\frac{n}{2}\right] + 2$, which yields the lower bound.

Upper bound: We discuss two cases according to the parity of n.

- When $n \geq 6$ is even. Let $W = \left\{v^1_{\frac{\kappa}{2}+1}, v^2_{\frac{\kappa}{2}+1}, v^{2i+1}_{\frac{\kappa}{2}+1}, v^n_{\kappa};\ \ 1 \leq i \leq \frac{n-2}{2}\right\}$. Then, with the similar reasoning given in the proof of Theorem 2.3 for the upper bound, the set W is a metric generator for $H_{n,\kappa}$.
- When $n \geq 3$ is odd. Let $S = \left\{v^1_{\frac{\kappa}{2}+1}, v^{2i}_{\frac{\kappa}{2}}, v^n_{\kappa};\ \ 1 \leq i \leq \frac{n-1}{2}\right\}$. Let $p = (v^r_t, v^m_u)$ be any arbitrary pair of vertices in $H_{n,\kappa}$. To prove that S is a metric generator, we have to show that there always a vertex in S which identifies the pair p. We will discuss three possibilities:

Possibility 1: When $r = m$, then we always have a vertex $s \in S$ such that $s \in B_{r+1}$ or $s \in B_{r-1}$ and s identifies the pair p.

Possibility 2: When $r \in \{m, m+1, m-1\}$, then there is a vertex s in S from the block B_r (or B_m) such that s identifies the pair p.

Possibility 3: When $r \notin \{m-1, m, m+1\}$, then at least one of the following two observations must true:

- S contains an element s from the block B_r (or B_m) such that s identifies the pair p.
- S contains an element s from the block B_{r+1} (or B_{m+1}) such that s identifies the pair p.

Hence, S is a metric generator for $H_{n,\kappa}$.

Lemma 2.5. For odd $\kappa \geq 5$ and any $n \geq 4$, if S is a minimum metric generator for $H_{n,\kappa}$, then $|S| \geq \left[\frac{n}{2}\right] + 1$.

Proof. Let $\left[\frac{n}{2}\right] + 1 = l$. By Lemma 2.1, S must have two vertices from both the end blocks of $H_{n,\kappa}$. Now, we have to show that S contains at least $l-2$ more vertices. Contrarily, assume that S contains $l-3$ more vertices. Then, with the similar reasoning given in the proof of Lemma 2.4, we get two consecutive blocks such that none of them contributes in the set S, which is a contradiction of Lemma 2.2. So, our supposition is wrong. Hence, S must contain at least $l-2$ more vertices, which implies that $|S| \geq l$.

Theorem 2.5. For odd $\kappa \geq 5$ and any $n \geq 4$, $dim\left(H_{n,\kappa}\right) = \left[\frac{n}{2}\right] + 1$.

Proof. $dim\left(H_{4,\kappa}\right) \geq \left[\frac{n}{2}\right] + 1$, by Lemma 2.5. Moreover, with the similar justification proposed for the proof of upper bound in Theorem 2.4, $H_{n,\kappa}$ has a metric generator $W = \left\{v^{2i-1}_{\frac{\kappa}{2}+1}, v^{n}_{\kappa};\ \ 1 \leq i \leq \frac{n}{2}\right\}$ when n is even, and has a metric generator $W = \left\{v^{2i-1}_{\frac{\kappa}{2}};\ \ 1 \leq i \leq \frac{n+1}{2}\right\}$ when n is odd. It follows that $dim\left(H_{n,\kappa}\right) \leq \left[\frac{n}{2}\right] + 1$.

2.2 $T_{n,\kappa}$ with Non Adjacent Cut Vertices

$R_{n,\kappa}$ denotes a chain cactus $T_{n,\kappa}$ such that the cut-vertices, lying in the same non-pendent polygon of $T_{n,\kappa}$, are not adjacent, see Fig. 3. We further classify $R_{n,\kappa}$ into three types:

- Whenever κ is even and the distance between cut vertices is $\dfrac{\kappa}{2}$, then we let $v^{j}_{\frac{\kappa}{2}+1} = v^{j+1}_{1} = v_j$ (a joint/cut vertex between jth and $(j+1)$th polygons/cycles) for $1 \leq j \leq n-1$.
- Whenever κ is odd and the distance between cut vertices is $\dfrac{\kappa-1}{2}$, then we let $v^{j}_{\frac{\kappa+1}{2}} = v^{j+1}_{1} = v_j$ (a joint/cut vertex between jth and $(j+1)$th polygons/cycles) for $1 \leq j \leq n-1$.
- Without loss of generality, we let $v^{j}_{\kappa-1} = v^{j+1}_{1} = v_j$, otherwise (a joint/cut vertex between jth and $(j+1)$th polygons/cycles) for $1 \leq j \leq n-1$.

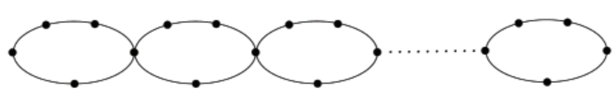

Figure 3: A 5-polygonal chain cactus with non-adjacent cut vertices

With the similar justification proposed for the proof of Lemma 2.2, we have the following result:

Lemma 2.6. For $\kappa > 5$ and $n \geq 3$, it is not possible that two consecutive blocks do not contribute to form any metric generator for $R_{n,\kappa}$.

Theorem 2.6. For odd $\kappa \geq 3$, $dim(R_{2,\kappa}) = 2$.

Proof. Since only a path graph has the metric dimension equals to 1, by Theorem 1.1, $dim(R_{2,\kappa}) \geq 2$. Now, we have to prove that $dim(R_{2,\kappa}) \leq 2$ by investigating a metric generator of cardinality 2. Let us consider the set of vertices $W = \left\{v_1^1, v_{\frac{\kappa+1}{2}}^2\right\}$. Then, metric vectors of all the vertices in $R_{2,\kappa}$ with respect to W are:

$$m_W\left(v_i^1\right) = \begin{cases} \left(i-1, i+\dfrac{\kappa-1}{2}\right), & 1 \leq i < \dfrac{\kappa-1}{2}, \\ \left(\dfrac{\kappa-3}{2}, \kappa-1\right), & i = \dfrac{\kappa-1}{2}, \\ \left(k-i+1, \dfrac{3\kappa-1}{2}-i\right), & \dfrac{\kappa-1}{2} < i \leq \kappa. \end{cases}$$

$$m_W\left(v_i^2\right) = \begin{cases} \left(i, \dfrac{\kappa+1}{2}-i\right), & 1 \leq i \leq \dfrac{\kappa+1}{2}, \\ \left(\kappa-i+2, i-\dfrac{\kappa+1}{2}\right), & \dfrac{\kappa+1}{2} < i \leq \kappa. \end{cases}$$

It can be easily verified that for every pair x, y of distinct vertices, we have $m_W(x) \neq m_W(y)$. So, W is a metric generator for $R_{2,\kappa}$, and $dim(R_{2,\kappa}) \leq 2$.

Theorem 2.7. For even $\kappa \geq 4$, $dim(R_{2,\kappa}) = 3$.

Proof. The proof follows from the following two claims:

Claim I: $(dim(R_{2,\kappa}) \geq 3)$

Suppose contrarily that $dim(R_{2,\kappa}) < 3$. Since any metric generator for $R_{2,\kappa}$ must contain a vertex from both end blocks, by Lemma 2.1, $dim(R_{2,\kappa}) \geq 2$. Let $S = \{x, y\}$ be a minimum metric generator for $R_{2,\kappa}$, where $x \in B_1$ and $y \in B_2$. Then, there are two possibilities:

1. Whenever $d\left(x, v_\kappa^1\right) = \dfrac{\kappa}{2}$ or $d\left(y, v_1^2\right) = \dfrac{\kappa}{2}$, then there are two vertices u_1 and u_2, lying in the same block and both are neighbors of the joint, such that $m_s(u_1) = m_s(u_2)$, a contradiction.
2. Whenever $d\left(x, v_\kappa^1\right) \neq \dfrac{\kappa}{2} \neq d\left(y, v_1^2\right)$, then there are two vertices w_1 (lying in the block B_1), and w_2 (lying in the block B_2) such that w_1, w_2 are neighbors of the joint and $m_s(w_1) = m_s(w_2)$, a contradiction.

Thus, according to these possibilities, S is not a metric generator. So, our supposition is wrong and $dim(R_{2,\kappa}) \geq 3$.

Claim II: $(dim(R_{2,\kappa}) \leq 3)$

Let us consider a set $S = \{v_1^1, v_2^2, v_{\kappa-2}^2\}$ of vertices. Then, metric vectors of the vertices of $R_{2,\kappa}$ with respect to S are:

$$m_S\left(v_i^1\right)=\begin{cases}(i-1,1+i,i+1), & 1\le i\le\frac{\kappa}{2}-2,\\ \left(\frac{\kappa}{2}-2,\frac{\kappa}{2},\frac{\kappa}{2}-1\right), & i=\frac{\kappa}{2}-1,\\ \left(\frac{\kappa}{2}-1,\frac{\kappa}{2}+1,\frac{\kappa}{2}-2\right), & i=\frac{\kappa}{2},\\ \left(\frac{\kappa}{2},\frac{\kappa}{2},\frac{\kappa}{2}-2\right), & i=\frac{\kappa}{2}+1,\\ (\kappa-i+1,\kappa-i+1,\kappa-i+2), & \frac{\kappa}{2}+1<i\le\kappa-2,\\ (2,2,1), & i=\kappa-1,\\ (1,1,2), & i=\kappa.\end{cases}$$

$$m_S\left(v_i^2\right)=\begin{cases}(1,1,2), & i=1,\\ (\kappa-i+2,i-2,1+i), & 2\le i\le\frac{\kappa}{2}+1,\\ \left(\frac{\kappa}{2}+2,\frac{\kappa}{2},\frac{\kappa}{2}+3\right), & i=\frac{\kappa}{2}+2,\\ \left(\frac{\kappa}{2}+2,\kappa-i+2,\kappa-i+1\right), & \frac{\kappa}{2}+2<i\le\kappa.\end{cases}$$

It can be seen that all the metric vectors are different, which implies that S is a metric generator for $R_{2,\kappa}$. Hence $dim(R_{2,\kappa})\ge 3$.

Let $u,v\in V(G)$ be any two vertices. Then, u,v are called twins if either $N[u]=N[v]$ or $N(u)=N(v)$. The relation of twins between vertices of G is an equivalence relation, which partitioned $V(G)$ into classes each of which is called a twin class. A twin class may be singleton [6]. The following results are useful tools to identify twins in a graph G.

Lemma 2.7. [6] If u and v are twins in a connected graph G, then no vertex, except u and v, of G identifies the vertices u and v.

Accordingly, we have the following remark:

Remark 2.1. If U is twin class in a connected graph G with $|U|=l\ge 2$, then every metric generator for G contains at least $l-1$ vertices from U.

Theorem 2.8. For $n\ge 3$, $\dim(R_{n,4})=n$.

Proof. We prove the result with two cases providing lower and upper bounds for the metric dimension of $R_{n,4}$.

Case-I (Lower bound)

In $R_{n,4}$, we obtain n twin classes each of them has cardinality 2. Now, if S is a minimum metric generator for $R_{n,4}$, then S must contain at least one vertex from each twin class, by Remark 2.1. This implies that $dim(R_{n,4})=|S|\ge n$.

Case-II (Upper bound)

Let $S=\{v_3^1,v_2^t:2\le t\le n\}\subset V(R_{n,4})$. Then, S is a metric generator for $R_{n,4}$, because all the vertices have distinct metric vectors with respect to S as listed below:

for fixed $1 \le j \le n$,

$m_S(v_1^1) = (a_{1j}, a_{2j}, \ldots, a_{nj})$, where the lth coordinate is

$a_{lj} = \{2l - 1, \quad \text{for } 1 \le l \le n.$

$m_S(v_2^j) = (a_{1j}, a_{2j}, \ldots, a_{nj})$, where the lth coordinate is

$$a_{lj} = \begin{cases} 2(j-l), & \text{whenever } 1 \le l < j, \\ 0, & \text{whenever } l = j, \\ 2(l-j), & \text{whenever } j < l \le n. \end{cases}$$

$m_S(v_3^j) = m_S(v_1^{j+1}) = (a_{1j}, a_{2j}, \ldots, a_{nj})$, where the lth coordinate is

$$a_{lj} = \begin{cases} 2j - 2l + 1, & \text{whenever } 1 \le l < j, \\ 1, & \text{whenever } l = j, \\ 2l - 2j - 1, & \text{whenever } j < l \le n. \end{cases}$$

$m_S(v_4^j) = (a_{1j}, a_{2j}, \ldots, a_{nj})$, where the lth coordinate is

$$a_{lj} = \begin{cases} 2(j-l), & \text{whenever } 1 \le l < j, \\ 2, & \text{whenever } l = j, \\ 2(l-2j), & \text{whenever } j < l \le n. \end{cases}$$

This implies that $dim(R_{n,4}) \le n$.

Lemma 2.8. For even $\kappa \ge 6$, if S is a minimum metric generator for $R_{3,\kappa}$, then $|S| \ge 3$.

Proof. By Lemma 2.1, S must contain a vertex from both the end blocks of $R_{3,\kappa}$. Suppose that a 2-element set $S = \{x, y\}$ is a metric generator for $R_{3,\kappa}$, where x lies in the block B_1 and y lies in the block B_3. We will discuss two possibilities:

1. Whenever $d(x, v_1) = \frac{\kappa}{2}$, then there are vertices u_1, w_1 in the block B_1 such that u_1 and w_1 are BDS and $u_1 \sim v_1 \sim w_1$. It follows that $m_S(u_1) = m_S(w_1)$, a contradiction to the fact that S is a metric generator. Similarly, if $d(y, v_2) = \frac{\kappa}{2}$, then again we get a contradiction.
2. Whenever $d(x, v_1) \ne \frac{\kappa}{2} \ne d(y, v_2)$, then there are two vertices, u_2 in B_1 and w_2 in B_2, such that $u_2 \sim v_1 \sim w_2$ and
   Hence $m_S(u_2) = m_S(w_2)$, a contradiction. Similarly, there are two vertices u_3, w_3 both are the neighbors of the joint v_2, such that $m_S(u_3) = m_S(w_3)$, a contradiction.

$d(u_2, x) = 1 + d(v_1, x),\ d(w_2, x) = 1 + d(v_1, x).$

It follows that our supposition is wrong, and no 2-element set is a metric generator for $R_{3,\kappa}$. Thus $|S| \ge 3$.

Theorem 2.9. For even $\kappa \ge 6$, $dim(R_{3,\kappa}) = 3$.

Proof. By Lemma 2.8, $dim(R_{3,\kappa}) \ge 3$. Moreover, $dim(R_{3,\kappa}) \le 3$, because the set $S = \{v_\kappa^1, v_2^3, v_2^2\}$ is a metric generator for $R_{3,\kappa}$ due to the following distinct metric vectors of the vertices with respect to S:

$$m_S\left(\nu_l^1\right) = \begin{cases} (l, l+4, l+2), & 1 \leq l \leq \dfrac{\kappa}{2} - 1, \\ (\kappa - l, \kappa - l + 2, \kappa - l), & \dfrac{\kappa}{2} - 1 < l \leq \kappa - 1, \\ (0, 4, 2), & l = \kappa. \end{cases}$$

$$m_S\left(\nu_l^2\right) = \begin{cases} (1, 3, 1), & l = 1, \\ (l, l+2, l-2), & 2 \leq l \leq \dfrac{\kappa}{2} - 1, \\ (l, \kappa - l, l-2), & \dfrac{\kappa}{2} - 1 < l \leq \dfrac{\kappa}{2} + 2, \\ (l, \kappa - l, \kappa - l + 2), & \dfrac{\kappa}{2} + 2 < l \leq \kappa - 2, \\ (\kappa - l + 2, \kappa - l, \kappa - l + 2), & l = \kappa - 1, \\ (\kappa - l + 2, 2, \kappa - l + 2), & l = \kappa. \end{cases}$$

$$m_S\left(\nu_l^3\right) = \begin{cases} (3, 1, 3), & l = 1, \\ (l+2, l-2, l+2), & 2 \leq l \leq \dfrac{\kappa}{2} + 1, \\ (\kappa - l + 4, \kappa - l + 2, \kappa - l + 4), & \dfrac{\kappa}{2} + 1 \leq l \leq \kappa. \end{cases}$$

Theorem 2.10. For even $\kappa \geq 6$, $dim(R_{4,\kappa}) = 4$.

Proof. Let $S = \{\nu_1^1, \nu_2^2, \nu_2^3, \nu_2^4\} \subset V(R_{4,\kappa})$. Then the metric vector of ν_l^j with respect to S is given below:

$$m_S\left(\nu_l^1\right) = \begin{cases} (l-1, l+2, l+4, l+6), & 1 \leq l \leq \dfrac{\kappa}{2} - 1, \\ (\kappa - l + 1, \kappa - l + 2, \kappa - l + 4), & \dfrac{\kappa}{2} - 1 < l \leq \kappa - 1, \\ (1, 2, 4, 6), & l = \kappa. \end{cases}$$

$$m_S\left(\nu_l^2\right) = \begin{cases} (2, 1, 3, 5), & l = 1, \\ (l+1, l-2, l+2, l+4), & 2 \leq l \leq \dfrac{\kappa}{2} - 1, \\ \left(\dfrac{\kappa}{2} + 1, \dfrac{\kappa}{2} + 1 - 2, \dfrac{\kappa}{2}, \dfrac{\kappa}{2} + 1 + 2\right), & l = \dfrac{\kappa}{2}, \\ (\kappa - l + 3, \kappa - l + 2, \kappa - l, \kappa - l + 2), & \dfrac{\kappa}{2} + 1 \leq l \leq \kappa - 1, \\ (3, 2, 2, 4), & l = \kappa. \end{cases}$$

$$m_S\left(v_l^3\right) = \begin{cases} (4,3,1,3), & l = 1, \\ (l+3, l+2, l-3, l+2), & 2 \le l \le \frac{\kappa}{2} - 1, \\ \left(\frac{\kappa}{2}+3, \frac{\kappa}{2}+2, \frac{\kappa}{2}-2\right), & l = \frac{\kappa}{2}, \\ (\kappa - l + 5, \kappa - l + 4, \kappa - l + 2, \kappa - l), & \frac{\kappa}{2} + 1 \le l \le \kappa - 1, \\ (5,4,2,2), & l = \kappa. \end{cases}$$

$$m_S\left(v_l^4\right) = \begin{cases} (6,5,3,1), & l = 1, \\ (5+l, 4+l, 2+l, 2-l), & 2 \le l\frac{\kappa}{2} + 1, \\ (\kappa - l + 7, \kappa - l + 6, \kappa - l + 4, \kappa - l + 2), & \frac{\kappa}{2} + 1 \le l \le \kappa. \end{cases}$$

It can easily verify that all metric vectors are distinct. Thus, S is a metric generator for $R_{4,\kappa}$ and $dim(R_{4,\kappa}) \le 4$.

Now, we claim that if S is a minimum metric generator for $R_{4,\kappa}$, then $|S| \ge 4$. Suppose contrarily that $|S| = 3$. By Lemma 2.1, S must contain one vertex from both the end blocks of $R_{4,\kappa}$. Let $S = \{x, y, z\}$, where x lies in the first block B_1 and y lies in the last block B_4. There are two cases to discuss:

1. If z lies in the block B_1 (or B_4), then there exist BDS, u_1 lies in the block B_2 and w_1 lies in the block B_3 with $u_1 \sim v_2 \sim w_1$, such that $m_S(u_1) = m_S(w_1)$, a contradiction.
2. If z lies in the block B_2 (or B_3) and, without loss of generality, we suppose that z lies in the block B_2, then there are two possibilities:
   - Whenever $d(x, v_1) = \frac{\kappa}{2}$ $\left(\text{or } d(y, v_3) = \frac{\kappa}{2}\right)$, then there are BDS, u_2 and w_2 lying in the block B_1(or in the block B_4), such that both are the neighbors of joint v_1 (or v_3) and $m_S(u_2) = m_S(w_2)$, a contradiction to the fact that S is metric generator.
   - Whenever $d(x, v_1) \ne \frac{\kappa}{2} \ne d(y, v_3)$, then there are BDS, u_3 lies in the block B_3 and w_3 lies in the block B_4 such that $u_3 \sim v_3 \sim u_3$ and $m_S(u_3) = m_S(v_3)$, we get a contradiction.

All these possibilities conclude that our supposition is wrong and $|S| \ge 4$. Hence, $dim(R_{4,\kappa}) \ge 4$.

Theorem 2.11. For $n \ge 3$, $dim(R_{n,5}) = 2$.

Proof. By Theorem 1.1, only a path graph has the metric dimension equals to 1. Therefore, $dim(R_{n,5}) \ge 2$. Now, consider a set $W = \{v_1^1, v_3^n\}$ of vertices of $R_{n,5}$. Then, metric vectors of the vertices with respect to W are:

$$m_W\left(v_l^1\right) = \begin{cases} (1-l, 3-l+2(n-1)), & l = 1, \\ (l-1, 3-l+2(n-1)), & 1 < l \le 3, \\ (6-l, l+2n-5), & 3 < l \le 5. \end{cases}$$

For $2 \le j \le n$,

$$m_W\left(v_l^j\right) = \begin{cases} (l + 2(j-2), 3 - l + 2(n-j)), & 1 \le l \le 3, \\ (3 - l + 2j, 2(n-j) + l - 3), & 3 < l \le 5. \end{cases}$$

It can easily verify that for each pair of distinct vertices (x, y) in $R_{3,\kappa}$, we have $m_W(x) \neq m_W(y)$. Thus, W is a metric generator for $R_{n,5}$ and $dim(R_{n,5}) \le 2$. It completes the proof.

According to the similar reasoning of the proofs of Theorems 2.4 and 2.5 we have the following two results for $R_{n,\kappa}$.

Theorem 2.12. For even $\kappa \ge 6$ and any $n \ge 5$, $dim\left(R_{n,\kappa}\right) = \left[\frac{n}{2}\right] + 2$.

Theorem 2.13. For odd $\kappa \ge 7$ and any $n \ge 3$, $dim\left(R_{n,\kappa}\right) = \left[\frac{n}{2}\right] + 1$.

Theorem 2.14. For even $\kappa \ge 6$ and any $n \ge 3$, whenever $d\left(v_i, v_{i+1}\right) = \dfrac{\kappa}{2}$ in $R_{n,\kappa}$ for each $1 \le i \le n-1$, then $dim(R_{n,\kappa}) = n$.

Proof. Let S be a minimum metric generator and assume that $|S| = n-1$. This implies that S does not have any vertex from at least one block B_t (say), then we have two vertices $u, w \in B_t$ such that u and w are BDS, $u \sim v_{t-1}$ $(or\ v_t)$, $w \sim v_{t-1}$ $(or\ v_t)$ and $m_S(u) = m_S(w)$. This is a contradiction to the fact that S is a metric generator for $R_{n,\kappa}$. Hence $dim(R_{n,\kappa}) = |S| \ge n$.

Now, let $S = \left\{v^i_{\frac{\kappa}{2}-1} : 1 \le i \le n\right\} \subset V\left(R_{n,\kappa}\right)$. Then, with the similar reasoning as given for the proof of upper bound in Theorem 2.3, S is a metric generator for $R_{n,\kappa}$. It follows that $dim(R_{n,\kappa}) \le n$.

Theorem 2.15. For odd $\kappa \ge 7$ and any $n \ge 3$, whenever $d\left(v_i, v_{i+1}\right) = \dfrac{\kappa-1}{2}$ in $R_{n,\kappa}$ for each $1 \le i \le n-1$, then $dim(R_{n,\kappa}) = 2$.

Proof. By Theorem 1.1, $dim(R_{n,\kappa}) \ge 2$, because $R_{n,\kappa}$ is not a path graph. Now, let $W = \left\{v^1_{\frac{\kappa-3}{2}}, v^n_{\frac{\kappa+1}{2}}\right\}$ and corresponding metric vectors of the vertices are:

$$m_W\left(v_i^1\right) = \begin{cases} \left(\dfrac{\kappa-5}{2} - i + 1, \dfrac{\kappa+1}{2} - i + \dfrac{\kappa-1}{2}(n-1) - \dfrac{\kappa-1}{2}(j-1)\right), & 1 \le i \le \dfrac{\kappa-3}{2}, \\ \left(i - \dfrac{\kappa-3}{2}, \dfrac{\kappa+1}{2} - i + \dfrac{\kappa-1}{2}(n-1) - \dfrac{\kappa-1}{2}(j-1)\right), & \dfrac{\kappa-3}{2} < i \le \dfrac{\kappa+1}{2}, \\ \left(i - \dfrac{\kappa-3}{2}, \dfrac{\kappa-3}{2} + i + \dfrac{\kappa-1}{2}(n-3) - \dfrac{\kappa-1}{2}(j-1)\right), & \dfrac{\kappa+1}{2} < i \le \kappa - 2, \\ \left(\dfrac{3\kappa-3}{2} - i, \dfrac{\kappa-3}{2} + i + \dfrac{\kappa-1}{2}(n-3) - \dfrac{\kappa-1}{2}(j-1)\right), & \kappa - 2 < i \le \kappa. \end{cases}$$

For $2 \le j \le n$,

$$m_W\left(v_i^j\right) = \begin{cases} \left(i + \dfrac{\kappa-3}{2} + \dfrac{\kappa-1}{2}(j-2) - 1\right), \dfrac{\kappa+1}{2} - i + \dfrac{\kappa-1}{2}(n-1) - \dfrac{\kappa-1}{2}(j-1)\Big), & 1 \le i \le \dfrac{\kappa+1}{2}, \\ \left(\dfrac{3\kappa-1}{2} - i + \dfrac{\kappa-1}{2}(j-2)\right), \dfrac{\kappa-3}{2} + i + \dfrac{\kappa-1}{2}(n-3) - \dfrac{\kappa-1}{2}(j-1)\Big), & \dfrac{\kappa+1}{2} < i \le \kappa. \end{cases}$$

It can easily verify that for every two distinct vertices x, y of $R_{n,\kappa}$, we have $m_W(x) \neq m_W(y)$. It follows that W is a metric generator for $R_{n,\kappa}$ and $dim(R_{n,\kappa}) \geq 2$. It concludes the proof.

2.3 Star Polygonal Cactus

A star polygonal cactus is a κ-polygonal cactus in which all polygons have a common cut vertex. It is denoted by $W_{n,\kappa}$, where n represents number of polygons C_κ. A star polygonal cactus contains exactly one vertex of degree $2n$ and all other vertices have degree two that is why $W_{n,\kappa}$ is considered to be a unique and special type of cactus graph. Mathematically, if $\nu_\kappa^1 = \nu_\kappa^2 = \ldots = \nu_\kappa^n = J$ (a cut-vertex/joint), then $V\left(W_{n,\kappa}\right) = \{J\} \bigcup_{j=1}^{n} \left(V\left(C_\kappa\right) - \left\{\nu_\kappa^j\right\}\right)$ and $E\left(W_{n,\kappa}\right) = \bigcup_{j=1}^{n} E\left(C_\kappa\right)$.

We have the following results on metric dimension problem regarding star cactus.

Lemma 2.9. For $\kappa \geq 3$ and $n \geq 3$, if S is a minimum metric generator for $W_{n,\kappa}$, then S must contain at least one vertex from each block.

Proof. Suppose contrarily that S does not contain a vertex from jth block B_j(say), then we have vertices y and x in B_j, where x and y are neighbors of the joint J, and $d(x,u) = d(u,J) + 1,\ d(y,u) = d(u,J) + 1$ for all $u \in S$. Thus $m_S(x) = m_S(y)$, a contradiction. Hence S must contain at least one vertex from each block of $W_{n,\kappa}$.

Lemma 2.10. For odd $\kappa \geq 3$ and $n \geq 3$, the set $S = \left\{\nu^1_{\frac{\kappa-1}{2}}, \nu^2_{\frac{\kappa-1}{2}}, \ldots, \nu^n_{\frac{\kappa-1}{2}}\right\}$ is a metric generator for $W_{n,\kappa}$.

Proof. For each fixed $1 \leq j \leq n$, the metric vectors of the ith vertex in jth block is:

$$m_S(\nu_i^j) = (a_{i1}^j, a_{i2}^j, \ldots, a_{in}^j), \tag{1}$$

where the lth coordinate a_{il}^j in (1) can be obtained as follows:

For $1 \leq i \leq \dfrac{\kappa-1}{2}$,

$$a_{il}^j = \begin{cases} \dfrac{\kappa-1}{2} + i, & \text{whenever } \ l \neq j, \\[2mm] \dfrac{\kappa-1}{2} - i, & \text{whenever } \ l = j. \end{cases}$$

For $\dfrac{\kappa-1}{2} < i \leq \kappa - 1$,

$$a_{il}^j = \begin{cases} \dfrac{3\kappa-1}{2} - i, & \text{whenever } \ l \neq j, \\[2mm] i - \dfrac{\kappa-1}{2}, & \text{whenever } \ l = j. \end{cases}$$

For $i = \kappa$ and $1 \leq j \leq n$, $\nu_\kappa^j = J$,

$$m_S(J) = \underbrace{\left(\frac{\kappa-1}{2}, \frac{\kappa-1}{2}, \ldots, \frac{\kappa-1}{2}\right)}_{n-times}.$$

It can be seen that all the metric vectors are distinct, which yields that S is a metric generator of $W_{n,\kappa}$.

Theorem 2.16. For odd $\kappa \geq 3$ and $n \geq 3$, $dim(W_{n,\kappa}) = n$.

Proof. Let S be a minimum metric generator. $W_{n,\kappa}$ has n blocks and S must contain a vertex from each block, by Lemma 2.9. So, $dim(W_{n,\kappa}) = |S| \geq n$. Moreover, Lemma 2.10 provides a metric generator for $W_{n,\kappa}$ of cardinality n, which yields that $dim(W_{n,\kappa}) \leq n$.

Lemma 2.11. For even $\kappa \geq 4$ and $n \geq 3$, if S is a minimum metric generator for $W_{n,\kappa}$, then S contains single vertex from only one block.

Proof. Suppose contrarily that S contain only one vertex from two blocks, vertex x from jth block B_j and vertex y from tth block B_t. There are two possibilities to discuss:

Possibility 1. If $d\,(x, J) = \frac{\kappa}{2}$, then for the neighbors u, v of J in B_j, we have, $d\,(u, x) = \frac{\kappa}{2} - 1 = d\,(v, x)$ and $d(u, s) = d(v, s)$ for each $s \in S - \{x\}$, because x and y are BDS. Thus $m_S(u) = m_S(v)$ and S is not a metric generator, a contradiction. Similarly, if $d\,(y, J) = \frac{\kappa}{2}$, then again we get a contradiction.

Possibility 2. If $d\,(J, x) \neq \frac{\kappa}{2} \neq d\,(J, y)$, then there are BDS w, z in $W_{n,\kappa}$ such that $w \in B_j$, $z \in B_t$. In this case

$d(u, x) = d(z, x) = 1 + d(J, x)$,

$d(u, y) = d(z, y) = 1 + d(J, y)$

and $d(w, s) = d(z, s)$ for each $s \in S - \{x, y\}$. Hence $m_S(w) = m_S(z)$, a contradiction. Therefore, S contains single vertex from only one block.

From the above Lemma, we have the following consequence:

Corollary 2.1. For even $\kappa \geq 4$ and $n \geq 3$, a minimum metric generator S for $W_{n,\kappa}$ must contain at least two vertices from each of $(n - 1)$ blocks.

Lemma 2.12. For even $\kappa \geq 4$ and $n \geq 3$, if S is a minimum metric generator for $W_{n,\kappa}$, then $|S| \geq 2n - 1$.

Proof. There are n blocks in $W_{n,\kappa}$ and S must contain a vertex from each block, by Lemma 2.9. So, S must contain one vertex from only one block and at least $2(n - 1)$ vertices from the remaining $(n - 1)$ blocks, by Lemma 2.11 and Corollary 2.1. Thus $|S| \geq 1 + 2(n - 1) = 2n - 1$.

Lemma 2.13. For even $\kappa \geq 4$ and $n \geq 3$, the set $S = \{v_2^1, v_2^j, v_{\kappa-1}^j : 1 \leq j \leq n\}$ is a metric generator for $W_{n,\kappa}$.

Proof. To prove that S is a metric generator, we need to show that for each pair (x, y) of vertices in $W_{n,\kappa}$, there is generally a vertex in S which identifies the pair (x, y). We consider the following cases:

Case-I Whenever both the vertices x and y are in the same block B_t of $W_{n,\kappa}$. Then there are two possibility:

1. If x and y are not BDS, then there is a vertex $s \in S$ such that $d(x, s) \neq d(y, s)$ and $m_S(x) \neq m_S(y)$. So, s identifies (x, y).
2. If x and y are not BDS, then $d(x, v_j) \neq d(y, v_j)$. So, there is a vertex $s \in S - \{v_i^t\}$ such that $d(x, s) = d(x, v_j) + d(v_j, s)$, $d(x, s) = d(x, v_j) + d(v_j, s)$ and $d(x, s) \neq d(y, s)$. Hence, $m_S(x) = m_S(y)$.

Case-II Whenever both x and y do not belong to the same block. Suppose that $x \in B_j$ and $y \in B_t$, where $t \neq j$. Then, we have two possibilities:

1. If $x,\ y$ are BDS, then there are two vertices u, v in S either $u, v \in B_t$ or $u, v \in B_j$ such that $d(x, u) \neq d(y, u)$ *or* $d(x, v) \neq d(y, v)$.
   So, the pair (x, y) must be identified.
2. If $x,\ y$ are not BDS, then there is a vertex $s \in S$ lying in the block containing x or y, such that $d(x, s) \neq d(y, s)$. So, s identifies the pair (x, y).

Case-III Whenever either x or y is a joint vertex, without any ambiguity, we assume that x is a joint vertex and y lies in any block B_j. If x and y are adjacent, then there is always a vertex $u \in S$, where u belongs to the block B_t, $t \neq j$ such that $d(u, x) = d(u, y) - 1$ and $d(u, y) = d(u, y) + 1$. Hence $m_S(x) \neq m_S(y)$. Otherwise, there is a vertex $s \in S$ such that $d(s, x) = d(s, y) - d(y, x)$. So, $d(s, x) \neq d(s, y)$. Hence s identifies the pair (x, y).

All these cases proved that S performs metric identification for $W_{n,\kappa}$. It completes the proof.

Theorem 2.17. For even $\kappa \geq 4$ and $n \geq 3$, $dim(W_{n,\kappa}) = 2n - 1$.

Proof. Let S be a minimum metric generator for $W_{n,\kappa}$. By Lemma 2.12, $|S| \geq 2n-1$, so $dim(W_{n,\kappa}) \geq 2n - 1$. Moreover, $W_{n,\kappa}$ has a metric generator of cardinality $2n - 1$, by Lemma 2.13, which implies that $dim(W_{n,\kappa}) \leq 2n - 1$. It completes the proof.

3 Concluding Remarks

A family of graphs has a constant metric dimension if *dim(G)* is finite and independent of the order of the graph in the family. If *dim(G)* varies and depends on the order of the graph, then the metric dimension is known as unbounded [9,21]. Two types of polygonal cacti are considered in the context of resolvability (metric identification) and computed the exact value of metric dimension. We analyzed that these families of cactus graphs possessed constant metric dimension, only in few cases. Precisely, we investigated that the family of star polygonal cactus $W_{n,\kappa}$ possessed the unbounded metric dimension, whereas the family of chain polygonal cactus possessed both the constant and unbounded metric dimensions in various cases, described as follows:

- The family $H_{n,\kappa}$ of chain polygonal cactus possessed the constant metric dimension whenever:
  - $H_{n,\kappa}$ consisted of more than two polygons of length 3.
  - there were only three polygons in $H_{n,\kappa}$ of odd length more than 3.
  - there were only four polygons in $H_{n,\kappa}$ of even length more than 2.
  - otherwise, $H_{n,\kappa}$ possessed the unbounded metric dimension.
- The family $R_{n,\kappa}$ of chain polygonal cactus possessed the constant metric dimension whenever:
  - $R_{n,\kappa}$ consisted of two, three and four polygons of length more than 2.
  - $R_{n,\kappa}$ consisted of more than two polygons of length 5.
  - $d(v_i, v_{i+1}) = \dfrac{\kappa - 1}{2}$ in $R_{n,\kappa}$ for odd $\kappa \geq 7$ and any $n \geq 3$.
  - otherwise, $R_{n,\kappa}$ possessed the unbounded constant metric dimension.

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