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ARTICLE

Multi-Attribute Group Decision-Making Method under Spherical Fuzzy Bipolar Soft Expert Framework with Its Application

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ABSTRACT

Spherical fuzzy soft expert set (SFSES) theory blends the perks of spherical fuzzy sets and group decision-making into a unified approach. It allows solutions to highly complicated uncertainties and ambiguities under the unbiased supervision and group decision-making of multiple experts. However, SFSES theory has some deficiencies such as the inability to interpret and portray the bipolarity of decision-parameters. This work highlights and overcomes these limitations by introducing the novel spherical fuzzy bipolar soft expert sets (SFBSESs) as a powerful hybridization of spherical fuzzy set theory with bipolar soft expert sets (BSESs). Followed by the development of certain set-theoretic operations and properties of the proposed model, important problems, including the selection of non-powered dam (NPD) sites for hydropower conversion are discussed and solved under the proposed approach. These problems mainly focus on the need for an efficient tool capable of considering the bipolarity of parameters, complicated ambiguities, and multiple opinions. Supporting the new approach by a detailed comparative analysis, it is concluded that the proposed model is more comprehensive and reliable for multi-attribute group decision-making (MAGDM) than the previous tools, particularly considering the bipolarity of parameters under SFSES environment.

KEYWORDS

Spherical fuzzy sets; bipolar soft expert sets; group decision-making; algorithm; non-powered dams

1 Introduction

Considering complicated issues like falling economies, fastly spreading epidemics, global carbon footprints, environmental pollution, severe weather conditions due to rapid climate changes, energy crises, and depleting resources, many uncertainties arise in making profitable policy decisions. Concerning these issues, including the decision for an optimum object, ranking the policies concerning their possible outcomes, and predicting based on available data, decision sciences offer tools capable of handling vagueness and uncertainties effectively. Following the very first uncertainty theory of probability that was developed during the 16^{th} century, several research scholars around the globe have made remarkable contributions to the decision-making process by introducing many powerful tools, which are capable of dealing with the uncertainties and complications of vast decision-making situations. Since the group decision-making procedure is getting much more complicated with ever-increasing ambiguities and global issues, continuous development in decision-making tools is necessary analogous to broadening decision requirements.

Concerning the uncertainties and partial information in the problems, classical set theories could not help in decision-making. To tackle these situations, Zadeh [1] revolutionized the decision sciences with his invention of fuzzy sets. These fuzzy sets allowed the solutions to uncertain problems by considering partiality of truth rather than perfect truth and perfect false. In contrast to classical sets with memberships 0 and 1 proclaiming only absolute conditions, Zadeh [1] assumed belongingness degrees for alternatives in the interval [0,1]. Therefore, the fuzzy sets provide information about the partial belongingness of things, contrary to their presence or absence. This revolutionary idea motivated many researchers to develop and base their decision-making models using fuzzy sets [2].

The fuzzy sets discuss only the membership of agreement about an object and therefore fail to solve problems that must consider both the belongingness and non-belongingness degrees. To incorporate disagreement degrees for such issues, Atanassov [3] came up with the intuitionistic fuzzy set (IFS) theory. These IFSs took both belongingness and non-belongingness degrees into consideration, corresponding to the decision-maker's opinion about agreement and disagreement with a particular perspective, including the restriction that the sum of both these degrees must not exceed unity. Later, this model proved unsuccessful in handling problems with these two membership degrees summing up to values above one. For such problems, Atanassov [4] extended his IFSs to IFSs of the second type (type-2 IFSs). These type-2 IFSs allowed the sums of the belongingness and non-belongingness degrees to exceed unity with the new constraint that the sum of squares of these two degrees must keep from 0 to 1. Identical to these type-2 IFSs, Yager [5] introduced the Pythagorean fuzzy set (PyFS) model as an extension of the IFSs. Later, many decision-makers embraced this generalized form to develop more powerful hybrid decision models. Deveci et al. [6] provided a survey on the applications of PyFSs. Recently, Habib et al. [7] used PyFS-based multi-criteria decision-making (MCDM) methods for diagnosing the risk of eight different types of cancer in children.

Adding to the above scenarios, the situations like elections, surveys, and polls, often ask for a neutral opinion as well. For instance, a survey may ask for positive, negative and neutral opinions about various points. These scenarios require a third degree, i.e., the neutral membership degree in addition to the two prior discussed evaluation degrees. Considering such situations, Coung et al. [8–10] proposed picture fuzzy sets (PFSs) by considering negative, positive, and neutral membership degrees reflecting the disagreement, agreement, and neutrality in the decision-maker's opinion. Currently, Singh et al. [11] studied applications of picture fuzzy correlation coefficients. A drawback of these PFSs is that they are not capable to tackle problems with negative, positive, and neutral memberships summing above unity. Considering the problems with this sum exceeding unity, Kahraman et al. [12] initiated the notion of spherical fuzzy sets (SFSs) by smoothening the restriction of the picture fuzzy sets with the condition that the sum of squares of positive, neutral, and negative memberships ranges from 0 to 1. Thus, it provided a generalization for the previous models. Later, Gündogdu et al. [13] proposed a TOPSIS (technique of order preference similarity to the ideal solution) approach for dealing with decision-making applications using generalized SFSs. Akram et al. [14] discussed a group decision-making mechanism based on the complex spherical fuzzy VIKOR method. Recently, Donyatalab et al. [15] introduced their new spherical fuzzy similarity and distance measures, and discussed their applicability in the diagnosis of diseases. Wang et al. [16] discussed a detailed site selection application for offshore wind power plants using SFSs under two-stage MCDM. Similarly, in the past few years, SFSs have been employed to tackle many decision-making situations [17–21]. Fig. 1 compares the IFSs, PyFSs, PFSs, and SFSs in a graphical manner.

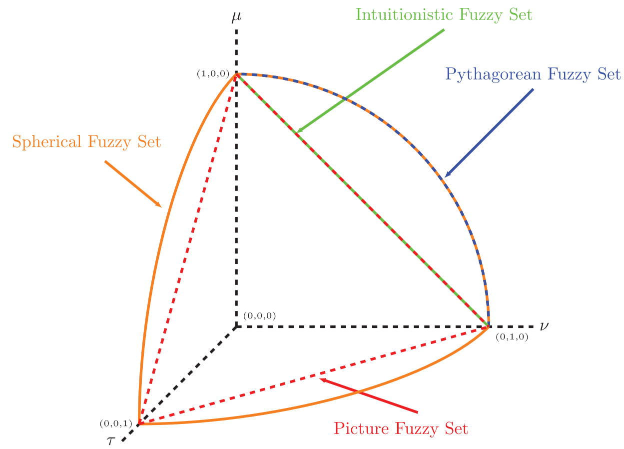

Figure 1: Pictorial representation of IFS, PyFS, PFS, and SFS

One common limitation of all the above-discussed models is that they consider opinions respective to only a single parameter. However, generally, the decision-making problems are based on choices keeping multiple parameters under consideration. To integrate multiple parameters in one place, Molodtsov [22] introduced soft sets as the parameterized families of the provided universe of discourse. Maji et al. [23] discussed the decision-making applications of soft sets. Inspired by the parameterizing capability of soft sets, many hybrid structures have been developed in the last two decades by the combination of multiple uncertainty theories with soft sets. For instance, Sun et al. [24] combined linguistic variables with soft set theory to form linguistic valued soft sets with linguistic values. Perveen et al. [25] developed a spherical fuzzy soft set (SFSS) model by combining SFSs with soft sets. Guleria et al. [26] introduced T-spherical fuzzy soft sets (T-SFSSs) as the hybridization of T-SFSs with soft sets. Akram et al. [27] provided TOPSIS approach for dealing with complex spherical fuzzy information. In addition, complex spherical Dombi fuzzy aggregation operators were initiated by Akram et al. [28]. Zahid et al. [29] introduced an ELECTRE-based method using complex spherical fuzzy information for group decision-making. Akram et al. [30] discussed the decision-making applications of complex spherical fuzzy N-soft sets. Recently, Yang et al. [31] introduced T-spherical fuzzy Bonferroni mean operators and discussed their applications in multi-criteria decision-making. Manna et al. [32] presented an MADM method under a complex neutrosophic environment. Adeel et al. [33] provided a decision-making analysis using the ELECTRE-I approach under hesitant fuzzy N-soft sets. Some other recent works include [34–38]. Very recently, Liu et al. [39] presented a fuzzy soft expert prediction system to diagnose the COVID-19 (Coronavirus disease 2019).

The highly demanding problems, particularly those affecting important future considerations, require group decision-making. The previously discussed models are restricted to a single expert's opinions and are not capable of making group decisions efficiently. To fulfill this requirement, Alkhazaleh et al. [40] developed soft expert sets (SESs), capable of dealing with multiple experts' opinions in one place. Later, Alkhazaleh et al. [41] combined fuzzy sets with SESs and presented fuzzy soft expert sets (FSESs). Considering the group decision-making capabilities of SESs, lots of recent works focused on the hybridizations and improvements based on these sets. Bashir et al. [42] introduced possibility fuzzy soft expert sets by providing possibility degrees along with the fuzzy memberships. Fuzzy parameterized soft expert sets developed by Bashir et al. [43] allowed preference of parameters by assigning fuzzy values to parameters. Fusing SFSs with SESs, Perveen et al. [44] established spherical fuzzy-SES (SFSES) model and discussed its MAGDM applications. Combining the rating-based decision qualities with SFSs and SESs, Akram et al. [45] presented spherical fuzzy N-SESs and explored their MAGDM problems in survey-based election prediction and product review.

In considering the uncertainties with multiple parameters, many decision-making problems arise with bipolar parameters interrelated symmetrically. For instance, a medical test can be sensitive or specific, a person can be young or old, and a student can be dull or intelligent. Such dual behavior of parameters requires tools eligible to manage and depict this bipolarity. Shabir et al. [46] introduced bipolar soft sets (BSSs) composed of two soft sets; one based on the set of parameters, and the other based upon the not-set of parameters (one set contains the parameters opposite to those in the other set). Ali et al. [47] presented bipolar-SESs (BSESs) as the fusion of BSSs with SESs, and discussed their decision-making applications. Further extending BSESs for uncertain situations, Ali et al. [48] initiated fuzzy BSESs (FBSESs) and discussed their MAGDM applications. Recently, Ali et al. [49] ranked the fish passage designs for a hydropower project based on spherical fuzzy-BSSs (SFBSSs). Many other models based on BSSs came out in the recent years, e.g., q-rung orthopair fuzzy BSSs (qROFBSSs) [50]. Table 1 provides the list of abbreviations used in the paper. Table 2 represents the list of frequently used symbols throughout the work.

Table 1: List of abbreviations

Abbreviation	Description
FSs	Fuzzy sets
IFSs	Intuitionistic fuzzy sets
PyFSs	Pythagorean fuzzy sets
PFSs	Picture fuzzy sets
SFSs	Spherical fuzzy sets
SSs	Soft sets
SFSSs	Spherical fuzzy soft sets
BSSs	Bipolar soft sets
SESs	Soft expert sets
NPDs	Non-powered dams
BSESs	Bipolar soft expert sets
FBSESs	Fuzzy bipolar soft expert sets
SFBSSs	Spherical fuzzy bipolar soft sets
qROFBSSs	q-Rung orthopair fuzzy bipolar soft sets
SFSESs	Spherical fuzzy soft expert sets
SFBSESs	Spherical fuzzy bipolar soft expert sets
MCDM	Multi-criteria decision-making
MAGDM	Multi-attribute group decision-making
COVID-19	Coronavirus disease 2019
TWh	Terawatt-hour

(Continued)

Table 1 (continued)

Abbreviation	Description
US	United States
ORNL	Oak Ridge national laboratory
MW	Megawatt

Table 2: List of symbols

Symbol	Meaning
Γ, Λ, Π	Spherical fuzzy bipolar soft expert sets
$\mathscr{Z}$	Universe of alternatives
$\mathfrak{z}$	Alternatives in $\mathscr{Z}$
$\mathscr{P}$	Collection of all parameters
$\mathscr{B}$	The set of parameters (favorable parameters)
$\neg\mathscr{B}$	The not set of parameters (nonfavorable parameters)
$\beta, \neg\beta$	Parameters in $\mathscr{B}$ and $\neg\mathscr{B}$
$\mathscr{E}$	Set of experts
u, v, w	Experts in $\mathscr{E}$
$\mathcal{O}$	Set of opinions
$\mathscr{Q} = \mathscr{P} \times \mathscr{E} \times \mathcal{O}$	Collection of all binary parameter-wise expert opinions
$\mathscr{L}$	Subset of $\mathscr{Q}$ corresponding to $\mathscr{B}$ (favorable)
$\neg\mathscr{L}$	Subset of $\mathscr{Q}$ corresponding to $\neg\mathscr{B}$ (non-favorable)
$\ell, \neg\ell$	Elements in $\mathscr{L}$ and $\neg\mathscr{L}$
$\mathbb{SF}(\mathscr{Z})$	Family of all SFSs on $\mathscr{Z}$
$\hat{\zeta}, \hat{\pi}$	SFS mapping from $\mathscr{L}$ to $\mathbb{SF}(\mathscr{Z})$
$\hat{\eta}, \hat{\psi}$	SFS mapping from $\neg\mathscr{L}$ to $\mathbb{SF}(\mathscr{Z})$
μ, τ, ν	Positive, neutral and negative membership degrees
Γ^c	Complement of Γ
$\hat{\Phi}$	Relative null SFBSES
$\hat{\mathfrak{U}}$	Relative absolute SFBSES
$\hat{\cup}_E, \hat{\cap}_E$	Extended SFBSE union and intersection
$\hat{\cup}_R, \hat{\cap}_R$	Restricted SFBSE union and intersection
$\hat{\vee}, \hat{\wedge}$	SFBSE-OR and SFBSE-AND operation
$\hat{\Gamma} = (\hat{\zeta}, \mathscr{L})$	Favor soft expert subset of Γ
$\mathring{\Gamma} = (\hat{\eta}, \neg\mathscr{L})$	Non-favor soft expert subset of Γ
$\lambda(mid, tbb, bbb, med)$	Level threshold function (mid level, top-bottom-bottom level, bottom-bottom-bottom level, medium level)
$\mathfrak{L}(\hat{\Gamma}, \lambda)$	Level favor SES of Γ under λ
$\mathfrak{L}(\mathring{\Gamma}, \lambda)$	Level non-favor SES of Γ under λ
$(\mathfrak{F}, level_\Gamma)$	Focus-level SES of Γ

The complications arising in the emerging decision problems considering the energy crisis and depleting energy resources ask for more efficient decision-making powered by much stronger and comprehensive tools. For instance, hydropower is a non-depleting carbon-free renewable energy resource bearing maximum potential as compared to all other renewable energy resources. However, all dams and water reservoirs are not dedicated to hydropower generation. Hadjerioua et al. [51] provided a detailed estimation of energy capacity at non-powered dams (NPDs) in the United States and their report signifies the importance of NPDs capable of generating electricity and adding to hydropower generation through proper installations and considerations. The point is that all NPDs cannot generate electricity, and the selection of NPDs capable of hydropower generation requires complicated MAGDM. This demands considering spherical fuzzy opinions of multiple experts (or teams) for tackling the uncertainties governed by the bipolar parameters depicting favoring and symmetrically non-favoring factors. Such a problem is unsolvable by the existing theories, including SFSESs, BSESs, SFBSSs, etc. That is why, this work focuses on the development of a novel hybrid model named spherical fuzzy bipolar soft expert sets or SFBSESs as a combination of SFSs [12] and BSESs [47], and then the ranking of the NPDs' suitability for hydropower installations under SFBSESs. Table 3 gives an overview of the considerations taken into account by multiple decision-models. It can be clearly observed that the innovative proposed model considers maximum dependencies affecting the decision-making, hence leaving negligible chances of error as compared to all of the pre-existing tools.

Table 3: Domains of the decision-making methods

Model	Disagreement	Neutrality	Parameterization	Multiple experts	Bipolarity
Fuzzy sets [1]	×	×	×	×	×
IFSs [3]	✓	×	×	×	×
PFSs [8–10]	✓	✓	×	×	×
SFSs [12]	✓	✓	×	×	×
Soft sets [22]	×	×	✓	×	×
SFSSs [25]	✓	✓	✓	×	×
SESs [40]	×	×	✓	✓	×
SFSESs [44]	✓	✓	✓	✓	×
BSSs [46]	×	×	✓	×	✓
BSESs [47]	×	×	✓	✓	✓
FBSESs [48]	×	×	✓	✓	✓
SFBSSs [49]	✓	✓	✓	×	✓
Proposed SFBSESs	✓	✓	✓	✓	✓

1.1 Motivations and Contributions

The motivations of this work are provided as below:

1. The possible contribution of NPDs in hydropower generation (as discussed in [51]) is very important in minimizing the power crisis. However, a lot of positive and negative factors considering the infrastructure, location, politics, etc., ask for efficient MAGDM decision-making under the expertise of multiple experts.

2. The existing models like PFSs [8–10], SFSSs [25] and SFSESs [44] offer reliable solutions to uncertainties by considering the degrees of positive, negative and neutral opinions. However, all of these are incapable to discuss the bipolarity of decision-parameters, therefore unreliable to solve problems governed by symmetrically opposing factors.
3. Models based on BSSs [46] like BSESs [47], FBSESs [48] and qROFBSSs [50] efficiently differentiate between the bipolar decision-parameters, however all these models fail to consider the degree of neutrality when solving uncertain problems.
4. The MAGDM capabilities of SESs [40] for dealing complex situations considering multiple experts offer more reliable and unbiased solutions. Most of the existing works as discussed in Table 3 lack this capability.

The main contributions of this paper are given as below:

1. Combining the uncertain decision-making properties of SFSs [12] with the group decision-making capabilities of BSESs [47], a hybrid model is presented with higher performance when examining MAGDM problems in spherical fuzzy environment under bipolar parameters. This hybridization allows group decision considerations of 3 dimensional opinions regarding the uncertainties in problems governed by symmetrically opposing parameters.
2. The algebraic characteristics and operations of the initiated structure are discussed and illustrated through numerical examples. These operations depict inclusion and aggregation techniques in multiple ways for combining and comparing between different sets of SFBSE information.
3. A powerful MAGDM algorithm under SFBSESs is proposed. This enhanced algorithm combines the previous algorithms (as in [44,48,49]) to tackle their limitations as discussed in the literature. Consequently, many problems unsolvable by the previous works due to the loss of information will now seek their solutions reliably.
4. Considering the importance of NPDs in improving the power generation and the complicated uncertainties arising in appropriate site selection procedure for hydropower conversion, a model NPDs site selection problem is modeled using SFBSES and solved under the proposed algorithm. This model application gives an insight about how equally complicated actual problems can be solved with the presented method.

1.2 Structure of the Paper

This paper is organized as:

- Section 2 recalls the preliminary definitions and related notions that will be utilized in the further development of this work.
- Section 3 presents SFBSESs along with their algebraic properties and fundamental operations, including subset-hood, complement, intersection (restricted and extended), union (restricted and extended), SFBSE-AND operation and SFBSE-OR operation.
- Section 4 explores an application of the SFBSESs, i.e., ranking of NPDs regarding suitabilities for hydropower generation, and its solution under the novel algorithm proposed for SFBSESs.
- Section 5 discusses the advantages of proposed MAGDM hybrid model by comparing it with the existing models, and debates the possible limitations.
- At last, Section 6 gives some conclusive remarks and possible future extensions of the presented method.

2 Preliminaries

This section recalls important notions useful in the development of this paper. Let us start with the definition of spherical fuzzy sets (or SFSs).

Definition 2.1. [12] For $\mathscr{Z}$ being the universe and μ, τ, ν being the fuzzy membership functions representing the positive, negative, and neutral values, respectively. A *spherical fuzzy set* (SFS) $\mathscr{S}$ over $\mathscr{Z}$ is represented as:

$$\mathscr{S} = \{(\mathfrak{z}, \mu_{\mathscr{S}}(\mathfrak{z}), \tau_{\mathscr{S}}(\mathfrak{z}), \nu_{\mathscr{S}}(\mathfrak{z}))|\mathfrak{z} \in \mathscr{Z}\}$$

where

$$0 \leq (\mu_{\mathscr{S}}(\mathfrak{z}))^2 + (\tau_{\mathscr{S}}(\mathfrak{z}))^2 + (\nu_{\mathscr{S}}(\mathfrak{z}))^2 \leq 1, \quad \forall \mathfrak{z} \in \mathscr{Z}.$$

The set $\mathbb{S}(\mathscr{Z})$ represents the family of all SFSs over $\mathscr{Z}$.

Some important properties and operations of the SFSs are listed as follows:

Definition 2.2. [12] Suppose $\mathscr{J}$ and $\mathscr{K}$ represent SFSs on the universal set $\mathscr{Z}$. Then

1. $\mathscr{J} \subseteq \mathscr{K}$ if $\forall \mathfrak{z} \in \mathscr{Z}$, $\mu_{\mathscr{J}}(\mathfrak{z}) \leq \mu_{\mathscr{K}}(\mathfrak{z}), \tau_{\mathscr{J}}(\mathfrak{z}) \leq \tau_{\mathscr{K}}(\mathfrak{z})$ and $\nu_{\mathscr{J}}(\mathfrak{z}) \geq \nu_{\mathscr{K}}(\mathfrak{z})$.
2. $\mathscr{J} = \mathscr{K} \iff \mathscr{J} \subseteq \mathscr{K}$ and $\mathscr{K} \subseteq \mathscr{J}$.
3. $\mathscr{J} \cup \mathscr{K} = \{(\mathfrak{z}, \max\{\mu_{\mathscr{J}}(\mathfrak{z}), \mu_{\mathscr{K}}(\mathfrak{z})\}, \min\{\tau_{\mathscr{J}}(\mathfrak{z}), \tau_{\mathscr{K}}(\mathfrak{z})\}, \min\{\nu_{\mathscr{J}}(\mathfrak{z}), \nu_{\mathscr{K}}(\mathfrak{z})\})|\mathfrak{z} \in \mathscr{Z}\}$.
4. $\mathscr{J} \cap \mathscr{K} = \{(\mathfrak{z}, \min\{\mu_{\mathscr{J}}(\mathfrak{z}), \mu_{\mathscr{K}}(\mathfrak{z})\}, \min\{\tau_{\mathscr{J}}(\mathfrak{z}), \tau_{\mathscr{K}}(\mathfrak{z})\}, \max\{\nu_{\mathscr{J}}(\mathfrak{z}), \nu_{\mathscr{K}}(\mathfrak{z})\})|\mathfrak{z} \in \mathscr{Z}\}$.
5. $\left(\mathscr{J}\right)^c = \{(\mathfrak{z}, \nu_{\mathscr{J}}(\mathfrak{z}), \tau_{\mathscr{J}}(\mathfrak{z}), \mu_{\mathscr{J}}(\mathfrak{z}))|\mathfrak{z} \in \mathscr{Z}\}$.

The next definition calls back the notion of spherical fuzzy soft sets (or SFSSs).

Definition 2.3. [25] Suppose $\mathscr{Z}$ represents the universe of objects, $\mathscr{P}$ represents the set of parameters, and $\mathscr{B} \subseteq \mathscr{P}$, then the set $\Theta = (\mathscr{K}, \mathscr{B})$ is called a *spherical fuzzy soft set* (SFSS) on $\mathscr{Z}$, such that $\forall \mathfrak{z} \in \mathscr{Z}$ and $\beta \in \mathscr{B}$, the mapping $\mathscr{K}: \mathscr{B} \to \mathbb{S}(\mathscr{Z})$ is provided by:

$$\mathscr{K}(\beta) = \{(\mathfrak{z}, \mu_{\mathscr{K}(\beta)}(\mathfrak{z}), \tau_{\mathscr{K}(\beta)}(\mathfrak{z}), \nu_{\mathscr{K}(\beta)}(\mathfrak{z}))\}$$

where $\mu_{\mathscr{K}(\beta)}, \tau_{\mathscr{K}(\beta)}$ and $\nu_{\mathscr{K}(\beta)}$ show the degrees of agreement, disagreement, and neutrality of the object $\mathfrak{z}$ with the condition:

$$0 \leq (\mu_{\mathscr{K}(\beta)}(\mathfrak{z}))^2 + (\tau_{\mathscr{K}(\beta)}(\mathfrak{z}))^2 + (\nu_{\mathscr{K}(\beta)}(\mathfrak{z}))^2 \leq 1, \quad \forall \mathfrak{z} \in \mathscr{Z}, \beta \in \mathscr{B}.$$

Some important definitions for SFSSs are recalled below, which will help in development of the new MAGDM algorithm.

Definition 2.4. [25] Consider a SFSS $\Theta = (\mathscr{K}, \mathscr{B})$ on the universal set $\mathscr{Z}$. Suppose the mapping $\lambda: \mathscr{B} \to [0, 1]^3$, where $\forall \beta \in \mathscr{B}$, $\lambda(\beta) = (\mathring{p}(\beta), \mathring{q}(\beta), \mathring{r}(\beta))$ such that $\mathring{p}, \mathring{q}, \mathring{r}: \mathscr{B} \to [0, 1]$. Then, a *level soft-set* $\mathfrak{L}(\Theta, \lambda) = (\mathscr{K}_\lambda, \mathscr{B})$ of Θ represents a classical set, which is given as:

$$\mathscr{K}_\lambda(\beta) = \{\mathfrak{z} \in \mathscr{Z}|\mu_{\mathscr{K}(\beta)}(\mathfrak{z}) \geq \mathring{p}(\beta), \tau_{\mathscr{K}(\beta)}(\mathfrak{z}) \leq \mathring{q}(\beta), \nu_{\mathscr{K}(\beta)}(\mathfrak{z}) \leq \mathring{r}(\beta)\}, \quad \forall \beta \in \mathscr{B}.$$

Definition 2.5. [25] Considering the SFSS $\Theta = (\mathscr{K}, \mathscr{B})$ over $\mathscr{Z}$, following are the definitions of the four commonly used threshold functions:

1. **Mid-level Threshold Function (mid_Θ):**
   The mapping $mid_\Theta: \mathscr{B} \to [0, 1]^3$ for the SFSS $\Theta = (\mathscr{K}, \mathscr{B})$ is defined by
   $mid_\Theta(\beta) = \left(\mathring{p}_{mid_\Theta}(\beta), \mathring{q}_{mid_\Theta}(\beta), \mathring{r}_{mid_\Theta}(\beta)\right) \quad \forall x \in \mathscr{B},$

such that

$$\mathring{p}_{mid_\Theta}(\beta) = \frac{1}{|\mathscr{Z}|}\sum_{\mathfrak{z}\in\mathscr{Z}}\mu_{\mathscr{K}(\beta)}(\mathfrak{z});$$

$$\mathring{q}_{mid_\Theta}(\beta) = \frac{1}{|\mathscr{Z}|}\sum_{\mathfrak{z}\in\mathscr{Z}}\tau_{\mathscr{K}(\beta)}(\mathfrak{z});$$

$$\mathring{r}_{mid_\Theta}(\beta) = \frac{1}{|\mathscr{Z}|}\sum_{\mathfrak{z}\in\mathscr{Z}}\nu_{\mathscr{K}(\beta)}(\mathfrak{z}).$$

The mid-level soft set of Θ with respect to mid_Θ is represented by $\mathfrak{L}(\Theta, mid_\Theta(\beta))$.

2. **Top-bottom-bottom-level Threshold Function (tbb_Θ):**
The mapping tbb_Θ: $\mathscr{B} \to [0,1]^3$ for the SFSS $\Theta = (\mathscr{K}, \mathscr{B})$ is provided as:
$tbb_\Theta(\beta) = \left(\mathring{p}_{tbb_\Theta}(\beta), \mathring{q}_{tbb_\Theta}(\beta), \mathring{r}_{tbb_\Theta}(\beta)\right) \quad \forall\beta \in \mathscr{B}$,
such that
$\mathring{p}_{tbb_\Theta}(\beta) = \max_{\mathfrak{z}\in\mathscr{Z}} \mu_{\mathscr{K}(\beta)}(\mathfrak{z}); \mathring{q}_{tbb_\Theta}(\beta) = \min_{\mathfrak{z}\in\mathscr{Z}} \tau_{\mathscr{K}(\beta)}(\mathfrak{z}); \mathring{r}_{tbb_\Theta}(\beta) = \min_{\mathfrak{z}\in\mathscr{Z}} \nu_{\mathscr{K}(\beta)}(\mathfrak{z}).$

The top-bottom-bottom-level (tbb-level) soft set of Θ regarding tbb_Θ is represented by $\mathfrak{L}(\Theta, tbb_\Theta(\beta))$.

3. **Bottom-bottom-bottom-level Threshold Function (bbb_Θ):**
The mapping bbb_Θ: $\mathscr{B} \to [0,1]^3$ for the SFSS $\Theta = (\mathscr{K}, \mathscr{B})$ is defined as:
$bbb_\Theta(\beta) = \left(\mathring{p}_{bbb_\Theta}(\beta), \mathring{q}_{bbb_\Theta}(\beta), \mathring{r}_{bbb_\Theta}(\beta)\right) \quad \forall\beta \in \mathscr{B}$,
such that
$\mathring{p}_{bbb_\Theta}(\beta) = \min_{\mathfrak{z}\in\mathscr{Z}} \mu_{\mathscr{K}(\beta)}(\mathfrak{z}); \mathring{q}_{bbb_\Theta}(\beta) = \min_{\mathfrak{z}\in\mathscr{Z}} \tau_{\mathscr{K}(\beta)}(\mathfrak{z}); \mathring{r}_{bbb_\Theta}(\beta) = \min_{\mathfrak{z}\in\mathscr{Z}} \nu_{\mathscr{K}(\beta)}(\mathfrak{z}).$
The bottom-bottom-bottom-level (bbb-level) soft set of Θ concerning bbb_Θ is represented by $\mathfrak{L}(\Theta, bbb_\Theta(\beta))$.

4. **Med-level Threshold Function (med_Θ):**
The mapping med_Θ: $\mathscr{B} \to [0,1]^3$ for the SFSS $\Theta = (\mathscr{K}, \mathscr{B})$ is provided by:
$med_\Theta(\beta) = \left(\mathring{p}_{med_\Theta}(\beta), \mathring{q}_{med_\Theta}(\beta), \mathring{r}_{med_\Theta}(\beta)\right) \quad \forall\beta \in \mathscr{B}$,
such that

$$\mathring{p}_{med_\Theta}(\beta) = \begin{cases} \mu_{\mathscr{K}(\beta)}\left(\mathfrak{z}_{\left(\frac{|\mathscr{Z}|+1}{2}\right)}\right), & if\ |\mathscr{Z}|\ is\ odd, \\ \dfrac{\mu_{\mathscr{K}(\beta)}\left(\mathfrak{z}_{\left(\frac{|\mathscr{Z}|}{2}\right)}\right) + \mu_{\mathscr{K}(\beta)}\left(\mathfrak{z}_{\left(\frac{|\mathscr{Z}|}{2}+1\right)}\right)}{2}, & if\ |\mathscr{Z}|\ is\ even. \end{cases}$$

$$\mathring{q}_{med_\Theta}(\beta) = \begin{cases} \tau_{\mathscr{K}(\beta)}\left(\mathfrak{z}_{\left(\frac{|\mathscr{X}|+1}{2}\right)}\right), & if\,|\mathscr{X}|\text{ is odd},\\ \dfrac{\tau_{\mathscr{K}(\beta)}\left(\mathfrak{z}_{\left(\frac{|\mathscr{X}|}{2}\right)}\right) + \tau_{\mathscr{K}(\beta)}\left(\mathfrak{z}_{\left(\frac{|\mathscr{X}|}{2}+1\right)}\right)}{2}, & if\ |\mathscr{X}|\text{ is even}.\end{cases}$$

$$\mathring{r}_{med_\Theta}(\beta) = \begin{cases} \nu_{\mathscr{K}(\beta)}\left(\mathfrak{z}_{\left(\frac{|\mathscr{X}|+1}{2}\right)}\right), & if\ |\mathscr{X}|\text{ is odd},\\ \dfrac{\nu_{\mathscr{K}(\beta)}\left(\mathfrak{z}_{\left(\frac{|\mathscr{X}|}{2}\right)}\right) + \nu_{\mathscr{K}(\beta)}\left(\mathfrak{z}_{\left(\frac{|\mathscr{X}|}{2}+1\right)}\right)}{2}, & if\ |\mathscr{X}|\text{ is even}.\end{cases}$$

Here $\mathring{p}_{med_\Theta}(\beta), \mathring{q}_{med_\Theta}(\beta)$, and $\mathring{r}_{med_\Theta}(\beta)$ serve as the medians, which are respectively computed by ascending (or descending) arrangement of the positive, negative and neutral values. Note that the med-level soft set of Θ regarding med_Θ is represented by $\mathfrak{L}(\Theta, med_\Theta(\beta))$.

To deal with MAGDM problems concerning spherical fuzzy information, the existing notion of spherical fuzzy soft expert set (or SFSES) is defined as below:

Definition 2.6. [44] Suppose $\mathscr{X}$ is the universe, $\mathscr{P}$ is the set of parameters, $\mathscr{E}$ is the set of experts and $\mathcal{O} = \{1 = agree, 0 = disagree\}$ is the set of their opinions. Consider the set $\mathscr{Q} = \mathscr{P} \times \mathscr{E} \times \mathcal{O}$ and let $\mathscr{L} \subseteq \mathscr{Q}$. Then, the set $\mathscr{S}_{\mathcal{E}} = (\mathscr{G}, \mathscr{L})$ is called the *spherical fuzzy soft expert set* (or SFSES) over $\mathscr{X}$, where $\mathscr{G}$ is a function from $\mathscr{L}$ to $\mathbb{S}(\mathscr{X})$.

The following definition reviews the notion of bipolar soft sets:

Definition 2.7. [46] Consider $\mathscr{X}$ represents the universe of alternatives, $\mathscr{P}$ represents the universe of parameters, then the set $\Delta = (\zeta, \psi, \mathscr{P})$ is said to be a *bipolar soft set* (BSS) on the universe $\mathscr{X}$, if the functions ζ and ψ are provided as:

$$\zeta\colon \mathscr{B} \to \mathbb{P}(\mathscr{X}), \quad \psi\colon \neg\mathscr{B} \to \mathbb{P}(\mathscr{X}).$$

The set $\neg\mathscr{B}$ serves as the not-set of $\mathscr{B}$ with the parameters opposite to the parameters in $\mathscr{B}$, where

$$\zeta(\beta) \cap \psi(\neg\beta) = \emptyset, \quad \forall\beta \in \mathscr{B} \text{ and } \neg\beta \in \neg\mathscr{B},$$

whereas $\mathbb{P}(\mathscr{X})$ represents the power set of $\mathscr{X}$.

The following definition recalls the concept of bipolar soft expert sets (or BSESs), which are capable to handle bipolarity in MAGDM situations:

Definition 2.8. [47] Consider $\mathscr{X}$ represents the universe of alternatives, $\mathscr{P}$ represents the universe of parameters, $\mathscr{E}$ is the set of experts and $\mathcal{O} = \{1 = agree, 0 = disagree\}$ is the set of their estimations. Consider the set $\mathscr{Q} = \mathscr{P} \times \mathscr{E} \times \mathcal{O}$ and let $\mathscr{L} \subseteq \mathscr{Q}$. Then, the triple $\Gamma = (\hat{\zeta}, \hat{\psi}, \mathscr{L})$ is referred to as a *bipolar soft expert set* (BSES) over $\mathscr{X}$, if the functions $\hat{\zeta}$ and $\hat{\psi}$ are respectively given by

$$\hat{\zeta}\colon \mathscr{L} \to \mathbb{P}(\mathscr{X}), \quad \hat{\psi}\colon \neg\mathscr{L} \to \mathbb{P}(\mathscr{X}).$$

Here, $\neg\mathscr{L}$ serves as the not-set of $\mathscr{L}$, which contains the opinions opposite to the opinions in $\mathscr{L}$ with the following constraint:

$\hat{\zeta}(\ell) \cap \hat{\psi}(\neg\ell) = \emptyset, \quad \forall \ell \in \mathscr{L}$ and $\neg\ell \in \neg\mathscr{L}$.

3 Spherical Fuzzy Bipolar Soft Expert Sets

This section first introduces the concepts of spherical fuzzy bipolar soft expert sets (or SFBSESs), and then discusses the necessary operations and properties. The following defines the novel SFBSESs.

Definition 3.1. Let $\mathscr{Z}$ be the universe of alternatives, $\mathscr{P}$ be the set of parameters and let $\mathscr{E}$ be the set of experts with the opinions of the form $\mathcal{O} = \{1 = agree, 0 = disagree\}$. Then for $\mathscr{Q} = \mathscr{P} \times \mathscr{E} \times \mathcal{O}$, a triple $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{L})$ is called a *spherical fuzzy bipolar soft expert set* (SFBSES), if for $\mathscr{L} \subseteq \mathscr{Q}$, the functions $\hat{\zeta}$ and $\hat{\eta}$ are given as:

$\hat{\zeta} : \mathscr{L} \to \mathbb{SF}(\mathscr{Z}), \quad \hat{\eta} : \neg\mathscr{L} \to \mathbb{SF}(\mathscr{Z})$.

Here, $\neg\mathscr{L}$ is the not-set of parameters $\mathscr{L}$ with the opinions opposite to the opinions in $\mathscr{L}$, whereas $\mathbb{SF}(\mathscr{Z})$ is the family of all SFSs on $\mathscr{Z}$.

$\forall \ell \in \mathscr{L}$ and $\neg\ell \in \neg\mathscr{L}$,

$\hat{\zeta}(\ell) = \{(\mathfrak{z}, \mu_{\hat{\zeta}(\ell)}(\mathfrak{z}), \tau_{\hat{\zeta}(\ell)}(\mathfrak{z}), \nu_{\hat{\zeta}(\ell)}(\mathfrak{z})) | \mathfrak{z} \in \mathscr{Z}\}$,

$\hat{\eta}(\neg\ell) = \{(\mathfrak{z}, \mu_{\hat{\eta}(\neg\ell)}(\mathfrak{z}), \tau_{\hat{\eta}(\neg\ell)}(\mathfrak{z}), \nu_{\hat{\eta}(\neg\ell)}(\mathfrak{z})) | \mathfrak{z} \in \mathscr{Z}\}$.

such that μ, τ, and ν represent the degrees of agreement, disagreement, and neutrality, where $\forall \mathfrak{z} \in \mathscr{Z}$:

$$0 \le (\mu_{\hat{\zeta}(\ell)}(\mathfrak{z}))^2 + (\mu_{\hat{\eta}(\neg\ell)}(\mathfrak{z}))^2 \le 1 \tag{1}$$

$$0 \le (\tau_{\hat{\zeta}(\ell)}(\mathfrak{z}))^2 + (\tau_{\hat{\eta}(\neg\ell)}(\mathfrak{z}))^2 \le 1 \tag{2}$$

$$0 \le (\nu_{\hat{\zeta}(\ell)}(\mathfrak{z}))^2 + (\nu_{\hat{\eta}(\neg\ell)}(\mathfrak{z}))^2 \le 1 \tag{3}$$

The following example illustrates how SFBSESs can be used in modeling and depicting complicated decision-making situations.

Example 3.1. Consider an international bank that is trying to increase its business by effectively implementing the foreign exchange strategies. To ensure it happens, the bank needs a strong analysis as well as a valuable and efficient prediction method for the currency exchanges. However, this is a difficult and complex task. To deal with this, the bank calls its market analysts and currency exchangers to ensure the maximum profit and least risk.

These forex experts propose three strategies already working in the market, comprising the set $\mathscr{Z} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3\}$. To compare these strategies, the experts consider the set $\mathscr{B} = \{\beta_1 = \text{accurate analysis}, \beta_2 = \text{strong prediction}, \beta_3 = \text{least risk}\}$ as the collection of favorable qualities (parameters), whereas the not-set $\neg\mathscr{B} = \{\neg\beta_1 = \text{poor analysis}, \neg\beta_2 = \text{weak prediction}, \neg\beta_3 = \text{high risk}\}$ as the set of dangerous parameters for the business. Let $\mathscr{E} = \{u, v, w\}$ represents the set of experts, and $\mathcal{O} = \{1 = \text{agree}, 0 = \text{disagree}\}$ the set of opinions. The experts make a SFBSES $(\hat{\zeta}, \hat{\eta}, \mathscr{L})$ (for $\mathscr{L} = \mathscr{B} \times \mathscr{E} \times \mathcal{O}$) after a thorough analyzation of the proposed strategies as shown below:

$$\hat{\zeta}(\beta_1, u, 1) = \{\langle \mathfrak{z}_1, (0.80, 0.05, 0.20)\rangle, \langle \mathfrak{z}_2, (0.60, 0.01, 0.30)\rangle, \langle \mathfrak{z}_3, (0.70, 0.03, 0.30)\rangle\}$$
$$\hat{\zeta}(\beta_1, v, 1) = \{\langle \mathfrak{z}_1, (0.76, 0.04, 0.36)\rangle, \langle \mathfrak{z}_2, (0.80, 0.03, 0.40)\rangle, \langle \mathfrak{z}_3, (0.80, 0.04, 0.25)\rangle\}$$
$$\hat{\zeta}(\beta_1, w, 1) = \{\langle \mathfrak{z}_1, (0.90, 0.02, 0.15)\rangle, \langle \mathfrak{z}_2, (0.70, 0.04, 0.30)\rangle, \langle \mathfrak{z}_3, (0.86, 0.04, 0.30)\rangle\}$$

$$\hat{\zeta}(\beta_2, u, 1) = \{\langle \mathfrak{z}_1, (0.30, 0.10, 0.79)\rangle, \langle \mathfrak{z}_2, (0.50, 0.09, 0.60)\rangle, \langle \mathfrak{z}_3, (0.34, 0.05, 0.50)\rangle\}$$
$$\hat{\zeta}(\beta_2, v, 1) = \{\langle \mathfrak{z}_1, (0.50, 0.08, 0.70)\rangle, \langle \mathfrak{z}_2, (0.60, 0.03, 0.40)\rangle, \langle \mathfrak{z}_3, (0.70, 0.09, 0.39)\rangle\}$$
$$\hat{\zeta}(\beta_2, w, 1) = \{\langle \mathfrak{z}_1, (0.35, 0.02, 0.75)\rangle, \langle \mathfrak{z}_2, (0.40, 0.04, 0.70)\rangle, \langle \mathfrak{z}_3, (0.62, 0.08, 0.40)\rangle\}$$

$$\hat{\zeta}(\beta_3, u, 1) = \{\langle \mathfrak{z}_1, (0.70, 0.06, 0.40)\rangle, \langle \mathfrak{z}_2, (0.86, 0.06, 0.30)\rangle, \langle \mathfrak{z}_3, (0.90, 0.03, 0.15)\rangle\}$$
$$\hat{\zeta}(\beta_3, v, 1) = \{\langle \mathfrak{z}_1, (0.67, 0.03, 0.25)\rangle, \langle \mathfrak{z}_2, (0.73, 0.07, 0.10)\rangle, \langle \mathfrak{z}_3, (0.75, 0.04, 0.35)\rangle\}$$
$$\hat{\zeta}(\beta_3, w, 1) = \{\langle \mathfrak{z}_1, (0.80, 0.10, 0.30)\rangle, \langle \mathfrak{z}_2, (0.90, 0.01, 0.10)\rangle, \langle \mathfrak{z}_3, (0.70, 0.10, 0.40)\rangle\}$$

$$\hat{\zeta}(\beta_1, u, 0) = \{\langle \mathfrak{z}_1, (0.15, 0.04, 0.66)\rangle, \langle \mathfrak{z}_2, (0.45, 0.02, 0.50)\rangle, \langle \mathfrak{z}_3, (0.30, 0.10, 0.70)\rangle\}$$
$$\hat{\zeta}(\beta_1, v, 0) = \{\langle \mathfrak{z}_1, (0.30, 0.03, 0.46)\rangle, \langle \mathfrak{z}_2, (0.23, 0.03, 0.70)\rangle, \langle \mathfrak{z}_3, (0.20, 0.07, 0.73)\rangle\}$$
$$\hat{\zeta}(\beta_1, w, 0) = \{\langle \mathfrak{z}_1, (0.10, 0.05, 0.80)\rangle, \langle \mathfrak{z}_2, (0.30, 0.04, 0.60)\rangle, \langle \mathfrak{z}_3, (0.15, 0.09, 0.60)\rangle\}$$

$$\hat{\zeta}(\beta_2, u, 0) = \{\langle \mathfrak{z}_1, (0.65, 0.02, 0.40)\rangle, \langle \mathfrak{z}_2, (0.60, 0.07, 0.50)\rangle, \langle \mathfrak{z}_3, (0.65, 0.02, 0.20)\rangle\}$$
$$\hat{\zeta}(\beta_2, v, 0) = \{\langle \mathfrak{z}_1, (0.40, 0.01, 0.60)\rangle, \langle \mathfrak{z}_2, (0.55, 0.01, 0.66)\rangle, \langle \mathfrak{z}_3, (0.30, 0.03, 0.45)\rangle\}$$
$$\hat{\zeta}(\beta_2, w, 0) = \{\langle \mathfrak{z}_1, (0.70, 0.06, 0.50)\rangle, \langle \mathfrak{z}_2, (0.38, 0.10, 0.40)\rangle, \langle \mathfrak{z}_3, (0.40, 0.05, 0.36)\rangle\}$$

$$\hat{\zeta}(\beta_3, u, 0) = \{\langle \mathfrak{z}_1, (0.30, 0.10, 0.60)\rangle, \langle \mathfrak{z}_2, (0.23, 0.06, 0.60)\rangle, \langle \mathfrak{z}_3, (0.22, 0.10, 0.80)\rangle\}$$
$$\hat{\zeta}(\beta_3, v, 0) = \{\langle \mathfrak{z}_1, (0.25, 0.03, 0.80)\rangle, \langle \mathfrak{z}_2, (0.10, 0.02, 0.91)\rangle, \langle \mathfrak{z}_3, (0.33, 0.06, 0.77)\rangle\}$$
$$\hat{\zeta}(\beta_3, w, 0) = \{\langle \mathfrak{z}_1, (0.20, 0.07, 0.76)\rangle, \langle \mathfrak{z}_2, (0.24, 0.01, 0.40)\rangle, \langle \mathfrak{z}_3, (0.40, 0.07, 0.35)\rangle\}$$

$$\hat{\eta}(\neg\beta_1, u, 1) = \{\langle \mathfrak{z}_1, (0.20, 0.02, 0.70)\rangle, \langle \mathfrak{z}_2, (0.40, 0.03, 0.60)\rangle, \langle \mathfrak{z}_3, (0.32, 0.01, 0.60)\rangle\}$$
$$\hat{\eta}(\neg\beta_1, v, 1) = \{\langle \mathfrak{z}_1, (0.30, 0.03, 0.60)\rangle, \langle \mathfrak{z}_2, (0.25, 0.04, 0.20)\rangle, \langle \mathfrak{z}_3, (0.30, 0.02, 0.70)\rangle\}$$
$$\hat{\eta}(\neg\beta_1, w, 1) = \{\langle \mathfrak{z}_1, (0.15, 0.04, 0.70)\rangle, \langle \mathfrak{z}_2, (0.13, 0.03, 0.73)\rangle, \langle \mathfrak{z}_3, (0.20, 0.05, 0.65)\rangle\}$$

$$\hat{\eta}(\neg\beta_2, u, 1) = \{\langle \mathfrak{z}_1, (0.60, 0.03, 0.36)\rangle, \langle \mathfrak{z}_2, (0.42, 0.04, 0.39)\rangle, \langle \mathfrak{z}_3, (0.62, 0.06, 0.42)\rangle\}$$
$$\hat{\eta}(\neg\beta_2, v, 1) = \{\langle \mathfrak{z}_1, (0.40, 0.07, 0.54)\rangle, \langle \mathfrak{z}_2, (0.70, 0.01, 0.42)\rangle, \langle \mathfrak{z}_3, (0.30, 0.09, 0.36)\rangle\}$$
$$\hat{\eta}(\neg\beta_2, w, 1) = \{\langle \mathfrak{z}_1, (0.30, 0.08, 0.65)\rangle, \langle \mathfrak{z}_2, (0.60, 0.10, 0.40)\rangle, \langle \mathfrak{z}_3, (0.30, 0.02, 0.70)\rangle\}$$

$$\hat{\eta}(\neg\beta_3, u, 1) = \{\langle\mathfrak{z}_1, (0.50, 0.06, 0.70)\rangle, \langle\mathfrak{z}_2, (0.40, 0.03, 0.70)\rangle, \langle\mathfrak{z}_3, (0.12, 0.06, 0.68)\rangle\}$$
$$\hat{\eta}(\neg\beta_3, v, 1) = \{\langle\mathfrak{z}_1, (0.32, 0.01, 0.72)\rangle, \langle\mathfrak{z}_2, (0.40, 0.05, 0.64)\rangle, \langle\mathfrak{z}_3, (0.22, 0.07, 0.73)\rangle\}$$
$$\hat{\eta}(\neg\beta_3, w, 1) = \{\langle\mathfrak{z}_1, (0.30, 0.02, 0.70)\rangle, \langle\mathfrak{z}_2, (0.15, 0.16, 0.89)\rangle, \langle\mathfrak{z}_3, (0.25, 0.02, 0.80)\rangle\}$$

$$\hat{\eta}(\neg\beta_1, u, 0) = \{\langle\mathfrak{z}_1, (0.80, 0.03, 0.15)\rangle, \langle\mathfrak{z}_2, (0.65, 0.06, 0.46)\rangle, \langle\mathfrak{z}_3, (0.70, 0.01, 0.30)\rangle\}$$
$$\hat{\eta}(\neg\beta_1, v, 0) = \{\langle\mathfrak{z}_1, (0.65, 0.04, 0.30)\rangle, \langle\mathfrak{z}_2, (0.70, 0.07, 0.24)\rangle, \langle\mathfrak{z}_3, (0.80, 0.01, 0.15)\rangle\}$$
$$\hat{\eta}(\neg\beta_1, w, 0) = \{\langle\mathfrak{z}_1, (0.78, 0.06, 0.20)\rangle, \langle\mathfrak{z}_2, (0.70, 0.03, 0.30)\rangle, \langle\mathfrak{z}_3, (0.60, 0.05, 0.28)\rangle\}$$

$$\hat{\eta}(\neg\beta_2, u, 0) = \{\langle\mathfrak{z}_1, (0.40, 0.05, 0.60)\rangle, \langle\mathfrak{z}_2, (0.60, 0.04, 0.32)\rangle, \langle\mathfrak{z}_3, (0.40, 0.03, 0.66)\rangle\}$$
$$\hat{\eta}(\neg\beta_2, v, 0) = \{\langle\mathfrak{z}_1, (0.50, 0.01, 0.65)\rangle, \langle\mathfrak{z}_2, (0.36, 0.05, 0.55)\rangle, \langle\mathfrak{z}_3, (0.70, 0.04, 0.32)\rangle\}$$
$$\hat{\eta}(\neg\beta_2, w, 0) = \{\langle\mathfrak{z}_1, (0.70, 0.01, 0.32)\rangle, \langle\mathfrak{z}_2, (0.30, 0.01, 0.72)\rangle, \langle\mathfrak{z}_3, (0.65, 0.09, 0.40)\rangle\}$$

$$\hat{\eta}(\neg\beta_3, u, 0) = \{\langle\mathfrak{z}_1, (0.50, 0.02, 0.60)\rangle, \langle\mathfrak{z}_2, (0.29, 0.02, 0.70)\rangle, \langle\mathfrak{z}_3, (0.80, 0.06, 0.21)\rangle\}$$
$$\hat{\eta}(\neg\beta_3, v, 0) = \{\langle\mathfrak{z}_1, (0.67, 0.10, 0.30)\rangle, \langle\mathfrak{z}_2, (0.60, 0.10, 0.30)\rangle, \langle\mathfrak{z}_3, (0.63, 0.08, 0.35)\rangle\}$$
$$\hat{\eta}(\neg\beta_3, w, 0) = \{\langle\mathfrak{z}_1, (0.59, 0.03, 0.41)\rangle, \langle\mathfrak{z}_2, (0.80, 0.06, 0.10)\rangle, \langle\mathfrak{z}_3, (0.70, 0.05, 0.28)\rangle\}$$

Tables 4 and 5 represent the formulated SFBSES in terms of $\hat{\zeta}$ and $\hat{\eta}$, where $(\hat{\zeta}, \mathscr{L})$ and $(\hat{\eta}, \neg\mathscr{L})$ are SFSESs. For $\ell_i \in \mathscr{L}$, $\neg\ell_i \in \neg\mathscr{L}$, and $\mathfrak{z}_j \in \mathscr{Z}$, Table 6 represents the SFBSES with entries.

Table 4: SFSES $(\hat{\zeta}, \mathscr{L})$ regarding set of parameters $\mathscr{L}$

$(\hat{\zeta}, \mathscr{L})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$(\beta_1, u, 1)$	(0.80, 0.05, 0.20)	(0.60, 0.01, 0.30)	(0.70, 0.03, 0.30)
$(\beta_1, v, 1)$	(0.76, 0.04, 0.36)	(0.80, 0.03, 0.40)	(0.80, 0.04, 0.25)
$(\beta_1, w, 1)$	(0.90, 0.02, 0.15)	(0.70, 0.04, 0.30)	(0.86, 0.04, 0.30)
$(\beta_2, u, 1)$	(0.30, 0.10, 0.79)	(0.50, 0.09, 0.60)	(0.34, 0.05, 0.50)
$(\beta_2, v, 1)$	(0.50, 0.08, 0.70)	(0.60, 0.03, 0.40)	(0.70, 0.09, 0.39)
$(\beta_2, w, 1)$	(0.35, 0.02, 0.75)	(0.40, 0.04, 0.70)	(0.62, 0.08, 0.40)
$(\beta_3, u, 1)$	(0.70, 0.06, 0.40)	(0.86, 0.06, 0.30)	(0.90, 0.03, 0.15)
$(\beta_3, v, 1)$	(0.67, 0.03, 0.25)	(0.73, 0.07, 0.10)	(0.75, 0.04, 0.35)
$(\beta_3, w, 1)$	(0.80, 0.10, 0.30)	(0.90, 0.01, 0.10)	(0.70, 0.10, 0.40)
$(\beta_1, u, 0)$	(0.15, 0.04, 0.66)	(0.45, 0.02, 0.50)	(0.30, 0.10, 0.70)
$(\beta_1, v, 0)$	(0.30, 0.03, 0.46)	(0.23, 0.03, 0.70)	(0.20, 0.07, 0.73)
$(\beta_1, w, 0)$	(0.10, 0.05, 0.80)	(0.30, 0.04, 0.60)	(0.15, 0.09, 0.60)
$(\beta_2, u, 0)$	(0.65, 0.02, 0.40)	(0.60, 0.07, 0.50)	(0.65, 0.02, 0.20)
$(\beta_2, v, 0)$	(0.40, 0.01, 0.60)	(0.55, 0.01, 0.66)	(0.30, 0.03, 0.45)
$(\beta_2, w, 0)$	(0.70, 0.06, 0.50)	(0.38, 0.10, 0.40)	(0.40, 0.05, 0.36)
$(\beta_3, u, 0)$	(0.30, 0.10, 0.60)	(0.23, 0.06, 0.60)	(0.22, 0.10, 0.80)
$(\beta_3, v, 0)$	(0.25, 0.03, 0.80)	(0.10, 0.02, 0.91)	(0.33, 0.06, 0.77)
$(\beta_3, w, 0)$	(0.20, 0.07, 0.76)	(0.24, 0.01, 0.40)	(0.40, 0.07, 0.35)

Table 5: SFSES $(\hat{\eta}, \neg\mathscr{L})$ concerning not-set of parameters $\neg\mathscr{L}$

$(\hat{\eta}, \neg\mathscr{L})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$(\neg\beta_1, u, 1)$	(0.20, 0.02, 0.70)	(0.40, 0.03, 0.60)	(0.32, 0.01, 0.60)
$(\neg\beta_1, v, 1)$	(0.30, 0.03, 0.60)	(0.25, 0.04, 0.20)	(0.30, 0.02, 0.70)
$(\neg\beta_1, w, 1)$	(0.15, 0.04, 0.70)	(0.13, 0.03, 0.73)	(0.20, 0.05, 0.65)
$(\neg\beta_2, u, 1)$	(0.60, 0.03, 0.36)	(0.42, 0.04, 0.39)	(0.62, 0.06, 0.42)
$(\neg\beta_2, v, 1)$	(0.40, 0.07, 0.54)	(0.70, 0.01, 0.42)	(0.30, 0.09, 0.36)
$(\neg\beta_2, w, 1)$	(0.30, 0.08, 0.65)	(0.60, 0.10, 0.40)	(0.30, 0.02, 0.70)
$(\neg\beta_3, u, 1)$	(0.50, 0.06, 0.70)	(0.40, 0.03, 0.70)	(0.12, 0.06, 0.68)
$(\neg\beta_3, v, 1)$	(0.32, 0.01, 0.72)	(0.40, 0.05, 0.64)	(0.22, 0.07, 0.73)
$(\neg\beta_3, w, 1)$	(0.30, 0.02, 0.70)	(0.15, 0.16, 0.89)	(0.25, 0.02, 0.80)
$(\neg\beta_1, u, 0)$	(0.80, 0.03, 0.15)	(0.65, 0.06, 0.46)	(0.70, 0.01, 0.30)
$(\neg\beta_1, v, 0)$	(0.65, 0.04, 0.30)	(0.70, 0.07, 0.24)	(0.80, 0.01, 0.15)
$(\neg\beta_1, w, 0)$	(0.78, 0.06, 0.20)	(0.70, 0.03, 0.30)	(0.60, 0.05, 0.28)
$(\neg\beta_2, u, 0)$	(0.40, 0.05, 0.60)	(0.60, 0.04, 0.32)	(0.40, 0.03, 0.66)
$(\neg\beta_2, v, 0)$	(0.50, 0.01, 0.65)	(0.36, 0.05, 0.55)	(0.70, 0.04, 0.32)
$(\neg\beta_2, w, 0)$	(0.70, 0.01, 0.32)	(0.30, 0.01, 0.72)	(0.65, 0.09, 0.40)
$(\neg\beta_3, u, 0)$	(0.50, 0.02, 0.60)	(0.29, 0.02, 0.70)	(0.80, 0.06, 0.21)
$(\neg\beta_3, v, 0)$	(0.67, 0.10, 0.30)	(0.60, 0.10, 0.30)	(0.63, 0.08, 0.35)
$(\neg\beta_3, w, 0)$	(0.59, 0.03, 0.41)	(0.80, 0.06, 0.10)	(0.70, 0.05, 0.28)

Table 6: SFBSES $(\hat{\zeta}, \hat{\eta}, \mathscr{L})$ in Example 3.1

$(\hat{\zeta}, \hat{\eta}, \mathscr{L})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$(\beta_1, u, 1)$	$\begin{pmatrix}(0.80, 0.05, 0.20)\\(0.20, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.01, 0.30)\\(0.40, 0.03, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.03, 0.30)\\(0.32, 0.01, 0.60)\end{pmatrix}$
$(\beta_1, v, 1)$	$\begin{pmatrix}(0.76, 0.04, 0.36)\\(0.30, 0.03, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.80, 0.03, 0.40)\\(0.25, 0.04, 0.20)\end{pmatrix}$	$\begin{pmatrix}(0.80, 0.04, 0.25)\\(0.30, 0.02, 0.70)\end{pmatrix}$
$(\beta_1, w, 1)$	$\begin{pmatrix}(0.90, 0.02, 0.15)\\(0.15, 0.04, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.04, 0.30)\\(0.13, 0.03, 0.73)\end{pmatrix}$	$\begin{pmatrix}(0.86, 0.04, 0.30)\\(0.20, 0.05, 0.65)\end{pmatrix}$
$(\beta_2, u, 1)$	$\begin{pmatrix}(0.30, 0.10, 0.79)\\(0.60, 0.03, 0.36)\end{pmatrix}$	$\begin{pmatrix}(0.50, 0.09, 0.60)\\(0.42, 0.04, 0.39)\end{pmatrix}$	$\begin{pmatrix}(0.34, 0.05, 0.50)\\(0.62, 0.06, 0.42)\end{pmatrix}$
$(\beta_2, v, 1)$	$\begin{pmatrix}(0.50, 0.08, 0.70)\\(0.40, 0.07, 0.54)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.03, 0.40)\\(0.70, 0.01, 0.42)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.09, 0.39)\\(0.30, 0.09, 0.36)\end{pmatrix}$
$(\beta_2, w, 1)$	$\begin{pmatrix}(0.35, 0.02, 0.75)\\(0.30, 0.08, 0.65)\end{pmatrix}$	$\begin{pmatrix}(0.40, 0.04, 0.70)\\(0.60, 0.10, 0.40)\end{pmatrix}$	$\begin{pmatrix}(0.62, 0.08, 0.40)\\(0.30, 0.02, 0.70)\end{pmatrix}$
$(\beta_3, u, 1)$	$\begin{pmatrix}(0.70, 0.06, 0.40)\\(0.50, 0.06, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.86, 0.06, 0.30)\\(0.40, 0.03, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.90, 0.03, 0.15)\\(0.12, 0.06, 0.68)\end{pmatrix}$
$(\beta_3, v, 1)$	$\begin{pmatrix}(0.67, 0.03, 0.25)\\(0.32, 0.01, 0.72)\end{pmatrix}$	$\begin{pmatrix}(0.73, 0.07, 0.10)\\(0.40, 0.05, 0.64)\end{pmatrix}$	$\begin{pmatrix}(0.75, 0.04, 0.35)\\(0.22, 0.07, 0.73)\end{pmatrix}$

(Continued)

Table 6 (continued)

$(\hat{\zeta}, \hat{\eta}, \mathscr{L})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$(\beta_3, w, 1)$	$\begin{pmatrix}(0.80, 0.10, 0.30)\\(0.30, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.90, 0.01, 0.10)\\(0.15, 0.16, 0.89)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.10, 0.40)\\(0.25, 0.02, 0.80)\end{pmatrix}$
$(\beta_1, u, 0)$	$\begin{pmatrix}(0.15, 0.04, 0.66)\\(0.80, 0.03, 0.15)\end{pmatrix}$	$\begin{pmatrix}(0.45, 0.02, 0.50)\\(0.65, 0.06, 0.46)\end{pmatrix}$	$\begin{pmatrix}(0.30, 0.10, 0.70)\\(0.70, 0.01, 0.30)\end{pmatrix}$
$(\beta_1, v, 0)$	$\begin{pmatrix}(0.30, 0.03, 0.46)\\(0.65, 0.04, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.23, 0.03, 0.70)\\(0.70, 0.07, 0.24)\end{pmatrix}$	$\begin{pmatrix}(0.20, 0.07, 0.73)\\(0.80, 0.01, 0.15)\end{pmatrix}$
$(\beta_1, w, 0)$	$\begin{pmatrix}(0.10, 0.05, 0.80)\\(0.78, 0.06, 0.20)\end{pmatrix}$	$\begin{pmatrix}(0.30, 0.04, 0.60)\\(0.70, 0.03, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.15, 0.09, 0.60)\\(0.60, 0.05, 0.28)\end{pmatrix}$
$(\beta_2, u, 0)$	$\begin{pmatrix}(0.65, 0.02, 0.40)\\(0.40, 0.05, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.07, 0.50)\\(0.60, 0.04, 0.32)\end{pmatrix}$	$\begin{pmatrix}(0.65, 0.02, 0.20)\\(0.40, 0.03, 0.66)\end{pmatrix}$
$(\beta_2, v, 0)$	$\begin{pmatrix}(0.40, 0.01, 0.60)\\(0.50, 0.01, 0.65)\end{pmatrix}$	$\begin{pmatrix}(0.55, 0.01, 0.66)\\(0.36, 0.05, 0.55)\end{pmatrix}$	$\begin{pmatrix}(0.30, 0.03, 0.45)\\(0.70, 0.04, 0.32)\end{pmatrix}$
$(\beta_2, w, 0)$	$\begin{pmatrix}(0.70, 0.06, 0.50)\\(0.70, 0.01, 0.32)\end{pmatrix}$	$\begin{pmatrix}(0.38, 0.10, 0.40)\\(0.30, 0.01, 0.72)\end{pmatrix}$	$\begin{pmatrix}(0.40, 0.05, 0.36)\\(0.65, 0.09, 0.40)\end{pmatrix}$
$(\beta_3, u, 0)$	$\begin{pmatrix}(0.30, 0.10, 0.60)\\(0.50, 0.02, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.23, 0.06, 0.60)\\(0.29, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.22, 0.10, 0.80)\\(0.80, 0.06, 0.21)\end{pmatrix}$
$(\beta_3, v, 0)$	$\begin{pmatrix}(0.25, 0.03, 0.80)\\(0.67, 0.10, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.10, 0.02, 0.91)\\(0.60, 0.10, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.33, 0.06, 0.77)\\(0.63, 0.08, 0.35)\end{pmatrix}$
$(\beta_3, w, 0)$	$\begin{pmatrix}(0.20, 0.07, 0.76)\\(0.59, 0.03, 0.41)\end{pmatrix}$	$\begin{pmatrix}(0.24, 0.01, 0.40)\\(0.80, 0.06, 0.10)\end{pmatrix}$	$\begin{pmatrix}(0.40, 0.07, 0.35)\\(0.70, 0.05, 0.28)\end{pmatrix}$

$$k_{ij} = \begin{pmatrix} \hat{\zeta}(\ell_i)(\mathfrak{z}_j) \\ \hat{\eta}(\neg\ell_i)(\mathfrak{z}_j) \end{pmatrix},$$

whereas Table 7 represents the general form of the single tabular format for SFBSESs. It can be observed from Table 6, how the opinions of the experts differ from each other according to their expertise. This flexible opinion set provides better insight and analysis for the complex scenario and hence offers unbiased and more reliable handling of the decision problem under spherical fuzzy information, while considering the bipolarity of the affecting parameters.

Table 7: General tabular representation of SFBSES $(\hat{\zeta}, \hat{\eta}, \mathscr{L})$

$(\hat{\zeta}, \hat{\eta}, \mathscr{L})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\cdots$	$\mathfrak{z}_n$
ℓ_1	$\begin{pmatrix}(\mu_{\hat{\zeta}(\ell_1)}, \tau_{\hat{\zeta}(\ell_1)}, \nu_{\hat{\zeta}(\ell_1)})\\(\mu_{\hat{\eta}(\neg\ell_1)}, \tau_{\hat{\eta}(\neg\ell_1)}, \nu_{\hat{\eta}(\neg\ell_1)})\end{pmatrix}_{11}$	$\begin{pmatrix}(\mu_{\hat{\zeta}(\ell_1)}, \tau_{\hat{\zeta}(\ell_1)}, \nu_{\hat{\zeta}(\ell_1)})\\(\mu_{\hat{\eta}(\neg\ell_1)}, \tau_{\hat{\eta}(\neg\ell_1)}, \nu_{\hat{\eta}(\neg\ell_1)})\end{pmatrix}_{12}$	$\cdots$	$\begin{pmatrix}(\mu_{\hat{\zeta}(\ell_1)}, \tau_{\hat{\zeta}(\ell_1)}, \nu_{\hat{\zeta}(\ell_1)})\\(\mu_{\hat{\eta}(\neg\ell_1)}, \tau_{\hat{\eta}(\neg\ell_1)}, \nu_{\hat{\eta}(\neg\ell_1)})\end{pmatrix}_{1n}$

(Continued)

Table 7 (continued)

$(\hat{\zeta}, \hat{\eta}, \mathscr{L})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\cdots$	$\mathfrak{z}_n$
ℓ_2	$\begin{pmatrix}(\mu_{\hat{\zeta}(\ell_2)}, \tau_{\hat{\zeta}(\ell_2)}, \nu_{\hat{\zeta}(\ell_2)}) \\ (\mu_{\hat{\eta}(\neg\ell_2)}, \tau_{\hat{\eta}(\neg\ell_2)}, \nu_{\hat{\eta}(\neg\ell_2)})\end{pmatrix}_{21}$	$\begin{pmatrix}(\mu_{\hat{\zeta}(\ell_2)}, \tau_{\hat{\zeta}(\ell_2)}, \nu_{\hat{\zeta}(\ell_2)}) \\ (\mu_{\hat{\eta}(\neg\ell_2)}, \tau_{\hat{\eta}(\neg\ell_2)}, \nu_{\hat{\eta}(\neg\ell_2)})\end{pmatrix}_{22}$	$\cdots$	$\begin{pmatrix}(\mu_{\hat{\zeta}(\ell_2)}, \tau_{\hat{\zeta}(\ell_2)}, \nu_{\hat{\zeta}(\ell_2)}) \\ (\mu_{\hat{\eta}(\neg\ell_2)}, \tau_{\hat{\eta}(\neg\ell_2)}, \nu_{\hat{\eta}(\neg\ell_2)})\end{pmatrix}_{2n}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ℓ_m	$\begin{pmatrix}(\mu_{\hat{\zeta}(\ell_m)}, \tau_{\hat{\zeta}(\ell_m)}, \nu_{\hat{\zeta}(\ell_m)}) \\ (\mu_{\hat{\eta}(\neg\ell_m)}, \tau_{\hat{\eta}(\neg\ell_m)}, \nu_{\hat{\eta}(\neg\ell_m)})\end{pmatrix}_{m1}$	$\begin{pmatrix}(\mu_{\hat{\zeta}(\ell_m)}, \tau_{\hat{\zeta}(\ell_m)}, \nu_{\hat{\zeta}(\ell_m)}) \\ (\mu_{\hat{\eta}(\neg\ell_m)}, \tau_{\hat{\eta}(\neg\ell_m)}, \nu_{\hat{\eta}(\neg\ell_m)})\end{pmatrix}_{m2}$	$\cdots$	$\begin{pmatrix}(\mu_{\hat{\zeta}(\ell_m)}, \tau_{\hat{\zeta}(\ell_m)}, \nu_{\hat{\zeta}(\ell_m)}) \\ (\mu_{\hat{\eta}(\neg\ell_m)}, \tau_{\hat{\eta}(\neg\ell_m)}, \nu_{\hat{\eta}(\neg\ell_m)})\end{pmatrix}_{mn}$

Remark 3.1. The above example demonstrates a problem that could not be fully represented by any of the previous tools. SFSESs fail to consider the bipolarity of parameters, BSESs fail to tackle the complicated spherical fuzzy uncertainties in the problem and SFBSSs cannot manage opinions of multiple experts in a single place. This highlights the need for the proposed work which unifies all these models into a single model providing superior uncertainty handling under multiple experts while considering the bipolar behavior of governing decision-parameters.

The coming definition defines the SFBSE subset.

Definition 3.2. A SFBSES $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{L})$ is called the *SFBSE subset* of a SFBSES $\Lambda = (\hat{\pi}, \hat{\psi}, \mathscr{M})$, if and only if:

1. $\mathscr{L} \subseteq \mathscr{M}$.
2. $(\hat{\zeta}, \mathscr{L})$ is a spherical fuzzy soft expert subset of $(\hat{\pi}, \mathscr{M})$.
3. $(\hat{\psi}, \neg\mathscr{M})$ is a spherical fuzzy soft expert subset of $(\hat{\eta}, \neg\mathscr{L})$.

$\Gamma \hat{\subseteq} \Lambda$ represents the above subset relation. In the same way, Λ is called the superset of Γ and the relation is represented by $\Lambda \hat{\supseteq} \Gamma$.

Remark 3.2. Analytically, a SFBSE-subset shows lesser suitability in a decision scenario as compared to that shown by the respective super-set. This lesser suitability or deprived favorability is expressed in the form of missing considerations (relatively fewer decision-parameters) accompanied by lower supporting membership degrees. For instance, two SFBSESs are reported for a firm's performance at different times with the second set being the subset of the first set. The subset relation then indicates a decline in the firm's performance with increasing time.

Example 3.2. Recall Example 3.1. Suppose that two SFBSESs $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{M})$ and $\Lambda = (\hat{\pi}, \hat{\psi}, \mathscr{N})$ are respectively presented by Tables 8 and 9 where $\mathscr{M} = \{(\beta_1, u, 1), (\beta_3, v, 1), (\beta_2, w, 0)\}$ and $\mathscr{N} = \{(\beta_1, u, 1), (\beta_3, v, 1), (\beta_2, u, 0), (\beta_2, w, 0)\}$.

Clearly, $\Gamma \hat{\subseteq} \Lambda$.

Table 8: SFBSES $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{M})$ in Example 3.2

$(\hat{\zeta}, \hat{\eta}, \mathscr{M})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$(\beta_1, u, 1)$	$\begin{pmatrix}(0.80, 0.05, 0.20) \\ (0.20, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.01, 0.30) \\ (0.40, 0.03, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.03, 0.30) \\ (0.32, 0.01, 0.60)\end{pmatrix}$

(Continued)

Table 8 (continued)

$(\hat{\zeta}, \hat{\eta}, \mathscr{M})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$(\beta_3, v, 1)$	$\begin{pmatrix}(0.67, 0.03, 0.25)\\(0.32, 0.01, 0.72)\end{pmatrix}$	$\begin{pmatrix}(0.73, 0.07, 0.10)\\(0.40, 0.05, 0.64)\end{pmatrix}$	$\begin{pmatrix}(0.75, 0.04, 0.35)\\(0.22, 0.07, 0.73)\end{pmatrix}$
$(\beta_2, w, 0)$	$\begin{pmatrix}(0.70, 0.06, 0.50)\\(0.70, 0.01, 0.32)\end{pmatrix}$	$\begin{pmatrix}(0.38, 0.10, 0.40)\\(0.30, 0.01, 0.72)\end{pmatrix}$	$\begin{pmatrix}(0.40, 0.05, 0.36)\\(0.65, 0.09, 0.40)\end{pmatrix}$

Table 9: SFBSES $\Lambda = (\hat{\pi}, \hat{\psi}, \mathscr{N})$ in Example 3.2

$(\hat{\pi}, \hat{\psi}, \mathscr{N})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$(\beta_1, u, 1)$	$\begin{pmatrix}(0.90, 0.05, 0.10)\\(0.10, 0.02, 0.80)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.01, 0.30)\\(0.35, 0.03, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.75, 0.03, 0.20)\\(0.30, 0.01, 0.65)\end{pmatrix}$
$(\beta_3, v, 1)$	$\begin{pmatrix}(0.73, 0.03, 0.20)\\(0.30, 0.01, 0.75)\end{pmatrix}$	$\begin{pmatrix}(0.80, 0.07, 0.10)\\(0.40, 0.05, 0.64)\end{pmatrix}$	$\begin{pmatrix}(0.82, 0.04, 0.30)\\(0.20, 0.07, 0.76)\end{pmatrix}$
$(\beta_2, u, 0)$	$\begin{pmatrix}(0.60, 0.05, 0.20)\\(0.30, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.01, 0.30)\\(0.40, 0.03, 0.50)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.03, 0.30)\\(0.32, 0.01, 0.70)\end{pmatrix}$
$(\beta_2, w, 0)$	$\begin{pmatrix}(0.75, 0.06, 0.30)\\(0.50, 0.01, 0.42)\end{pmatrix}$	$\begin{pmatrix}(0.50, 0.10, 0.30)\\(0.20, 0.01, 0.73)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.05, 0.16)\\(0.60, 0.09, 0.45)\end{pmatrix}$

Definition 3.3. Any two SFBSESs Γ and Λ are *equal*, if and only if they are both SFBSE subsets of each other. That is,

$\Gamma = \Lambda \iff \Gamma \hat{\subseteq} \Lambda$ and $\Lambda \hat{\subseteq} \Gamma$.

Based on the concept of complement of SFSs, the complement of a SFBSES is defined as:

Definition 3.4. The *complement of SFBSES* $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{L})$ is given by $\Gamma^c = (\hat{\zeta}^c, \hat{\eta}^c, \mathscr{L})$ where $\forall \ell \in \mathscr{L}$ and $\neg\ell \in \neg\mathscr{L}$, the mappings $\hat{\zeta}^c \colon \mathscr{L} \to \mathbb{SF}(\mathscr{Z})$ and $\hat{\eta}^c \colon \neg\mathscr{L} \to \mathbb{SF}(\mathscr{Z})$ are provided by:

$$\hat{\zeta}^c(\ell) = \{(\mathfrak{z}, \nu_{\hat{\zeta}(\ell)}(\mathfrak{z}), \tau_{\hat{\zeta}(\ell)}(\mathfrak{z}), \mu_{\hat{\zeta}(\ell)}(\mathfrak{z})) | \mathfrak{z} \in \mathscr{Z}\},$$

$$\hat{\eta}^c(\neg\ell) = \{(\mathfrak{z}, \nu_{\hat{\eta}(\neg\ell)}(\mathfrak{z}), \tau_{\hat{\eta}(\neg\ell)}(\mathfrak{z}), \mu_{\hat{\eta}(\neg\ell)}(\mathfrak{z})) | \mathfrak{z} \in \mathscr{Z}\}.$$

Example 3.3. Consider the SFBSES $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{M})$ as in Example 3.2. By Definition 3.4, its complement $\Gamma^c = (\hat{\zeta}^c, \hat{\eta}^c, \mathscr{M})$ is given in Table 10.

Table 10: SFBSE Complement of $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{M})$ in Example 3.3

$(\hat{\zeta}^c, \hat{\eta}^c, \mathscr{M})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$(\beta_1, u, 1)$	$\begin{pmatrix}(0.20, 0.05, 0.80)\\(0.70, 0.02, 0.20)\end{pmatrix}$	$\begin{pmatrix}(0.30, 0.01, 0.60)\\(0.60, 0.03, 0.40)\end{pmatrix}$	$\begin{pmatrix}(0.30, 0.03, 0.70)\\(0.60, 0.01, 0.32)\end{pmatrix}$
$(\beta_3, v, 1)$	$\begin{pmatrix}(0.25, 0.03, 0.67)\\(0.72, 0.01, 0.32)\end{pmatrix}$	$\begin{pmatrix}(0.10, 0.07, 0.73)\\(0.64, 0.05, 0.40)\end{pmatrix}$	$\begin{pmatrix}(0.35, 0.04, 0.75)\\(0.73, 0.07, 0.22)\end{pmatrix}$
$(\beta_2, w, 0)$	$\begin{pmatrix}(0.50, 0.06, 0.70)\\(0.32, 0.01, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.40, 0.10, 0.38)\\(0.72, 0.01, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.36, 0.05, 0.40)\\(0.40, 0.09, 0.65)\end{pmatrix}$

Proposition 3.1. Let $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{L})$ be a SFBSES on $\mathscr{Z}$, then

1. $(\Gamma^c)^c = \Gamma$

Proof.

1. By Definition 3.4, $\Gamma^c = (\hat{\zeta}, \hat{\eta}, \mathscr{L})^c = (\hat{\zeta}^c, \hat{\eta}^c, \mathscr{L})$ such that

$$\hat{\zeta}^c(\ell) = \{(\mathfrak{z}, \nu_{\hat{\zeta}(\ell)}(\mathfrak{z}), \tau_{\hat{\zeta}(\ell)}(\mathfrak{z}), \mu_{\hat{\zeta}(\ell)}(\mathfrak{z})) | \mathfrak{z} \in \mathscr{Z}\},$$

$$\hat{\eta}^c(\neg\ell) = \{(\mathfrak{z}, \nu_{\hat{\eta}(\neg\ell)}(\mathfrak{z}), \tau_{\hat{\eta}(\neg\ell)}(\mathfrak{z}), \mu_{\hat{\eta}(\neg\ell)}(\mathfrak{z})) | \mathfrak{z} \in \mathscr{Z}\},$$

for all $\ell \in \mathscr{L}$ and $\neg\ell \in \neg\mathscr{L}$. This infers that

$$(\hat{\zeta}^c(\ell))^c = \{(\mathfrak{z}, \mu_{\hat{\zeta}(\ell)}(\mathfrak{z}), \tau_{\hat{\zeta}(\ell)}(\mathfrak{z}), \nu_{\hat{\zeta}(\ell)}(\mathfrak{z})) | \mathfrak{z} \in \mathscr{Z}\} = \hat{\zeta},$$

$$(\hat{\eta}^c(\neg\ell))^c = \{(\mathfrak{z}, \mu_{\hat{\eta}(\neg\ell)}(\mathfrak{z}), \tau_{\hat{\eta}(\neg\ell)}(\mathfrak{z}), \nu_{\hat{\eta}(\neg\ell)}(\mathfrak{z})) | \mathfrak{z} \in \mathscr{Z}\} = \hat{\eta}.$$

Thus, $((\hat{\zeta}, \hat{\eta}, \mathscr{L})^c)^c = (\hat{\zeta}, \hat{\eta}, \mathscr{L})$, or $(\Gamma^c)^c = \Gamma$.

The relative null and relative absolute SFBSESs over $\mathscr{Z}$ are provided in the following Definition 3.5.

Definition 3.5. A SFBSES $\hat{\Phi} = (\hat{\phi}, \hat{U}, \mathscr{L})$ is called a *relative null SFBSES* on the universe $\mathscr{Z}$, if $\forall \ell \in \mathscr{L}$,

$\hat{\phi}(\ell) = \{(\mathfrak{z}, 0, 0, 1) | \mathfrak{z} \in \mathscr{Z}\}$,

and $\forall \neg\ell \in \neg\mathscr{L}$,

$\hat{U}(\neg\ell) = \{(\mathfrak{z}, 1, 0, 0) | \mathfrak{z} \in \mathscr{Z}\}$.

Similarly, a SFBSES $\hat{\mathfrak{U}} = (\hat{U}, \hat{\phi}, \mathscr{L})$ is called a *relative absolute SFBSES* on universe $\mathscr{Z}$, if $\forall \ell \in \mathscr{L}$,

$\hat{U}(\ell) = \{(\mathfrak{z}, 1, 0, 0) | \mathfrak{z} \in \mathscr{Z}\}$,

and $\forall \neg\ell \in \neg\mathscr{L}$,

$\hat{\phi}(\neg\ell) = \{(\mathfrak{z}, 0, 0, 1) | \mathfrak{z} \in \mathscr{Z}\}$.

Proposition 3.2. Consider the relative null and relative absolute SFBSESs, i.e., $\hat{\Phi}$ and $\hat{\mathfrak{U}}$, respectively, then

1. $\hat{\Phi}^c = \hat{\mathfrak{U}}$
2. $\hat{\mathfrak{U}}^c = \hat{\Phi}$

Proof. The proofs are obvious from the Definitions 3.4 and 3.5.

The following Definition 3.6 explains the concepts of agree and disagree SFBSESs:

Definition 3.6. The *agree SFBSES* $(\hat{\zeta}, \hat{\eta}, \mathscr{L})_1$ over the universe $\mathscr{Z}$ is the SFBSE subset of $(\hat{\zeta}, \hat{\eta}, \mathscr{L})$ defined as:

$(\hat{\zeta}, \hat{\eta}, \mathscr{L})_1 = \hat{\zeta}(\ell) \cup \hat{\eta}(\neg\ell),$

for all $\ell \in \mathscr{B} \times \mathscr{E} \times \{1\}$ and $\neg\ell \in \neg\mathscr{B} \times \mathscr{E} \times \{1\}$. Similarly, the disagree SFBSES $(\hat{\zeta}, \hat{\eta}, \mathscr{L})_0$ over the universe $\mathscr{Z}$ is the SFBSE subset of $(\hat{\zeta}, \hat{\eta}, \mathscr{L})$ defined as:

$(\hat{\zeta}, \hat{\eta}, \mathscr{L})_0 = \hat{\zeta}(\ell) \cup \hat{\eta}(\neg\ell),$

for all $\ell \in \mathscr{B} \times \mathscr{E} \times \{0\}$ and $\neg\ell \in \neg\mathscr{B} \times \mathscr{E} \times \{0\}$.

Example 3.4. Consider the SFBSES $(\hat{\zeta}, \hat{\eta}, \mathscr{L})$ in Example 3.1. The corresponding agree and disagree SFBSESs are respectively shown by Tables 11 and 12.

Table 11: Agree SFBSES of $(\hat{\zeta}, \hat{\eta}, \mathscr{L})$ in Example 3.4

$(\hat{\zeta}, \hat{\eta}, \mathscr{L})_1$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
(β_1, u)	$\begin{pmatrix}(0.80, 0.05, 0.20)\\(0.20, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.01, 0.30)\\(0.40, 0.03, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.03, 0.30)\\(0.32, 0.01, 0.60)\end{pmatrix}$
(β_1, v)	$\begin{pmatrix}(0.76, 0.04, 0.36)\\(0.30, 0.03, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.80, 0.03, 0.40)\\(0.25, 0.04, 0.20)\end{pmatrix}$	$\begin{pmatrix}(0.80, 0.04, 0.25)\\(0.30, 0.02, 0.70)\end{pmatrix}$
(β_1, w)	$\begin{pmatrix}(0.90, 0.02, 0.15)\\(0.15, 0.04, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.04, 0.30)\\(0.13, 0.03, 0.73)\end{pmatrix}$	$\begin{pmatrix}(0.86, 0.04, 0.30)\\(0.20, 0.05, 0.65)\end{pmatrix}$
(β_2, u)	$\begin{pmatrix}(0.30, 0.10, 0.79)\\(0.60, 0.03, 0.36)\end{pmatrix}$	$\begin{pmatrix}(0.50, 0.09, 0.60)\\(0.42, 0.04, 0.39)\end{pmatrix}$	$\begin{pmatrix}(0.34, 0.05, 0.50)\\(0.62, 0.06, 0.42)\end{pmatrix}$
(β_2, v)	$\begin{pmatrix}(0.50, 0.08, 0.70)\\(0.40, 0.07, 0.54)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.03, 0.40)\\(0.70, 0.01, 0.42)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.09, 0.39)\\(0.30, 0.09, 0.36)\end{pmatrix}$
(β_2, w)	$\begin{pmatrix}(0.35, 0.02, 0.75)\\(0.30, 0.08, 0.65)\end{pmatrix}$	$\begin{pmatrix}(0.40, 0.04, 0.70)\\(0.60, 0.10, 0.40)\end{pmatrix}$	$\begin{pmatrix}(0.62, 0.08, 0.40)\\(0.30, 0.02, 0.70)\end{pmatrix}$
(β_3, u)	$\begin{pmatrix}(0.70, 0.06, 0.40)\\(0.50, 0.06, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.86, 0.06, 0.30)\\(0.40, 0.03, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.90, 0.03, 0.15)\\(0.12, 0.06, 0.68)\end{pmatrix}$
(β_3, v)	$\begin{pmatrix}(0.67, 0.03, 0.25)\\(0.32, 0.01, 0.72)\end{pmatrix}$	$\begin{pmatrix}(0.73, 0.07, 0.10)\\(0.40, 0.05, 0.64)\end{pmatrix}$	$\begin{pmatrix}(0.75, 0.04, 0.35)\\(0.22, 0.07, 0.73)\end{pmatrix}$
(β_3, w)	$\begin{pmatrix}(0.80, 0.10, 0.30)\\(0.30, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.90, 0.01, 0.10)\\(0.15, 0.16, 0.89)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.10, 0.40)\\(0.25, 0.02, 0.80)\end{pmatrix}$

Table 12: Disagree SFBSES of $(\hat{\zeta}, \hat{\eta}, \mathscr{L})$ in Example 3.4

$(\hat{\zeta}, \hat{\eta}, \mathscr{L})_0$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
(β_1, u)	$\begin{pmatrix}(0.15, 0.04, 0.66)\\(0.80, 0.03, 0.15)\end{pmatrix}$	$\begin{pmatrix}(0.45, 0.02, 0.50)\\(0.65, 0.06, 0.46)\end{pmatrix}$	$\begin{pmatrix}(0.30, 0.10, 0.70)\\(0.70, 0.01, 0.30)\end{pmatrix}$
(β_1, v)	$\begin{pmatrix}(0.30, 0.03, 0.46)\\(0.65, 0.04, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.23, 0.03, 0.70)\\(0.70, 0.07, 0.24)\end{pmatrix}$	$\begin{pmatrix}(0.20, 0.07, 0.73)\\(0.80, 0.01, 0.15)\end{pmatrix}$
(β_1, w)	$\begin{pmatrix}(0.10, 0.05, 0.80)\\(0.78, 0.06, 0.20)\end{pmatrix}$	$\begin{pmatrix}(0.30, 0.04, 0.60)\\(0.70, 0.03, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.15, 0.09, 0.60)\\(0.60, 0.05, 0.28)\end{pmatrix}$
(β_2, u)	$\begin{pmatrix}(0.65, 0.02, 0.40)\\(0.40, 0.05, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.07, 0.50)\\(0.60, 0.04, 0.32)\end{pmatrix}$	$\begin{pmatrix}(0.65, 0.02, 0.20)\\(0.40, 0.03, 0.66)\end{pmatrix}$
(β_2, v)	$\begin{pmatrix}(0.40, 0.01, 0.60)\\(0.50, 0.01, 0.65)\end{pmatrix}$	$\begin{pmatrix}(0.55, 0.01, 0.66)\\(0.36, 0.05, 0.55)\end{pmatrix}$	$\begin{pmatrix}(0.30, 0.03, 0.45)\\(0.70, 0.04, 0.32)\end{pmatrix}$
(β_2, w)	$\begin{pmatrix}(0.70, 0.06, 0.50)\\(0.70, 0.01, 0.32)\end{pmatrix}$	$\begin{pmatrix}(0.38, 0.10, 0.40)\\(0.30, 0.01, 0.72)\end{pmatrix}$	$\begin{pmatrix}(0.40, 0.05, 0.36)\\(0.65, 0.09, 0.40)\end{pmatrix}$
(β_3, u)	$\begin{pmatrix}(0.30, 0.10, 0.60)\\(0.50, 0.02, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.23, 0.06, 0.60)\\(0.29, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.22, 0.10, 0.80)\\(0.80, 0.06, 0.21)\end{pmatrix}$
(β_3, v)	$\begin{pmatrix}(0.25, 0.03, 0.80)\\(0.67, 0.10, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.10, 0.02, 0.91)\\(0.60, 0.10, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.33, 0.06, 0.77)\\(0.63, 0.08, 0.35)\end{pmatrix}$
(β_3, w)	$\begin{pmatrix}(0.20, 0.07, 0.76)\\(0.59, 0.03, 0.41)\end{pmatrix}$	$\begin{pmatrix}(0.24, 0.01, 0.40)\\(0.80, 0.06, 0.10)\end{pmatrix}$	$\begin{pmatrix}(0.40, 0.07, 0.35)\\(0.70, 0.05, 0.28)\end{pmatrix}$

The following definition provide the idea of AND operation between SFBSESs:

Definition 3.7. For two SFBSESs $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{L})$ and $\Lambda = (\hat{\pi}, \hat{\psi}, \mathscr{M})$ on the universal set $\mathscr{Z}$, the *AND operation* denoted as a SFBSES $\Gamma \hat{\wedge} \Lambda$, is defined by:

$\Gamma \hat{\wedge} \Lambda = (\hat{\rho}, \hat{\theta}, \mathscr{L} \times \mathscr{M})$,

such that $\forall (\ell, m) \in \mathscr{L} \times \mathscr{M}$ and $(\neg\ell, \neg m) \in \neg\mathscr{L} \times \neg\mathscr{M}$, $\hat{\rho}\colon \mathscr{L} \times \mathscr{M} \to \mathbb{SF}(\mathscr{Z})$ and $\hat{\theta}\colon \neg\mathscr{L} \times \neg\mathscr{M} \to \mathbb{SF}(\mathscr{Z})$ are functions defined as:

$\hat{\rho}(\ell, m) = \hat{\zeta}(\ell) \cap \hat{\pi}(m), \quad \hat{\theta}(\neg\ell, \neg m) = \hat{\eta}(\neg\ell) \cup \hat{\psi}(\neg m).$

The next definition gives the OR operation between SFBSESs.

Definition 3.8. For two SFBSESs $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{L})$ and $\Lambda = (\hat{\pi}, \hat{\psi}, \mathscr{M})$ on the universal set $\mathscr{Z}$, the *OR operation* represented as a SFBSES $\Gamma \hat{\vee} \Lambda$ is defined by:

$\Gamma \hat{\vee} \Lambda = (\hat{\gamma}, \hat{\lambda}, \mathscr{L} \times \mathscr{M})$, such that $\forall(\ell, m) \in \mathscr{L} \times \mathscr{M}$ and $(\neg\ell, \neg m) \in \neg\mathscr{L} \times \neg\mathscr{M}$, $\hat{\gamma}: \mathscr{L} \times \mathscr{M} \to \mathbb{SF}(\mathscr{Z})$ and $\hat{\lambda}: \neg\mathscr{L} \times \neg\mathscr{M} \to \mathbb{SF}(\mathscr{Z})$ are functions defined as:

$$\hat{\gamma}(\ell, m) = \hat{\zeta}(\ell) \cup \hat{\pi}(m), \quad \hat{\lambda}(\neg\ell, \neg m) = \hat{\eta}(\neg\ell) \cap \hat{\psi}(\neg m).$$

The following example provides an illustration of the AND and OR operations between two SFBSESs:

Example 3.5. Reconsider Example 3.1. For $\mathscr{M} = \{(\beta_1, u, 1), (\beta_2, v, 1), (\beta_3, u, 0), (\beta_3, v, 0)\}$, and $\mathscr{N} = \{(\beta_1, u, 1), (\beta_3, v, 1)\}$, the corresponding SFBSESs are shown in Tables 13 and 14, respectively. Using Definition 3.7, the AND operation $\Gamma \hat{\wedge} \Lambda$ between these two SFBSESs is displayed in Table 15. Similarly, using Definition 3.8, the OR operation $\Gamma \hat{\vee} \Lambda$ between the given SFBSESs is calculated as shown in Table 16.

Table 13: SFBSES $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{M})$ in Example 3.5

$(\hat{\zeta}, \hat{\eta}, \mathscr{M})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$(\beta_1, u, 1)$	$\begin{pmatrix}(0.80, 0.05, 0.20)\\(0.20, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.01, 0.30)\\(0.40, 0.03, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.03, 0.30)\\(0.32, 0.01, 0.60)\end{pmatrix}$
$(\beta_2, v, 1)$	$\begin{pmatrix}(0.50, 0.08, 0.70)\\(0.40, 0.07, 0.54)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.03, 0.40)\\(0.70, 0.01, 0.42)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.09, 0.39)\\(0.30, 0.09, 0.36)\end{pmatrix}$
$(\beta_3, u, 0)$	$\begin{pmatrix}(0.30, 0.10, 0.60)\\(0.50, 0.02, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.23, 0.06, 0.60)\\(0.29, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.22, 0.10, 0.80)\\(0.80, 0.06, 0.21)\end{pmatrix}$
$(\beta_3, v, 0)$	$\begin{pmatrix}(0.25, 0.03, 0.80)\\(0.67, 0.10, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.10, 0.02, 0.91)\\(0.60, 0.10, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.33, 0.06, 0.77)\\(0.63, 0.08, 0.35)\end{pmatrix}$

Table 14: SFBSES $\Lambda = (\hat{\pi}, \hat{\psi}, \mathscr{N})$ in Example 3.5

$(\hat{\pi}, \hat{\psi}, \mathscr{N})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$(\beta_1, u, 1)$	$\begin{pmatrix}(0.90, 0.05, 0.10)\\(0.10, 0.02, 0.80)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.01, 0.30)\\(0.35, 0.03, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.75, 0.03, 0.20)\\(0.30, 0.01, 0.65)\end{pmatrix}$
$(\beta_3, v, 1)$	$\begin{pmatrix}(0.73, 0.03, 0.20)\\(0.30, 0.01, 0.75)\end{pmatrix}$	$\begin{pmatrix}(0.80, 0.07, 0.10)\\(0.40, 0.05, 0.64)\end{pmatrix}$	$\begin{pmatrix}(0.82, 0.04, 0.30)\\(0.20, 0.07, 0.76)\end{pmatrix}$

Table 15: The AND operation between SFBSESs Γ and Λ in Example 3.5

$\Gamma \hat{\wedge} \Lambda$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$((\beta_1, u, 1), (\beta_1, u, 1))$	$\begin{pmatrix}(0.80, 0.05, 0.20)\\(0.20, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.01, 0.30)\\(0.40, 0.03, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.03, 0.30)\\(0.32, 0.01, 0.60)\end{pmatrix}$

(Continued)

Table 15 (continued)

$\Gamma\hat{\wedge}\Lambda$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$((\beta_1, u, 1), (\beta_3, v, 1))$	$\begin{pmatrix}(0.73, 0.03, 0.20)\\(0.30, 0.01, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.01, 0.30)\\(0.40, 0.03, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.03, 0.30)\\(0.32, 0.01, 0.60)\end{pmatrix}$
$((\beta_2, v, 1), (\beta_1, u, 1))$	$\begin{pmatrix}(0.50, 0.05, 0.70)\\(0.40, 0.02, 0.54)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.01, 0.40)\\(0.70, 0.01, 0.42)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.03, 0.39)\\(0.30, 0.01, 0.36)\end{pmatrix}$
$((\beta_2, v, 1), (\beta_3, v, 1))$	$\begin{pmatrix}(0.50, 0.03, 0.70)\\(0.40, 0.01, 0.54)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.03, 0.40)\\(0.70, 0.01, 0.42)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.04, 0.39)\\(0.30, 0.07, 0.36)\end{pmatrix}$
$((\beta_3, u, 0), (\beta_1, u, 1))$	$\begin{pmatrix}(0.30, 0.05, 0.60)\\(0.50, 0.02, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.23, 0.01, 0.60)\\(0.35, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.22, 0.03, 0.80)\\(0.80, 0.01, 0.21)\end{pmatrix}$
$((\beta_3, u, 0), (\beta_3, v, 1))$	$\begin{pmatrix}(0.30, 0.03, 0.60)\\(0.50, 0.01, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.23, 0.06, 0.60)\\(0.40, 0.02, 0.64)\end{pmatrix}$	$\begin{pmatrix}(0.22, 0.04, 0.80)\\(0.80, 0.06, 0.21)\end{pmatrix}$
$((\beta_3, v, 0), (\beta_1, u, 1))$	$\begin{pmatrix}(0.25, 0.03, 0.80)\\(0.67, 0.02, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.10, 0.01, 0.91)\\(0.60, 0.03, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.33, 0.03, 0.77)\\(0.63, 0.01, 0.35)\end{pmatrix}$
$((\beta_3, v, 0), (\beta_3, v, 1))$	$\begin{pmatrix}(0.25, 0.03, 0.80)\\(0.67, 0.01, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.10, 0.02, 0.91)\\(0.60, 0.05, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.33, 0.04, 0.77)\\(0.63, 0.07, 0.35)\end{pmatrix}$

Table 16: The OR operation between SFBSESs Γ and Λ in Example 3.5

$\Gamma\hat{\vee}\Lambda$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$((\beta_1, u, 1), (\beta_1, u, 1))$	$\begin{pmatrix}(0.90, 0.05, 0.10)\\(0.10, 0.02, 0.80)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.01, 0.30)\\(0.35, 0.03, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.75, 0.03, 0.20)\\(0.30, 0.01, 0.65)\end{pmatrix}$
$((\beta_1, u, 1), (\beta_3, v, 1))$	$\begin{pmatrix}(0.80, 0.03, 0.20)\\(0.20, 0.01, 0.75)\end{pmatrix}$	$\begin{pmatrix}(0.80, 0.01, 0.10)\\(0.40, 0.03, 0.64)\end{pmatrix}$	$\begin{pmatrix}(0.82, 0.03, 0.30)\\(0.20, 0.01, 0.76)\end{pmatrix}$
$((\beta_2, v, 1), (\beta_1, u, 1))$	$\begin{pmatrix}(0.90, 0.05, 0.10)\\(0.10, 0.02, 0.80)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.01, 0.30)\\(0.35, 0.01, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.75, 0.03, 0.20)\\(0.30, 0.01, 0.65)\end{pmatrix}$
$((\beta_2, v, 1), (\beta_3, v, 1))$	$\begin{pmatrix}(0.73, 0.03, 0.20)\\(0.30, 0.01, 0.75)\end{pmatrix}$	$\begin{pmatrix}(0.80, 0.03, 0.10)\\(0.40, 0.01, 0.42)\end{pmatrix}$	$\begin{pmatrix}(0.82, 0.04, 0.30)\\(0.20, 0.07, 0.76)\end{pmatrix}$
$((\beta_3, u, 0), (\beta_1, u, 1))$	$\begin{pmatrix}(0.90, 0.05, 0.10)\\(0.10, 0.02, 0.80)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.01, 0.30)\\(0.29, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.75, 0.03, 0.20)\\(0.80, 0.01, 0.21)\end{pmatrix}$
$((\beta_3, u, 0), (\beta_3, v, 1))$	$\begin{pmatrix}(0.73, 0.03, 0.20)\\(0.30, 0.01, 0.75)\end{pmatrix}$	$\begin{pmatrix}(0.80, 0.06, 0.10)\\(0.29, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.82, 0.04, 0.30)\\(0.20, 0.06, 0.76)\end{pmatrix}$
$((\beta_3, v, 0), (\beta_1, u, 1))$	$\begin{pmatrix}(0.90, 0.03, 0.10)\\(0.10, 0.02, 0.80)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.01, 0.30)\\(0.35, 0.03, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.75, 0.03, 0.20)\\(0.30, 0.01, 0.65)\end{pmatrix}$
$((\beta_3, v, 0), (\beta_3, v, 1))$	$\begin{pmatrix}(0.73, 0.03, 0.20)\\(0.30, 0.01, 0.75)\end{pmatrix}$	$\begin{pmatrix}(0.80, 0.02, 0.10)\\(0.40, 0.05, 0.64)\end{pmatrix}$	$\begin{pmatrix}(0.82, 0.04, 0.30)\\(0.20, 0.07, 0.76)\end{pmatrix}$

Proposition 3.3. Let $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{L})$ and $\Lambda = (\hat{\zeta}_1, \hat{\eta}_1, \mathscr{M})$ be the SFBSESs on $\mathscr{X}$, then

1. $(\Gamma \hat{\wedge} \Lambda)^c = \Gamma^c \hat{\vee} \Lambda^c$
2. $(\Gamma \hat{\vee} \Lambda)^c = \Gamma^c \hat{\wedge} \Lambda^c$

Proof.

1. Let $\Gamma \hat{\wedge} \Lambda = (\hat{\zeta}, \hat{\eta}, \mathscr{L}) \hat{\wedge} (\hat{\zeta}_1, \hat{\eta}_1, \mathscr{M}) = (\hat{\rho}, \hat{\theta}, \mathscr{L} \times \mathscr{M})$. Then for all $(\ell, m) \in \mathscr{L} \times \mathscr{M}$,
$(\Gamma \hat{\wedge} \Lambda)^c = (\hat{\rho}, \hat{\theta}, \mathscr{L} \times \mathscr{M})^c = (\hat{\rho}^c, \hat{\theta}^c, \mathscr{L} \times \mathscr{M})$
where $\hat{\rho}^c = (\hat{\zeta}(\ell) \cap \hat{\zeta}_1(m))^c = \hat{\zeta}^c(\ell) \cup \hat{\zeta}_1^c(m)$
and $\hat{\theta}^c = (\hat{\eta}(\neg\ell) \cup \hat{\eta}_1(\neg m))^c = \hat{\eta}^c(\neg\ell) \cap \hat{\eta}_1^c(\neg m)$.
For all $(\ell, m) \in \mathscr{L} \times \mathscr{M}$,
$\Gamma^c \hat{\vee} \Lambda^c = (\hat{\zeta}, \hat{\eta}, \mathscr{L})^c \hat{\vee} (\hat{\zeta}_1, \hat{\eta}_1, \mathscr{M})^c = (\hat{\gamma}, \hat{\lambda}, \mathscr{L} \times \mathscr{M})$
where $\hat{\gamma} = \hat{\zeta}^c(\ell) \cup \hat{\zeta}_1^c(m)$ and $\hat{\lambda} = \hat{\eta}^c(\neg\ell) \cap \hat{\eta}_1^c(\neg m)$
Thus, $\hat{\gamma} = \hat{\rho}^c$ and $\hat{\lambda} = \hat{\theta}^c$. Hence $(\Gamma \hat{\wedge} \Lambda)^c = \Gamma^c \hat{\vee} \Lambda^c$.

2. Let $\Gamma \hat{\vee} \Lambda = (\hat{\zeta}, \hat{\eta}, \mathscr{L}) \hat{\vee} (\hat{\zeta}_1, \hat{\eta}_1, \mathscr{M}) = (\hat{\gamma}, \hat{\lambda}, \mathscr{L} \times \mathscr{M})$. Then for all $(\ell, m) \in \mathscr{L} \times \mathscr{M}$,
$(\Gamma \hat{\wedge} \Lambda)^c = (\hat{\gamma}, \hat{\lambda}, \mathscr{L} \times \mathscr{M})^c = (\hat{\gamma}^c, \hat{\lambda}^c, \mathscr{L} \times \mathscr{M})$
where $\hat{\gamma}^c = (\hat{\zeta}(\ell) \cup \hat{\zeta}_1(m))^c = \hat{\zeta}^c(\ell) \cap \hat{\zeta}_1^c(m)$
and $\hat{\lambda}^c = (\hat{\eta}(\neg\ell) \cap \hat{\eta}_1(\neg m))^c = \hat{\eta}^c(\neg\ell) \cup \hat{\eta}_1^c(\neg m)$.
For all $(\ell, m) \in \mathscr{L} \times \mathscr{M}$,
$\Gamma^c \hat{\wedge} \Lambda^c = (\hat{\zeta}, \hat{\eta}, \mathscr{L})^c \hat{\wedge} (\hat{\zeta}_1, \hat{\eta}_1, \mathscr{M})^c = (\hat{\rho}, \hat{\theta}, \mathscr{L} \times \mathscr{M})$
where $\hat{\rho} = \hat{\zeta}^c(\ell) \cap \hat{\zeta}_1^c(m)$ and $\hat{\theta} = \hat{\eta}^c(\neg\ell) \cup \hat{\eta}_1^c(\neg m)$
Thus, $\hat{\rho} = \hat{\gamma}^c$ and $\hat{\theta} = \hat{\lambda}^c$. Hence $(\Gamma \hat{\vee} \Lambda)^c = \Gamma^c \hat{\wedge} \Lambda^c$.

Proposition 3.4. Let $\Gamma = (\hat{\zeta}_1, \hat{\eta}_1, \mathscr{L})$, $\Lambda = (\hat{\zeta}_2, \hat{\eta}_2, \mathscr{M})$, and $\Pi = (\hat{\zeta}_3, \hat{\eta}_3, \mathscr{N})$ be three SFBSESs over $\mathscr{X}$, then

1. $\Gamma \hat{\wedge} (\Lambda \hat{\wedge} \Pi) = (\Gamma \hat{\wedge} \Lambda) \hat{\wedge} \Pi$.
2. $\Gamma \hat{\vee} (\Lambda \hat{\vee} \Pi) = (\Gamma \hat{\vee} \Lambda) \hat{\vee} \Pi$.
3. $\Gamma \hat{\wedge} (\Lambda \hat{\vee} \Pi) = (\Gamma \hat{\wedge} \Lambda) \hat{\vee} (\Gamma \hat{\wedge} \Pi)$.
4. $\Gamma \hat{\vee} (\Lambda \hat{\wedge} \Pi) = (\Gamma \hat{\vee} \Lambda) \hat{\wedge} (\Gamma \hat{\vee} \Pi)$.

Proof.

1. By Definition 3.7, we have

$$\begin{aligned}
\Gamma \hat{\wedge} (\Lambda \hat{\wedge} \Pi) &= (\hat{\zeta}_1(\ell), \hat{\eta}_1(\neg\ell), \mathscr{L}) \hat{\wedge} \big(\hat{\zeta}_2(m) \cap \hat{\zeta}_3(n), \hat{\eta}_2(\neg m) \cup \hat{\eta}_3(\neg n), \mathscr{M} \times \mathscr{N}\big) \\
&= \big(\hat{\zeta}_1(\ell) \cap (\hat{\zeta}_2(m) \cap \hat{\zeta}_3(n)), \hat{\eta}_1(\neg\ell) \cup (\hat{\eta}_2(\neg m) \cup \hat{\eta}_3(\neg n)), \mathscr{L} \times (\mathscr{M} \times \mathscr{N})\big) \\
&= \big((\hat{\zeta}_1(\ell) \cap \hat{\zeta}_2(m)) \cap \hat{\zeta}_3(n), (\hat{\eta}_1(\neg\ell) \cup \hat{\eta}_2(\neg m)) \cup \hat{\eta}_3(\neg n), (\mathscr{L} \times \mathscr{M}) \times \mathscr{N}\big) \\
&= \big(\hat{\zeta}_1(\ell) \cap \hat{\zeta}_2(m), \hat{\eta}_1(\neg\ell) \cup \hat{\eta}_2(m), \mathscr{L} \times \mathscr{M}\big) \hat{\wedge} (\hat{\zeta}_3(n), \hat{\eta}_3(\neg n), \mathscr{N}) \\
&= (\Gamma \hat{\wedge} \Lambda) \hat{\wedge} \Pi.
\end{aligned}$$

Hence proved.

2. By Definition 3.8, we have

$$\begin{aligned}\Gamma\hat{\vee}(\Lambda\hat{\vee}\Pi) &= (\hat{\zeta}_1(\ell), \hat{\eta}_1(\neg\ell), \mathscr{L})\hat{\vee}\big(\hat{\zeta}_2(m)\cup\hat{\zeta}_3(n), \hat{\eta}_2(\neg m)\cap\hat{\eta}_3(\neg n), \mathscr{M}\times\mathscr{N}\big)\\ &= \big(\hat{\zeta}_1(\ell)\cup(\hat{\zeta}_2(m)\cup\hat{\zeta}_3(n)), \hat{\eta}_1(\neg\ell)\cap(\hat{\eta}_2(\neg m)\cap\hat{\eta}_3(\neg n)), \mathscr{L}\times(\mathscr{M}\times\mathscr{N})\big)\\ &= \big((\hat{\zeta}_1(\ell)\cup\hat{\zeta}_2(m))\cup\hat{\zeta}_3(n), (\hat{\eta}_1(\neg\ell)\cap\hat{\eta}_2(\neg m))\cap\hat{\eta}_3(\neg n), (\mathscr{L}\times\mathscr{M})\times\mathscr{N}\big)\\ &= \big(\hat{\zeta}_1(\ell)\cup\hat{\zeta}_2(m), \hat{\eta}_1(\neg\ell)\cap\hat{\eta}_2(m), \mathscr{L}\times\mathscr{M}\big)\hat{\vee}(\hat{\zeta}_3(n), \hat{\eta}_3(\neg n), \mathscr{N})\\ &= (\Gamma\hat{\vee}\Lambda)\hat{\vee}\Pi.\end{aligned}$$

Hence proved.

The remaining proofs are similar as above parts.

The SFBSE restricted intersection and union are defined in the following Definitions 3.9 and 3.10, respectively.

Definition 3.9. The SFBSE *restricted intersection* among the SFBSESs $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{L})$ and $\Lambda = (\hat{\pi}, \hat{\psi}, \mathscr{M})$ denoted by a SFBSES $\Gamma\hat{\cap}_R\Lambda$ is defined as:

$\Gamma\hat{\cap}_R\Lambda = (\hat{\rho}, \hat{\varpi}, \mathscr{W})$,

such that $\forall\omega \in \mathscr{W} = \mathscr{L}\cap\mathscr{M} \neq \emptyset$ and $\neg\mathscr{W} = \neg\mathscr{L}\cap\neg\mathscr{M} \neq \emptyset$, the mappings $\hat{\rho}\colon \mathscr{W} \to \mathbb{SF}(\mathscr{Z})$ and $\hat{\varpi}\colon \neg\mathscr{W} \to \mathbb{SF}(\mathscr{Z})$ are given by

$\hat{\rho}(\omega) = \hat{\zeta}(\omega)\cap\hat{\pi}(\omega), \quad \hat{\varpi}(\neg\omega) = \hat{\eta}(\neg\omega)\cup\hat{\psi}(\neg\omega).$

Definition 3.10. The SFBSE *restricted union* among the SFBSESs $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{L})$ and $\Lambda = (\hat{\pi}, \hat{\psi}, \mathscr{M})$ denoted by a SFBSES $\Gamma\hat{\cup}_R\Lambda$ is defined as

$\Gamma\hat{\cup}_R\Lambda = (\hat{\varsigma}, \hat{\vartheta}, \mathscr{W})$,

such that $\forall\,\omega \in \mathscr{W} = \mathscr{L}\cap\mathscr{M} \neq \emptyset$ and $\neg\mathscr{W} = \neg\mathscr{L}\cap\neg\mathscr{M} \neq \emptyset$, the mappings $\hat{\varsigma}\colon \mathscr{W} \to \mathbb{SF}(\mathscr{Z})$ and $\hat{\vartheta}\colon \neg\mathscr{W} \to \mathbb{SF}(\mathscr{Z})$ are provided by

$\hat{\varsigma}(\omega) = \hat{\zeta}(\omega)\cup\hat{\pi}(\omega), \quad \hat{\vartheta}(\neg\omega) = \hat{\eta}(\neg\omega)\cap\hat{\psi}(\neg\omega).$

Example 3.6. Reconsider Example 3.1 again. Consider the SFBSESs corresponding to the sets $\mathscr{M} = \{(\beta_1, u, 1), (\beta_2, v, 1), (\beta_3, w, 1), (\beta_1, v, 0)\}$ and $\mathscr{N} = \{(\beta_1, v, 1), (\beta_2, v, 1), (\beta_3, w, 1), (\beta_1, v, 0)\}$ are respectively presented by Tables 17 and 18. Then, by Definition 3.9, the restricted intersection between the SFBSESs Γ and Λ is shown in Table 19. Similarly, their restricted union is represented by Table 20, using the Definition 3.10.

Table 17: SFBSES $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{M})$ in Example 3.6

$(\hat{\zeta}, \hat{\eta}, \mathscr{M})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$(\beta_1, u, 1)$	$\begin{pmatrix}(0.80, 0.05, 0.20)\\(0.20, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.01, 0.30)\\(0.40, 0.03, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.03, 0.30)\\(0.32, 0.01, 0.60)\end{pmatrix}$

(Continued)

Table 17 (continued)

$(\hat{\zeta}, \hat{\eta}, \mathscr{M})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$(\beta_2, v, 1)$	$\begin{pmatrix}(0.50, 0.08, 0.70)\\(0.40, 0.07, 0.54)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.03, 0.40)\\(0.70, 0.01, 0.42)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.09, 0.39)\\(0.30, 0.09, 0.36)\end{pmatrix}$
$(\beta_3, w, 1)$	$\begin{pmatrix}(0.80, 0.10, 0.30)\\(0.30, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.90, 0.01, 0.10)\\(0.15, 0.16, 0.89)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.10, 0.40)\\(0.25, 0.02, 0.80)\end{pmatrix}$
$(\beta_1, v, 0)$	$\begin{pmatrix}(0.30, 0.03, 0.46)\\(0.65, 0.04, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.23, 0.03, 0.70)\\(0.70, 0.07, 0.24)\end{pmatrix}$	$\begin{pmatrix}(0.20, 0.07, 0.73)\\(0.80, 0.01, 0.15)\end{pmatrix}$

Table 18: SFBSES $\Lambda = (\hat{\pi}, \hat{\psi}, \mathscr{N})$ in Example 3.6

$(\hat{\pi}, \hat{\psi}, \mathscr{N})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$(\beta_1, v, 1)$	$\begin{pmatrix}(0.70, 0.05, 0.10)\\(0.30, 0.02, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.75, 0.01, 0.20)\\(0.33, 0.03, 0.68)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.03, 0.40)\\(0.10, 0.04, 0.85)\end{pmatrix}$
$(\beta_2, v, 1)$	$\begin{pmatrix}(0.30, 0.03, 0.70)\\(0.60, 0.01, 0.45)\end{pmatrix}$	$\begin{pmatrix}(0.50, 0.07, 0.25)\\(0.50, 0.05, 0.40)\end{pmatrix}$	$\begin{pmatrix}(0.82, 0.04, 0.30)\\(0.20, 0.07, 0.76)\end{pmatrix}$
$(\beta_3, w, 1)$	$\begin{pmatrix}(0.40, 0.05, 0.60)\\(0.70, 0.02, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.65, 0.01, 0.28)\\(0.30, 0.03, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.03, 0.30)\\(0.22, 0.01, 0.80)\end{pmatrix}$
$(\beta_1, v, 0)$	$\begin{pmatrix}(0.35, 0.06, 0.50)\\(0.70, 0.01, 0.45)\end{pmatrix}$	$\begin{pmatrix}(0.80, 0.10, 0.23)\\(0.18, 0.02, 0.83)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.05, 0.36)\\(0.36, 0.07, 0.65)\end{pmatrix}$

Table 19: Restricted intersection between SFBSESs Γ and Λ in Example 3.6

$\Gamma \hat{\cap}_R \Lambda$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$(\beta_2, v, 1)$	$\begin{pmatrix}(0.30, 0.03, 0.70)\\(0.60, 0.01, 0.45)\end{pmatrix}$	$\begin{pmatrix}(0.50, 0.03, 0.40)\\(0.70, 0.01, 0.40)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.04, 0.39)\\(0.30, 0.07, 0.36)\end{pmatrix}$
$(\beta_3, w, 1)$	$\begin{pmatrix}(0.40, 0.05, 0.60)\\(0.70, 0.02, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.65, 0.01, 0.28)\\(0.30, 0.03, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.03, 0.40)\\(0.25, 0.01, 0.80)\end{pmatrix}$
$(\beta_1, v, 0)$	$\begin{pmatrix}(0.30, 0.03, 0.50)\\(0.70, 0.01, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.23, 0.03, 0.70)\\(0.70, 0.02, 0.24)\end{pmatrix}$	$\begin{pmatrix}(0.20, 0.05, 0.73)\\(0.80, 0.01, 0.15)\end{pmatrix}$

Table 20: Restricted union between SFBSESs Γ and Λ in Example 3.6

$\Gamma \hat{\cup}_R \Lambda$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$(\beta_2, v, 1)$	$\begin{pmatrix}(0.50, 0.03, 0.70)\\(0.40, 0.01, 0.54)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.03, 0.25)\\(0.50, 0.01, 0.42)\end{pmatrix}$	$\begin{pmatrix}(0.82, 0.04, 0.30)\\(0.20, 0.07, 0.76)\end{pmatrix}$
$(\beta_3, w, 1)$	$\begin{pmatrix}(0.80, 0.05, 0.30)\\(0.30, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.90, 0.01, 0.10)\\(0.15, 0.03, 0.89)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.03, 0.30)\\(0.22, 0.01, 0.80)\end{pmatrix}$
$(\beta_1, v, 0)$	$\begin{pmatrix}(0.35, 0.03, 0.46)\\(0.65, 0.01, 0.45)\end{pmatrix}$	$\begin{pmatrix}(0.80, 0.03, 0.23)\\(0.18, 0.02, 0.83)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.05, 0.36)\\(0.36, 0.01, 0.65)\end{pmatrix}$

Proposition 3.5. Let $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{L})$ and $\Lambda = (\hat{\zeta}_1, \hat{\eta}_1, \mathscr{M})$ be the SFBSESs on $\mathscr{Z}$, then

1. $(\Gamma \hat{\cap}_R \Lambda)^c = \Gamma^c \hat{\cup}_R \Lambda^c$
2. $(\Gamma \hat{\cup}_R \Lambda)^c = \Gamma^c \hat{\cap}_R \Lambda^c$

Proof.

1. Consider $\Gamma \hat{\cap}_R \Lambda = (\hat{\zeta}, \hat{\eta}, \mathscr{L}) \hat{\cap}_R (\hat{\zeta}_1, \hat{\eta}_1, \mathscr{M}) = (\hat{\varkappa}, \hat{\varpi}, \mathscr{L} \times \mathscr{M})$. Then $\forall \omega \in \mathscr{W} = \mathscr{L} \cap \mathscr{M} \neq \emptyset$,
   $(\Gamma \hat{\cap}_R \Lambda)^c = (\hat{\varkappa}, \hat{\varpi}, \mathscr{W})^c = (\hat{\varkappa}^c, \hat{\varpi}^c, \mathscr{W})$
   such that $\hat{\varkappa}^c = (\hat{\zeta}(\omega) \hat{\cap} \hat{\zeta}_1(\omega))^c = \hat{\zeta}^c(\omega) \hat{\cup} \hat{\zeta}_1^c(\omega)$ and $\hat{\varpi}^c = (\hat{\eta}(\neg\omega) \cup \hat{\eta}_1(\neg\omega))^c = \hat{\eta}^c(\neg\omega) \cap \hat{\eta}_1^c(\neg\omega)$.
   Similarly, $\forall \omega \in \mathscr{W}$,
   $\Gamma^c \hat{\cup}_R \Lambda^c = (\hat{\chi}, \hat{\psi}, \mathscr{W})$
   such that $\hat{\chi} = \hat{\zeta}^c(\omega) \cup \hat{\zeta}_1^c(\omega)$ and $\hat{\psi} = \hat{\eta}^c(\neg\omega) \cap \hat{\eta}_1^c(\neg\omega)$.
   This implies that $\hat{\chi} = \hat{\varkappa}^c$ and $\hat{\psi} = \hat{\varpi}^c$. Hence $(\Gamma \hat{\cap}_R \Lambda)^c = \Gamma^c \hat{\cup}_R \Lambda^c$.
2. Consider $\Gamma \hat{\cup}_R \Lambda = (\hat{\zeta}, \hat{\eta}, \mathscr{L}) \hat{\cup}_R (\hat{\zeta}_1, \hat{\eta}_1, \mathscr{M}) = (\hat{\varkappa}, \hat{\varpi}, \mathscr{W})$. Then $\forall \omega \in \mathscr{W} = \mathscr{L} \cap \mathscr{M} \neq \emptyset$,
   $(\Gamma \hat{\cup}_R \Lambda)^c = (\hat{\varkappa}, \hat{\varpi}, \mathscr{W})^c = (\hat{\varkappa}^c, \hat{\varpi}^c, \mathscr{W})$
   such that $\hat{\varkappa}^c = (\hat{\zeta}(\omega) \cup \hat{\zeta}_1(\omega))^c = \hat{\zeta}^c(\omega) \cap \hat{\zeta}_1^c(\omega)$ and $\hat{\varpi}^c = (\hat{\eta}(\neg\omega) \cap \hat{\eta}_1(\neg\omega))^c = \hat{\eta}^c(\neg\omega) \cup \hat{\eta}_1^c(\neg\omega)$.
   Similarly, $\forall \omega \in \mathscr{W}$,
   $\Gamma^c \hat{\cap}_R \Lambda^c = (\hat{\chi}, \hat{\psi}, \mathscr{W})$
   such that $\hat{\chi} = \hat{\zeta}^c(\omega) \cap \hat{\zeta}_1^c(\omega)$ and $\hat{\psi} = \hat{\eta}^c(\neg\omega) \cup \hat{\eta}_1^c(\neg\omega)$
   This implies that $\hat{\chi} = \hat{\varkappa}^c$ and $\hat{\psi} = \hat{\varpi}^c$. Hence $(\Gamma \hat{\cup}_R \Lambda)^c = \Gamma^c \hat{\cap}_R \Lambda^c$.

Definition 3.11. The SFBSE *extended intersection* among the SFBSESs $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{L})$ and $\Lambda = (\hat{\pi}, \hat{\psi}, \mathscr{M})$, denoted by a SFBSES $\Gamma \hat{\cap}_E \Lambda$, is defined as:

$\Gamma \hat{\cap}_E \Lambda = (\check{\rho}, \check{\varpi}, \mathscr{W})$,

such that $\forall \omega \in \mathscr{W} = \mathscr{L} \cup \mathscr{M}$, the mappings $\check{\rho}: \mathscr{W} \to \mathbb{SF}(\mathscr{Z})$ and $\check{\varpi}: \neg\mathscr{W} \to \mathbb{SF}(\mathscr{Z})$ are given by

$$\check{\rho} = \begin{cases} \hat{\zeta}(\omega) & \text{if } \omega \in \mathscr{L} - \mathscr{M}, \\ \hat{\pi}(\omega) & \text{if } \omega \in \mathscr{M} - \mathscr{L}, \\ \hat{\zeta}(\omega) \cap \hat{\pi}(\omega) & \text{if } \omega \in \mathscr{L} \cap \mathscr{M}. \end{cases}$$

$$\breve{\varpi} = \begin{cases} \hat{\eta}(\neg\omega) & \text{if } \neg\omega \in \neg\mathscr{L} - \neg\mathscr{M}, \\ \hat{\psi}(\neg\omega) & \text{if } \neg\omega \in \neg\mathscr{M} - \neg\mathscr{L}, \\ \hat{\eta}(\neg\omega) \cup \hat{\psi}(\neg\omega) & \text{if } \neg\omega \in \neg\mathscr{L} \cap \neg\mathscr{M}. \end{cases}$$

Definition 3.12. The SFBSE *extended union* among the SFBSESs $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{L})$ and $\Lambda = (\hat{\pi}, \hat{\psi}, \mathscr{M})$, denoted by a SFBSES $\Gamma \hat{\cup}_E \Lambda$, is defined as:

$\Gamma \hat{\cup}_E \Lambda = (\breve{\zeta}, \breve{\vartheta}, \mathscr{W})$,

such that $\forall \omega \in \mathscr{W} = \mathscr{L} \cup \mathscr{M}$, the mappings $\breve{\zeta}\colon \mathscr{W} \to \mathbb{SF}(\mathscr{Z})$ and $\breve{\vartheta}\colon \neg\mathscr{W} \to \mathbb{SF}(\mathscr{Z})$ are given by:

$$\breve{\zeta} = \begin{cases} \hat{\zeta}(\omega) & \text{if } \omega \in \mathscr{L} - \mathscr{M} \\ \hat{\pi}(\omega) & \text{if } \omega \in \mathscr{M} - \mathscr{L} \\ \hat{\zeta}(\omega) \cup \hat{\pi}(\omega) & \text{if } \omega \in \mathscr{L} \cap \mathscr{M} \end{cases}$$

$$\breve{\vartheta} = \begin{cases} \hat{\eta}(\neg\omega) & \text{if } \neg\omega \in \neg\mathscr{L} - \neg\mathscr{M} \\ \hat{\psi}(\neg\omega) & \text{if } \neg\omega \in \neg\mathscr{M} - \neg\mathscr{L} \\ \hat{\eta}(\neg\omega) \cap \hat{\psi}(\neg\omega) & \text{if } \neg\omega \in \neg\mathscr{L} \cap \neg\mathscr{M}. \end{cases}$$

Example 3.7. Reconsider the SFBSESs Γ and Λ in Example 3.6. By Definition 3.11, their extended intersection $\Gamma \hat{\cap}_E \Lambda$ is given in Table 21. In a similar manner, using Definition 3.12, the SFBSE extended union $\Gamma \hat{\cup}_E \Lambda$ is shown in Table 22.

Table 21: Extended intersection between SFBSESs Γ and Λ in Example 3.7

$\Gamma \hat{\cap}_E \Lambda$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$(\beta_1, u, 1)$	$\begin{pmatrix}(0.80, 0.05, 0.20)\\(0.20, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.01, 0.30)\\(0.40, 0.03, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.03, 0.30)\\(0.32, 0.01, 0.60)\end{pmatrix}$
$(\beta_1, v, 1)$	$\begin{pmatrix}(0.70, 0.05, 0.10)\\(0.30, 0.02, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.75, 0.01, 0.20)\\(0.33, 0.03, 0.68)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.03, 0.40)\\(0.10, 0.04, 0.85)\end{pmatrix}$
$(\beta_2, v, 1)$	$\begin{pmatrix}(0.30, 0.03, 0.70)\\(0.60, 0.01, 0.45)\end{pmatrix}$	$\begin{pmatrix}(0.50, 0.03, 0.40)\\(0.70, 0.01, 0.40)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.04, 0.39)\\(0.30, 0.07, 0.36)\end{pmatrix}$
$(\beta_3, w, 1)$	$\begin{pmatrix}(0.40, 0.05, 0.60)\\(0.70, 0.02, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.65, 0.01, 0.28)\\(0.30, 0.03, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.03, 0.40)\\(0.25, 0.01, 0.80)\end{pmatrix}$
$(\beta_1, v, 0)$	$\begin{pmatrix}(0.30, 0.03, 0.50)\\(0.70, 0.01, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.23, 0.03, 0.70)\\(0.70, 0.02, 0.24)\end{pmatrix}$	$\begin{pmatrix}(0.20, 0.05, 0.73)\\(0.80, 0.01, 0.15)\end{pmatrix}$

Table 22: Extended union between SFBSESs Γ and Λ in Example 3.7

$\Gamma\hat{\cup}_E\Lambda$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$(\beta_1, u, 1)$	$\begin{pmatrix}(0.80, 0.05, 0.20)\\(0.20, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.01, 0.30)\\(0.40, 0.03, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.03, 0.30)\\(0.32, 0.01, 0.60)\end{pmatrix}$
$(\beta_1, v, 1)$	$\begin{pmatrix}(0.70, 0.05, 0.10)\\(0.30, 0.02, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.75, 0.01, 0.20)\\(0.33, 0.03, 0.68)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.03, 0.40)\\(0.10, 0.04, 0.85)\end{pmatrix}$
$(\beta_2, v, 1)$	$\begin{pmatrix}(0.50, 0.03, 0.70)\\(0.40, 0.01, 0.54)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.03, 0.25)\\(0.50, 0.01, 0.42)\end{pmatrix}$	$\begin{pmatrix}(0.82, 0.04, 0.30)\\(0.20, 0.07, 0.76)\end{pmatrix}$
$(\beta_3, w, 1)$	$\begin{pmatrix}(0.80, 0.05, 0.30)\\(0.30, 0.02, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.90, 0.01, 0.10)\\(0.15, 0.03, 0.89)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.03, 0.30)\\(0.22, 0.01, 0.80)\end{pmatrix}$
$(\beta_1, v, 0)$	$\begin{pmatrix}(0.35, 0.03, 0.46)\\(0.65, 0.01, 0.45)\end{pmatrix}$	$\begin{pmatrix}(0.80, 0.03, 0.23)\\(0.18, 0.02, 0.83)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.05, 0.36)\\(0.36, 0.01, 0.65)\end{pmatrix}$

Proposition 3.6. Let $\Gamma = (\hat{\zeta}_1, \hat{\eta}_1, \mathscr{L})$, $\Lambda = (\hat{\zeta}_2, \hat{\eta}_2, \mathscr{M})$, and $\Pi = (\hat{\zeta}_3, \hat{\eta}_3, \mathscr{N})$ be the SFBSESs on the universe $\mathscr{Z}$, then

1. $\Gamma\hat{\cap}_E\Lambda = \Lambda\hat{\cap}_E\Gamma$
2. $\Gamma\hat{\cup}_E\Lambda = \Lambda\hat{\cup}_E\Gamma$
3. $\Gamma\hat{\cap}_E(\Lambda\hat{\cap}_E\Pi) = (\Gamma\hat{\cap}_E\Lambda)\hat{\cap}_E\Pi$.
4. $\Gamma\hat{\cup}_E(\Lambda\hat{\cup}_E\Pi) = (\Gamma\hat{\cup}_E\Lambda)\hat{\cup}_E\Pi$.
5. $\Gamma\hat{\cap}_E(\Lambda\hat{\cup}_E\Pi) = (\Gamma\hat{\cap}_E\Lambda)\hat{\cup}_E(\Gamma\hat{\cap}_E\Pi)$.
6. $\Gamma\hat{\cup}_E(\Lambda\hat{\cap}_E\Pi) = (\Gamma\hat{\cup}_E\Lambda)\hat{\cap}_E(\Gamma\hat{\cup}_E\Pi)$.

Proof.

1. Let $\Gamma\hat{\cap}_E\Lambda = (\check{\rho}, \check{\varpi}, \mathscr{L} \cup \mathscr{M})$. By Definition 3.11, $\forall\omega \in \mathscr{L} \cup \mathscr{M}$, we have

$$\check{\rho}(\omega) = \begin{cases} \hat{\zeta}_1(\omega) & \text{if } \omega \in \mathscr{L} - \mathscr{M} \\ \hat{\zeta}_2(\omega) & \text{if } \omega \in \mathscr{M} - \mathscr{L} \\ \hat{\zeta}_1(\omega) \cap \hat{\zeta}_2(\omega) & \text{if } \omega \in \mathscr{L} \cap \mathscr{M} \end{cases}$$

$$\check{\varpi}(\neg\omega) = \begin{cases} \hat{\eta}_1(\neg\omega) & \text{if } \neg\omega \in (\neg\mathscr{L}) - (\neg\mathscr{M}) \\ \hat{\eta}_2(\neg\omega) & \text{if } \neg\omega \in (\neg\mathscr{M}) - (\neg\mathscr{L}) \\ \hat{\eta}_1(\neg\omega) \cup \hat{\eta}_2(\neg\omega)) & \text{if } \neg\omega \in (\neg\mathscr{L}) \cap (\neg\mathscr{M}) \end{cases}$$

Eliminating the trivial cases, and considering only $\omega \in \mathscr{L} \cap \mathscr{M}$,

$$\begin{aligned}\hat{\zeta}_1(\omega)\hat{\cap}_E\hat{\zeta}_2(\omega) = \check{\rho}(\omega) &= \hat{\zeta}_1(\omega) \cap \hat{\zeta}_2(\omega)\\ &= \hat{\zeta}_2(\omega) \cap \hat{\zeta}_1(\omega)\\ &= \hat{\zeta}_2(\omega)\hat{\cap}_E\hat{\zeta}_1(\omega)\end{aligned}$$

Similarly,

$$\begin{aligned}\hat{\eta}_1(\neg\omega)\hat{\cap}_E\hat{\eta}_2(\neg\omega) = \check{\varpi}(\neg\omega) &= \hat{\eta}_1(\neg\omega) \cup \hat{\eta}_2(\neg\omega)\\ &= \hat{\eta}_2(\neg\omega) \cup \hat{\eta}_1(\neg\omega)\\ &= \hat{\eta}_2(\neg\omega)\hat{\cap}_E\hat{\eta}_1(\neg\omega)\end{aligned}$$

This implies that $(\hat{\zeta}_1, \hat{\eta}_1, \mathscr{L})\hat{\cap}_E(\hat{\zeta}_2, \hat{\eta}_2, \mathscr{M}) = (\hat{\zeta}_2, \hat{\eta}_2, \mathscr{M})\hat{\cap}_E(\hat{\zeta}_1, \hat{\eta}_1, \mathscr{L})$. Hence $\Gamma\hat{\cap}_E\Lambda = \Lambda\hat{\cap}_E\Gamma$.

The remaining parts can be proved using similar arguments.

Proposition 3.7. Let $\Gamma = (\hat{\zeta}_1, \hat{\eta}_1, \mathscr{L})$ and $\Lambda = (\hat{\zeta}_2, \hat{\eta}_2, \mathscr{L})$ be two SFBSESs over $\mathscr{X}$, then

1. $\Gamma\hat{\vee}\Gamma = \Gamma\hat{\wedge}\Gamma$
2. $\Gamma\hat{\cap}_E\Lambda = \Gamma\hat{\cap}_R\Lambda$
3. $\Gamma\hat{\cup}_E\Lambda = \Gamma\hat{\cup}_R\Lambda$
4. $\Gamma\hat{\cap}_R\Gamma = \Gamma\hat{\cup}_R\Gamma = \Gamma$
5. $\Gamma\hat{\cap}_E\Gamma = \Gamma\hat{\cup}_E\Gamma = \Gamma$

Proof. The proofs can be directly deduced from Definitions 3.8 to 3.12.

4 Application of SFBSESs in MAGDM Environmental Problem

In this section, we present an application of SFBSESs regarding the ranking of non-powered dams to be powered in the future. Before going to the application and presenting the algorithm, we define some important notions as follows:

Remark 4.1. Note that the SFSESs $\hat{\Gamma} = (\hat{\zeta}, \mathscr{L})$ and $\mathring{\Gamma} = (\hat{\eta}, \neg\mathscr{L})$ corresponding to the SFBSES $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{L})$ are respectively called the favor SFSES and non-favor SFSES.

Remark 4.2. The SES tables representing the level favor SES $\mathfrak{L}(\hat{\Gamma}, \lambda) = (\mathscr{C}_\lambda, \mathscr{L})$ and the level non-favor SES $\mathfrak{L}(\mathring{\Gamma}, \lambda) = (\mathscr{D}_\lambda, \neg\mathscr{L})$ are obtained by using Definition 2.4 having elements a_{ij} and b_{ij}, respectively.

Remark 4.3. The SES table representing the focus-level SES $(\mathfrak{F}, level_\Gamma)$ is obtained as a difference of level favor and level non-favor SESs with entries

$c_{ij} = a_{ij} - b_{ij}$.

The agree and disagree subsets of the focus-level SES are defined as follows:

Definition 4.1. The *agree focus-level SES* $(\mathfrak{F}, level_\Gamma)_1$ is the soft expert subset of $(\mathfrak{F}, level_\Gamma)$ defined as:

$(\mathfrak{F}, level_\Gamma)_1 = \{\mathfrak{F}(\ell): \ell \in \mathscr{P} \times \mathscr{E} \times \{1\}\}$.

Similarly, the *disagree focus-level SES* $(\mathfrak{F}, level_\Gamma)_0$ is the soft expert subset of $(\mathfrak{F}, level_\Gamma)$, which is provided by

$(\mathfrak{F}, level_\Gamma)_0 = \{\mathfrak{F}(\ell): \ell \in \mathscr{P} \times \mathscr{E} \times \{0\}\}$.

We now present an algorithm based on SFBSES for ranking alternatives considering an MAGDM problem as follows:

Algorithm 1: Ranking objects under SFBSES information.

1) **Input:**
   (a) The universe $\mathscr{Z}$ of objects,
   (b) The set $\mathscr{P}$ of parameters and corresponding not-set $\neg\mathscr{P}$ of parameters,
   (c) The set $\mathscr{E}$ of experts,
   (d) For $\mathscr{L} = \mathscr{P} \times \mathscr{E} \times \mathcal{O}$ and $\neg\mathscr{L} = \neg\mathscr{P} \times \mathscr{E} \times \mathcal{O}$, insert the SFBSES $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{L})$ using the experts' opinions.
2) Calculate the favor SFSES $\hat{\Gamma} = (\hat{\zeta}, \mathscr{L})$ and the non-favor SFSES $\mathring{\Gamma} = (\hat{\eta}, \neg\mathscr{L})$ for Γ.
3) On the basis of threshold function λ (mid, tbb, bbb, or med-threshold functions), find the favor threshold and the non-favor threshold.
4) Find the level favor SES using the level favor threshold of $\hat{\Gamma}$.
5) Find the level non-favor SES using the level non-favor threshold of $\mathring{\Gamma}$.
6) Calculate the focus level SES as a difference of level favor and level non-favor SESs.
7) Find the agree focus-level SES and disagree focus-level SES tables with entries ag_{ij} and dag_{ij}, respectively.
8) Find the F-agree scores $FA_j = \sum_i ag_{ij}$.
9) Find the F-disagree scores $FD_j = \sum_i dag_{ij}$.
10) Calculate the decision scores $d_j = FA_j - FD_j$.
11) Find s such that $d_s = \max(d_j)$.

Output: $\mathfrak{z}_s$ is the optimal choice analogous to s in the last step. For ascending values of d_js, corresponding $\mathfrak{z}_j$s give the ranking of the alternatives.

Fig. 2 gives a flowchart for the above algorithm.

Example 4.1. Ranking of non-powered dams' suitability for a hydroelectric project:

With the continuously increasing population, adoption of technologies, electricity-powered appliances, industries, vehicles, and more, the need for power production is ever increasing, leading to massive usage of resources consumed in the power-generation process. For many years, this power has been generated over the globe using power-generation plants based on non-renewable resources such as natural gas, oil, coal, etc. Unfortunately, these resources are non-renewable and depleting at rates faster than ever before. In this scenario, consideration of an alternative is mandatory to keep the power-generation procedure intact while minimizing the usage of non-renewable resources.

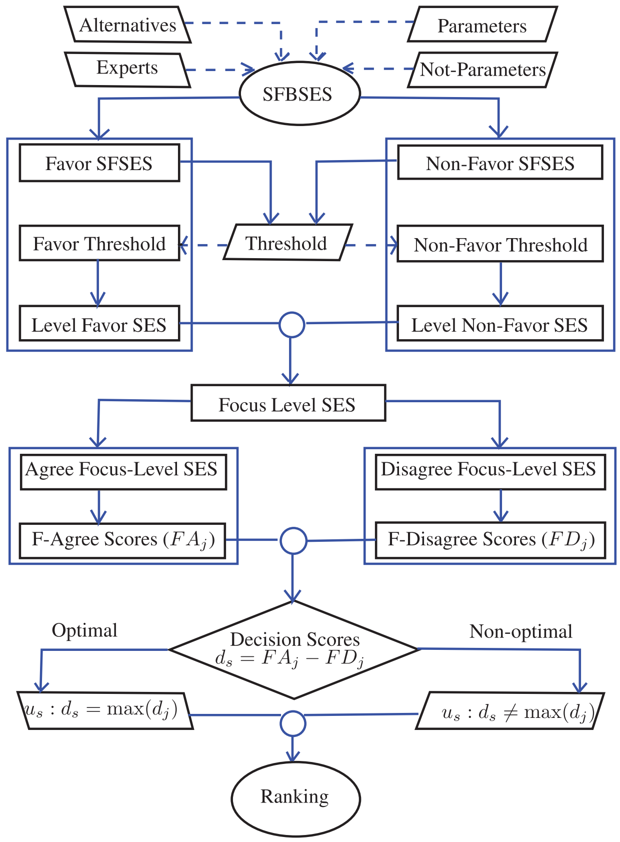

Figure 2: Multi-attribute group decision-making under SFBSESs

Renewable resources based on the sun, water, earth, wind, and nuclear energy prove very effective alternatives for the discussed scenario. These resources include solar power, wind energy, geothermal energy, nuclear power, hydropower, tidal power, bio-gas, etc. All these resources have advantages and limitations based on their setup costs, geological requirements, possible hazards, sustainability, dependence, efficiency, and more. According to recent trends, governments all over the globe are planning and working to maximize the usage of renewable resources and minimize the usage of fossil fuels for power generation. On analyzing the global renewable energy generation and consumption, it clarifies that hydropower is the most common and widely used renewable energy resource all over the globe. Due to the massive amount of water on planet Earth, this particular renewable resource can benefit more widely and efficiently by considering critical measures and improvements. As of 2020, according to the international energy agency[1], hydropower reached 4418 Terawatt-hour (TWh) by generating more power than all other renewable energy resources in total.

[1] https://www.iea.org/reports/hydropower.

Hydropower mainly generates using dams all over the globe, equipped with special power-generation equipment including turbines, pumps, etc. Apart from these hydroelectric projects, numerous watersheds, dams, and reservoirs do not yet intend to produce electricity. Such structures are mainly devoted to water storage, flood control, recreational activities, and navigation. These non-powered dams (NPDs) are much more in number than powered dams and constitute a substantial part of the big picture. Considering only United States (US), there are around 90,000 documented dams, of which only 3% generate electricity. However, according to organizations like Oak Ridge National Laboratory (ORNL)[2], a lot of these NPDs can be effectively used for hydropower generation. From the energy potential assessment made at US NPDs in 2012 by Hadjerioua et al. [51] under the US Department of Energy carried out by ORNL, Fig. 3 represents the NPDs in the US with hydropower potential exceeding one megawatt (MW). The blue circles indicate the NPDs have electricity generation potential with different radii corresponding to distinct capacities. It shows that NPDs can add to the power generation capacities by necessary measures and installations.

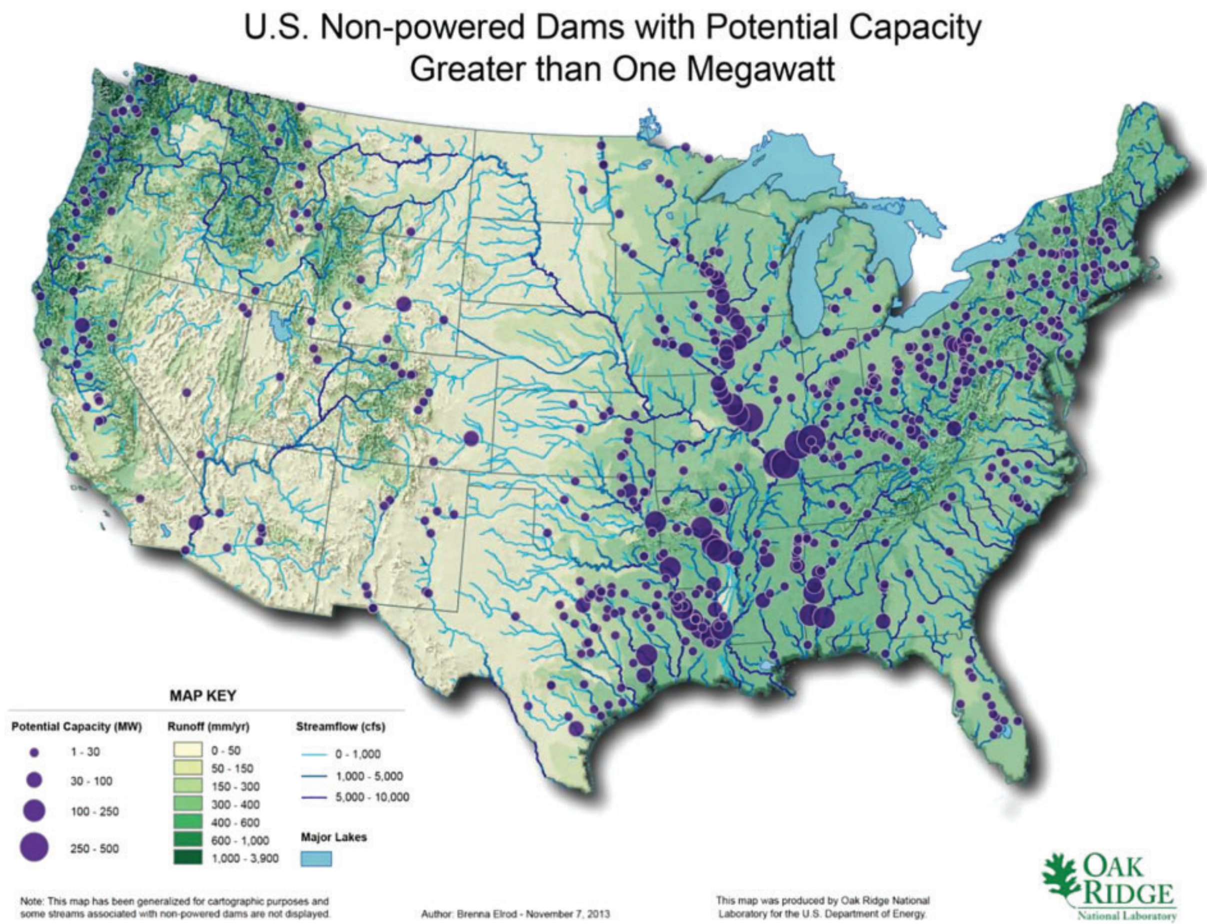

Figure 3: US NPDs with energy potential greater than 1 MW[3]

[2] https://smh.ornl.gov/.
[3] https://hydrosource.ornl.gov/maps.

Every dam has its characteristics, geology, structure, and environmental and political impact. Therefore, a clear understanding of all these aspects is compulsory before hydroelectric generation. Turning NPDs into hydroelectric generators saves a great deal counting money, resources, space, and infrastructure. However, considering which NPDs are eligible for power generation and comparatively better than the other competing alternatives is demanding and requires effective decision-making. Motivated by this, the following example will consider the ranking of four NPDs with varying infrastructures, climates, geographies, and political considerations.

Consider there are four NPDs constituting the set $\mathscr{X} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3, \mathfrak{z}_4\}$ in a country with distinct locations, structures, and environmental and functional characteristics. In order to address the energy crisis, the government is willing to power one of these dams subject to the dominating capability of the dam and the low-price budget. The government assigns this job to the national energy committee to critically analyze the available dams and provide a ranking of the dams based on their suitability and efficiency for hydropower generation. The committee makes two teams, each containing experts from engineering, power generation, environment control, economy, and international relations. These two teams gather important data for these dams, including their climate, nearby populations, water quality, water inflow and outflow, infrastructure, political clashes, expenses, etc. The energy committee advises the teams to rank the NPDs by considering parameters based on the pre-generation decision parameters considered by ORNL[2]. Considering the committee's advice, teams collectively consider two sets of parameters: the set of parameters, i.e., favorable parameters, and the not-set of parameters, i.e., the non-favorable parameters, as shown in Fig. 4. The favorable parameters are represented by the set $\mathscr{P} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$, such that:

- β_1: Good design includes appropriate infrastructure, sound water transportation, good dam condition, easy access to the location, etc.
- β_2: Efficient functionality includes functional eligibility, good storage, safety measures implementation, effective water discharge to turn turbines, etc.
- β_3: Suitable environment includes favorable geolocation, reliable climate, less impact on wildlife, good water quality, fewer sediments, disruptions, etc.
- β_4: Socio-economic serenity includes local ownership by the government, minimum political clashes, etc.

Similarly, the non-favorable parameters form the set $\neg\mathscr{P} = \{\neg\beta_1, \neg\beta_2, \neg\beta_3, \neg\beta_4\}$, such that:

- $\neg\beta_1$: Bad design includes unreliable infrastructure unsuitable for setting hydropower tools, poor water transportation, bad dam condition, remote location, etc.
- $\neg\beta_2$: Inefficient functionality includes functional ineligibility, minimal storage, no safety measures implemented, ineffective water discharge discouraging turbines operation, etc.
- $\neg\beta_3$: Unsuitable environment includes unfavorable geolocation, nonreliable climate, adverse effects on wildlife, poor water quality, high quantity of sediments, etc.
- $\neg\beta_4$: Socio-economic disruptions include ownership issues and conflicts, political clashes, impact on neighboring countries, etc.

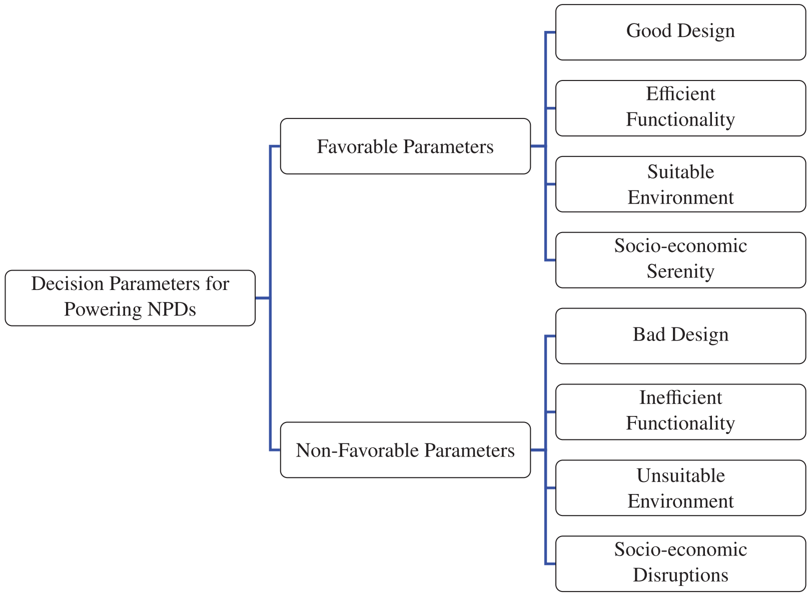

Figure 4: Decision parameters powering non-powered dams

Consider the two teams u and v act as the two experts collectively for the decision-making process, and generate a comprehensive report based on the collective team opinions in the form of a SFBSES $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{L})$ represented in Table 23, where $\mathscr{L} = \mathscr{P} \times \mathscr{E} \times \mathcal{O}$, such that $\mathscr{E}$ represents the set of experts and $\mathcal{O} = \{1 = agree, 0 = disagree\}$ represents the set of corresponding opinions.

Table 23: SFBSES $\Gamma = (\hat{\zeta}, \hat{\eta}, \mathscr{L})$ in Example 4.1

$(\hat{\zeta}, \hat{\eta}, \mathscr{L})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$	$\mathfrak{z}_4$
$(\beta_1, u, 1)$	$\begin{pmatrix}(0.75, 0.15, 0.40)\\(0.45, 0.20, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.80, 0.08, 0.35)\\(0.30, 0.05, 0.80)\end{pmatrix}$	$\begin{pmatrix}(0.83, 0.03, 0.43)\\(0.46, 0.10, 0.76)\end{pmatrix}$	$\begin{pmatrix}(0.78, 0.03, 0.55)\\(0.50, 0.07, 0.70)\end{pmatrix}$
$(\beta_1, v, 1)$	$\begin{pmatrix}(0.83, 0.10, 0.36)\\(0.40, 0.08, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.76, 0.13, 0.60)\\(0.40, 0.14, 0.74)\end{pmatrix}$	$\begin{pmatrix}(0.79, 0.08, 0.35)\\(0.38, 0.02, 0.86)\end{pmatrix}$	$\begin{pmatrix}(0.63, 0.04, 0.45)\\(0.54, 0.12, 0.73)\end{pmatrix}$
$(\beta_2, u, 1)$	$\begin{pmatrix}(0.80, 0.10, 0.29)\\(0.40, 0.13, 0.63)\end{pmatrix}$	$\begin{pmatrix}(0.90, 0.09, 0.10)\\(0.12, 0.04, 0.79)\end{pmatrix}$	$\begin{pmatrix}(0.73, 0.05, 0.50)\\(0.53, 0.10, 0.74)\end{pmatrix}$	$\begin{pmatrix}(0.71, 0.05, 0.50)\\(0.52, 0.06, 0.62)\end{pmatrix}$
$(\beta_2, v, 1)$	$\begin{pmatrix}(0.76, 0.08, 0.40)\\(0.32, 0.03, 0.80)\end{pmatrix}$	$\begin{pmatrix}(0.69, 0.09, 0.70)\\(0.60, 0.14, 0.64)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.09, 0.39)\\(0.30, 0.09, 0.36)\end{pmatrix}$	$\begin{pmatrix}(0.72, 0.13, 0.46)\\(0.52, 0.12, 0.69)\end{pmatrix}$
$(\beta_3, u, 1)$	$\begin{pmatrix}(0.60, 0.06, 0.45)\\(0.53, 0.08, 0.68)\end{pmatrix}$	$\begin{pmatrix}(0.89, 0.06, 0.31)\\(0.39, 0.03, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.54, 0.03, 0.57)\\(0.60, 0.06, 0.49)\end{pmatrix}$	$\begin{pmatrix}(0.79, 0.03, 0.35)\\(0.29, 0.06, 0.83)\end{pmatrix}$
$(\beta_3, v, 1)$	$\begin{pmatrix}(0.73, 0.03, 0.35)\\(0.32, 0.01, 0.72)\end{pmatrix}$	$\begin{pmatrix}(0.80, 0.09, 0.30)\\(0.40, 0.05, 0.64)\end{pmatrix}$	$\begin{pmatrix}(0.43, 0.03, 0.75)\\(0.22, 0.07, 0.73)\end{pmatrix}$	$\begin{pmatrix}(0.68, 0.10, 0.37)\\(0.22, 0.07, 0.73)\end{pmatrix}$
$(\beta_4, u, 1)$	$\begin{pmatrix}(0.40, 0.36, 0.60)\\(0.70, 0.26, 0.40)\end{pmatrix}$	$\begin{pmatrix}(0.86, 0.06, 0.30)\\(0.40, 0.03, 0.70)\end{pmatrix}$	$\begin{pmatrix}(0.90, 0.03, 0.15)\\(0.12, 0.06, 0.68)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.28, 0.35)\\(0.42, 0.46, 0.38)\end{pmatrix}$
$(\beta_4, v, 1)$	$\begin{pmatrix}(0.53, 0.03, 0.39)\\(0.32, 0.01, 0.72)\end{pmatrix}$	$\begin{pmatrix}(0.75, 0.07, 0.40)\\(0.40, 0.05, 0.64)\end{pmatrix}$	$\begin{pmatrix}(0.75, 0.24, 0.35)\\(0.43, 0.17, 0.63)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.20, 0.40)\\(0.45, 0.18, 0.50)\end{pmatrix}$

(Continued)

Table 23 (continued)

$(\hat{\zeta}, \hat{\eta}, \mathscr{L})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$	$\mathfrak{z}_4$
$(\beta_1, u, 0)$	$\begin{pmatrix}(0.43, 0.10, 0.70)\\(0.80, 0.13, 0.35)\end{pmatrix}$	$\begin{pmatrix}(0.38, 0.09, 0.75)\\(0.76, 0.06, 0.41)\end{pmatrix}$	$\begin{pmatrix}(0.36, 0.06, 0.80)\\(0.70, 0.01, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.50, 0.10, 0.80)\\(0.70, 0.04, 0.50)\end{pmatrix}$
$(\beta_1, v, 0)$	$\begin{pmatrix}(0.30, 0.03, 0.76)\\(0.65, 0.04, 0.43)\end{pmatrix}$	$\begin{pmatrix}(0.50, 0.13, 0.80)\\(0.70, 0.10, 0.43)\end{pmatrix}$	$\begin{pmatrix}(0.40, 0.07, 0.70)\\(0.80, 0.15, 0.35)\end{pmatrix}$	$\begin{pmatrix}(0.40, 0.07, 0.55)\\(0.80, 0.16, 0.40)\end{pmatrix}$
$(\beta_2, u, 0)$	$\begin{pmatrix}(0.30, 0.12, 0.73)\\(0.60, 0.15, 0.39)\end{pmatrix}$	$\begin{pmatrix}(0.20, 0.07, 0.85)\\(0.80, 0.04, 0.08)\end{pmatrix}$	$\begin{pmatrix}(0.50, 0.02, 0.68)\\(0.70, 0.03, 0.56)\end{pmatrix}$	$\begin{pmatrix}(0.50, 0.02, 0.70)\\(0.60, 0.08, 0.45)\end{pmatrix}$
$(\beta_2, v, 0)$	$\begin{pmatrix}(0.40, 0.10, 0.80)\\(0.65, 0.11, 0.45)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.01, 0.66)\\(0.60, 0.05, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.45, 0.03, 0.70)\\(0.35, 0.04, 0.35)\end{pmatrix}$	$\begin{pmatrix}(0.50, 0.03, 0.65)\\(0.70, 0.04, 0.42)\end{pmatrix}$
$(\beta_3, u, 0)$	$\begin{pmatrix}(0.50, 0.10, 0.60)\\(0.70, 0.02, 0.50)\end{pmatrix}$	$\begin{pmatrix}(0.25, 0.06, 0.83)\\(0.79, 0.02, 0.20)\end{pmatrix}$	$\begin{pmatrix}(0.60, 0.10, 0.50)\\(0.50, 0.06, 0.60)\end{pmatrix}$	$\begin{pmatrix}(0.40, 0.10, 0.80)\\(0.70, 0.06, 0.51)\end{pmatrix}$
$(\beta_3, v, 0)$	$\begin{pmatrix}(0.35, 0.03, 0.69)\\(0.67, 0.10, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.20, 0.02, 0.90)\\(0.60, 0.10, 0.40)\end{pmatrix}$	$\begin{pmatrix}(0.70, 0.06, 0.50)\\(0.70, 0.08, 0.18)\end{pmatrix}$	$\begin{pmatrix}(0.33, 0.06, 0.77)\\(0.68, 0.08, 0.25)\end{pmatrix}$
$(\beta_4, u, 0)$	$\begin{pmatrix}(0.60, 0.10, 0.43)\\(0.50, 0.12, 0.63)\end{pmatrix}$	$\begin{pmatrix}(0.23, 0.16, 0.76)\\(0.64, 0.02, 0.40)\end{pmatrix}$	$\begin{pmatrix}(0.10, 0.10, 0.90)\\(0.83, 0.06, 0.26)\end{pmatrix}$	$\begin{pmatrix}(0.40, 0.10, 0.63)\\(0.70, 0.06, 0.36)\end{pmatrix}$
$(\beta_4, v, 0)$	$\begin{pmatrix}(0.46, 0.03, 0.55)\\(0.70, 0.10, 0.30)\end{pmatrix}$	$\begin{pmatrix}(0.40, 0.02, 0.70)\\(0.76, 0.10, 0.38)\end{pmatrix}$	$\begin{pmatrix}(0.36, 0.06, 0.47)\\(0.50, 0.08, 0.45)\end{pmatrix}$	$\begin{pmatrix}(0.33, 0.08, 0.77)\\(0.73, 0.08, 0.30)\end{pmatrix}$

Based on the above report, the NPDs are decided to be ranked using Algorithm 1. The corresponding favor SFSES and non-favor SFSES are shown in Tables 24 and 25, respectively. Committee further decides to consider the med-level threshold for decision-making. The favor and non-favor med-level thresholds correspond to Tables 24 and 25.

Table 24: $\hat{\Gamma} = (\hat{\zeta}, \mathscr{L})$ in Example 4.1

$(\hat{\zeta}, \mathscr{L})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$	$\mathfrak{z}_4$
$(\beta_1, u, 1)$	(0.75, 0.15, 0.40)	(0.80, 0.08, 0.35)	(0.83, 0.03, 0.43)	(0.78, 0.03, 0.55)
$(\beta_1, v, 1)$	(0.83, 0.10, 0.36)	(0.76, 0.13, 0.60)	(0.79, 0.08, 0.35)	(0.63, 0.04, 0.45)
$(\beta_2, u, 1)$	(0.80, 0.10, 0.29)	(0.90, 0.09, 0.10)	(0.73, 0.05, 0.50)	(0.71, 0.05, 0.50)
$(\beta_2, v, 1)$	(0.76, 0.08, 0.40)	(0.69, 0.09, 0.70)	(0.70, 0.09, 0.39)	(0.72, 0.13, 0.46)
$(\beta_3, u, 1)$	(0.60, 0.06, 0.45)	(0.89, 0.06, 0.31)	(0.54, 0.03, 0.57)	(0.79, 0.03, 0.35)
$(\beta_3, v, 1)$	(0.73, 0.03, 0.35)	(0.80, 0.09, 0.30)	(0.43, 0.03, 0.75)	(0.68, 0.10, 0.37)
$(\beta_4, u, 1)$	(0.40, 0.36, 0.60)	(0.86, 0.06, 0.30)	(0.90, 0.03, 0.15)	(0.70, 0.28, 0.35)
$(\beta_4, v, 1)$	(0.53, 0.03, 0.39)	(0.75, 0.07, 0.40)	(0.75, 0.24, 0.35)	(0.70, 0.20, 0.40)
$(\beta_1, u, 0)$	(0.43, 0.10, 0.70)	(0.38, 0.09, 0.75)	(0.36, 0.06, 0.80)	(0.50, 0.10, 0.80)
$(\beta_1, v, 0)$	(0.30, 0.03, 0.76)	(0.50, 0.13, 0.80)	(0.40, 0.07, 0.70)	(0.40, 0.07, 0.55)
$(\beta_2, u, 0)$	(0.30, 0.12, 0.73)	(0.20, 0.07, 0.85)	(0.50, 0.02, 0.68)	(0.50, 0.02, 0.70)
$(\beta_2, v, 0)$	(0.40, 0.10, 0.80)	(0.60, 0.01, 0.66)	(0.45, 0.03, 0.70)	(0.50, 0.03, 0.65)
$(\beta_3, u, 0)$	(0.50, 0.10, 0.60)	(0.25, 0.06, 0.83)	(0.60, 0.10, 0.50)	(0.40, 0.10, 0.80)
$(\beta_3, v, 0)$	(0.35, 0.03, 0.69)	(0.20, 0.02, 0.90)	(0.70, 0.06, 0.50)	(0.33, 0.06, 0.77)

(Continued)

Table 24 (continued)

$(\hat{\zeta}, \mathscr{L})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$	$\mathfrak{z}_4$
$(\beta_4, u, 0)$	(0.60, 0.10, 0.43)	(0.23, 0.16, 0.76)	(0.10, 0.10, 0.90)	(0.40, 0.10, 0.63)
$(\beta_4, v, 0)$	(0.46, 0.03, 0.55)	(0.40, 0.02, 0.70)	(0.36, 0.06, 0.47)	(0.33, 0.08, 0.77)

Table 25: $\mathring{\Gamma} = (\hat{\eta}, \neg\mathscr{L})$ in Example 4.1

$(\hat{\eta}, \neg\mathscr{L})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$	$\mathfrak{z}_4$
$(\neg\beta_1, u, 1)$	(0.45, 0.20, 0.70)	(0.30, 0.05, 0.80)	(0.46, 0.10, 0.76)	(0.50, 0.07, 0.70)
$(\neg\beta_1, v, 1)$	(0.40, 0.08, 0.70)	(0.40, 0.14, 0.74)	(0.38, 0.02, 0.86)	(0.54, 0.12, 0.73)
$(\neg\beta_2, u, 1)$	(0.40, 0.13, 0.63)	(0.12, 0.04, 0.79)	(0.53, 0.10, 0.74)	(0.52, 0.06, 0.62)
$(\neg\beta_2, v, 1)$	(0.32, 0.03, 0.80)	(0.60, 0.14, 0.64)	(0.30, 0.09, 0.36)	(0.52, 0.12, 0.69)
$(\neg\beta_3, u, 1)$	(0.53, 0.08, 0.68)	(0.39, 0.03, 0.70)	(0.60, 0.06, 0.49)	(0.29, 0.06, 0.83)
$(\neg\beta_3, v, 1)$	(0.32, 0.01, 0.72)	(0.40, 0.05, 0.64)	(0.22, 0.07, 0.73)	(0.22, 0.07, 0.73)
$(\neg\beta_4, u, 1)$	(0.70, 0.26, 0.40)	(0.40, 0.03, 0.70)	(0.12, 0.06, 0.68)	(0.42, 0.46, 0.38)
$(\neg\beta_4, v, 1)$	(0.32, 0.01, 0.72)	(0.40, 0.05, 0.64)	(0.43, 0.17, 0.63)	(0.45, 0.18, 0.50)
$(\neg\beta_1, u, 0)$	(0.80, 0.13, 0.35)	(0.76, 0.06, 0.41)	(0.70, 0.01, 0.30)	(0.70, 0.04, 0.50)
$(\neg\beta_1, v, 0)$	(0.65, 0.04, 0.43)	(0.70, 0.10, 0.43)	(0.80, 0.15, 0.35)	(0.80, 0.16, 0.40)
$(\neg\beta_2, u, 0)$	(0.60, 0.15, 0.39)	(0.80, 0.04, 0.08)	(0.70, 0.03, 0.56)	(0.60, 0.08, 0.45)
$(\neg\beta_2, v, 0)$	(0.65, 0.11, 0.45)	(0.60, 0.05, 0.60)	(0.35, 0.04, 0.35)	(0.70, 0.04, 0.42)
$(\neg\beta_3, u, 0)$	(0.70, 0.02, 0.50)	(0.79, 0.02, 0.20)	(0.50, 0.06, 0.60)	(0.70, 0.06, 0.51)
$(\neg\beta_3, v, 0)$	(0.67, 0.10, 0.30)	(0.60, 0.10, 0.40)	(0.70, 0.08, 0.18)	(0.68, 0.08, 0.25)
$(\neg\beta_4, u, 0)$	(0.50, 0.12, 0.63)	(0.64, 0.02, 0.40)	(0.83, 0.06, 0.26)	(0.70, 0.06, 0.36)
$(\neg\beta_4, v, 0)$	(0.70, 0.10, 0.30)	(0.76, 0.10, 0.38)	(0.50, 0.08, 0.45)	(0.73, 0.08, 0.30)

$$med_{\hat{\Gamma}} = \begin{Bmatrix} \langle(\beta_1, u, 1), (0.790, 0.055, 0.415)\rangle, \\ \langle(\beta_1, v, 1), (0.775, 0.090, 0.405)\rangle, \\ \langle(\beta_2, u, 1), (0.765, 0.070, 0.395)\rangle, \\ \langle(\beta_2, v, 1), (0.710, 0.090, 0.430)\rangle, \\ \langle(\beta_3, u, 1), (0.695, 0.045, 0.400)\rangle, \\ \langle(\beta_3, v, 1), (0.705, 0.060, 0.360)\rangle, \\ \langle(\beta_4, u, 1), (0.780, 0.170, 0.325)\rangle, \\ \langle(\beta_4, v, 1), (0.725, 0.135, 0.395)\rangle, \\ \langle(\beta_1, u, 0), (0.405, 0.095, 0.775)\rangle, \\ \langle(\beta_1, v, 0), (0.400, 0.070, 0.730)\rangle, \\ \langle(\beta_2, u, 0), (0.400, 0.045, 0.715)\rangle, \\ \langle(\beta_2, v, 0), (0.475, 0.030, 0.680)\rangle, \\ \langle(\beta_3, u, 0), (0.450, 0.100, 0.700)\rangle, \\ \langle(\beta_3, v, 0), (0.340, 0.045, 0.730)\rangle, \\ \langle(\beta_4, u, 0), (0.315, 0.100, 0.695)\rangle, \\ \langle(\beta_4, v, 0), (0.380, 0.045, 0.625)\rangle. \end{Bmatrix} \quad med_{\mathring{\Gamma}} = \begin{Bmatrix} \langle(\neg\beta_1, u, 1), (0.455, 0.085, 0.730)\rangle, \\ \langle(\neg\beta_1, v, 1), (0.400, 0.100, 0.735)\rangle, \\ \langle(\neg\beta_2, u, 1), (0.460, 0.080, 0.685)\rangle, \\ \langle(\neg\beta_2, v, 1), (0.420, 0.105, 0.665)\rangle, \\ \langle(\neg\beta_3, u, 1), (0.460, 0.060, 0.690)\rangle, \\ \langle(\neg\beta_3, v, 1), (0.270, 0.060, 0.725)\rangle, \\ \langle(\neg\beta_4, u, 1), (0.410, 0.160, 0.540)\rangle, \\ \langle(\neg\beta_4, v, 1), (0.415, 0.110, 0.635)\rangle, \\ \langle(\neg\beta_1, u, 0), (0.730, 0.050, 0.380)\rangle, \\ \langle(\neg\beta_1, v, 0), (0.750, 0.125, 0.415)\rangle, \\ \langle(\neg\beta_2, u, 0), (0.650, 0.060, 0.420)\rangle, \\ \langle(\neg\beta_2, v, 0), (0.625, 0.045, 0.435)\rangle, \\ \langle(\neg\beta_3, u, 0), (0.700, 0.040, 0.505)\rangle, \\ \langle(\neg\beta_3, v, 0), (0.675, 0.090, 0.275)\rangle, \\ \langle(\neg\beta_4, u, 0), (0.670, 0.060, 0.380)\rangle, \\ \langle(\neg\beta_4, v, 0), (0.715, 0.090, 0.340)\rangle. \end{Bmatrix}$$

Using the above thresholds, Tables 26 and 27 represent the med-level favor SES and the med-level non-favor SES, respectively. Table 28 represents the focus med-level SES as a difference between the above two Tables. Further, the agree and disagree subsets of this focus med-level SES are shown in Tables 29 and 30, respectively.

Table 26: Med-Level favor SES $\mathfrak{L}(\hat{\Gamma}; med_{\hat{\Gamma}})$ of $\hat{\Gamma}$

$\mathfrak{L}(\hat{\Gamma}; med_{\hat{\Gamma}})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$	$\mathfrak{z}_4$
$(\beta_1, u, 1)$	0	0	0	0
$(\beta_1, v, 1)$	0	0	1	0
$(\beta_2, u, 1)$	0	0	0	0
$(\beta_2, v, 1)$	1	0	0	0
$(\beta_3, u, 1)$	0	0	0	1
$(\beta_3, v, 1)$	1	0	0	0
$(\beta_4, u, 1)$	0	1	1	0
$(\beta_4, v, 1)$	0	0	0	0
$(\beta_1, u, 0)$	0	0	0	0
$(\beta_1, v, 0)$	0	0	1	1
$(\beta_2, u, 0)$	0	0	1	1
$(\beta_2, v, 0)$	0	1	0	1
$(\beta_3, u, 0)$	1	0	1	0
$(\beta_3, v, 0)$	1	0	0	0
$(\beta_4, u, 0)$	1	0	0	1
$(\beta_4, v, 0)$	1	0	0	0

Table 27: Med-Level non-favor SES $\mathfrak{L}(\mathring{\Gamma}; med_{\mathring{\Gamma}})$ of $\mathring{\Gamma}$

$\mathfrak{L}(\mathring{\Gamma}; med_{\mathring{\Gamma}})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$	$\mathfrak{z}_4$
$(\neg\beta_1, u, 1)$	0	0	0	1
$(\neg\beta_1, v, 1)$	1	0	0	0
$(\neg\beta_2, u, 1)$	0	0	0	1
$(\neg\beta_2, v, 1)$	0	0	0	0
$(\neg\beta_3, u, 1)$	0	0	1	0
$(\neg\beta_3, v, 1)$	1	1	0	0
$(\neg\beta_4, u, 1)$	0	0	0	0
$(\neg\beta_4, v, 1)$	0	0	0	0
$(\neg\beta_1, u, 0)$	0	0	0	0
$(\neg\beta_1, v, 0)$	0	0	0	0
$(\neg\beta_2, u, 0)$	0	1	0	0
$(\neg\beta_2, v, 0)$	0	0	0	1
$(\neg\beta_3, u, 0)$	1	1	0	0

(Continued)

Table 27 (continued)

$\mathfrak{L}(\mathring{\Gamma}; med_{\mathring{\Gamma}})$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$	$\mathfrak{z}_4$
$(\neg\beta_3, v, 0)$	0	0	1	1
$(\neg\beta_4, u, 0)$	0	0	1	1
$(\neg\beta_4, v, 0)$	0	0	0	1

Table 28: Focus Med-Level SES $(\mathfrak{F}; med_\Gamma)$ for Γ in Example 4.1

$(\mathfrak{F}; med_\Gamma)$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$	$\mathfrak{z}_4$
$(\beta_1, u, 1)$	0	0	0	− 1
$(\beta_1, v, 1)$	− 1	0	1	0
$(\beta_2, u, 1)$	0	0	0	− 1
$(\beta_2, v, 1)$	1	0	0	0
$(\beta_3, u, 1)$	0	0	− 1	1
$(\beta_3, v, 1)$	0	− 1	0	0
$(\beta_4, u, 1)$	0	1	1	0
$(\beta_4, v, 1)$	0	0	0	0
$(\beta_1, u, 0)$	0	0	0	0
$(\beta_1, v, 0)$	0	0	1	1
$(\beta_2, u, 0)$	0	− 1	1	1
$(\beta_2, v, 0)$	0	1	0	0
$(\beta_3, u, 0)$	0	− 1	1	0
$(\beta_3, v, 0)$	1	0	− 1	− 1
$(\beta_4, u, 0)$	1	0	− 1	0
$(\beta_4, v, 0)$	1	0	0	− 1

Table 29: Agree Focus Med-Level SES $(\mathfrak{F}; med_\Gamma)_1$ for Γ in Example 4.1

$(\mathfrak{F}; med_\Gamma)_1$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$	$\mathfrak{z}_4$
(β_1, u)	0	0	0	− 1
(β_1, v)	− 1	0	1	0
(β_2, u)	0	0	0	− 1
(β_2, v)	1	0	0	0
(β_3, u)	0	0	− 1	1
(β_3, v)	0	− 1	0	0
(β_4, u)	0	1	1	0
(β_4, v)	0	0	0	0
$FA_j = \sum_i ag_{ij}$	0	0	1	− 1

Table 30: Disagree Focus Med-Level SES $(\mathfrak{F}; med_\Gamma)_0$ for Γ in Example 4.1

$(\mathfrak{F}; med_\Gamma)_0$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$	$\mathfrak{z}_4$
(β_1, u)	0	0	0	0
(β_1, v)	0	0	1	1
(β_2, u)	0	− 1	1	1
(β_2, v)	0	1	0	0
(β_3, u)	0	− 1	1	0
(β_3, v)	1	0	− 1	− 1
(β_4, u)	1	0	− 1	0
(β_4, v)	1	0	0	− 1
$FD_j = \sum_i dag_{ij}$	3	− 1	1	0

Based on the F-agree and F-disagree scores calculated in Tables 29 and 30, Table 31 represents the decision scores basing the final ranking of the NPDs as:

$\mathfrak{z}_2 > \mathfrak{z}_3 > \mathfrak{z}_4 > \mathfrak{z}_1$

Hence, $\mathfrak{z}_2$ is the most suitable dam to work on, while $\mathfrak{z}_1$ is the worst alternative.

Table 31: Decision scores

$\mathscr{Z}$	F-Agree score	F-Disagree score	Decision score
	FA_j	FD_j	$d_j = FA_j - FD_j$
$\mathfrak{z}_1$	0	3	− 3
$\mathfrak{z}_2$	0	− 1	1
$\mathfrak{z}_3$	1	1	0
$\mathfrak{z}_4$	− 1	0	− 1

5 Comparative Analysis

This section debates the advantages and disadvantages of the proposed model, corresponding to the existing research gaps and limitations. Further, this section examines a comparative analysis of the initiated model with some recent models and proves its dominance.

5.1 Advantages

Many complicated problems concerning power generation, environmental pollution, political and geological conflicts, and complex infrastructures, arise each coming day. These add to the uncertainties requiring intricate handling under corresponding experts and needing appropriate decision-making tools. Analogous to the complexity and requirements of the problems, researchers that contribute to decision-making are developing powerful decision models capable of handling specific issues unsolvable by the already existing methods. These hybridizations of different models stand very important in filling the gaps and limitations in the decision-making process and provide efficient generalized structures for the existing tools. This work initiates SFBSESs as a powerful generalization to many structures including SFSs [12], SFSSs [25], SFSESs [44], BSSs [46], BSESs [47], FBSESs [48], and SFBSSs [49]. Being the generalization, SFBSESs combine the characteristics of the antecedent models and are applicable in a much larger domain. The following list the main advantages of proposed SFBSESs:

1. The positive, neutral, and negative opinions for alternatives in the spherical fuzzy environment guarantee proper handling of complicated uncertainties.
2. The bipolar parameters depicting favorable and symmetrically non-favorable dependencies are properly distinguished in two distinct sets and then considered parallel to each other in the decision-making process.
3. Multiple experts put in their opinions at the same time minimizing the chances of biased decisions and enhancing the overall process by the merger of multiple expertise at once.

5.2 Comparison

In solving the complicated MAGDM problems, both SFSESs [44] and FBSESs [48] are efficient in their respective domains. Being the combination of SFSs [12] and SESs [40], SFSESs offer solutions to MAGDM problems considering the positive, neutral, and negative opinions of decision-makers in the soft expert environment. However, SFSESs cannot distinguish between the bipolar parameters in decision-making and therefore are not very suitable for the corresponding problems. On the other hand, FBSESs as a combination of BSSs [46] and FSESs [41] offer solutions to uncertain MAGDM problems by taking the parameters' bipolarity into account efficiently. But FBSESs fail to consider the negative and neutral memberships in the experts' opinions. In addition, the MADM model SFBSSs [49] being a combination of SFSs [12] and BSSs [46] allows the dealing of spherical fuzzy information while considering the bipolarity of parameters. However, it is restricted to a single expert only. As the generalization of SFSESs, FBSESs, SFBSSs, and SFBSESs efficiently handle MAGDM problems by considering both the bipolarity of parameters and the neutrality and disagreement of the experts' opinions in one place. Fig. 5 explains this difference graphically (Venn diagrams of the four sets covering the four triangular sections representing the qualities including parameterization, bipolarity, spherical fuzzy information, and multiple experts' opinions). Table 32 represents the comparison between SFSESs, FBSESs, SFBSSs, and proposed SFBSESs by reconsidering Example 4.1 and calculating the final scores as follows:

- Using SFSESs (only considering the favorable parameters) and reducing focus level SES to level favor SES in Algorithm 1.
- Using FBSESs (only considering the degrees of agreement) and calculating the focus level SES as the difference of fuzzy SESs for favorable and non-favorable parameters in Algorithm 1.
- Using SFBSSs (fixing $(u, 1)$ as the single expert opinion for reducing the SESs to soft sets) under the Algorithm based on SFBSSs [49].

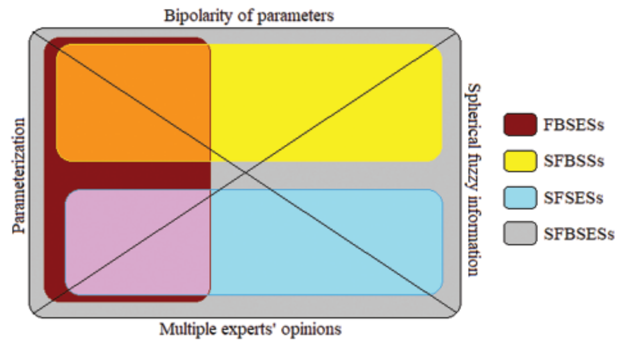

Figure 5: SFBSESs *vs.* FBSESs, SFSESs, and SFBSSs

Table 32: Score-comparison for Example 4.1

Objects	SFSESs [44]	FBSESs [48]	SFBSSs [49]	Proposed SFBSESs
$\mathfrak{z}_1$	− 2	3.89	0	− 3
$\mathfrak{z}_2$	0	6.33	1	1
$\mathfrak{z}_3$	− 1	4.24	0	0
$\mathfrak{z}_4$	− 3	4.50	− 1	− 1

Based on the calculated final scores, the ranking orders of the alternatives are provided in Table 33. Further from Fig. 6, it is clear that ranking results are different by applying SFSESs, FBSESs, SFBSSs, and proposed SFBSESs. When using SFSESs [44], the ranking orders deviate from those, obtained by SFBSESs in the 3rd and 4th positions. With FBSESs [48], the 2nd and 3rd ranks get shuffled compared to those with SFBSESs. In addition, the SFBSSs [49] fail to make a distinction between the 2nd and 3rd rank. These deviations from the results obtained in Example 4.1 indicate the possibility of errors in making decisions by neglecting the bipolarity, ignoring the neutrality and disagreement memberships, or basing the decisions on a single expert's opinions. Hence, it advocates that the proposed SFBSESs are more applicable and accurate than the previous models. The reduction of SFBSESs to SFSESs, FBSESs, and SFBSSs yields the same optimal solution, whereas the overall ranking provided by SFBSESs is relatively more clear and comprehensive. This proves that the proposed work is more robust and reliable relatively.

Table 33: Alternative rankings for Example 4.1 obtained by different models

Decision model	Ranking
SFSESs [44]	$\mathfrak{z}_2 > \mathfrak{z}_3 > \mathfrak{z}_1 > \mathfrak{z}_4$
FBSESs [48]	$\mathfrak{z}_2 > \mathfrak{z}_4 > \mathfrak{z}_3 > \mathfrak{z}_1$
SFBSSs [49]	$\mathfrak{z}_2 > \mathfrak{z}_1 = \mathfrak{z}_3 > \mathfrak{z}_4$
Proposed SFBSESs	$\mathfrak{z}_2 > \mathfrak{z}_3 > \mathfrak{z}_4 > \mathfrak{z}_1$

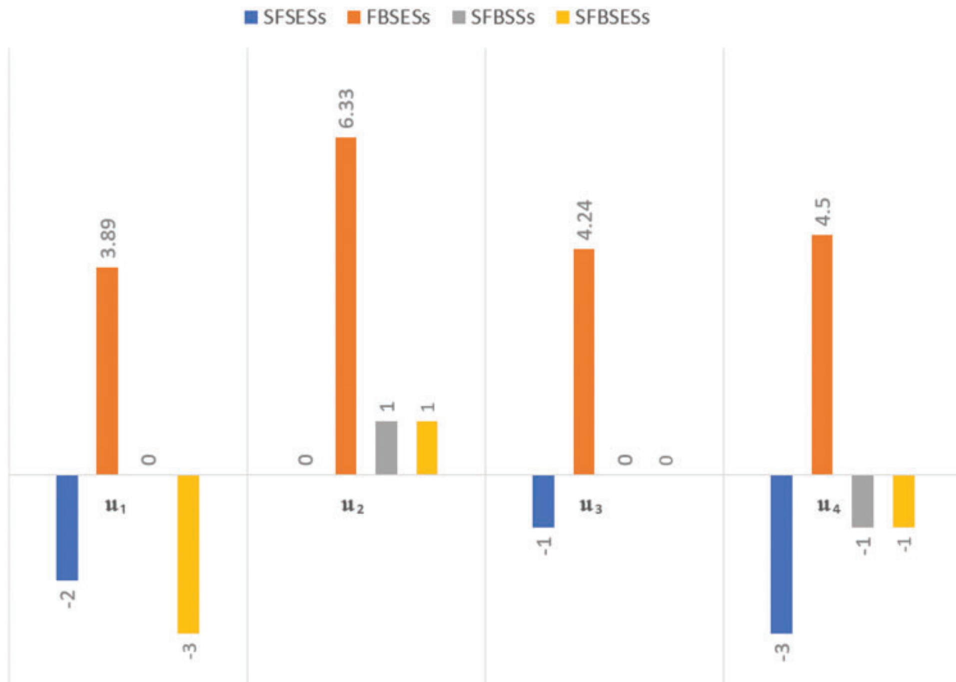

Figure 6: Scores comparison by SFSESs, FBSESs and SFBSESs

5.3 Limitations

This subsection discusses the limitations and difficulties in the decision-making process by using the initiated model and algorithm. One of the flaws of the proposed model comes from its parent model SFSs, i.e., the sum of square of the degrees of agreement, disagreement, and neutrality must lie between 0 and 1. It limits the proposed model from operating in problems with this sum exceeding unity. This limitation can be managed by further extending the model using T-spherical fuzzy sets. Another ambiguity lies in the selection of threshold functions, where one may give a relatively different outcome than the others. However, the optimal selections generally remain the same. For this purpose, the decision-makers should choose the threshold functions in relevance to the data variations. Further, the lengthy algorithm and massive data make the decision process tiresome. The computing software like MATLAB and Mathematica, can be used to facilitate the implementation of the proposed algorithm. Similar to all other tools, the proposed work is also not completely perfect in itself, and particular extensions and alterations in the future may address the imperfections specifically.

6 Conclusive Remarks and Future Directions

Complicated problems, including global warming, energy crises, environmental pollution, depleting resources, and unstable economies require stakeholders, experts, and administrators to group together for seeking reliable solutions. Spherical fuzzy soft expert sets (SFSESs) [44] can efficiently deal with such uncertain MAGDM problems by considering the positive, neutral, and negative membership degrees of multiple experts' opinions in one place. However, SFSESs fail to deal with the bipolarity of decision-parameters and hence fail to handle situations affected by parameters depicting bipolar behavior. This limitation asks for tools capable of dealing with bipolar nature problems requiring the high complexity addressable by SFSES theory. To address this requirement, we presented the novel hybrid spherical fuzzy bipolar soft expert sets (SFBSESs) by the fusion of spherical fuzzy sets [12] and bipolar soft expert sets [47]. We discussed the algebraic properties and operations of the proposed model and demonstrated these operations with numerical examples. Considering the importance of hydropower in global energy requirements and the possibility to convert certain NPD (non-powered dam) sites to hydropower dams, we presented a SFBSES-based ranking application with four NPDs advised by the two teams regarding important bipolar decision parameters in Example 4.1. The comparison section provided a detailed sensitivity analysis between FBSESs [48], SFSESs [44], and proposed SFBSESs, proving our work to be more reliable, accurate, and applicable than the pre-existing works. Being capable of dealing with maximum conditions depicting the problems and as a generalization of the previous models, SFBSESs are more robust, efficient, and comprehensive. In a nutshell, SFBSESs offer more reliable solutions to society's complicated group decision-making problems.

Opposed to the discussed advantages, the limitations portion indicated the difficulties and restrictions concerned with the proposed model, i.e., the applicability domain restricted by the memberships' sum condition, and the selection of the most suitable threshold function. In the future, we look forward to extending this work to the following:

- Linguistic spherical fuzzy bipolar soft expert sets using [24].
- Complex neutrosophic bipolar soft expert sets using [32].
- Possibility spherical fuzzy bipolar soft expert sets using [42].
- Fuzzy parameterized spherical fuzzy soft expert sets using [43].
- Spherical fuzzy bipolar N-soft expert sets using [45].

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