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ARTICLE

On Fractional Differential Inclusion for an Epidemic Model via L-Fuzzy Fixed Point Results

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ABSTRACT

The real world is filled with uncertainty, vagueness, and imprecision. The concepts we meet in everyday life are vague rather than precise. In real-world situations, if a model requires that conclusions drawn from it have some bearings on reality, then two major problems immediately arise, viz. real situations are not usually crisp and deterministic; complete descriptions of real systems often require more comprehensive data than human beings could recognize simultaneously, process and understand. Conventional mathematical tools which require all inferences to be exact, are not always efficient to handle imprecisions in a wide variety of practical situations. Following the latter development, a lot of attention has been paid to examining novel L-fuzzy analogues of conventional functional equations and their various applications. In this paper, new coincidence point results for single-valued mappings and an L-fuzzy set-valued map in metric spaces are proposed. Regarding novelty and generality, the obtained invariant point notions are compared with some well-known related concepts via non-trivial examples. It is observed that our principal results subsume and refine some important ones in the corresponding domains. As an application, one of our results is utilized to discuss more general existence conditions for realizing the solutions of a non-integer order inclusion model for COVID-19.

KEYWORDS

Hausdorff metric; L-fuzzy set; L-fuzzy set-valued map; Caputo fractional differential inclusion; COVID-19

1 Introduction

From the beginning of the universe, man has been exerting great efforts in understanding nature and then coming up with a good connection between life and what it requires. This struggle is broken down into three phases, namely, understanding the surrounding ambient, acknowledgment of creativity, and preparing for the future. In these strives, a lot of challenges like linguistic interpretation, characterization of inter-connected phenomena into suitable categories, application of non-liberal ideas, vagueness in data analysis, and more than a handful of others, affect the precision of results. These problems, common with everyday activities can be avoided by using the concepts of fuzzy sets because they are more flexible than crisp sets. Numerous fields of mathematics, the social sciences, and engineering have undergone enormous upheavals since Zadeh [1] introduced fuzzy sets.

The fundamental ideas of fuzzy sets have been refined and used in several contexts. In 1981, Heilpern [2] proved an invariant point theorem for fuzzy contraction mappings, which is a fuzzy analogue of invariant point theorems due to Nadler [3] and Banach [4]. Following [2], a number of authors have studied the existence of invariant points of fuzzy set-valued maps; for example, see [5–9]. Initiated by Goguen [10], L-fuzzy sets are a particularly intriguing development of the fuzzy set notion that substitutes a complete distributive lattice for the range set's interval [0,1].

A recent study by Rashid et al. [11] introduced the idea of L-fuzzy mappings (Lmap) and examined a pair of Lmaps that are β_{FL}-admissible in order to prove a common invariant point theorem. Rashid et al. [12] established the ideas of D_{τ_L} and μ_L^{∞} distances for L-fuzzy sets and generalized the existing invariant point theorems for fuzzy and multi-valued mappings as an improvement of the notion of Hausdorff distance and μ_∞-metric for fuzzy sets.

On the other hand, fractional differential inclusions arise in different problems in mathematical physics, bio-mathematics, control theory, critical point theory for non-smooth energy functionals, differential variational inequalities, fuzzy set arithmetic, traffic theory, and so on. Usually, the first most investigated problem in the study of differential inclusion is the criteria for the existence of its solutions. In this context, several authors have applied different invariant point techniques and topological methods to establish the existence results of differential inclusions in abstract spaces. In the current literature, we can find much work on fractional-order models coming-up with different measures for curbing the novel corona virus (COVID-19). Lately, Rahman et al. [13] investigated a fractional non-integer order fuzzy dynamical system and established an epidemic model for COVID-19. Their proposed model is examined for solvability, using an invariant point method. On close developments, we can cite [14–18].

Following the existing findings, we notice that L-fuzzy invariant point results using the characterizations of $\mathscr{MT}$-function and $\mathscr{D}$-function are yet to be adequately examined. As a result of the latter observation, it becomes clear that applications of such ideas in the areas of analyzing solvability conditions of epidemic models are undoubtedly missing in the literature. Hence, by utilizing the recently established auxiliary functions, under the name $\mathscr{D}$-function, this paper introduces new contractive inequalities for Lmaps and then examines criteria for the existence of L-fuzzy invariant points for such mappings. In line with the awareness that non-integer order differential inclusions are more suitable for analyzing the situation with non-statistical uncertainties, general existence conditions for obtaining the solutions of a new Caputo non-integer order inclusion model for COVID-19 are studied, availing one of the proposed L-fuzzy contractions. Since non-classical analysis is a known better tool for understanding and managing epidemic models, the idea developed herein is hoped to cause a spike in the use of fuzzy analysis to study various physical phenomena. Within the setting of differential equations and invariant point theory with metrics, the idea proposed in this work unifies and extends a few important findings in the corresponding literature.

The rest of the paper is structured as follows: Section 2 contains the fundamentals needed to establish our principal proposal. Results and discussion are presented in Section 3. Section 4 is concerned with the application of one of the proposed notions in the fractional differential inclusion model for COVID-19. The summary and conclusion of our obtained key notions are presented in Section 5.

2 Preliminaries

We collate, herein, a few fundamentals which are useful to our principal results. These basic concepts are picked from [1–3,8,9,19,20]. Throughout this paper, the sets $\mathbb{R}$, $\digamma_+$ and $\mathbb{N}$, represent the sets of real numbers, nonnegative real numbers and the set of natural numbers, respectively.

Let $(\mho, \mu)$ be a metric space (MS). Denote by $CB(\mho)$, the collection of all nonempty closed and bounded subsets of $\mho$. For $A, B \in CB(\mho)$, the mapping $\Upsilon : CB(\mho) \times CB(\mho) \longrightarrow \digamma_+$ defined as

$$\Upsilon(A, B) = \max\left\{\sup_{\varsigma\in B} \mu(\varsigma, A), \sup_{\omega\in A} \mu(\omega, B)\right\},$$

where $\mu(\varsigma, A) = \inf_{\omega\in A} \mu(\varsigma, \omega)$, is recognized as the Hausdorff metric.

Definition 2.1. [21] The mapping $\rho_{\mathscr{M}\mathscr{T}}$: $(0, \infty) \longrightarrow [0, 1)$ is called an $\mathscr{M}\mathscr{T}$-function if it satisfies the Mizoguchi-Takahashi's condition.

Definition 2.2. [22] The mapping ρ: $\digamma_+ \longrightarrow [0, 1)$ is termed a function of contractive factor, if for any nonincreasing sequence $\{\varsigma_x\}_{x\geq 1}$ in $\digamma_+$, we have $0 \leq \sup_{x\in\mathbb{N}} \rho(\varsigma_x) < 1$.

Recently, Monairah et al. [23] came up with variants of Definitions 2.1 and 2.2 in the fashion given hereunder.

Definition 2.3 [23]**.** The mapping ρ: $\digamma_+ \longrightarrow \left[0, \frac{1}{k}\right)$ is termed a $\mathscr{D}$-function if it satisfies the condition: for every $t \in \digamma_+$, there exists $k \in (1, \infty)$: $\limsup_{r\longrightarrow t^+} \rho(r) < \frac{1}{k}$.

Definition 2.4. [23] ρ: $\digamma_+ \longrightarrow \left[0, \frac{1}{k}\right)$ is called a function of $\frac{1}{k}$-contractive factor, if for any sequence $\{\varsigma_x\}_{x\geq 1}$ in $\digamma_+$ from and after some fixed terms, it is nonincreasing and $0 \leq \sup_{x\in\mathbb{N}} \rho(\varsigma_x) < \frac{1}{k}$, for some $k \in (1, \infty)$.

The following two lemmas are essential to the discussion of our principal findings:

Lemma 2.1. [23] Let ρ: $\digamma_+ \longrightarrow \left[0, \frac{1}{k}\right)$ be a $\mathscr{D}$-function. Then, ρ: $\digamma_+ \longrightarrow \left[0, \frac{1}{k}\right)$ defined as $\rho(t) = \frac{\rho(t) + \frac{1}{k}}{2}$ is also a $\mathscr{D}$-function for every $t \in \digamma_+$ and some $k \in (1, \infty)$.

Lemma 2.2. [23] Let ρ: $\digamma_+ \longrightarrow \left[0, \frac{1}{k}\right)$, $k \in (1, \infty)$. Then, the statements hereunder are equivalent:

(i) ρ is a $\mathscr{D}$-function.

(ii) For each $t \in \digamma_+$, there exist $\sigma_t^{(1)} \in \left[0, \frac{1}{k}\right)$ and $\delta_t^{(1)} > 0$ such that $\rho(s) \leq \sigma_t^{(1)}$ for all $s \in (t, t+\delta_t^{(1)})$.

(iii) For each $t \in \digamma_+$, there exist $\sigma_t^{(2)} \in \left[0, \frac{1}{k}\right)$ and $\delta_t^{(2)} > 0$ such that $\rho(s) \leq \sigma_t^{(2)}$ for all $s \in [t, t+\delta_t^{(2)}]$.

(iv) For each $t \in \digamma_+$, there exist $\sigma_t^{(3)} \in \left[0, \frac{1}{k}\right)$ and $\delta_t^{(3)} > 0$ such that $\rho(s) \leq \sigma_t^{(3)}$ for all $s \in (t, t+\delta_t^{(3)}]$.

(v) For each $t \in \digamma_+$, there exist $\sigma_t^{(4)} \in \left[0, \frac{1}{k}\right)$ and $\delta_t^{(4)} > 0$ such that $\rho(s) \leq \sigma_t^{(4)}$ for all $s \in [t, t+\delta_t^{(4)}]$.

(vi) For any sequence $\{\varsigma_x\}_{x\geq 1}$ in $\digamma_+$, from and after certain term, it is decreasing and $0 \leq \sup_{x\in\mathbb{N}} \rho(\varsigma_x) < \frac{1}{k}$.

(vii) ρ is a function of $\frac{1}{k}$-contractive factor, that is, for any sequence $\{\varsigma_x\}_{x\geq 1}$ in $\digamma_+$, from and after certain term, it is decreasing and $0 \leq \sup_{x\in\mathbb{N}} \rho(\varsigma_x) < \frac{1}{k}$.

Definition 2.5. [24] A relation $\preceq$ on a nonempty set L is called a partial order if it is

(i) reflexive;

(ii) antisymmetric;

(iii) transitive.

A set L together with a partial ordering $\preceq$ is called a partially ordered set (or pset) and is denoted by $(L, \preceq_L)$.

Definition 2.6. [24] Let L be a nonempty set and $(L, \preceq)$ be a pset. Then, any two elements $\varsigma, \omega \in L$ are said to be comparable if either $\varsigma \preceq \omega$ or $\omega \preceq \varsigma$.

Definition 2.7. [24] A pset $(L, \preceq_L)$ is called:

(i) a lattice, if $\varsigma \vee \omega \in L$, $\varsigma \wedge \omega \in L$ for any $\varsigma, \omega \in L$;

(ii) a complete lattice, if $\bigvee \nabla \in L$, $\bigwedge \nabla \in L$ for any $\nabla \subseteq L$;

(iii) distributive lattice if $\varsigma \vee (\omega \wedge \xi) = (\varsigma \vee \omega) \wedge (\varsigma \vee \xi)$, $\varsigma \wedge (\omega \vee \xi) = (\varsigma \wedge \omega) \vee (\varsigma \wedge \xi)$, for any $\varsigma, \omega, \xi \in L$.

A pset L is called a complete lattice if for every doubleton $\{\varsigma, \omega\}$ in L, either $\sup\{\varsigma, \omega\} = \varsigma \bigvee \omega$ or $\inf\{\varsigma, \omega\} = \varsigma \bigwedge \omega$ exists.

Definition 2.8. [10] An L-fuzzy set (L-fset) ∇ on a nonempty set $\mho$ is a function with domain $\mho$ and whose range lies in a complete distributive lattice L with top and bottom elements 1_L and 0_L, respectively.

Denote the class of all L-fuzzy sets on a nonempty set $\mho$ by $L^{\mho}$ (to depict a mapping : $\mho \longrightarrow L$).

Definition 2.9. [10] The τ_L-level set of an L-fset ∇ is denoted by $[\nabla]_{\tau_L}$ and is defined as follows:

$$[\nabla]_{\tau_L} = \begin{cases} \overline{\{\varsigma \in \mho\colon 0_L \preceq_L \nabla(\varsigma)\}}, & \text{if} \quad \tau_L = 0_L \\ \{\varsigma \in \mho\colon \tau_L \preceq_L \nabla(\varsigma)\}, & \text{if} \quad \tau_L \in L \setminus \{0_L\}. \end{cases}$$

Definition 2.10. [11,12] Let $\mho$ be a nonempty set and Y a MS. Then, $\Psi\colon \mho \longrightarrow L^Y$ is called an Lmap. The function value $\Psi(\varsigma)(\omega)$ is called the degree of membership of ω in $\Psi(\varsigma)$. For any two Lmaps $\Upsilon, \Psi\colon \mho \longrightarrow L^Y$, a point $u \in \mho$ is called an L-fuzzy invariant point of Υ if there exists $\tau_L \in L \setminus \{0_L\}$ such that $u \in [\Upsilon u]_{\tau_L}$. A point u is known as a common L-fuzzy invariant point of Υ and Ψ if $u \in [\Upsilon u]_{\tau_L} \cap [\Psi u]_{\tau_L}$.

Consistent with Rashid et al. [11,12], let $(\mho, \mu)$ be a MS and consider $\tau_L \in L \setminus \{0_L\}$ such that $[\nabla]_{\tau_L}, [\triangle]_{\tau_L} \in CB(\mho)$. Then,

$$p_{\tau_L}(\nabla, \triangle) = \inf_{\varsigma \in [\nabla]_{\tau_L}, \omega \in [\triangle]_{\tau_L}} \mu(\varsigma, \omega).$$

$$D_{\tau_L}(\nabla, \triangle) = \Upsilon([\nabla]_{\tau_L}, [\triangle]_{\tau_L}).$$

$$p(\nabla, \triangle) = \sup_{\tau_L} p_{\tau_L}(\nabla, \triangle).$$

$$\mu_L^{\infty}(\nabla, \triangle) = \sup_{\tau_L} D_{\tau_L}(\nabla, \triangle).$$

Definition 2.11. [19,25] Let $\Delta, \Theta, \Lambda \colon \mho \longrightarrow \mho$ be self-mappings and $\Psi \colon \mho \longrightarrow L^{\mho}$ be an Lmap. An element u is called a coincidence point of Δ, Θ, Λ, and Ψ if $\Delta u = \Theta u = \Lambda u \in [\Psi u]_{\tau_L}$. If $\Delta = \Theta = \Lambda = I_{\mho}$ (the identity mapping on $\mho$), then $u = \Delta u = \Theta u = \Lambda u \in [\Psi u]_{\tau_L}$ gives an L-fuzzy invariant point of Ψ.

We represent the set of all L-fuzzy invariant points of Ψ and the family of coincidence points of Δ, Θ, Λ and Ψ by $\mathscr{LF}_{ix}(\Psi)$, and $\mathscr{COP}(\Delta, \Theta, \Lambda, \Psi)$, respectively.

3 Main Results

We begin this section by introducing the idea of coincidence point results for an Lmap and single-valued mappings.

Theorem 3.1. Let $(\mho, \mu)$ be a complete MS, $\Psi \colon \mho \longrightarrow L^{\mho}$ an Lmap, $\Delta, \Theta, \Lambda \colon \mho \longrightarrow \mho$ be continuous self-mappings and $\rho \colon F_+ \longrightarrow [0, 1) = I \setminus \{1\}$ be a $\mathscr{D}$-function. Suppose that:

(ax_1) for every $\varsigma \in \mho$, there exists $\tau_L \in L \setminus \{0_L\}$ such that $[\Psi\varsigma]_{\tau_L}$ is a nonempty closed and bounded subset of $\mho$;

(ax_2) for every $\varsigma \in \mho$, $\{\Delta\omega = \Theta\omega = \Lambda\omega \text{ for all } \omega \in [\Psi\varsigma]_{\tau_L}\} \subseteq [\Psi\varsigma]_{\tau_L}$;

(ax_3) there exist mappings $\delta, \nu, h \colon \mho \longrightarrow F_+$ such that

$$\begin{aligned}\Upsilon([\Psi\varsigma]_{\tau_L}, [\Psi\omega]_{\tau_L}) \leq{}& \rho(\mu(\varsigma, \omega))[\zeta_1\mu(\varsigma, \omega) + \zeta_2\mu(\varsigma, [\Psi\varsigma]_{\tau_L}) + \zeta_3\mu(\omega, [\Psi\omega]_{\tau_L})] \\ &+ \delta(\Delta\omega)\mu(\Delta\omega, [\Psi\varsigma]_{\tau_L}) + \nu(\Theta\omega)\mu(\Theta\omega, [\Psi\varsigma]_{\tau_L}) \\ &+ h(\Lambda\omega)\mu(\Lambda\omega, [\Psi\varsigma]_{\tau_L}),\end{aligned}$$

for all $\varsigma, \omega \in \mho$, where $\zeta_1, \zeta_2, \zeta_3 \in F_+$ with $\zeta_1 + \zeta_2 + \zeta_3 < 1$.

Then, $\mathscr{COP}(\Delta, \Theta, \Lambda, \Psi) \cap \mathscr{LF}_{ix}(\Psi) \neq \emptyset$.

Proof. By (ax_2), we notice that for every $\varsigma \in \mho$, $\mu(\Delta\omega, [\Psi\varsigma]_{\tau_L}) = \mu(\Theta\omega, [\Psi\varsigma]_{\tau_L}) = \mu(\Lambda\omega, [\Psi\varsigma]_{\tau_L}) = 0$ for all $\omega \in [\Psi\varsigma]_{\tau_L}$. So, for every $\varsigma \in \mho$, it if follows from (ax_3) that for all $\omega \in [\Psi\varsigma]_{\tau_L}$,

$$\Upsilon([\Psi\varsigma]_{\tau_L}, [\Psi\omega]_{\tau_L}) \leq \rho(\mu(\varsigma, \omega))[\zeta_1\mu(\varsigma, \omega) + \zeta_2\mu(\varsigma, [\Psi\varsigma]_{\tau_L}) + \zeta_3\mu(\omega, [\Psi\omega]_{\tau_L}). \tag{1}$$

Further, for every $\omega \in [\Psi\varsigma]_{\tau_L}$, $\mu(\omega, [\Psi\omega]_{\tau_L}) \le \Upsilon([\Psi\varsigma]_{\tau_L}, [\Psi\omega]_{\tau_L})$. Hence, for each $\varsigma \in \mho$, (1) gives

$$\begin{aligned}\mu(\omega, [\Psi\omega]_{\tau_L}) &\le \rho(\mu(\varsigma, \omega))[\zeta_1\mu(\varsigma, \omega) + \zeta_2\mu(\varsigma, [\Psi\varsigma]_{\tau_L}) + \zeta_3\mu(\omega, [\Psi\omega]_{\tau_L})\\ &\le \frac{\rho(\mu(\varsigma, \omega))[\zeta_1\mu(\varsigma, \omega) + \zeta_2\mu(\varsigma, [\Psi\varsigma]_{\tau_L})]}{1 - \zeta_3\rho(\mu(\varsigma, \omega))}\\ &\le \rho(\mu(\varsigma, \omega))[\zeta_1\mu(\varsigma, \omega) + \zeta_2\mu(\varsigma, [\Psi\varsigma]_{\tau_L})].\end{aligned} \tag{2}$$

Let $\varsigma_0 \in \mho$ and choose $\varsigma_1 \in [\Psi\varsigma_0]_{\tau_L}$. If $\mu(\varsigma_0, \varsigma_1) = 0$, then $\varsigma_0 = \varsigma_1 \in [\Psi\varsigma_0]_{\tau_L}$, that is, $\varsigma_0 \in \mathscr{L}\mathscr{F}_{ix}(\Psi)$ and the proof is finished. Otherwise, if $\mu(\varsigma_0, \varsigma_1) > 0$, then consider a function ρ: $F_+ \longrightarrow I\backslash\{1\}$ defined as $\rho(t) = \dfrac{1 + \rho(t)}{2}$. By Lemma 2.1, ρ is a $\mathscr{D}$-function and $0 \le \rho(t) < \rho(t) < 1$ for all $t \in F_+$. From (2), it follows that

$$\begin{aligned}\mu(\varsigma_1, [\Psi\varsigma_1]_{\tau_L}) &\le \rho(\mu(\varsigma_0, \varsigma_1))[\zeta_1\mu(\varsigma_0, \varsigma_1) + \zeta_2\mu(\varsigma_0, [\Psi\varsigma_0]_{\tau_L})]\\ &< \rho(\mu(\varsigma_0, \varsigma_1))[\zeta_1\mu(\varsigma_0, \varsigma_1) + \zeta_2\mu(\varsigma_0, \varsigma_1)]\\ &= \rho(\mu(\varsigma_0, \varsigma_1))[(\zeta_1 + \zeta_2)\mu(\varsigma_0, \varsigma_1)].\end{aligned} \tag{3}$$

Given that $\zeta_1 + \zeta_2 + \zeta_3 < 1$, there exists $\eta \in (0, 1)$ such that $\zeta_1 + \zeta_2 < \eta = 1 - \zeta_3 < 1$. Thus, (3) can be represented as:

$$\begin{aligned}\mu(\varsigma_1, [\Psi\varsigma_1]_{\tau_L}) &< \eta\rho(\mu(\varsigma_0, \varsigma_1))\mu(\varsigma_0, \varsigma_1)\\ &< \rho(\mu(\varsigma_0, \varsigma_1))\mu(\varsigma_0, \varsigma_1).\end{aligned} \tag{4}$$

From (4), we claim that there exists $\varsigma_2 \in [\Psi\varsigma_1]_{\tau_L}$ such that

$$\mu(\varsigma_1, \varsigma_2) < \rho(\mu(\varsigma_0, \varsigma_1))\mu(\varsigma_0, \varsigma_1). \tag{5}$$

Assume that this claim is not true, that is, $\mu(\varsigma_1, \varsigma_2) \ge \rho(\mu(\varsigma_0, \varsigma_1))\mu(\varsigma_0, \varsigma_1)$. Then, we get

$$\mu(\varsigma_1, \varsigma_2) \ge \inf_{\gamma \in [\Psi\varsigma_1]_{\tau_L}} \mu(\varsigma_1, \gamma) \ge \rho(\mu(\varsigma_0, \varsigma_1))\mu(\varsigma_0, \varsigma_1),$$

that is, $\mu(\varsigma_1, [\Psi\varsigma_1]_{\tau_L}) \ge \rho(\mu(\varsigma_0, \varsigma_1))\mu(\varsigma_0, \varsigma_1)$, contradicting (4). Now, if $\mu(\varsigma_1, \varsigma_2) = 0$, then $\varsigma_1 = \varsigma_2 \in \Psi\varsigma_1$ and so $\varsigma_1 \in \mathscr{L}\mathscr{F}_{ix}(\Psi)$. Otherwise, there exists $\varsigma_3 \in [\Psi\varsigma_2]_{\tau_L}$ such that

$$\mu(\varsigma_2, \varsigma_3) < \rho(\mu(\varsigma_1, \varsigma_2))\mu(\varsigma_1, \varsigma_2). \tag{6}$$

Let $\tau_x = \mu(\varsigma_{x-1}, \varsigma_x)$ for every $x \in \mathbb{N}$. On similar steps as above, we can construct a sequence $\{\varsigma_x\}_{x\in\mathbb{N}}$ in $\mho$ with $\varsigma_x \in [\Psi\varsigma_{x-1}]_{\tau_L}$ for every $x \in \mathbb{N}$ and

$$\tau_{x+1} < \rho(\tau_x) < \tau_x. \tag{7}$$

Given that ρ is a $\mathscr{D}$-function, then, utilizing Lemma 2.2, yields $0 \le \sup_{x\in\mathbb{N}} \rho(\tau_x) < \sup_{x\in\mathbb{N}} \rho(\tau_x) < 1$. Hence, $0 < \sup_{x\in\mathbb{N}} \rho(\tau_x) = \sup\left\{\dfrac{1 + \rho(\tau_x)}{2} \colon x \in \mathbb{N}\right\} < 1$. Take $\xi \colon = \sup_{x\in\mathbb{N}} \rho(\tau_x)$, then $0 < \xi < 1$. Since $\rho(t) < 1$ for all $t \in F_+$, then, by (7), $\{\tau_x\}_{x\in\mathbb{N}}$ is a nonincreasing sequence. Whence, for every $x \in \mathbb{N}$, we see that

$$\tau_{x+1} < \rho(\tau_x) \le \xi\tau_x. \tag{8}$$

Hence, it if follows from (8) that

$$\mu(\varsigma_x, \varsigma_{x+1}) = \tau_{x+1} \leq \xi \tau_x \leq \cdots \leq \xi^x \tau_1 = \xi^x \mu(\varsigma_0, \varsigma_1). \tag{9}$$

For any $m, x, x_0 \in \mathbb{N}$ with $m > x > x_0$, by (9), we get

$$\begin{aligned}\mu(\varsigma_m, \varsigma_x) &\leq \sum_{j=x}^{m-1} \mu(\varsigma_j, \varsigma_{j+1}) \leq \sum_{j=x}^{m-1} \xi^j \mu(\varsigma_0, \varsigma_1) \\ &\leq \sum_{j=x}^{\infty} \xi^j \mu(\varsigma_0, \varsigma_1) \leq \frac{\xi^x}{1-\xi}\mu(\varsigma_0, \varsigma_1) \\ &\longrightarrow 0 \ \ (\text{as} \ \ x \longrightarrow \infty).\end{aligned}$$

Thus, $\lim_{x\longrightarrow\infty} \{\mu(\varsigma_m, \varsigma_x): m > x\} = 0$. This proves that $\{\varsigma_x\}_{x\in\mathbb{N}}$ is a Cauchy sequence in $\mho$. Whence, there exists $u \in \mho$ such that $\varsigma_x \longrightarrow u$ as $x \longrightarrow \infty$. Since $\varsigma_x \in [\Psi\varsigma_{x-1}]_{\tau_L}$ for every $x \in \mathbb{N}$, it if follows from condition (ax_2) that for every $x \in \mathbb{N}$,

$$\Delta\varsigma_x = \Theta\varsigma_x = \Lambda\varsigma_x \in [\Psi\varsigma_{x-1}]_{\tau_L}. \tag{10}$$

Employing the defining property of Δ, Θ and Λ, leads to

$$\begin{aligned}u &= \lim_{x\longrightarrow\infty} \Delta\varsigma_x = \lim_{x\longrightarrow\infty} \Theta\varsigma_x = \lim_{x\longrightarrow\infty} \Lambda\varsigma_x \\ &= \lim_{x\longrightarrow\infty} \Delta u = \lim_{x\longrightarrow\infty} \Theta u = \lim_{x\longrightarrow\infty} \Lambda u.\end{aligned}$$

We propose that $u \in [\Psi u]_{\tau_L}$. Suppose the contrary that $\mu(u, [\Psi u]_{\tau_L}) > 0$. Since the function $\varsigma \longmapsto \mu(\varsigma, [\Psi u]_{\tau_L})$ is continuous, then, from condition (ax_3), we obtain

$$\begin{aligned}\mu(u, [\Psi u]_{\tau_L}) &= \lim_{x\longrightarrow\infty} \mu(\varsigma_x, [\Psi u]_{\tau_L}) \\ &\leq \lim_{x\longrightarrow\infty} \Upsilon([\Psi\varsigma_{x-1}]_{\tau_L}, [\Psi u]_{\tau_L}) \\ &\leq \lim_{x\longrightarrow\infty} \Big\{ \rho(\mu(\varsigma_{x-1}, u))[\zeta_1\mu(\varsigma_{x-1}, u) + \zeta_2\mu(\varsigma_{x-1}, \Psi\varsigma_{x-1}) + \zeta_3\mu(u, \Psi u)] \\ &\quad + \delta(\Delta u)\mu(\Delta u, [\Psi\varsigma_{x-1}]_{\tau_L}) + \nu(\Theta u)\mu(\Theta u, [\Psi\varsigma_{x-1}]_{\tau_L}) \\ &\quad + h(\Lambda u)\mu(\Lambda u, [\Psi\varsigma_{x-1}]_{\tau_L}) \Big\} \\ &< \lim_{x\longrightarrow\infty} \Big\{ \rho(\mu(\varsigma_{x-1}, u))[\zeta_1\mu(\varsigma_{x-1}, u) + \zeta_2\mu(\varsigma_{x-1}, \varsigma_x) + \zeta_3\mu(u, [\Psi u]_{\tau_L})] \\ &\quad + \delta(\Delta u)\mu(\Delta u, \varsigma_x) + \nu(\Theta u)\mu(\Theta u, \varsigma_x) \\ &\quad + h(\Lambda u)\mu(\Lambda u, \varsigma_x) \Big\} \\ &< \zeta_3\rho(\mu(u, u))\mu(u, [\Psi u]_{\tau_L}) \\ &< \zeta_3\mu(u, [\Psi u]_{\tau_L}),\end{aligned}$$

a contradiction for all $\zeta_3 \in (0,1)$. Hence, $\mu(u,[\Psi u]_{\tau_L}) = 0$. Since $[\Psi u]_{\tau_L}$ is closed, it must be the case that $u \in [\Psi u]_{\tau_L}$. By condition (ax_2), $\Delta u = \Theta u = \Lambda u \in [\Psi u]_{\tau_L}$. Consequently, $u \in \mathscr{COP}(\Delta,\Theta,\Lambda,\Psi) \cap \mathscr{LF}_{ix}(\Psi)$.

Definition 3.1. [26] A nonempty subset ∇ of $\mho$ is called proximal if, for every $\varsigma \in \mho$, there exists $\zeta_1 \in \nabla$ such that $\mu(\varsigma,\zeta_1) = \mu(\varsigma,\nabla)$.

We denote the class of all bounded proximal subsets of $\mho$ by $\mathscr{P}_b^r(\mho)$. Since every proximal set is closed (see [26]), it follows that $\mathscr{P}_b^r(\mho) \subseteq CB(\mho)$. Now, take

$$\mathscr{C}_{\mathscr{P}}(\mho) = \{\nabla \in L^{\mho}\colon [\nabla]_{\tau_L} \in \mathscr{P}_b^r(\mho), \quad \text{for each} \quad \tau_L \in L \setminus \{0_L\}\}.$$

In what follows, we study another coincidence point result in connection with μ_L^∞-metric for L-fuzzy sets. It is pertinent to point out that the L-fuzzy invariant point results in the setting of μ_L^∞-metric is of great importance in analyzing Hausdorff dimensions. These dimensions help us to understand the notions of ε^∞ -space which is of tremendous importance in higher energy physics (see, e.g., [27,28]).

Theorem 3.2. Let $(\mho,\mu)$ be a complete MS, $\Psi\colon \mho \longrightarrow \mathscr{C}_{\mathscr{P}}(\mho)$ an Lmap, $\Delta,\Theta,\Lambda\colon \mho \longrightarrow \mho$ be continuous self-mappings and $\rho\colon \digamma_+ \longrightarrow I \setminus \{1\}$ be a $\mathscr{D}$-function. Assume that:

(ax_1) for every $\varsigma \in \mho$, $\{\Delta\omega = \Theta\omega = \Lambda\omega \;\text{ for all }\; \omega \in \Psi\varsigma\} \subseteq \Psi\varsigma$;

(ax_2) there exist three mappings $\delta,\nu,h\colon \mho \longrightarrow \digamma_+$ such that

$$\begin{aligned}\mu_L^\infty(\Psi\varsigma,\Psi\omega) \le{}& \rho(\mu(\varsigma,\omega))[\zeta_1\mu(\varsigma,\omega) + \zeta_2 p(\varsigma,\Psi\varsigma) + \zeta_3 p(\omega,\Psi\omega)]\\ &+ \delta(\Delta\omega)p(\Delta\omega,\Psi\varsigma) + \nu(\Theta\omega)p(\Theta\omega,\Psi\varsigma)\\ &+ h(\Lambda\omega)p(\Lambda\omega,\Psi\varsigma),\end{aligned}$$

for all $\varsigma,\omega \in \mho$, where $\zeta_1,\zeta_2,\zeta_3 \in \digamma_+$ with $\zeta_1+\zeta_2+\zeta_3 < 1$.

Then, $\mathscr{COP}(\Delta,\Theta,\Lambda,\Psi) \cap \mathscr{LF}_{ix}(\Psi) \neq \emptyset$.

Proof. Let $\varsigma \in \mho$ be arbitrary, and define a function $\tau_L\colon \mho \longrightarrow L\setminus\{0_L\}$ by $\tau_L(\varsigma) := \tau_L = 1_L$. Then, by hypothesis, $[\Psi\varsigma]_{1_L} \in CB(\mho)$. Whence, for all $\varsigma,\omega \in \mho$, we see that

$$\begin{aligned}D_{1_L}(\Psi\varsigma,\Psi\omega) &\le \mu_L^\infty(\Psi\varsigma,\Psi\omega)\\ &\le \rho(\mu(\varsigma,\omega))[\zeta_1\mu(\varsigma,\omega) + \zeta_2 p(\varsigma,\Psi\varsigma) + \zeta_3 p(\omega,\Psi\omega)]\\ &\quad + \delta(\Delta\omega)p(\Delta\omega,\Psi\varsigma) + \nu(\Theta\omega)p(\Theta\omega,\Psi\varsigma)\\ &\quad + h(\Lambda\omega)p(\Lambda\omega,\Psi\varsigma).\end{aligned}$$

Since $[\Psi\varsigma]_{1_L} \subseteq [\Psi\varsigma]_{\tau_L} \in CB(\mho)$, it follows that $\mu(\varsigma,[\Psi\varsigma]_{\tau_L}) \le \mu(\varsigma,[\Psi\varsigma]_{1_L})$ for every $\tau_L \in L\setminus\{0_L\}$. Hence, $p(\varsigma,\Psi\varsigma) \le \mu(\varsigma,[\Psi\varsigma]_{1_L})$ for all $\varsigma \in \mho$. Consequently,

$$\begin{aligned}\Upsilon([\Psi\varsigma]_{1_L},[\Psi\omega]_{1_L}) \le{}& \rho(\mu(\varsigma,\omega))[\zeta_1\mu(\varsigma,\omega) + \zeta_2\mu(\varsigma,[\Psi\varsigma]_{1_L}) + \zeta_3\mu(\omega,[\Psi\omega]_{1_L})]\\ &+ \delta(\Delta\omega)\mu(\Delta\omega,[\Psi\varsigma]_{1_L}) + \nu(\Theta\omega)\mu(\Theta\omega,[\Psi\varsigma]_{1_L})\\ &+ h(\Lambda\omega)\mu(\Lambda\omega,[\Psi\varsigma]_{1_L}).\end{aligned}$$

Hence, Theorem 3.1 can be applied to find $u \in \mho$ such that $u \in \mathscr{COP}(\Delta,\Theta,\Lambda,\Psi) \cap \mathscr{LF}_{ix}(\Psi)$.

We provide the following example to support the hypotheses of Theorem 3.1.

Example 3.1. Let $L = \{p, q, r, v, s, m, x, w\}$ be such that $p \preceq_L, s \preceq_L r \preceq_L w, p \preceq_L v \preceq_L q \preceq_L w$, $s \preceq_L m \preceq_L w, v \preceq_L m \preceq_L w, x \preceq_L q \preceq_L w$; and each element of the doubletons $\{r, m\}, \{m, q\}, \{s, x\}, \{x, v\}$ are not comparable. Then, $(L, \preceq_L)$ is a complete distributive lattice. Let η^∞ be the space of all bounded sequences endowed with the supremum norm $\|.\|_\infty$, and let $\{e_x\}$ be the canonical basis of η^∞. Let $\{\zeta_x\}_{x\in\mathbb{N}}$ be a sequence of positive real numbers satisfying $\zeta_1 = \zeta_2$ and $\zeta_{x+1} < \zeta_x$ for all $x \geq 2$ (for example, take $\zeta_1 = \frac{1}{2}$ and $\zeta_x = \frac{1}{x}, x \geq 2$). It follows that $\{\zeta_x\}_{x\in\mathbb{N}}$ is convergent. Put $v_x = \zeta_x e_x$ for all $x \in \mathbb{N}$ and let $\mho = \{v_x\}_{x\in\mathbb{N}}$ be a bounded and complete subset of η^∞. Then $(\mho, \|.\|_\infty)$ is a complete MS, and $\|v_x - v_m\|_\infty = \zeta_x$ if $m > x$.

Consider an Lmap $\Psi: \mho \longrightarrow L^\mho$ defined as follows:

$$\Psi(v_1)(t) = \Psi(v_2)(t) = \begin{cases} w, & \text{if } \quad t \in \{v_1, v_2\} \\ p, & \text{if } \quad t = v_{x+1}, x > 2, \end{cases}$$

$$\Psi(v_3)(t) = \Psi(v_4)(t) = \cdots = \Psi(v_x)(t) = \begin{cases} r, & \text{if } \quad t \in \{v_1, v_2\} \\ w, & \text{if } \quad t = v_{x+1}. \end{cases}$$

Then, for every $x \in \mathbb{N}$ and $\zeta_L(v_x) = w$, we see that

$$[\Psi v_x]_{\zeta_L} = \begin{cases} \{v_1, v_2\}, & \text{if } \quad w \in \{v_1, v_2\} \\ \{v_{x+1}\}, & \text{if } \quad w \notin \{v_1, v_2\}. \end{cases}$$

Further, define the mappings $\Delta, \Theta, \Lambda: \mho \longrightarrow \mho$ as

$$\Delta v_x = \Theta v_x = \Lambda v_x = \begin{cases} v_2, & \text{if } \quad x \in \{1, 2\} \\ v_{x+1}, & \text{if } \quad x > 2. \end{cases}$$

Then, we observe that the following hold:

(ax_1) for every $\varsigma \in \mho$, there exists $\zeta_L \in L \setminus \{0_L\}$: $[\Psi\varsigma]_{\zeta_L} \in CB(\mho)$;

(ax_2) for every $\varsigma \in \mho$, $\{\Delta\omega = \Theta\omega = \Lambda\omega \in [\Psi\varsigma]_{\zeta_L}\} \subseteq [\Psi\varsigma]_{\zeta_L}$;

and $\mathscr{COP}(\Delta, \Theta, \Lambda, \Psi) \cap \mathscr{LF}_{ix}(\Psi) = \{v_1, v_2\}$.

To prove that Δ, Θ and Λ are continuous, it is enough to show the nonexpansiveness of Δ, Θ and Λ. So, we check the cases

(i) $\|\Delta v_1 - \Delta v_2\|_\infty = 0 < \zeta_1 = \|v_1 - v_2\|_\infty$;

(iii) $\|\Delta v_1 - \Delta v_m\|_\infty = \zeta_2 = \zeta_1 = \|v_1 - v_m\|_\infty$ for any $m > 2$;

(iv) $\|\Delta v_2 - \Delta v_m\|_\infty = \zeta_2 = \|v_2 - v_m\|_\infty$ for any $m > 2$;

(vi) $\|\Delta v_x - \Delta v_m\|_\infty = \zeta_{x+1} < \zeta_x = \|v_x - v_m\|_\infty$ for any $m > 2$ and $m > x$.

This shows that $\|\Delta\varsigma - \Delta\omega\|_\infty \leq \|\varsigma - \omega\|_\infty$ for all $\varsigma, \omega \in \mho$, indicating the nonexpansiveness of Δ. Since $\Delta = \Theta = \Lambda$, then Δ, Θ and Λ are continuous.

Define $\rho: F_+ \longrightarrow I \setminus \{1\}$ as

$$\rho(t) = \begin{cases} \frac{\zeta_{x+2}}{\zeta_x}, & \text{if } \quad t = \zeta_x \quad \text{for some } \quad x \in \mathbb{N} \\ \frac{1}{5}, & \text{otherwise.} \end{cases}$$

Since $\lim_{r\longrightarrow t^+}\sup \rho(r) = \frac{1}{5} < 1$ for all $t \in \digamma_+$, then ρ is a $\mathscr{D}$-function. Also, define the mappings $\delta, \nu, h\colon \mho \longrightarrow \mho$ by

$$\delta(v_x) = \nu(v_x) = h(v_x) = \begin{cases} 0, & \text{if} \quad x \in \{1,2\} \\ x, & \text{if} \quad x > 2. \end{cases}$$

Now, we claim that

$$\begin{aligned}\Upsilon_\infty([\Psi\varsigma]_{\zeta_L},[\Psi\omega]_{\zeta_L}) \le \rho(\|\varsigma-\omega\|_\infty)[\zeta_1\|\varsigma-\omega\|_\infty + \zeta_2\|\varsigma-[\Psi\varsigma]_{\zeta_L}\|_\infty \\ + \zeta_3\|\omega-[\Psi\omega]_{\zeta_L}\|_\infty] + \delta(\Delta\omega)\|\Delta\omega-[\Psi\varsigma]_{\zeta_L}\|_\infty \\ + \nu(\Theta\omega)\|\Theta\omega-[\Psi\varsigma]_{\zeta_L}\|_\infty + h(\Lambda\omega)\|\Lambda\omega-[\Psi\varsigma]_{\zeta_L}\|_\infty\end{aligned} \tag{11}$$

for all $\varsigma, \omega \in \mho$ and $\zeta_1, \zeta_2, \zeta_3 \in \digamma_+$ with $\zeta_1+\zeta_2+\zeta_3 < 1$, where Υ_∞ is the Hausdorff metric induced by the norm $\|.\|_\infty$.

To verify (11), we check the possibilities:

Case 1. If $x = 1, m = 2$ and $\zeta_1 = \frac{1}{6},\ \zeta_2 = \zeta_3 = 0$, we see that

$$\begin{aligned}&\rho(\|v_1-v_2\|_\infty)\left(\zeta_1\|v_1-v_2\|_\infty + \zeta_2\|v_1-[\Psi v_1]_{\zeta_L}\|_\infty + \zeta_3\|v_2-[\Psi v_2]_{\zeta_L}\|_\infty\right) \\ &\quad + \delta(\Delta v_2)\|\Delta v_2-[\Psi v_1]_{\zeta_L}\|_\infty + \nu(\Theta v_2)\|\Theta v_2-[\Psi v_1]_{\zeta_L}\|_\infty \\ &\quad + h(\Lambda v_2)\|\Lambda v_2-[\Psi v_1]_{\zeta_L}\|_\infty \\ &= \frac{\zeta_3}{6} > 0 = \Upsilon_\infty([\Psi v_1]_{\zeta_L},[\Psi v_2]_{\zeta_L}).\end{aligned}$$

Case 2. For $x = 1, m > 2$ and $\zeta_1 = \frac{1}{3},\ \zeta_2 = \zeta_3 = 0$, we see that

$$\begin{aligned}&\rho(\|v_1-v_m\|_\infty)\left(\zeta_1\|v_1-v_m\|_\infty + \zeta_2\|v_1-[\Psi v_1]_{\zeta_L}\|_\infty + \zeta_3\|v_m-[\Psi v_m]_{\zeta_L}\|_\infty\right) \\ &\quad + \delta(\Delta v_m)\|\Delta v_m-[\Psi v_1]_{\zeta_L}\|_\infty + \nu(\Theta v_m)\|\Theta v_m-[\Psi v_1]_{\zeta_L}\|_\infty \\ &\quad + h(\Lambda v_m)\|\Lambda v_m-[\Psi v_1]_{\zeta_L}\|_\infty \\ &= \frac{\zeta_3}{3} + 3(m+1)\zeta_1 > \zeta_1 = \Upsilon_\infty([\Psi v_1]_{\zeta_L},[\Psi v_m]_{\zeta_L}).\end{aligned}$$

Case 3. For $x = 2, m > 2$ and $\zeta_1 = \frac{1}{8},\ \zeta_2 = \zeta_3 = 0$, we see that

$$\begin{aligned}&\rho(\|v_2-v_m\|_\infty)\left(\zeta_1\|v_2-v_m\|_\infty + \zeta_2\|v_2-[\Psi v_2]_{\zeta_L}\|_\infty + \zeta_3\|v_m-[\Psi v_m]_{\zeta_L}\|_\infty\right) \\ &\quad + \delta(\Delta v_m)\|\Delta v_m-[\Psi v_2]_{\zeta_L}\|_\infty + \nu(\Theta v_m)\|\Theta v_m-[\Psi v_2]_{\zeta_L}\|_\infty \\ &\quad + h(\Lambda v_m)\|\Lambda v_m-[\Psi v_2]_{\zeta_L}\|_\infty \\ &= \frac{\zeta_4}{8} + 3(m+1)\zeta_2 > \zeta_2 = \Upsilon_\infty([\Psi v_2]_{\zeta_L},[\Psi v_m]_{\zeta_L}).\end{aligned}$$

Case 4. For $x > 2$, $m > x$ and $\zeta_1 = \frac{1}{4}$, $\zeta_2 = \zeta_3 = 0$, we see that

$$\rho(\|v_x - v_m\|_\infty)\left(\zeta_1\|v_x - v_m\|_\infty + \zeta_2\|v_x - [\Psi v_x]_{\zeta_L}\|_\infty + \zeta_3\|v_m - [\Psi v_m]_{\zeta_L}\|_\infty\right)$$
$$+ \delta(\Delta v_m)\|\Delta v_m - [\Psi v_x]_{\zeta_L}\|_\infty + \nu(\Theta v_m)\|\Theta v_m - [\Psi v_x]_{\zeta_L}\|_\infty$$
$$+ h(\Lambda v_m)\|\Lambda v_m - [\Psi v_x]_{\zeta_L}\|_\infty$$
$$= \frac{\zeta_{x+2}}{4} + 3(m+2)\zeta_{x+1} > \zeta_{x+1} = \Upsilon_\infty([\Psi v_x]_{\zeta_L}, [\Psi v_m]_{\zeta_L}).$$

Thus, following cases (1)–(4), we have demonstrated that (11) is valid. Hence, all the assumptions of Theorem 3.1 are agreed with. Whence, $\mathscr{COP}(\Delta, \Theta, \Lambda, \Psi) \cap \mathscr{LF}_{ix}(\Psi) \neq \emptyset$.

Now, notice that if we take $L = [0,1]$ and $w = 1$,

$$\mu_1^\infty(\Psi(v_1), \Psi(v_m)) = \zeta_1 > \lambda\zeta_1 = \lambda\|v_1 - v_m\|_\infty$$

for all $\lambda \in (0, 1)$ and $m > 2$. Hence, the main result of Heilpern [2] is not useable in this example to obtain any fuzzy invariant point of Ψ.

Similarly, let $\zeta_L = w$ and consider a multivalued mapping $F: \mho \longrightarrow CB(\mho)$ defined by $Fv = [\Psi v]_w$ for all $v \in \mho$. Then, we see that

$$\Upsilon(\Psi v_1, \Psi v_m) = \Upsilon([\Psi v_1]_w, [\Psi v_m]_w)$$
$$= \zeta_1 > \lambda\zeta_1 = \lambda\|v_1 - v_m\|_\infty$$

for all $\lambda \in (0, 1)$ and $m > 2$. Whence, the principal result of Nadler [3] is not useful here to obtain the invariant point of Ψ.

Also, we note that

$$\Upsilon([\Psi v_1]_{\zeta_L}, [\Psi v_m]_{\zeta_L}) = \zeta_1 > \zeta_3 = \rho(\|v_1 - v_m\|_\infty)\|v_1 - v_m\|_\infty$$

for all $m > 2$. Thus, all the Mizoguchi-Takahashi type results are not valid here.

Fig. 1 represents the Lattice in Example 3.1.

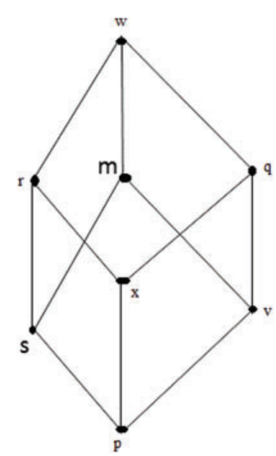

Figure 1: The lattice in example 3.1

Consider a nonempty subset A of $\mho$ and the mapping $\Psi: \mho \longrightarrow \mho$. We recall that A is Ψ-invariant if $\Psi(A) \subseteq A$. Now, we come up with some observations from Theorems 3.1 and 3.2.

Corollary 3.1. Let $(\mho, \mu)$ be a complete MS, $\Psi: \mho \longrightarrow L^{\mho}$ be an Lmap, $\Delta: \mho \longrightarrow \mho$ be a continuous self-mapping and $\rho: \digamma_{+} \longrightarrow I \setminus \{1\}$ be a $\mathscr{D}$-function. Suppose that:

(i) for every $\varsigma \in \mho$, there exists $\tau_L \in L \setminus \{0_L\}$ such that $[\Psi\varsigma]_{\tau_L} \in CB(\mho)$;

(ii) $[\Psi\varsigma]_{\tau_L}$ is Δ-invariant (i.e. $\Delta([\Psi\varsigma]_{\tau_L}) \subseteq [\Psi\varsigma]_{\tau_L}$) for every $\varsigma \in \mho$;

(iii) there exists $\delta: \mho \longrightarrow \digamma_{+}$ such that

$$\begin{aligned}\Upsilon([\Psi\varsigma]_{\tau_L}, [\Psi\omega]_{\tau_L}) \leq{}& \rho(\mu(\varsigma, \omega))[\zeta_1\mu(\varsigma, \omega) + \zeta_2\mu(\varsigma, [\Psi\varsigma]_{\tau_L}) + \zeta_3\mu(\omega, [\Psi\omega]_{\tau_L})]\\ &+ \delta(\Delta\omega)\mu(\Delta\omega, [\Psi\varsigma]_{\tau_L}),\end{aligned}$$

for all $\varsigma, \omega \in \mho$ and $\zeta_1, \zeta_2, \zeta_3 \in \digamma_{+}$ with $\zeta_1 + \zeta_2 + \zeta_3 < 1$.

Then, $\mathscr{COP}(\Delta, \Psi) \cap \mathscr{LF}_{ix}(\Psi) \neq \emptyset$.

Proof. Define the mappings $\nu, h: \mho \longrightarrow \digamma_{+}$ as $\nu(\varsigma) = h(\varsigma) = 0$ for all $\varsigma \in \mho$ in Theorem 3.1.

The next observation follows from Corollary 3.1.

Corollary 3.2. Let $(\mho, \mu)$ be a complete MS, $\Psi: \mho \longrightarrow L^{\mho}$ be an Lmap, $\Delta: \mho \longrightarrow \mho$ be continuous and $\rho: \digamma_{+} \longrightarrow I \setminus \{1\}$ be a $\mathscr{D}$-function. Suppose that

(i) for every $\varsigma \in \mho$, there exists $\tau_L \in L \setminus \{0_L\}$ such that $[\Psi\varsigma]_{\tau_L} \in CB(\mho)$;

(ii) $[\Psi\varsigma]_{\tau_L}$ is Δ-invariant (i.e. $\Delta([\Psi\varsigma]_{\tau_L}) \subseteq [\Psi\varsigma]_{\tau_L}$) for every $\varsigma \in \mho$;

(iii) there exist $\xi \geq 0$ and $\hat{\delta}: \mho \longrightarrow [0, \xi]$ such that

$$\begin{aligned}\Upsilon([\Psi\varsigma]_{\tau_L}, [\Psi\omega]_{\tau_L}) \leq{}& \rho(\mu(\varsigma, \omega))[\zeta_1\mu(\varsigma, \omega) + \zeta_2\mu(\varsigma, [\Psi\varsigma]_{\tau_L}) + \zeta_3\mu(\omega, [\Psi\omega]_{\tau_L})]\\ &+ \hat{\delta}(\Delta\omega)\mu(\Delta\omega, [\Psi\varsigma]_{\tau_L}),\end{aligned}$$

for all $\varsigma, \omega \in \mho$ and $\zeta_1, \zeta_2, \zeta_3 \in \digamma_{+}$ with $\zeta_1 + \zeta_2 + \zeta_3 < 1$.

Then, $\mathscr{COP}(\Delta, \Psi) \cap \mathscr{LF}_{ix}(\Psi) \neq \emptyset$.

Corollary 3.3. Let $(\mho, \mu)$ be a complete MS, $\Psi: \mho \longrightarrow L^{\mho}$ be an Lmap, $\Delta: \mho \longrightarrow \mho$ be continuous and $\rho: \digamma_{+} \longrightarrow I \setminus \{1\}$ be a $\mathscr{D}$-function. Suppose that

(i) for every $\varsigma \in \mho$, there exists $\tau_L \in L \setminus \{0_L\}$ such that $[\Psi\varsigma]_{\tau_L} \in CB(\mho)$;

(ii) $[\Psi\varsigma]_{\tau_L}$ is Δ-invariant (i.e., $\Delta([\Psi\varsigma]_{\tau_L}) \subseteq [\Psi\varsigma]_{\tau_L}$) for every $\varsigma \in \mho$;

(iii) there exists $\xi \geq 0$ such that

$$\begin{aligned}\Upsilon([\Psi\varsigma]_{\tau_L}, [\Psi\omega]_{\tau_L}) \leq{}& \rho(\mu(\varsigma, \omega))[\zeta_1\mu(\varsigma, \omega) + \zeta_2\mu(\varsigma, [\Psi\varsigma]_{\tau_L}) + \zeta_3\mu(\omega, [\Psi\omega]_{\tau_L})]\\ &+ \xi\mu(\Delta\omega, [\Psi\varsigma]_{\tau_L}),\end{aligned}$$

for all $\varsigma, \omega \in \mho$ and $\zeta_1, \zeta_2, \zeta_3 \in \digamma_{+}$ with $\zeta_1 + \zeta_2 + \zeta_3 < 1$.

Then, $\mathscr{COP}(\Delta, \Psi) \cap \mathscr{LF}_{ix}(\Psi) \neq \emptyset$.

Proof. Take $\hat{\delta}: \mho \longrightarrow [0, \xi]$ as $\hat{\delta}(\varsigma) = \xi$ for all $\varsigma \in \mho$ in Corollary 3.2.

Corollary 3.4. Let $(\mho, \mu)$ be a complete MS, $\Psi: \mho \longrightarrow L^{\mho}$ be an Lmap and $\rho: F_+ \longrightarrow I \setminus \{1\}$ be a $\mathscr{D}$-function. Suppose that there exists $\delta: \mho \longrightarrow F_+$ such that for every $\varsigma \in \mho$, there exists $\tau_L \in L \setminus \{0_L\}$ with $[\Psi\varsigma]_{\tau_L} \in CB(\mho)$, and

$$\begin{aligned}\Upsilon([\Psi\varsigma]_{\tau_L}, [\Psi\omega]_{\tau_L}) \leq \rho(\mu(\varsigma, \omega))[\zeta_1\mu(\varsigma, \omega) + \zeta_2\mu(\varsigma, [\Psi\varsigma]_{\tau_L}) + \zeta_3\mu(\omega, [\Psi\omega]_{\tau_L})] \\ + \delta(\omega)\mu(\omega, [\Psi\varsigma]_{\tau_L}),\end{aligned} \tag{12}$$

for all $\varsigma, \omega \in \mho$ and $\zeta_1, \zeta_2, \zeta_3 \in F_+$ with $\zeta_1 + \zeta_2 + \zeta_3 < 1$.

Then, $\mathscr{L}\mathscr{F}_{ix}(\Psi) \neq \emptyset$.

Proof. Take $\Delta := I_{\mho}$, the identity mapping on $\mho$ in Corollary 3.1.

Corollary 3.5. Let $(\mho, \mu)$ be a complete MS, $\Psi: \mho \longrightarrow \mathscr{C}_{\mathscr{P}}(\mho)$ be an Lmap. Suppose that

$$\mu_L^{\infty}(\Psi(\varsigma), \Psi(\omega)) \leq \gamma\mu(\varsigma, \omega), \tag{13}$$

for all $\varsigma, \omega \in \mho$ and for some $\gamma \in (0, 1)$. Then, $\mathscr{L}\mathscr{F}_{ix}(\Psi) \neq \emptyset$.

Proof. First, note that $\Upsilon([\Psi\varsigma]_{\tau_L}, [\Psi\omega]_{\tau_L}) \leq \mu_L^{\infty}(\Psi(\varsigma), \Psi(\omega))$ for all $\varsigma, \omega \in \mho$ and $\tau_L \in L$. Now, taking $\zeta_2 = \zeta_3 = 0$, $\rho(t) = \dfrac{t}{\zeta_1(1+t)}$, $\zeta_1 \in (0, 1)$ and $\delta(t) = 0$ for all $t \in \Omega$ in Corollary 3.4 completes the proof.

4 An Application to Fractional Differential Inclusion for an Epidemic Model

Of recent, Ahmed et al. [14] examined the importance of lock-down in managing the escalation of COVID-19, using the following non-integer order epidemic model:

$$\begin{cases} {}^C D_{0+}^{\rho} W(t) = \Lambda^{\rho} - \beta^{v} WI - \lambda_1 WK - \bar{\mu}^{v} W + \gamma_1^{\rho} I + \gamma_2^{\rho} I_K + \theta_1^{v} W_K, \\ {}^C D_{0+}^{\rho} W_K(t) = \lambda_1^{v} WK - \bar{\mu}^{v} W_K - \theta_1^{v} W_K, \\ {}^C D_{0+}^{\rho} I(t) = \beta^{v} WI - \gamma_1^{\rho} - \tau_{K1}^{\rho} - \bar{\mu}^{v} I + \lambda_2^{v} IK + \theta_2^{v} I_K, \\ {}^C D_{0+}^{\rho} I_K(t) = \lambda_2^{v} IK - \bar{\mu}^{v} I_K - \theta_2^{\rho} - \gamma_2^{\rho} - \tau_{K2}^{v} I_K, \\ {}^C D_{0+}^{\rho} K(t) = \mu^{v} I - \phi^{v} K, \end{cases} \tag{14}$$

where the total population under study, $P(t)$ is partitioned into four units, viz. a susceptible population that is free from lock-down $W(t)$, a susceptible population that is not free from lock-down $W_K(t)$, an infective population that is free from lock-down $I(t)$, an infective population that is not free from lock-down $I_K(t)$, and a cumulative density of the lock-down program $K(t)$. For the rest of the parameters and numerical simulations in addition to a few new existence results of (14), one can refer to [14,23]. The above model is reformulated as:

$$\begin{cases} {}^C D_{0+}^{\rho} W(t) = \Theta_1(t, W, W, W_K, I, I_K, K), \\ {}^C D_{0+}^{\rho} W_K(t) = \Theta_2(t, W, W_K, I, I_K, K), \\ {}^C D_{0+}^{\rho} I(t) = \Theta_3(t, W, W_K, I, I_K, K), \\ {}^C D_{0+}^{\rho} I_K(t) = \Theta_4(t, W, W_K, I, I_K, K), \\ {}^C D_{0+}^{\rho} K(t) = \Theta_5(t, W, W_K, I, I_K, K), \end{cases} \tag{15}$$

where

$$\begin{cases} \Theta_1(t, W, W, W_K, I, I_K, K) = \Lambda^\rho - \beta^v WI - \lambda_1 WK - \bar{\mu}^v W + \gamma_1^\rho I + \gamma_2^\rho I_K + \theta_1^v W_K, \\ \Theta_2(t, W, W_K, I, I_K, K) = \lambda_1^v WK - \bar{\mu}^v W_K - \theta_1^v W_K, \\ \Theta_3(t, W, W_K, I, I_K, K) = \beta^v WI - \gamma_1^\rho - \tau_{K1}^{\ \rho} - \bar{\mu}^v I + \lambda_2^v IK + \theta_2^v I_K, \\ \Theta_4(t, W, W_K, I, I_K, K) = \lambda_2^v IK - \bar{\mu}^v I_K - \theta_2^\rho - \gamma_2^\rho - \tau_{K_2^v} I_K, \\ \Theta_5(t, W, W_K, I, I_K, K) = \mu^v I - \phi^v K. \end{cases} \tag{16}$$

Consequently, the model (14) takes the form:

$$\begin{cases} {}^C D_0^\rho \vartheta(t) = \nu(t, \vartheta(t)), \quad t \in \Omega = [0, b], \quad 0 < \rho < 1 \\ \vartheta(0) = \vartheta_0 \geq 0, \end{cases} \tag{17}$$

with the conditions:

$$\begin{cases} \vartheta(t) = (W, W_K, I, I_K, K)^{tr}, \\ \vartheta(0) = (W_0, W_{K_0}, I_0, I_{K_0}, K_0)^{tr}, \\ \nu(t, \vartheta(t)) = (\Theta_i(t, W, W_K, I, I_K, K))^{tr}, \quad i = 1, \ldots, 5, \end{cases} \tag{18}$$

where $(.)^{tr}$ denotes the transpose operation.

It is a fact that in general, differential equations are not efficient tools to analyze non-statistical uncertainties, since the derivative of a solution to any differential equation automatically enjoys all the regularity properties of the concerned mapping and of the solution itself. This hereditary property is not found under the setting of differential inclusions. With this information, we extend problem (14) to its set-valued version given as

$$\begin{cases} {}^C D_0^\rho \vartheta(t) \in V(t, \vartheta(t)), t \in \Omega = (0, \delta) \\ \vartheta(0) = \vartheta_0 \geq 0, \end{cases} \tag{19}$$

where $V\colon \Omega \times \mathbb{R} \longrightarrow P(\mathbb{R})$ is a set-valued map ($P(\mathbb{R})$ is the power set of $\mathbb{R}$). We establish existence conditions for solutions to the inclusion problem (19) for which the right-hand side is non-convex by applying an invariant point theorem for Lmaps. For some related recent applications of crisp mathematical techniques, we refer to [29–32] and some citations in there. We now recall some needed concepts of fractional calculus and set-valued analysis as follows.

Definition 4.1. [33] Let $\rho > 0$ and $\delta \in L'([0, \delta], \mathbb{R})$. Then, the Riemann-Liouville fractional integral order ρ for a function δ is defined as

$$I_{0+}^\rho \delta(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t - \tau)^{\rho - 1} \mu\tau, \quad t > 0,$$

where $\Gamma(.)$ is the gamma function given by $\Gamma(\rho) = \int_0^\infty \tau^{\rho-1} e^{-\tau} \mu\tau$.

Definition 4.2. [33] Let $x - 1 < \rho < x, x \in \mathbb{N}$, and $\delta \in C^x(0, \delta)$. Then, the Caputo fractional derivative of order ρ for a function δ is defined as

$${}^C D_{0+}^\rho \delta(t) = \frac{1}{\Gamma(x - \rho)} \int_0^t (t - \tau)^{x - \rho - 1} \delta^x(\tau) \mu\tau, t > 0.$$

Lemma 4.1. [33] Let $\Re(\rho) > 0, x = [\Re(\rho)] + 1$, and $\delta \in AC^x(0, \delta)$. Then

$$\left(I^{\rho}_{0+}\,{}^{C}D^{\rho}_{0+}\delta\right)(t) = \delta(t) - \frac{\sum_{k=1}^{m}(D^{k}_{0+}\delta)(0^{+})}{k!}.$$

In particular, if $0 < \rho \leq 1$, then $\left(I^{\rho}_{0+}\,{}^{C}D^{\rho}_{0+}\delta\right)(t) = \delta(t) - \delta(0)$.

Given Lemma 4.1, the integral reformulation of problem (17) which is equivalent to the model (14) is given by

$$\begin{aligned} \vartheta(t) &= \vartheta_0 + I^{\rho}_{0+}\nu(t, \vartheta(t)) \\ &= \vartheta_0 + \frac{1}{\Gamma(\rho)}\int_0^t (t-\tau)^{\rho-1}\nu(\tau, \vartheta(\tau))\mu\tau. \end{aligned} \tag{20}$$

Let $\mho = C(\Omega, \mathbb{R})$ denotes the Banach space of all continuous functions ϑ from Ω to $\mathbb{R}$ equipped with the norm defined as

$$\|\vartheta\| = \sup\{|\vartheta(t)|: t \in \Omega = [0, \delta]\},$$

where

$$|\vartheta(t)| = |W(t)| + |W_K(t)| + |I(t)| + |I_K(t)| + |K(t)|$$

and $W, W_K, I, I_K, K \in \mho$.

Definition 4.3. [34] Let $\mho$ be a nonempty set. A single-valued mapping $\delta\colon \mho \longrightarrow \mho$ is called a selection of a set-valued mapping $V\colon \mho \longrightarrow P(\mho)$, if $\delta(\vartheta) \in V(\vartheta)$ for every $\vartheta \in \mho$.

For each $\vartheta \in \mho$, we take the set of all selections of V by

$$W_{V,\vartheta} = \{\delta \in L'(\Omega, \mathbb{R}): \delta(t) \in V(t, \vartheta(t)) \quad \text{for almost every } (a.e.) \quad t \in \Omega\}.$$

Let $(L, \preceq_L)$ be a complete distributive lattice and $\tau_{L_V}\colon \mho \longrightarrow L \setminus \{0_L\}$ be an arbitrary mapping, where $V\colon \Omega\times\mathbb{R} \longrightarrow P(\mathbb{R})$ is a crisp set-valued map. For each $\vartheta \in \mho$, define an Lmap $\Psi(\vartheta)\colon \mho \longrightarrow L$ as

$$\Psi(\vartheta)(t) = \begin{cases} \tau_{L_V}(\vartheta), & \text{if } t \in V(t, \vartheta(t)) \\ 0_L, & \text{otherwise.} \end{cases}$$

Then, the set of all selections of V can be regarded as the set of all selections of an Lmap Ψ, denoted by $W_{L,\vartheta}$, and is defined as

$$W_{L,\vartheta} = \{\delta \in L'(\Omega, \mathbb{R}): \delta(t) \in [\Psi\vartheta]_{\tau_L} \quad \text{for } a.e. t \in \Omega\}.$$

Definition 4.4. A function $\vartheta \in C'(\Omega, \mathbb{R})$ is a solution of problem (19) if there is a function $\varphi \in L'(\Omega, \mathbb{R})$ with $\varphi(t) \in V(t, \vartheta(t)) a.e.$ on Ω such that

$$\vartheta(t) = \vartheta_0 + \frac{1}{\Gamma(\rho)}\int_0^t (t-\tau)^{\rho-1}\varphi(\tau)\mu\tau \tag{21}$$

and $\vartheta(0) = \vartheta_0 \geq 0$.

Definition 4.5. [34] A set-valued mapping $V\colon \Omega \longrightarrow P(\mathbb{R})$ with a nonempty compact convex values is said to be Lebesgue measurable, if for every $\varpi \in \mathbb{R}$, the function $t \longmapsto \mu(\varpi, V(t)) = \inf\{|\varpi - \zeta|: \zeta \in V(t)\}$ is Lebesgue measurable.

Definition 4.6. An Lmap $\Psi\colon \Omega \longrightarrow L^{\Omega}$ with nonempty compact convex cut set is said to be Lebesgue measurable, if for every $\tau_L \in L$, the function $t \longmapsto \mu(\varpi, [\Psi(t)]_{\tau_L}) = \inf\{|\varpi - \zeta|\colon \zeta \in [\Psi(t)]_{\tau_L}\}$ is Lebesgue measurable.

Let $\mathscr{K}(\mho)$ represents the family of all nonempty compact subsets of $\mho$. The following is our main contribution in this section.

Theorem 4.1. Assume that the following conditions are satisfied:

(N_1) $V\colon \Omega \times \mathbb{R} \longrightarrow \mathscr{K}(\mathbb{R})$ is such that $V(.,\vartheta)\colon \Omega \longrightarrow \mathscr{K}(\mathbb{R})$ is Lebesgue measurable for every $\vartheta \in \mathbb{R}$;

(N_2) there exists a continuous function $h\colon \Omega \longrightarrow F_+$ such that for all $\vartheta, \omega \in \mathbb{R}$,

$$\Upsilon(V(t,\vartheta), V(t,\omega)) \le h(t)|\vartheta - \omega|,$$

for almost all $t \in \Omega$ and $\mu(0, V(t,0)) \le h(t)$ for almost all $t \in \Omega$.

Then, the differential inclusion (19) has at least one solution in Ω, provided that $\Phi\|h\| < 1$, where $\Phi = \dfrac{b^{\rho}}{\Gamma(\rho+1)}$ and $\Gamma(.)$ is the gamma function.

Proof. We start by resolving (19) into an L-fuzzy invariant point problem. Accordingly, let $\mho = C(\Omega,\mathbb{R})$ and $\tau_L\colon \mho \longrightarrow L \setminus \{0_L\}$ be a mapping. For each $\vartheta \in \mho$, let the mapping $\beth_{\vartheta}\colon \Omega \longrightarrow \mathbb{R}$ be given as

$$\beth_{\vartheta}(t) = \vartheta_0 + \frac{1}{\Gamma(\rho)}\int_0^t (t-\tau)^{\rho-1}\varphi(\tau)\mu\tau, \quad \varphi \in W_{L,\vartheta}.$$

Then, define an Lmap $\Theta_V\colon \mho \longrightarrow L^{\mho}$ as follows:

$$\Theta_V(\vartheta)(l) = \begin{cases} \tau_L(\vartheta), & \text{if } \ l(t) = \beth_{\vartheta}(t) \\ 0_L, & \text{otherwise.} \end{cases}$$

By setting $\tau_L(\vartheta) := \tau_L$ for all $\vartheta \in \mho$, there exists $\tau_L \in L \setminus \{0_L\}$ such that

$$[\Theta_V(\vartheta)]_{\tau_L} = \left\{ \begin{aligned} & l \in \mho\colon \ l(t) = \beth_{\vartheta}(t) \\ & = \vartheta_0 + \frac{1}{\Gamma(\rho)}\int_0^t (t-\tau)^{\rho-1}\varphi(\tau)\mu\tau, \ \varphi \in W_{L,\vartheta} \end{aligned} \right\}.$$

It is clear that the L-fuzzy invariant points of Θ_V are solutions to problem (19). Now, we prove that Θ_V satisfies all the conditions of Corollary 3.5. For this, the following cases are examined:

Case I. $[\Theta_V(\vartheta)]_{\tau_L}$ is nonempty and closed for every $\varphi \in W_{L,\vartheta}$. Given that $V(.,\vartheta(.))$ is Lebesgue measurable, then $\Theta_V(\vartheta)$ is also Lebesgue measurable. Whence, by the Lebesgue measurable selection theorem (see, e.g., [34]), $\Theta_V(\vartheta)$ admits a Lebesgue measurable selection $\varphi\colon \Omega \longrightarrow \mathbb{R}$. Furthermore, by condition (N_2), $|\varphi(t)| \le h(t) + h(t)|\vartheta(t)|$, that is, $\varphi \in L'(\Omega,\mathbb{R})$ and hence V is integrably bounded. Thus, $W_{V,\vartheta}$ and $W_{L,\vartheta}$ are nonempty. Now, we show that $[\Theta_V(\vartheta)]_{\tau_L}$ is closed for every $\vartheta \in \mho$. Let $\{\vartheta_x\}_{x\in\mathbb{N}} \subseteq [\Theta_V(\vartheta)]_{\tau_L}$ be such that $\vartheta_x \longrightarrow u(x \longrightarrow \infty)$ in $\mho$. Then $u \in \mho$ and there exists $\varphi_x \in W_{L,\vartheta_x}$ such that for every $t \in \Omega$,

$$\vartheta_x(t) = \vartheta_0 + \frac{1}{\Gamma(\rho)}\int_0^t (t-\tau)^{\rho-1}\varphi_x(\tau)\mu\tau.$$

Since V has compact values, it follows that $[\Theta_V(\vartheta)]_{\tau_L} \in \mathscr{K}(\mho)$ for every $\tau_L \in L$. Then, we move onto a subsequence to obtain that φ_x converges to $u \in L'(\Omega, \mathbb{R})$. Whence, $u \in W_{L,\vartheta}$ and for every $t \in \Omega$, we see that

$$\vartheta_x(t) \longrightarrow u(t) = \vartheta_0 + \frac{1}{\Gamma(\rho)} \int_0^t (t-\tau)^{\rho-1} \varphi(\tau)\mu\tau.$$

Hence, $u \in [\Theta_V(\vartheta)]_{\tau_L}$.

Case 2. Next, we prove that (13) is satisfied. Recall that for every $t \in \Omega$ and $\tau_L \in L \setminus \{0_L\}$, we see that $V(t, \vartheta(t)) = [\Theta_V(\vartheta)]_{\tau_L}$. Accordingly, for every $\vartheta, \omega \in \mho$, let $\vartheta, \omega \in \mho$ and $l_1 \in [\Theta_V(\vartheta)]_{\tau_L}$. Then, there exists $\varphi_1(t) \in [\Theta_V(\vartheta)]_{\tau_L}$ for some $\tau_L \in L$ such that for every $t \in \Omega$,

$$l_1(t) = \vartheta_0 + \frac{1}{\Gamma(\rho)} + \int_0^t (t-\tau)^{\rho-1} \varphi_1(\tau)\mu\tau. \tag{22}$$

By (N_2), $\Upsilon([\Theta_V(\vartheta)]_{\tau_L}, [\Theta_V(\omega)]_{\tau_L}) \leq h(t)\|\vartheta - \omega\|$. Whence, there exists $\rho \in [\Theta_V(\omega)]_{\tau_L}$ for some $\tau_L \in L$ such that

$$|l_1(t) - \rho(t)| \leq h(t)|\vartheta(t) - \omega(t)|, \quad t \in \Omega.$$

Define $\Xi \colon \Omega \longrightarrow P(\mathbb{R})$ by $\Xi(t) = \{t \in \mathbb{R} \colon |l_1(t) - \rho(t)| \leq h(t)|\vartheta(t) - \omega(t)|\}$. Since the crisp set-valued map $\Xi(t) \cap [\Theta_V(\omega)]_{\tau_L}$ is Lebesgue measurable for every $\tau_L \in L \setminus \{0_L\}$ (see [34]), there exists a function φ_2 which is a Lebesgue measurable selection of Ξ. Thus, $\varphi_2(t) \in [\Theta_V(\omega)]_{\tau_L}$, and for every $t \in \Omega$, we see that $|\varphi_1(t) - \varphi_2(t)| \leq h(t)|\vartheta(t) - \omega(t)|$. For each $t \in \Omega$, take

$$l_2(t) = \vartheta_0 + \frac{1}{\Gamma(\rho)} \int_0^t (t-\tau)^{\rho-1} \varphi_2(\tau)\mu\tau. \tag{23}$$

Then, from (22) and (23), we obtain

$$\begin{aligned}
|l_1(t) - l_2(t)| &\leq \frac{1}{\Gamma(\rho)} \int_0^t (t-\tau)^{\rho-1} \left[|\varphi_1(\tau) - \varphi_2(\tau)|\right] \mu\tau \\
&\leq \frac{1}{\Gamma(\rho)} \int_0^t (t-\tau)^{\rho-1} [h(t)|\vartheta(t) - \omega(t)|||]\mu\tau \\
&\leq \frac{b^\rho}{\Gamma(\rho+1)} \|h\|\|\vartheta - \omega\| \\
&= \Phi\|h\|\|\vartheta - \omega\|.
\end{aligned}$$

Whence, $\|l_1 - l_2\| \leq \Phi\|h\|\|\vartheta - \omega\|$. On similar steps, interchanging the roles of ϑ and ω, we obtain

$$\Upsilon([\Theta_V(\vartheta)]_{\tau_L}, [\Theta_V(\omega)]_{\tau_L}) \leq \Phi\|h\|\|\vartheta - \omega\| = \gamma\|\vartheta - \omega\|. \tag{24}$$

Taking supremum over all of $\tau_L \in L \setminus \{0_L\}$ in (24), gives

$$\mu_L^\infty(\Psi(\vartheta), \Psi(\omega)) \leq \gamma\|\vartheta - \omega\| = \gamma\mu(\vartheta, \omega),$$

for all $\vartheta, \omega \in \mho$. Thus, all the hypotheses of Corollary 3.5 are satisfied. It follows that Θ_V has at least one L-fuzzy invariant point in $\mho$, which corresponds to the solution of problem 19.

5 Conclusions

This article established new coincidence point results for single-valued mappings and an Lmap (Theorems 3.1 and 3.2) by using a modified version of an $\mathscr{MT}$-function. Theorem 3.1 is an L-fuzzy extension of the invariant point results studied by Berinde-Berinde [35], Du [22], Mizoguchi-Takahashi [36], Nadler [3], Reich [37], Rus [38], and a handful of others in the related domains. In addition to the aforementioned work, Theorem 3.2 is an improvement of the principal idea of Heilpern [2]. As a consequence of the theorems obtained herein, a few corollaries can be derived from existing results. A comparative example (Example 3.1) is provided to support our abstractions and indicate the preeminence of the proposed ideas. In Theorem 14, an existence result for a nonlinear fractional differential inclusion model for COVID-19 was presented by utilizing the idea of μ_L^∞-metric for L-fuzzy sets.

The results of this paper, examined in metric space, are indeed fundamental. It follows that an ample amount of future work can be highlighted. Accordingly, the underlying space can be taken to other generalized, pseudo or quasi-metric spaces, such as b-MS, metric-like space, and fuzzy MS. On the flip side, the involved Lmap can be extended to some hybrid set-valued maps, such as fuzzy soft set-valued maps, intuitionistic maps, L-fuzzy soft set-valued maps, and so on. As a result of these suggested modifications, the contractive inequalities obtained herein will be modified. The latter possible variants will pave the way for better applications.

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