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ARTICLE

Pythagorean Fuzzy Einstein Aggregation Operators with Z-Numbers: Application in Complex Decision Aid Systems

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ABSTRACT

The primary goal of this research is to determine the optimal agricultural field selection that would most effectively support manufacturing producers in manufacturing production while accounting for unpredictability and reliability in their decision-making. The PFS is known to address the levels of participation and non-participation. To begin, we introduce the novel concept of a PFZN, which is a hybrid structure of Pythagorean fuzzy sets and the ZN. The PFZN is graded in terms of membership and non-membership, as well as reliability, which provides a strong advice in real-world decision support concerns. The PFZN is a useful tool for dealing with uncertainty in decision-aid problems. The PFZN is a practical way for dealing with such uncertainties in decision-aid problems. The list of aggregation operators: PFZN Einstein weighted averaging and PFZN Einstein weighted geometric, is established under the novel Pythagorean fuzzy ZNs. It is a more precise mathematical instrument for dealing with precision and uncertainty. The core of this research is to develop a numerical algorithm to tackle the uncertainty in real-life problems using PFZNs. To show the applicability and effectiveness of the proposed algorithm, we illustrate the numerical case study related to determining the optimal agricultural field. The main purpose of this work is to describe the extended EDAS approach, then compare the proposed methodology with many other methodologies now in use, and then demonstrate how the suggested methodology may be applied to real-world problems. In addition, the final ranking results that were obtained by the devised techniques were more efficient and dependable in comparison to the results provided by other methods presented in the literature.

KEYWORDS

Pythagorean fuzzy Z-number Einstein weighted averaging; Pythagorean fuzzy Z-number Einstein weighted geometric; decision making

List of Abbreviations

FS	Fuzzy set
PFS	Pythagorean fuzzy set
IFS	Intuitionistic FS
CWW	Computing with words

PF	Pythagorean fuzzy
ZN	Z-number
FN	Fuzzy number
AO	Aggregation operator
PFZNE	PFZN Einstein
IVIFS	Interval-valued IFS
PFZNW	PFZN weighted
ASC	Appraisal score
CT	Conventional tillage
SPDA	Sum of PDA
SNDA	Sum of NDA
ZTB	Zero tillage with bed
AvS	Average solution
PFZNE	PFZN Einstein
RT	Reduce tillage
PFA	PF averaging
PFZN	Pythagorean fuzzy Z-number
MAGDM	Multi-attribute group decision-making
OWA	Ordered weighted average
MCDM	Multi-criteria decision-making
PFZNEWA	PFZN Einstein weighted averaging
PFZNEWGA	PFZNE weighted geometric averaging
PFN	Pythagorean fuzzy number
PFZNWG	PFZN weighted geometric
IVPFSs	Interval-valued PFSs
PFZNEOGA	PFZNE ordered weighting averaging
PFZNEGA	PFZNE geometric averaging
PFZNEWA	PFZNE weighted averaging
PFZNEOWA	PFZNE ordered weighted averaging
CTB	Conventional tillage with bed planting
RTB	Reduce tillage with bed planting
PFZNWGA	PFZN weighted geometric averaging
NDA	Negative distance from average
PFZNEOWGA	PFZNE ordered weighting geometric averaging
PFZNOWA	PFZN ordered weighted averaging
PDA	Positive distance from average

1 Introduction

The idea of FSs has been introduced since 1965 in a variety of ways and across many academic fields. Logic, computer science, medicine, decision theory, and robotics are a few fields where this theory has many applications. Mathematical innovations have reached a very high level and continue to be made today. MAGDM is a challenge in management, engineering, economics, and various other fields. People often think that the options for data access based on need and weight are given in real numbers. However, most desirable values are tainted by ambiguity, making it challenging for those in control of decision-making to identify the optimal alternative as the system gets more complicated every day. An informative assessment for the PFS has been developed in [1], together with a justification for its validity and a discussion of the performance of the anticipated information measure.

In [2], Zeng et al. provided an innovative IFS that prevents information loss from participation and non-participation degrees. Sen et al. [3] provided sustainable supplier selection from an IFS decision-making viewpoint in order to address the question of how to acquire supplier selection. They talked about applying for the classic FN and converting ZNs to conventional FNs, and they provided a supplier selection example to show how useful the suggested process is. In [4], Rahman et al. examined a number of fundamental and significant definitions of PFSs, a number of operations on PFSs, and a number of algebraic laws relating to PFSs. In [5], Wei et al. devised plenty of PF power AOs: e.g., the PF average operator, the PF power geometric operator, the PF power ordered weighted average operator, the PF power hybrid average operator, etc.

Ejegwa investigates the idea of PFSs and draws some conclusions about how the score and accuracy functions work in [6]. PFSs have several characteristics that have been described. Pythagorean fuzzy relation is a concept that is developed in a PFS setting using numerical examples to support the developed relation. In [7], a stochastic EDAS strategy is provided in order to cope with the situation when the performance of the alternative values for each criterion follows a normal distribution. Using the suggested methodology, they evaluated options and noted the ambiguity of the information used to make decisions by obtaining optimistic and pessimistic assessment ratings. Oz et al. provided risk assessment for the clearing and grading process of a natural gas pipeline project: An extended TOPSIS model with PFSs for prioritizing hazards in [8]. Yager et al. in [9] discussed the notions of PF subsets and Pythagorean participation grades, which are related concepts, and our attention was also drawn to the negation's connection to the Pythagorean theorem. For the instance of PF subsets, they examined the fundamental set operations and complex numbers, and Pythagorean participation grades were shown to be related. In [10], Wang et al. investigated MCDM techniques using linguistic ZNs. In addition to defining and describing linguistic ZNs, this study also presented a comparison technique and a distance metric. Then they also introduced an expanded TODIM technique that relies on the Choquet integral for linguistic ZNs MCDM issues, taking into account the limited rationale of decision-making and the interactivity of criteria.

Tian et al. provided the procedure for calculating ZN relies on OWA weights and maximal entropy in [11], which is a simpler explanation of what ZN means. TOPSIS strategy simplifies MCDM situations, which rely on the idea of ZNs. Furthermore, Jia et al. [12] established a novel solution for ZNs based on complex FSs and its application in Decision-Making System.

Atanassov [13] added a second degree, known as the degree of non-participation, to the concept of the FS in 1986 to depict hesitancy and doubt on the degree of participation. For the first time, Aliev et al. in [14], present a broad strategy for building such functions that relies on the extension idea used with ZNs. The proposed method is useful for limiting the increase in uncertainty when computing the values of Z-valued functions, and it also takes into account a few ZN function characteristics.

In [15], Poleshchuk described a method to multicriteria decision making under Z-information. This method applied the anticipated utility paradigm to a standard economic decision-making issue. They also created an expanded TODIM strategy that relies on the Choquet integral for MCDM issues with linguistic ZNs. Jiang et al. [16] proffered a novel approach on the basis of Z-Network model based on Bayesian Network and ZN in which they expressed an application of the strategy to problems connected to cognitive and aesthetic concerns inherently defined by imperfect data, such as work satisfaction assessment and educational accomplishment of students appraisal. Kang et al. offered an environmental assessment under uncertainty using Dempster-Shafer theory and ZN in [17].

Internationally renowned tools, such as the Academic Motivation Scale, the Test of Attention (D2 Test), and Spielberger's Anxiety Test completed by students, are used to measure psychological factors. These Articles take into account a multi-criteria supplier selection dilemma where all the alternative parts are characterized by Z-information. They employed utility theories to solve these difficulties, and after evaluating the options, they chose the best one [18]. Abiyev et al. furnished ZN based fuzzy system for control of omnidirectional robot in [19]. Pal et al. offered a thorough analysis of the ZN method for CWW in [20]. CWW simulations with ZNs, make a ZN-based operator for figuring out how much compliance is needed, and give an algorithm for CWW with ZNs. In particular, they presented a summary of what we know about the generic design philosophy, mechanism, and hurdles that underlie CWW in general. Finally, they discussed the benefits and drawbacks of ZNs and provided some recommendations for improving this technology. To address the linear goal programming with equally desired minima issue, Ding et al. [21] created an enhanced version of QUALIFLEX based on linguistic ZNs; they called it the linguistic Z-QUALIFLEX approach. Linguistic ZNs are first used in order to represent the judgements of the decision-makers, which may more accurately describe the views that are inherently held by the decision-makers. Kang et al. in [22] proposed a methodology for ZN-based supplier selection that necessitates the transformation of information. This paper was divided into two parts: the first part addresses the problem of converting a ZN to a traditional FN in accordance with a fuzzy expectation; the second part addresses the issue of obtaining the best priority weight for supplier selection using a genetic algorithm, which is a quick and convenient way to determine the priority weight of the judgement matrix. In [23], Ren et al. suggested an MCDM strategy that relies on generalized ZNs and the Dempster-Shafer theory. To do so, they increase the ZN to a larger version that is more influenced by mortal affirmation tendencies and intrigued by the concept of a hesitant fuzzy linguistic word set. Interval-valued PFSs ranking order has been proffered by Garg in [24], who has improved the score function. In [25], Garg introduced a new generalized improved score function of IVIFSs. The goal of that paper was split into two categories. First, by taking into account the concept of a weighted average of the level of uncertainty between their participation functions, a new generalized enhanced scoring function has been introduced from the perspective of IVIFSs. Second, an IVIFSs-based strategy was used to solve the MCDM problem. For aggregating uncertain data, in [26], Riaz et al. presented the Pythagorean m-polar fuzzy weighted averaging, Pythagorean m-polar fuzzy weighted geometric, and symmetric Pythagorean m-polar fuzzy weighted averaging and symmetric Pythagorean m-polar fuzzy weighted geometric operations. They created a class of non-standard PF subsets with participation grades (a, b) that satisfy the condition $a^2 + b^2 \leq 1$ by focusing on the Pythagorean complement. Garg developed several aggregation methods for PFS in [27]. In [28], Du et al. analyzed a strategy as a generalization of the ZN and the neutrosophic set. This study suggested the idea of a neutrosophic ZN set, which was a strategic platform of neutrosophic values with the neutrosophic measures of dependability. A significant amount of work on MCDM has been done in recent years by a variety of researchers using PFS, the picture fuzzy set, N-soft set in [29–35].

The following outline will be used to summarize this article. In Section 2, we define key terms and discuss major aspects of relevant ideologies in support of our primary arguments. For a complete description of the PFZN Einstein operational law, including its definition, properties, and related theorems, see Section 3. We build a PFZN with an Einstein weighted aggregation operator and discuss its formulation, properties, and related theorems in detail in Section 4. In Section 5, we define and characterize PFZNs with an Einstein weighted aggregation operator, and we prove and analyze the proofs of various related theorems. Section 6 defines PFZNs with Einstein ordered weighted averaging, Einstein weighted geometric averaging, and Einstein ordered weighted geometric averaging aggregation operators and describes their properties. It also proves various related theorems. We divide Section 7 into two subsections. In Subsection 7.1, we developed MCDM approach using the PFZNEWA, PFZNEOWA, PFZNEWGA, and PFZNEOWGA operators, and in Subsection 7.2, we presented an example how to implemented these operators also provided comparison between these operators. In Section 8, we provided an EDAS method for the PFZNE operator and a numerical example for selecting agricultural fields. Finally, we provide a summary and some recommendations for future research in this field.

The extended EDAS approach is a novel concept that should be considered when looking for ways to deal with the truthness of membership and non-membership claims. The previously available approaches were incapable of managing this sort of data; hence, there was a vacuum in the market that needed to be addressed. As compared to the methods that were previously in use, the outcome obtained by this extended EDAS methodology yielded results that were more precise for the specific sort of data that was being examined.

The major goals of our study are:

1. To introduce a new approach for dealing with Pythagorean fuzzy Z-numbers using Einstein aggregation operators and an extended version of the EDAS method.

2. To demonstrate the effectiveness and robustness of the extended EDAS method in handling decision-making problems under uncertainty and imprecision.

3. To compare the proposed method with other existing operators and evaluate its superiority in terms of accuracy and performance.

4. To provide a comprehensive numerical example to illustrate the application of the proposed method in practical decision-making scenarios.

5. To contribute to the field of decision-making under uncertainty and imprecision by introducing a new method that can handle Pythagorean fuzzy Z-numbers effectively, and potentially be extended to other related areas of research.

Like any other model or approach, the proposed Pythagorean fuzzy Z-numbers with Einstein aggregation operators and extended EDAS method also has some limitations that need to be considered:

1. The proposed method assumes that the criteria weights are fixed and do not change over time or with changing conditions. However, in some cases, the weights may be dynamic, and the proposed method may not be suitable for such scenarios.

2. The method relies on subjective inputs from decision-makers, such as the membership values of the Pythagorean fuzzy Z-numbers and the preference weights of the criteria. These subjective inputs may introduce bias and uncertainty in the decision-making process.

3. The proposed method may not be appropriate for situations where there are a large number of alternatives and criteria, as it can become computationally expensive and time-consuming.

4. The method assumes that the Pythagorean fuzzy Z-numbers are independent of each other. However, in some cases, the relationships between the Pythagorean fuzzy Z-numbers may be correlated, and the method may not accurately reflect these relationships.

5. The method does not explicitly consider the possibility of incomplete or inconsistent information, which may occur in some decision-making scenarios.

Overall, the proposed method provides a useful framework for decision-making under uncertainty and imprecision. However, it is important to acknowledge its limitations and carefully consider the appropriateness of the method in specific decision-making situations.

2 Preliminaries

The following description and symbols have been abbreviated for time considerations in this article.

Definition 2.1. [36] Suppose that X is a nonempty set, and X has a participation function that is A. Where $\mu_A : X \to [0, 1]$, the function defines the level of participation of the element, $\wp \in X$.

That is: In X, a FS A is an object of the following form: $A = \{\langle \wp, \mu_A(\wp)\rangle \,|\wp \in X\}$.

Definition 2.2. In 2011, Zadeh [37] was the first person to propose the notion of the ZN. The ZN is discussed as, taking order pair of FNs $Z = (S, T)$, where S is a fuzzy limitation on the values of M, and T is the dependability for S, with M being a universal set.

Definition 2.3. [8] Assume that B is the PFS and here M is a universal set which described as

$$B = \{\wp, \mu_B(\wp), \nu_B(\wp) \mid \wp \in M\},$$

where the mapping $\mu_B(\wp) : M \to [0, 1]$ and $\nu_B(\wp) : M \to [0, 1]$ are the level of participation and the level of non-participation respectively, which satisfies the following requirement:

$$0 \le (\mu_B(\wp))^2 + (\nu_B(\wp))^2 \le 1.$$

To make things easier, Oz et al. [8] denoted a *PFN* by $(\mu_P(\wp), \nu_P(\wp))$, $P = (\mu_P, \nu_P)$. Consider three PFNs $\alpha = (\mu, \nu)$, $\alpha_1 = (\mu_1, \nu_1)$, and $\alpha_2 = (\mu_2, \nu_2)$, Yager et al. [9] revealed the fundamental operations, which are:

(1) $\alpha^- = [\nu, \mu]$;

(2) $\alpha_1 \vee \alpha_2 = [max\{\mu_1, \mu_2\}, min\{\nu_1, \nu_2\}]$;

(3) $\alpha_1 \wedge \alpha_2 = [min\{\mu_1, \mu_2\}, max\{\nu_1, \nu_2\}]$;

(4) $\alpha_1 \oplus \alpha_2 = [\sqrt{\mu_{21} + \mu_{22} - \mu_{21}\mu_{22}}, \nu_1\nu_2]$;

(5) $\alpha_1 \otimes \alpha_2 = \left[\mu_1\mu_2, \sqrt{\nu_{12} + \nu_{22} - \nu_{12}\nu_{22}}\right]$;

(6) $\tilde{I}.\alpha = [\sqrt{1 - (1 - \mu^2)^{\tilde{I}}}, \nu^{\tilde{I}}, \tilde{I} > 0]$;

(7) $\alpha^{\tilde{I}} = [\mu^{\tilde{I}}, \sqrt{1 - (1 - \nu^2)^{\tilde{I}}}, \tilde{I} > 0]$.

3 Pythagorean Fuzzy Z-Number

Definition 3.1. Suggest the PFZN to be Gz, and M be the universal set:

$$Gz = \{\wp, \mu(S, T)(\wp), \nu(S, T)(\wp) \,|\wp \in M\}$$

where the mapping $\mu(S, T)(\wp) : M \to [0, 1]$ and $\nu(S, T)(\wp) : M \to [0, 1]$ are constructed as follows:

$$Gz = \{(\mu(S, T)), \nu(S, T)\} = \{(\mu_S, \mu_T), (\nu_S, \nu_T)\}.$$

It meets the following requirements:

$$0 \le (\mu_S(\wp))^2 + (\nu_S(\wp))^2 \le 1,$$

$$0 \le (\mu_T(\wp))^2 + (\nu_T(\wp))^2 \le 1.$$

The characteristics of PFZNs will now be discussed, which already derive in Definition 3.1.

Definition 3.2. Let $Gz_1 = \{(\mu_1(S,T)), \nu_1(S,T)\} = \{(\mu_{S_1}, \mu_{T_1}), (\nu_{S_1}, \nu_{T_1})\}$ and $Gz_2 = \{(\mu_2(S,T)), \nu_2(S,T))\} = \{(\mu_{S_2}, \mu_{T_2}), (\nu_{S_2}, \nu_{T_2})\}$ be two PFZNs, which satisfies the following characteristics:

(1) $Gz_1 \supseteq Gz_2$ if and only if $\mu_{S_1} \geq \mu_{S_2}, \mu_{T_1} \geq \mu_{T_2}$ and $\nu_{S_1} \leq \nu_{S_2}, \nu_{T_1} \leq \nu_{T_2}$.

(2) $Gz_1 = Gz_2$ if and only if $Gz_1 \supseteq Gz_2$ and $Gz_1 \subseteq Gz_2$,

(3) $Gz_1 \cup Gz_2 = \{(\mu_{S_1} \vee \mu_{S_2}, \mu_{T_1} \vee \mu_{T_2}), (\nu_{S_1} \wedge \nu_{S_2}, \nu_{T_1} \wedge \nu_{T_2})\}$,

(4) $Gz_1 \cap Gz_2 = \{(\mu_{S_1} \wedge \mu_{S_2}, \mu_{T_1} \wedge \mu_{T_2}), (\nu_{S_1} \vee \nu_{S_2}, \nu_{T_1} \vee \nu_{T_2})\}$,

(5) $(Gz_1)^c = \{(\nu_{S_1}, \nu_{T_1}), (\mu_{S_1}, \mu_{T_1})\}$,

(6) $Gz_1 \oplus Gz_2 = \left\{ \begin{pmatrix} \sqrt{\mu_{S_1}^2 + \mu_{S_2}^2 - \mu_{S_1}^2\mu_{S_2}^2}, \\ \sqrt{\mu_{T_1}^2 + \mu_{T_2}^2 - \mu_{T_1}^2\mu_{T_2}^2} \end{pmatrix} (\nu_{S_1}\nu_{S_2}, \nu_{T_1}\nu_{T_2}) \right\}$,

(7) $Gz_1 \otimes Gz_2 = \left\{ \begin{matrix} (\mu_{S_1}\mu_{S_2}, \mu_{T_1}\mu_{T_2}), \\ \left(\sqrt{\nu_{S_1}^2 + \nu_{S_2}^2 - \nu_{S_1}^2\nu_{S_2}^2}, \sqrt{\nu_{T_1}^2 + \nu_{T_2}^2 - \nu_{T_1}^2\nu_{T_2}^2}\right) \end{matrix} \right\}$,

(8) $\tilde{I}Gz_1 = \left\{ \left(\sqrt{1 - (1 - \mu_{S_1}^2)^{\tilde{I}}}, \sqrt{1 - \left(1 - \mu_{T_1}^2\right)^{\tilde{I}}} \right), \left(\nu_{S_1}^{\tilde{I}}, \nu_{T_1}^{\tilde{I}}\right) \right\}$,

(9) $G^{\tilde{I}}z_1 = \left\{ \left(\mu_{S_1}^{\tilde{I}} \mu_{T_1}^{\tilde{I}}\right), \left(\sqrt{1 - (1 - \nu_{S_1}^2)^{\tilde{I}}}, \sqrt{1 - \left(1 - \nu_{T_1}^2\right)^{\tilde{I}}} \right) \right\}$.

Definition 3.3. Let $Gz_1 = \{(\mu_{S_1}, \mu_{T_1}), (\nu_{S_1}, \nu_{T_1})\}$ and $Gz_2 = \{(\mu_{S_2}, \mu_{T_2}), (\nu_{S_2}, \nu_{T_2})\} \in PFZN_S$. This leads to the following formulation of the scoring function:

$$J(Gz_{\grave{\imath}}) = \frac{1 + \mu_{S_{\grave{\imath}}}\mu_{T_{\grave{\imath}}} - \nu_{S_{\grave{\imath}}}\nu_{T_{\grave{\imath}}}}{2}. \tag{1}$$

where $J(Gz_{\grave{\imath}}) \in [0, 1]$. The ranking of $Gz_1 \geq Gz_2$, then there is $J(Gz_1) \geq J(Gz_2)$.

Definition 3.4. Let $Gz_{\grave{\imath}} = \{(\mu_{S_{\grave{\imath}}}, \mu_{T_{\grave{\imath}}}), (\nu_{S_{\grave{\imath}}}, \nu_{T_{\grave{\imath}}})\}, (\grave{\imath} = 1, 2, ..., \tilde{n})$ be a catalogue that contains of *PFZNs* and $\tilde{I}_{\grave{\imath}}$ is the weight of $\tilde{I}_{\grave{\imath}}(\grave{\imath} = 1, 2, ..., \tilde{n})$ such that $\tilde{I}_{\grave{\imath}} \in [0, 1]$ and $\sum_{\grave{\imath}=1}^{\tilde{n}} \tilde{I}_{\grave{\imath}} = 1$ then, a *PFZNEWA* mapping signified by the operator of dimension n $PFZNEWA : \varpi^{\tilde{n}} \rightarrow \varpi$, and

$$PFZNEWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \tilde{I}_{1.\epsilon} Gz_1 \oplus_\epsilon \tilde{I}_{2.\epsilon} Gz_2 \oplus_\epsilon ... \oplus_\epsilon \tilde{I}_{\tilde{n}.\epsilon} Gz_{\tilde{n}} \tag{2}$$

where ϖ is the catalogue that contains of all PFNs. In instance, if $\tilde{I}_{\grave{\imath}} = \frac{1}{\tilde{n}}, \forall \grave{\imath}$, then *PFZNEWA* operator simplified to PF averaging operator.

$$PFA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \frac{1}{\tilde{n}}_{.\epsilon} (Gz_1 \oplus_\epsilon Gz_2 \oplus_\epsilon ... \oplus_\epsilon Gz_{\tilde{n}})$$

Example 1. *Consider two PFZN as* $Gz_1 = \{(0.6, 0.8), (0.1, 0.3)\}$ *and* $Gz_2 = \{(0.5, 0.7), (0.2, 0.4)\}$. *As a result, the following is the definition of the score function used to rank a given PFZN by using Eq. (1),*

$$J(Gz_1) = \left(\frac{1 + (0.6 \times 0.8) - (0.1 \times 0.3)}{2}\right) = 0.725,$$
$$J(Gz_2) = \left(\frac{1 + (0.5 \times 0.7) - (0.2 \times 0.4)}{2}\right) = 0.595.$$

Hence, the ranking of $Gz_1 \geq Gz_2$*, then there is* $J(Gz_1) \geq J(Gz_2)$*. Using the operation (6) and (8) in Definition 3.2, we give the PFZNWGA operator of PFZNs.*

Definition 3.5. Let $Gz_{\grave{\imath}} = \left\{\left(\mu_{S_{\grave{\imath}}}, \mu_{T_{\grave{\imath}}}\right), \left(\nu_{S_{\grave{\imath}}}, \nu_{T_{\grave{\imath}}}\right)\right\}$ $(\grave{\imath} = 1, 2, ..., \tilde{n})$ be a group of *PFZNs* and $PFZNWA\colon \varpi^{\tilde{n}} \to \varpi$. Then, we will be able to categorize the *PFZNWA* operator as

$$PFZNWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \sum_{\grave{\imath}=1}^{\tilde{n}} \tilde{I}_{\grave{\imath}} Gz_{\grave{\imath}},$$

where $\tilde{I}_{\grave{\imath}}(\grave{\imath} = 1, 2, ..., \tilde{n})$ is the weight vector with $0 \leq \tilde{I}_{\grave{\imath}} \leq 1$ and $\sum_{\grave{\imath}=1}^{\tilde{n}} \tilde{I}_{\grave{\imath}} = 1$.

Theorem 3.1. Let $Gz_{\grave{\imath}} = \left\{\left(\mu_{S_{\grave{\imath}}}, \mu_{T_{\grave{\imath}}}\right), \left(\nu_{S_{\grave{\imath}}}, \nu_{T_{\grave{\imath}}}\right)\right\}$ $(\grave{\imath} = 1, 2, ..., \tilde{n})$ be a group of *PFZNs*.

Then, the acquire value of the *PFZNWA* operator is a *PFZN*, which is deduced using this formula:

$$PFZNWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \sum_{\grave{\imath}=1}^{\tilde{n}} \tilde{I}_{\grave{\imath}} Gz_{\grave{\imath}}$$
$$= \left\{\begin{array}{c} \left(\begin{array}{c}\sqrt{1 - \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1 - \mu_{S_1}^2\right)^{\tilde{I}_{\grave{\imath}}}}, \\ \sqrt{1 - \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1 - \mu_{T_1}^2\right)^{\tilde{I}_{\grave{\imath}}}}\end{array}\right), \\ \left(\Pi_{\grave{\imath}=1}^{\tilde{n}} \nu_{S_1}^{\tilde{I}_{\grave{\imath}}}, \Pi_{\grave{\imath}=1}^{\tilde{n}} \nu_{T_1}^{\tilde{I}_{\grave{\imath}}}\right)\end{array}\right\}$$

where $\tilde{I}_{\grave{\imath}}(\grave{\imath} = 1, 2, ..., \tilde{n})$ is the weight vector with $0 \leq \tilde{I}_{\grave{\imath}} \leq 1$ and $\sum_{\grave{\imath}=1}^{\tilde{n}} \tilde{I}_{\grave{\imath}} = 1$. Using the operation (7) and (9) in Definition 3.2, we give the *PFZNWGA* operator of *PFZNs*.

Definition 3.6. Let $Gz_{\grave{\imath}} = \left\{\left(\mu_{S_{\grave{\imath}}}, \mu_{T_{\grave{\imath}}}\right), \left(\nu_{S_{\grave{\imath}}}, \nu_{T_{\grave{\imath}}}\right)\right\}$ $(\grave{\imath} = 1, 2, ..., \tilde{n})$ be a group of *PFZNs*. Then the $PFZNWGA\colon \varpi^{\tilde{n}} \to \varpi$ operator is defined as

$$PFZNWGA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \Pi_{\grave{\imath}=1}^{\tilde{n}} G^{\tilde{I}_{\grave{\imath}}} z_{\grave{\imath}},$$

where $\tilde{I}_{\grave{\imath}}(\grave{\imath} = 1, 2, ..., \tilde{n})$ with $0 \leq \tilde{I}_{\grave{\imath}} \leq 1$ and $\sum_{\grave{\imath}=1}^{\tilde{n}} \tilde{I}_{\grave{\imath}} = 1$.

Theorem 3.2. Let $Gz_{\grave{\imath}} = \left\{\left(\mu_{S_{\grave{\imath}}}, \mu_{T_{\grave{\imath}}}\right), \left(\nu_{S_{\grave{\imath}}}, \nu_{T_{\grave{\imath}}}\right)\right\}$ $(\grave{\imath} = 1, 2, ..., \tilde{n})$ be a group of *PFZNs*. Then, the collected value of the *PFZNWGA* operator is a *PFZN*, which is obtained by the following formula:

$$PFZNWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \Pi_{\grave{\imath}=1}^{\tilde{n}} G^{\tilde{I}_{\grave{\imath}}} z_{\grave{\imath}}$$
$$= \left\{\begin{array}{c} (\Pi_{\grave{\imath}=1}^{\tilde{n}} \mu_{S_{\grave{\imath}}}^{\tilde{I}_{\grave{\imath}}}, \Pi_{\grave{\imath}=1}^{\tilde{n}} \mu_{T_{\grave{\imath}}}^{\tilde{I}_{\grave{\imath}}}), \\ \left(\sqrt{1 - \Pi_{\grave{\imath}=1}^{\tilde{n}}(1 - \nu_{S_{\grave{\imath}}}^2)^{\tilde{I}_{\grave{\imath}}}}, \sqrt{1 - \Pi_{\grave{\imath}=1}^{\tilde{n}}(1 - \nu_{T_{\grave{\imath}}}^2)^{\tilde{I}_{\grave{\imath}}}}\right)\end{array}\right\}$$

where $\tilde{I}_{\grave{\imath}}(\grave{\imath} = 1, 2, ..., \tilde{n})$ with $0 \leq \tilde{I}_{\grave{\imath}} \leq 1$ and $\sum_{\grave{\imath}=1}^{\tilde{n}} \tilde{I}_{\grave{\imath}} = 1$.

Definition 3.7. Let $\alpha_i = \left\{(\mu_{S_{\alpha_i}}, \mu_{T_{\alpha_i}})(\nu_{S_{\alpha_i}}, \nu_{T_{\alpha_i}})\right\}$ $(\imath = 1, 2, ..., \tilde{n})$ be a catalogue of *PFZNs*, then the PFZN order weighted averaging aggregation operator is defined as

$PFZNOWA(\alpha_1, \alpha_2, ..., \alpha_{\tilde{n}}) = \tilde{I}_1\alpha_{\sigma(1)} \oplus \tilde{I}_2\alpha_{\sigma(2)} \oplus ... \oplus \tilde{I}_{\tilde{n}}\alpha_{\sigma(\tilde{n})}$,

where $\tilde{I}_i(\imath = 1, 2, ..., \tilde{n})$ is the weighted vector of $\alpha_i(\imath = 1, 2, ..., \tilde{n})$ with $0 \leq \tilde{I}_i \leq 1$ and $\sum_{i=1}^{\tilde{n}} \tilde{I}_i = 1$.

Theorem 3.3. Let $\alpha_i = \left\{(\mu_{S_{\alpha_i}}, \mu_{T_{\alpha_i}})(\nu_{S_{\alpha_i}}, \nu_{T_{\alpha_i}})\right\}$ $(\imath = 1, 2, ..., \tilde{n})$ be a collection of *PFZNs*, then their aggregated value by using *PFZNOWA* operators as

$$PFZNOWA(\alpha_1, \alpha_2, ..., \alpha_{\tilde{n}}) = \left\{ \begin{array}{c} \left(\begin{array}{c} \sqrt{1 - \Pi_{i=1}^{\tilde{n}} \left(1 - \mu^2_{S\alpha_{\sigma(i)}}\right)^{\tilde{I}i}}, \\ \sqrt{1 - \Pi_{i=1}^{\tilde{n}} \left(1 - \mu^2_{T_{\alpha_{\sigma(i)}}}\right)^{\tilde{I}i}} \end{array} \right), \\ \left(\Pi_{i=1}^{\tilde{n}} \nu^{\tilde{I}i}_{S\alpha_{\sigma(i)}}, \Pi_{i=1}^{\tilde{n}} \nu^{\tilde{I}i}_{T\alpha_{\sigma(i)}} \right) \end{array} \right\}$$

where $\tilde{I}_i(\imath = 1, 2, ..., \tilde{n})$ is the weighted vector of $\alpha_i(\imath = 1, 2, ..., \tilde{n})$ with $0 \leq \tilde{I}_i \leq 1$ and $\sum_{i=1}^{\tilde{n}} \tilde{I}_i = 1$.

4 Einstein Operational Law of PFZNs

In this part of the article, we will cover Einstein operations on PFNs and look at plenty of the benefits associated with using them. In [35], Deshrijver et al. defined the generalized intersection of PFNs H and K, symbolized by "$\wedge$", and the generalized union of PFNs H and K, symbolized by "$\vee$".

$H \vee K = \{(\wp, T(\mu_H(\wp), \mu_K(\wp)), S(\mu_H(\wp), \mu_K(\wp)))|\wp \epsilon X\}$

$H \wedge K = \{(\wp, S(\mu_H(\wp), \mu_K(\wp)), T(\mu_H(\wp), \mu_K(\wp)))|\ \wp \epsilon X\}$

To express related intersections and unions, one can choose from a variety of t-norms and t-conorms groupings. They typically give nearly identical smooth estimation as the algebraic product and the algebraic sum, correspondingly, Einstein product $\otimes_\epsilon$ and Einstein sum $\oplus_\epsilon$ are two such families that make good alternatives which are specified as follows in the PF framework:

$$S_\epsilon(h, k) = \sqrt{\frac{h^2 + k^2}{1 + h^2_{.\epsilon}k^2}}, T_\epsilon(h, k) = \frac{h_{.\epsilon}k}{\sqrt{1 + (1 - h^2)._\epsilon(1 - k^2)}}$$

Definition 4.1. Assuming $Gz_1 = \left\{\left(\mu_{S_1}, \mu_{T_1}\right), \left(\nu_{S_1}, \nu_{T_1}\right)\right\}$ and $Gz_2 = \left\{\left(\mu_{S_2}, \mu_{T_2}\right), \left(\nu_{S_2}, \nu_{T_2}\right)\right\}$ be two PFZNs, which satisfies the following characteristics:

(1) $Gz_1 \supseteq Gz_2$ if and only if $\mu_{S_1} \geq \mu_{S_2}, \mu_{T_1} \geq \mu_{T_2}$ and $\nu_{S_1} \leq \nu_{S_2}, \nu_{T_1} \leq \nu_{T_2}$.

(2) $Gz_1 = Gz_2$ if and only if $Gz_1 \supseteq Gz_2$ and $Gz_1 \subseteq Gz_2$,

(3) $Gz_1 \cup Gz_2 = \left\{\left(\mu_{S_1} \vee \mu_{S_2}, \mu_{T_1} \vee \mu_{T_2}\right), \left(\nu_{S_1} \wedge \nu_{S_2}, \nu_{T_1} \wedge \nu_{T_2}\right)\right\}$,

(4) $Gz_1 \cap Gz_2 = \left\{\left(\mu_{S_1} \wedge \mu_{S_2}, \mu_{T_1} \wedge \mu_{T_2}\right), \left(\nu_{S_1} \vee \nu_{S_2}, \nu_{T_1} \vee \nu_{T_2}\right)\right\}$,

(5) $(Gz_1)^c = \left\{\left(\nu_{S_1}, \nu_{T_1}\right), \left(\mu_{S_1}, \mu_{T_1}\right)\right\}$,

$$(6)\ Gz_1 \oplus Gz_2 = \left[\begin{array}{c} \left(\sqrt{\frac{\mu_{S_1}^2 + \mu_{S_2}^2}{1+\mu_{S_1}^2 \cdot \in \mu_{S_2}^2}}, \sqrt{\frac{\mu_{T_1}^2 \mu_{T_2}^2}{1+\mu_{T_1}^2 \cdot \in \mu_{T_2}^2}}\right), \\ \left(\frac{\nu_{S_1}^2 \cdot \in \nu_{S_2}^2}{\sqrt{1+\left(1-\nu_{S_1}^2\right) \cdot \in \left(1-\nu_{S_2}^2\right)}}, \frac{\nu_{T_1}^{2.\in} \nu_{T_2}^2}{\sqrt{1+\left(1-\nu_{T_1}^2\right) \cdot \in \left(1-\nu_{T_2}^2\right)}}\right) \end{array} \right]$$

$$(7)\ Gz_1 \otimes Gz_2 = \left[\begin{array}{c} \left(\frac{\mu_{S_1}^2 \cdot_{\in} \mu_{S_2}^2}{\sqrt{1+\left(1-\mu_{S_1}^2\right) \cdot_{\in} \left(1-\mu_{S_2}^2\right)}}, \frac{\mu_{T_1}^2 \cdot_{\in} \mu_{T_2}^2}{\sqrt{1+\left(1-\mu_{T_1}^2\right) \cdot_{\in} \left(1-\mu_{T_2}^2\right)}}\right), \\ \left(\sqrt{\frac{\nu_{S_1}^2 + \nu_{S_2}^2}{1+\nu_{S_1}^2 \cdot_{\in} \nu_{S_2}^2}}, \sqrt{\frac{\nu_{T_1}^2 + \nu_{T_2}^2}{1+\nu_{T_1}^2 \cdot_{\in} \nu_{T_2}^2}}\right) \end{array} \right]$$

$$(8)\ \tilde{I}.Gz_1 = \left[\begin{array}{c} \left(\sqrt{\frac{\left(1+\mu_{S_1}^2\right)^{\tilde{I}} - \left(1-\mu_{S_2}^2\right)^{\tilde{I}}}{\left(1+\mu_{S_1}^2\right)^{\tilde{I}} + \left(1-\mu_{S_2}^2\right)^{\tilde{I}}}}, \sqrt{\frac{\left(1+\mu_{T_1}^2\right)^{\tilde{I}} - \left(1-\mu_{T_2}^2\right)^{\tilde{I}}}{\left(1+\mu_{T_1}^2\right)^{\tilde{I}} + \left(1-\mu_{T_2}^2\right)^{\tilde{I}}}}\right), \\ \left(\frac{\sqrt{2}(\nu_{S_1})^{\tilde{I}}}{\sqrt{(2-\nu_{S_1}^2)^{\tilde{I}} + (\nu_{S_1}^2)^{\tilde{I}}}}, \frac{\sqrt{2}(\nu_{T_1})^{\tilde{I}}}{\sqrt{(2-\nu_{T_1}^2)^{\tilde{I}} + (\nu_{S_{T_1}}^2)^{\tilde{I}}}}\right) \end{array} \right]$$

$$(9)\ (Gz_1)^{\tilde{I}} = \left[\begin{array}{c} \left(\frac{\sqrt{2}(\mu_{S_1})^{\tilde{I}}}{\sqrt{(2-\mu_{S_1}^2)^{\tilde{I}} + (\mu_{S_1}^2)^{\tilde{I}}}}, \frac{\sqrt{2}(\mu_{T_1})^{\tilde{I}}}{\sqrt{(2-\mu_{T_1}^2)^{\tilde{I}} + (\mu_{S_{T_1}}^2)^{\tilde{I}}}}\right), \\ \left(\sqrt{\frac{\left(1+\nu_{S_1}^2\right)^{\tilde{I}} - \left(1-\nu_{S_2}^2\right)^{\tilde{I}}}{\left(1+\nu_{S_1}^2\right)^{\tilde{I}} + \left(1-\nu_{S_2}^2\right)^{\tilde{I}}}}, \sqrt{\frac{\left(1+\nu_{T_1}^2\right)^{\tilde{I}} - \left(1-\nu_{T_2}^2\right)^{\tilde{I}}}{\left(1+\nu_{T_1}^2\right)^{\tilde{I}} + \left(1-\nu_{T_2}^2\right)^{\tilde{I}}}}.\right) \end{array} \right]$$

Theorem 4.1. Let $Gz_1 = \{(\mu_{S_1}, \mu_{T_1}), (\nu_{S_1}, \nu_{T_1})\}$, $Gz_2 = \{(\mu_{S_2}, \mu_{T_2}), (\nu_{S_2}, \nu_{T_2})\}$, and $Gz = \{(\mu_S, \mu_T), (\nu_S, \nu_T)\}$ be three PFZNs, then both $Gz_3 = Gz_1 \oplus_{\epsilon} Gz_2$ and $Gz_4 = \tilde{I}_{.\epsilon} Gz(\tilde{I} > 0)$ are also *PFZNs*.

Proof. As $\tilde{I}$ be an any positive real number and Gz be *PFZNs*, then $0 \le \mu_S \le 1, 0 \le \mu_T \le 1, 0 \le \nu_S \le 1$, $0 \le \nu_T \le 1, 0 \le (\mu(S)(\wp))^2 + (\nu(S)(\wp))^2 \le 1$, and $0 \le (\mu(T)(\wp))^2 + (\nu(T)(\wp))^2 \le 1$, then $1 - (\mu(T)(\wp))^2 \ge (\nu(T)(\wp))^2 \ge 0, 1 - (\nu(T)(\wp))^2 \ge (\mu(T)(\wp))^2 \ge 0$, and $\left(1 - (\mu(T)(\wp))^2\right)^{\tilde{I}} \ge \left((\nu(T)(\wp))^2\right)^{\tilde{I}}, \left(1 - (\mu(S)(\wp))^2\right)^{\tilde{I}} \ge \left((\nu(S)(\wp))^2\right)^{\tilde{I}}$ we get

$$\sqrt{\frac{\left(1+\left(\mu_{S_1}(\wp)\right)^2\right)^{\tilde{I}} - \left(1-\left(\mu_{S_1}(\wp)\right)^2\right)^{\tilde{I}}}{\left(1+\left(\mu_{S_1}(\wp)\right)^2\right)^{\tilde{I}} + \left(1-\left(\mu_{S_1}(\wp)\right)^2\right)^{\tilde{I}}}} \le \sqrt{\frac{\left(1+\left(\mu_{S_1}(\wp)\right)^2\right)^{\tilde{I}} - \left((\nu_{S_1}(\wp))^2\right)^{\tilde{I}}}{\left(1+\left(\mu_{S_1}(\wp)\right)^2\right)^{\tilde{I}} + \left((\nu_{S_1}(\wp))^2\right)^{\tilde{I}}}},$$

$$\sqrt{\frac{\left(1+\left(\mu_{T_1}(\wp)\right)^2\right)^{\tilde{I}}-\left(1-\left(\mu_{T_1}(\wp)\right)^2\right)^{\tilde{I}}}{\left(1+\left(\mu_{T_1}(\wp)\right)^2\right)^{\tilde{I}}+\left(1-\left(\mu_{T_1}(\wp)\right)^2\right)^{\tilde{I}}}} \leq \sqrt{\frac{\left(1+\left(\mu_{T_1}(\wp)\right)^2\right)^{\tilde{I}}-\left((\nu_{T_1}(\wp))^2\right)^{\tilde{I}}}{\left(1+\left(\mu_{T_1}(\wp)\right)^2\right)^{\tilde{I}}+\left((\nu_{T_1}(\wp))^2\right)^{\tilde{I}}}}.$$

and

$$\frac{\sqrt{2}(\nu_{S_1}(\wp))^{\tilde{I}}}{\sqrt{(2-(\nu_{S_1}(\wp))^2)^{\tilde{I}}+((\nu_{S_1}(\wp))^2)^{\tilde{I}}}} \leq \frac{\sqrt{2}(\nu_{S_1}(\wp))^{\tilde{I}}}{\sqrt{(1+(\mu_{S_1}(\wp))^2)^{\tilde{I}}+((\nu_{S_1}(\wp))^2)^{\tilde{I}}}},$$

$$\frac{\sqrt{2}(\nu_{T_1}(\wp))^{\tilde{I}}}{\sqrt{(2-(\nu_{T_1}(\wp))^2)^{\tilde{I}}+((\nu_{T_1}(\wp))^2)^{\tilde{I}}}} \leq \frac{\sqrt{2}(\nu_{T_1}(\wp))^{\tilde{I}}}{\sqrt{(1+(\mu_{T_1}(\wp))^2)^{\tilde{I}}+((\nu_{T_1}(\wp))^2)^{\tilde{I}}}}.$$

Thus,

$$\left(\sqrt{\frac{\left(1+\left(\mu_{S_1}(\wp)\right)^2\right)^{\tilde{I}}-\left(1-\left(\mu_{S_1}(\wp)\right)^2\right)^{\tilde{I}}}{\left(1+\left(\mu_{S_1}(\wp)\right)^2\right)^{\tilde{I}}+\left(1-\left(\mu_{S_1}(\wp)\right)^2\right)^{\tilde{I}}}}\right)^2+\left(\frac{\sqrt{2}(\nu_{S_1}(\wp))^{\tilde{I}}}{\sqrt{(2-(\nu_{S_1}(\wp))^2)^{\tilde{I}}+((\nu_{S_1}(\wp))^2)^{\tilde{I}}}}\right)^2 \leq 1,$$

$$\left(\sqrt{\frac{\left(1+\left(\mu_{T_1}(\wp)\right)^2\right)^{\tilde{I}}-\left(1-\left(\mu_{T_1}(\wp)\right)^2\right)^{\tilde{I}}}{\left(1+\left(\mu_{T_1}(\wp)\right)^2\right)^{\tilde{I}}+\left(1-\left(\mu_{T_1}(\wp)\right)^2\right)^{\tilde{I}}}}\right)^2+\left(\frac{\sqrt{2}(\nu_{T_1}(\wp))^{\tilde{I}}}{\sqrt{(2-(\nu_{T_1}(\wp))^2)^{\tilde{I}}+((\nu_{T_1}(\wp))^2)^{\tilde{I}}}}\right)^2 \leq 1.$$

Furthermore,

$$\left(\sqrt{\frac{\left(1+\left(\mu_{S_1}(\wp)\right)^2\right)^{\tilde{I}}-\left(1-\left(\mu_{S_1}(\wp)\right)^2\right)^{\tilde{I}}}{\left(1+\left(\mu_{S_1}(\wp)\right)^2\right)^{\tilde{I}}+\left(1-\left(\mu_{S_1}(\wp)\right)^2\right)^{\tilde{I}}}}\right)^2+\left(\frac{\sqrt{2}(\nu_{S_1}(\wp))^{\tilde{I}}}{\sqrt{(2-(\nu_{S_1}(\wp))^2)^{\tilde{I}}+((\nu_{S_1}(\wp))^2)^{\tilde{I}}}}\right)^2 = 0,$$

$$\left(\sqrt{\frac{\left(1+\left(\mu_{T_1}(\wp)\right)^2\right)^{\tilde{I}}-\left(1-\left(\mu_{T_1}(\wp)\right)^2\right)^{\tilde{I}}}{\left(1+\left(\mu_{T_1}(\wp)\right)^2\right)^{\tilde{I}}+\left(1-\left(\mu_{T_1}(\wp)\right)^2\right)^{\tilde{I}}}}\right)^2+\left(\frac{\sqrt{2}(\nu_{T_1}(\wp))^{\tilde{I}}}{\sqrt{(2-(\nu_{T_1}(\wp))^2)^{\tilde{I}}+((\nu_{T_1}(\wp))^2)^{\tilde{I}}}}\right)^2 = 0.$$

if and only if $\mu_{S_1}(\wp)=\nu_{S_1}(\wp)=0, \mu_{T_1}(\wp)=\nu_{T_1}(\wp)=0$. and

$$\left(\sqrt{\frac{\left(1+\left(\mu_{S_1}(\wp)\right)^2\right)^{\tilde{I}}-\left(1-\left(\mu_{S_1}(\wp)\right)^2\right)^{\tilde{I}}}{\left(1+\left(\mu_{S_1}(\wp)\right)^2\right)^{\tilde{I}}+\left(1-\left(\mu_{S_1}(\wp)\right)^2\right)^{\tilde{I}}}}\right)^2+\left(\frac{\sqrt{2}(\nu_{S_1}(\wp))^{\tilde{I}}}{\sqrt{(2-(\nu_{S_1}(\wp))^2)^{\tilde{I}}+((\nu_{S_1}(\wp))^2)^{\tilde{I}}}}\right)^2 = 1,$$

$$\left(\sqrt{\frac{\left(1+\left(\mu_{T_1}(\wp)\right)^2\right)^{\tilde{I}}-\left(1-\left(\mu_{T_1}(\wp)\right)^2\right)^{\tilde{I}}}{\left(1+\left(\mu_{T_1}(\wp)\right)^2\right)^{\tilde{I}}+\left(1-\left(\mu_{T_1}(\wp)\right)^2\right)^{\tilde{I}}}}\right)^2+\left(\frac{\sqrt{2}(\nu_{T_1}(\wp))^{\tilde{I}}}{\sqrt{(2-(\nu_{T_1}(\wp))^2)^{\tilde{I}}+((\nu_{T_1}(\wp))^2)^{\tilde{I}}}}\right)^2 = 1.$$

if and only if $\left(\mu_{S_1}(\wp)\right)^2 + \left(\nu_{S_1}(\wp)\right)^2 = 1, \left(\mu_{T_1}(\wp)\right)^2 + \left(\nu_{T_1}(\wp)\right)^2 = 1$. Thus, $Gz_4 = \tilde{I}_{.\epsilon} Gz$ is *PFZN* for any positive real number $\tilde{I}$.

Remark. *Now let us looking into $\tilde{I}_{.\epsilon} Gz$ and $Gz^{\tilde{I}}$ for a few particular situations of $\tilde{I}$ and Gz in the following. (a) If $Gz = \{(\mu_S, \mu_T), (\nu_S, \nu_T)\} = (1, 0)$, that is $\mu_S = \mu_T = 1, \nu_S = \nu_T = 0$, then*

$$\tilde{I}_{.\epsilon} Gz = \left[\begin{array}{c} \left(\sqrt{\frac{\left(1+\mu_S^2\right)^{\tilde{I}} - \left(1-\mu_S^2\right)^{\tilde{I}}}{\left(1+\mu_S^2\right)^{\tilde{I}} + \left(1-\mu_S^2\right)^{\tilde{I}}}}, \sqrt{\frac{\left(1+\mu_T^2\right)^{\tilde{I}} - \left(1-\mu_T^2\right)^{\tilde{I}}}{\left(1+\mu_T^2\right)^{\tilde{I}} + \left(1-\mu_T^2\right)^{\tilde{I}}}}\right), \\ \left(\frac{\sqrt{2}(\nu_S)^{\tilde{I}}}{\sqrt{(2-\nu_S^2)^{\tilde{I}} + (\nu_S^2)^{\tilde{I}}}}, \frac{\sqrt{2}(\nu_T)^{\tilde{I}}}{\sqrt{(2-\nu_T^2)^{\tilde{I}} + (\nu_{S_T}^2)^{\tilde{I}}}}\right) \end{array}\right] = (1, 0)$$

$$(Gz_1)^{\tilde{I}} = \left[\begin{array}{c} \left(\frac{\sqrt{2}(\mu_S)^{\tilde{I}}}{\sqrt{(2-\mu_S^2)^{\tilde{I}} + (\mu_S^2)^{\tilde{I}}}}, \frac{\sqrt{2}(\mu_T)^{\tilde{I}}}{\sqrt{(2-\mu_T^2)^{\tilde{I}} + (\mu_{S_T}^2)^{\tilde{I}}}}\right), \\ \left(\sqrt{\frac{\left(1+\nu_S^2\right)^{\tilde{I}} - \left(1-\nu_S^2\right)^{\tilde{I}}}{\left(1+\nu_S^2\right)^{\tilde{I}} + \left(1-\nu_S^2\right)^{\tilde{I}}}}, \sqrt{\frac{\left(1+\nu_T^2\right)^{\tilde{I}} - \left(1-\nu_T^2\right)^{\tilde{I}}}{\left(1+\nu_T^2\right)^{\tilde{I}} + \left(1-\nu_T^2\right)^{\tilde{I}}}}\right) \end{array}\right] = (1, 0)$$

i.e., $\tilde{I}_{.\epsilon}(1, 0) = (1, 0)$ *and* $(1, 0)^{\tilde{I}} = (1, 0)$

(b) If $Gz = \{(\mu_S, \mu_T), (\nu_S, \nu_T)\} = (0, 1)$, then

$$\tilde{I}_{.\epsilon} Gz = \left[\begin{array}{c} \left(\sqrt{\frac{\left(1+\mu_S^2\right)^{\tilde{I}} - \left(1-\mu_S^2\right)^{\tilde{I}}}{\left(1+\mu_S^2\right)^{\tilde{I}} + \left(1-\mu_S^2\right)^{\tilde{I}}}}, \sqrt{\frac{\left(1+\mu_T^2\right)^{\tilde{I}} - \left(1-\mu_T^2\right)^{\tilde{I}}}{\left(1+\mu_T^2\right)^{\tilde{I}} + \left(1-\mu_T^2\right)^{\tilde{I}}}}\right), \\ \left(\frac{\sqrt{2}(\nu_S)^{\tilde{I}}}{\sqrt{(2-\nu_S^2)^{\tilde{I}} + (\nu_S^2)^{\tilde{I}}}}, \frac{\sqrt{2}(\nu_T)^{\tilde{I}}}{\sqrt{(2-\nu_T^2)^{\tilde{I}} + (\nu_{S_T}^2)^{\tilde{I}}}}\right) \end{array}\right] = (0, 1)$$

$$(Gz_1)^{\tilde{I}} = \left[\begin{array}{c} \left(\frac{\sqrt{2}(\mu_S)^{\tilde{I}}}{\sqrt{(2-\mu_S^2)^{\tilde{I}} + (\mu_S^2)^{\tilde{I}}}}, \frac{\sqrt{2}(\mu_T)^{\tilde{I}}}{\sqrt{(2-\mu_T^2)^{\tilde{I}} + (\mu_{S_T}^2)^{\tilde{I}}}}\right), \\ \left(\sqrt{\frac{\left(1+\nu_S^2\right)^{\tilde{I}} - \left(1-\nu_S^2\right)^{\tilde{I}}}{\left(1+\nu_S^2\right)^{\tilde{I}} + \left(1-\nu_S^2\right)^{\tilde{I}}}}, \sqrt{\frac{\left(1+\nu_T^2\right)^{\tilde{I}} - \left(1-\nu_T^2\right)^{\tilde{I}}}{\left(1+\nu_T^2\right)^{\tilde{I}} + \left(1-\nu_T^2\right)^{\tilde{I}}}}\right) \end{array}\right] = (0, 1)$$

(c) If $\tilde{I} \longrightarrow 0$ *and* $0 < \mu_S, \nu_S < 1$, *then*

$$\tilde{I}_{.\epsilon} Gz = \begin{bmatrix} \left(\sqrt{\frac{\left(1+\mu_S^2\right)^{\tilde{I}} - \left(1-\mu_S^2\right)^{\tilde{I}}}{\left(1+\mu_S^2\right)^{\tilde{I}} + \left(1-\mu_S^2\right)^{\tilde{I}}}}, \sqrt{\frac{\left(1+\mu_T^2\right)^{\tilde{I}} - \left(1-\mu_T^2\right)^{\tilde{I}}}{\left(1+\mu_T^2\right)^{\tilde{I}} + \left(1-\mu_T^2\right)^{\tilde{I}}}} \right), \\ \left(\frac{\sqrt{2}(\nu_S)^{\tilde{I}}}{\sqrt{(2-\nu_S^2)^{\tilde{I}} + (\nu_S^2)^{\tilde{I}}}}, \frac{\sqrt{2}(\nu_T)^{\tilde{I}}}{\sqrt{(2-\nu_T^2)^{\tilde{I}} + (\nu_{S_T}^2)^{\tilde{I}}}} \right) \end{bmatrix} \longrightarrow (0,1)$$

$$(Gz_1)^{\tilde{I}} = \begin{bmatrix} \left(\frac{\sqrt{2}(\mu_S)^{\tilde{I}}}{\sqrt{(2-\mu_S^2)^{\tilde{I}} + (\mu_S^2)^{\tilde{I}}}}, \frac{\sqrt{2}(\mu_T)^{\tilde{I}}}{\sqrt{(2-\mu_T^2)^{\tilde{I}} + (\mu_{S_T}^2)^{\tilde{I}}}} \right), \\ \left(\sqrt{\frac{\left(1+\nu_S^2\right)^{\tilde{I}} - \left(1-\nu_S^2\right)^{\tilde{I}}}{\left(1+\nu_S^2\right)^{\tilde{I}} + \left(1-\nu_S^2\right)^{\tilde{I}}}}, \sqrt{\frac{\left(1+\nu_T^2\right)^{\tilde{I}} - \left(1-\nu_T^2\right)^{\tilde{I}}}{\left(1+\nu_T^2\right)^{\tilde{I}} + \left(1-\nu_T^2\right)^{\tilde{I}}}} \right) \end{bmatrix} \longrightarrow (1,0)$$

(d) If $\tilde{I} \longrightarrow +\infty$ *and* $0 < \mu_S, \nu_S < 1$, *then*

$$\tilde{I}_{.\epsilon} Gz = \begin{bmatrix} \left(\sqrt{\frac{\left(1+\mu_S^2\right)^{\tilde{I}} - \left(1-\mu_S^2\right)^{\tilde{I}}}{\left(1+\mu_S^2\right)^{\tilde{I}} + \left(1-\mu_S^2\right)^{\tilde{I}}}}, \sqrt{\frac{\left(1+\mu_T^2\right)^{\tilde{I}} - \left(1-\mu_T^2\right)^{\tilde{I}}}{\left(1+\mu_T^2\right)^{\tilde{I}} + \left(1-\mu_T^2\right)^{\tilde{I}}}} \right), \\ \left(\frac{\sqrt{2}(\nu_S)^{\tilde{I}}}{\sqrt{(2-\nu_S^2)^{\tilde{I}} + (\nu_S^2)^{\tilde{I}}}}, \frac{\sqrt{2}(\nu_T)^{\tilde{I}}}{\sqrt{(2-\nu_T^2)^{\tilde{I}} + (\nu_{S_T}^2)^{\tilde{I}}}} \right) \end{bmatrix} \longrightarrow (0,1)$$

$$= \begin{bmatrix} \left(\sqrt{\frac{(1)^{\tilde{I}} - \left(\frac{1-\mu_S^2}{1+\mu_S^2}\right)^{\tilde{I}}}{(1)^{\tilde{I}} + \left(\frac{1-\mu_S^2}{1+\mu_S^2}\right)^{\tilde{I}}}}, \sqrt{\frac{(1)^{\tilde{I}} - \left(\frac{1-\mu_T^2}{1+\mu_T^2}\right)^{\tilde{I}}}{(1)^{\tilde{I}} + \left(\frac{1-\mu_T^2}{1+\mu_T^2}\right)^{\tilde{I}}}} \right), \\ \left(\frac{\sqrt{2}(1)^{\tilde{I}}}{\sqrt{(\frac{2-\nu_S^2}{\nu_S})^{\tilde{I}} + (1)^{\tilde{I}}}}, \frac{\sqrt{2}(1)^{\tilde{I}}}{\sqrt{(\frac{2-\nu_T^2}{\nu_T})^{\tilde{I}} + (1)^{\tilde{I}}}} \right) \end{bmatrix}$$

$$\longrightarrow (0,1)$$

since $0 \leq \nu_S < 1, (0 \leq \nu_T < 1) \iff \nu_S < 2 - \nu_S \iff 1 < \frac{2-\nu_S}{\nu_S}$. *Thus* $\left(\frac{2-\nu_S}{\nu_S}\right)^{\tilde{I}} = +\infty$ *as* $\tilde{I} \longrightarrow \infty$

$$(Gz_1)^{\tilde{I}} = \begin{bmatrix} \left(\frac{\sqrt{2}(\mu_S)^{\tilde{I}}}{\sqrt{(2-\mu_S^2)^{\tilde{I}} + (\mu_S^2)^{\tilde{I}}}}, \frac{\sqrt{2}(\mu_T)^{\tilde{I}}}{\sqrt{(2-\mu_T^2)^{\tilde{I}} + (\mu_{S_T}^2)^{\tilde{I}}}} \right), \\ \left(\sqrt{\frac{\left(1+\nu_S^2\right)^{\tilde{I}} - \left(1-\nu_S^2\right)^{\tilde{I}}}{\left(1+\nu_S^2\right)^{\tilde{I}} + \left(1-\nu_S^2\right)^{\tilde{I}}}}, \sqrt{\frac{\left(1+\nu_T^2\right)^{\tilde{I}} - \left(1-\nu_T^2\right)^{\tilde{I}}}{\left(1+\nu_T^2\right)^{\tilde{I}} + \left(1-\nu_T^2\right)^{\tilde{I}}}} \right) \end{bmatrix} \longrightarrow (1,0)$$

(d) If $\tilde{I} = 1$, *then*

$$\tilde{I}_{.\epsilon} Gz = \left[\begin{array}{c} \left(\sqrt{\frac{\left(1+\mu_S^2\right)^{\tilde{I}} - \left(1-\mu_S^2\right)^{\tilde{I}}}{\left(1+\mu_S^2\right)^{\tilde{I}} + \left(1-\mu_S^2\right)^{\tilde{I}}}}, \sqrt{\frac{\left(1+\mu_T^2\right)^{\tilde{I}} - \left(1-\mu_T^2\right)^{\tilde{I}}}{\left(1+\mu_T^2\right)^{\tilde{I}} + \left(1-\mu_T^2\right)^{\tilde{I}}}}\right), \\ \left(\frac{\sqrt{2}(\nu_S)^{\tilde{I}}}{\sqrt{(2-\nu_S^2)^{\tilde{I}} + (\nu_S^2)^{\tilde{I}}}}, \frac{\sqrt{2}(\nu_T)^{\tilde{I}}}{\sqrt{(2-\nu_T^2)^{\tilde{I}} + (\nu_{S_T}^2)^{\tilde{I}}}}\right) \end{array}\right]$$

$$= \left[\begin{array}{c} \left(\sqrt{\frac{\left(1+\mu_S^2\right) - \left(1-\mu_S^2\right)}{\left(1+\mu_S^2\right) + \left(1-\mu_S^2\right)}}, \sqrt{\frac{\left(1+\mu_T^2\right) - \left(1-\mu_T^2\right)}{\left(1+\mu_T^2\right) + \left(1-\mu_T^2\right)}}\right), \\ \left(\frac{\sqrt{2}(\nu_S)}{\sqrt{(2-\nu_S^2) + (\nu_S^2)}}, \frac{\sqrt{2}(\nu_T)}{\sqrt{(2-\nu_T^2) + (\nu_{S_T}^2)}}\right) \end{array}\right]$$

$$= \{(\mu_S, \mu_T), (\nu_S, \nu_T)\}$$

i.e., $\tilde{I}_{.\epsilon} = Gz$

Theorem 4.2. Let $\tilde{I}, \tilde{I}_1, \tilde{I}_2 \geq 0$, then

(1) $Gz_1 \oplus_\epsilon Gz_2 = Gz_2 \oplus_\epsilon Gz_1$

(2) $Gz_1 \otimes_\epsilon Gz_2 = Gz_2 \otimes_\epsilon Gz_1$

(3) $\tilde{I}_{.\epsilon} (Gz_1 \oplus_\epsilon Gz_2) = \tilde{I}_{.\epsilon} Gz_2 \oplus_\epsilon \tilde{I}_{.\epsilon} Gz_1$

(4) $(Gz_1 \otimes_\epsilon Gz_2)^{\tilde{I}} = (Gz_2)^{\tilde{I}} \otimes_\epsilon (Gz_1)^{\tilde{I}}$

(5) $\tilde{I}_{1.\epsilon} Gz \oplus \tilde{I}_{2.\epsilon} Gz = (\tilde{I}_1 + \tilde{I}_2)_{.\epsilon} Gz$

(6) $(Gz)^{\tilde{I}_1} \otimes (Gz)^{\tilde{I}_2} = (Gz)^{\tilde{I}_1 + \tilde{I}_2}$

Proof. We prove the part (1), (3), and (5) and hence similar for other.

(1)

$$Gz_1 \oplus Gz_2 = \left[\begin{array}{c} \left(\sqrt{\frac{\mu_{S_1}^2 + \mu_{S_2}^2}{1 + \mu_{S_1}^2 \cdot_\in \mu_{S_2}^2}}, \sqrt{\frac{\mu_{T_1}^2 \mu_{T_2}^2}{1 + \mu_{T_1}^2 \cdot_\in \mu_{T_2}^2}}\right), \\ \left(\frac{\nu_{S_1}^2 \cdot_\in \nu_{S_2}^2}{\sqrt{1 + \left(1-\nu_{S_1}^2\right) \cdot_\in \left(1-\nu_{S_2}^2\right)}}, \frac{\nu_{T_1}^2 \cdot_\in \nu_{T_2}^2}{\sqrt{1 + \left(1-\nu_{T_1}^2\right) \cdot_\in \left(1-\nu_{T_2}^2\right)}}\right) \end{array}\right]$$

$$= \left[\begin{array}{c} \left(\sqrt{\frac{\mu_{S_2}^2 + \mu_{S_1}^2}{1 + \mu_{S_2}^2 \cdot_\epsilon \mu_{S_1}^2}}, \sqrt{\frac{\mu_{T_2}^2 \mu_{T_1}^2}{1 + \mu_{T_2}^2 \cdot_\epsilon \mu_{T_1}^2}} \right), \\ \left(\frac{\nu_{S_2}^2 \cdot_\epsilon \nu_{S_1}^2}{\sqrt{1 + \left(1 - \nu_{S_2}^2\right) \cdot_\epsilon \left(1 - \nu_{S_1}^2\right)}}, \frac{\nu_{T_2}^2 \cdot_\epsilon \nu_{T_1}^2}{\sqrt{1 + \left(1 - \nu_{T_2}^2\right) \cdot_\epsilon \left(1 - \nu_{T_1}^2\right)}} \right) \end{array} \right]$$

$$= Gz_2 \oplus Gz_1$$

(3)

$$Gz_1 \oplus_\epsilon Gz_2 = \left[\begin{array}{c} \left(\sqrt{\frac{\mu_{S_1}^2 + \mu_{S_2}^2}{1 + \mu_{S_1}^2 \cdot_\epsilon \mu_{S_2}^2}}, \sqrt{\frac{\mu_{T_1}^2 \mu_{T_2}^2}{1 + \mu_{T_1}^2 \cdot_\epsilon \mu_{T_2}^2}} \right), \\ \left(\frac{\nu_{S_1}^2 \cdot_\epsilon \nu_{S_2}^2}{\sqrt{1 + \left(1 - \nu_{S_1}^2\right) \cdot_\epsilon \left(1 - \nu_{S_2}^2\right)}}, \frac{\nu_{T_1}^2 \cdot_\epsilon \nu_{T_2}^2}{\sqrt{1 + \left(1 - \nu_{T_1}^2\right) \cdot_\epsilon \left(1 - \nu_{T_2}^2\right)}} \right) \end{array} \right]$$

is equivalent to

$$Gz_1 \oplus_\epsilon Gz_2 = \left[\begin{array}{c} \left(\begin{array}{c} \sqrt{\frac{\left(1 + \mu_{S_1}^2\right) \cdot \epsilon \left(1 + \mu_{S_2}^2\right) - \left(1 - \mu_{S_1}^2\right) \cdot \epsilon \left(1 - \mu_{S_2}^2\right)}{\left(1 + \mu_{S_1}^2\right) \cdot \epsilon \left(1 + \mu_{S_2}^2\right) + \left(1 - \mu_{S_1}^2\right) \cdot \epsilon \left(1 - \mu_{S_2}^2\right)}}, \\ \sqrt{\frac{\left(1 + \mu_{T_1}^2\right) \cdot \epsilon \left(1 + \mu_{T_2}^2\right) - \left(1 - \mu_{T_1}^2\right) \cdot \epsilon \left(1 - \mu_{T_2}^2\right)}{\left(1 + \mu_{T_1}^2\right) \cdot \epsilon \left(1 + \mu_{S_{T_2}}^2\right) + \left(1 - \mu_{T_1}^2\right) \cdot \epsilon \left(1 - \mu_{T_2}^2\right)}}, \end{array} \right), \\ \left(\frac{\sqrt{2}\nu_{S_1} \cdot_\epsilon \nu_{S_2}}{\sqrt{\left(2 - \nu_{S_1}^2\right) \cdot_\epsilon \left(2 - \nu_{S_2}^2\right) + \nu_{S_1}^2 \cdot_\epsilon \nu_{S_2}^2}}, \frac{\sqrt{2}\nu_{T_1} \cdot_\epsilon \nu_{T_2}}{\sqrt{\left(2 - \nu_{T_1}^2\right) \cdot_\epsilon \left(2 - \nu_{T_2}^2\right) + \nu_{T_1}^2 \cdot_\epsilon \nu_{T_2}^2}} \right) \end{array} \right]$$

Take $a_1 = \left(1 + \mu_{S_1}^2\right) \cdot \epsilon \left(1 + \mu_{S_2}^2\right)$, $b_1 = \left(1 - \mu_{S_1}^2\right) \cdot \epsilon \left(1 - \mu_{S_2}^2\right)$, $a_2 = \left(1 + \mu_{T_1}^2\right) \cdot \epsilon \left(1 + \mu_{T_2}^2\right)$, $b_2 = \left(1 - \mu_{T_1}^2\right) \cdot \epsilon \left(1 - \mu_{T_2}^2\right)$, $c_1 = \nu_{S_1} \cdot_\epsilon \nu_{S_2}$, $d_1 = \left(2 - \nu_{S_1}^2\right) \cdot_\epsilon \left(2 - \nu_{S_2}^2\right)$, $c_2 = \nu_{T_1} \cdot_\epsilon \nu_{T_2}$, and $d_2 = \left(2 - \nu_{T_1}^2\right) \cdot_\epsilon \left(2 - \nu_{T_2}^2\right)$, then

$$Gz_1 \oplus_\epsilon Gz_2 = \left[\left(\sqrt{\frac{a_1 - b_1}{a_1 + b_1}}, \sqrt{\frac{a_2 - b_2}{a_2 + b_2}} \right), \left(\frac{\sqrt{2}c_1}{\sqrt{d_1 + c_1}}, \frac{\sqrt{2}c_2}{\sqrt{d_2 + c_2}} \right) \right]$$

It arises from the Einstein Pythagorean law that

$$\tilde{I}_{.\epsilon}\left(Gz_1 \oplus_\epsilon Gz_2\right) = \tilde{I}_{.\epsilon}\begin{bmatrix}\left(\sqrt{\dfrac{a_1-b_1}{a_1+b_1}}, \sqrt{\dfrac{a_2-b_2}{a_2+b_2}}\right), \\ \left(\dfrac{\sqrt{2}c_1}{\sqrt{d_1+c_1}}, \dfrac{\sqrt{2}c_2}{\sqrt{d_2+c_2}}\right)\end{bmatrix}$$

$$= \begin{bmatrix}\left(\begin{array}{c}\sqrt{\dfrac{\left(1+\frac{a_1-b_1}{a_1+b_1}\right)^{\tilde{I}} - \left(1-\frac{a_1-b_1}{a_1+b_1}\right)^{\tilde{I}}}{\left(1+\frac{a_1-b_1}{a_1+b_1}\right)^{\tilde{I}} + \left(1-\frac{a_1-b_1}{a_1+b_1}\right)^{\tilde{I}}}}, \\ \sqrt{\dfrac{\left(1+\frac{a_2-b_2}{a_2+b_2}\right)^{\tilde{I}} - \left(1-\frac{a_2-b_2}{a_2+b_2}\right)^{\tilde{I}}}{\left(1+\frac{a_2-b_2}{a_2+b_2}\right)^{\tilde{I}} + \left(1-\frac{a_2-b_2}{a_2+b_2}\right)^{\tilde{I}}}}\end{array}\right) \\ \left(\dfrac{\left(\sqrt{2}.\left(\frac{\sqrt{2}c_1}{\sqrt{d_1+c_1}}\right)^{\tilde{I}}\right)}{\sqrt{\left(2-\frac{2c_1}{d_1+c_1}\right)^{\tilde{I}} + \left(\frac{2c_1}{d_1+c_1}\right)^{\tilde{I}}}}, \dfrac{\left(\sqrt{2}.\left(\frac{\sqrt{2}c_2}{\sqrt{d_2+c_2}}\right)^{\tilde{I}}\right)}{\sqrt{\left(2-\frac{2c_2}{d_2+c_2}\right)^{\tilde{I}} + \left(\frac{2c_2}{d_2+c_2}\right)^{\tilde{I}}}}\right)\end{bmatrix}$$

$$= \begin{bmatrix}\left(\sqrt{\dfrac{(a_1)^{\tilde{I}} - (b_1)^{\tilde{I}}}{(a_1)^{\tilde{I}} + (b_1)^{\tilde{I}}}}, \sqrt{\dfrac{(a_2)^{\tilde{I}} - (b_2)^{\tilde{I}}}{(a_2)^{\tilde{I}} + (b_2)^{\tilde{I}}}}\right), \\ \left(\dfrac{\sqrt{2}\,(c_1)^{\tilde{I}}}{\sqrt{(d_1)^{\tilde{I}} + (c_1)^{\tilde{I}}}}, \dfrac{\sqrt{2}\,(c_2)^{\tilde{I}}}{\sqrt{(d_2)^{\tilde{I}} + (c_2)^{\tilde{I}}}}\right)\end{bmatrix}$$

$$= \begin{bmatrix}\left(\begin{array}{c}\sqrt{\dfrac{\left(1+\mu_{S_1}^2\right)^{\tilde{I}} \cdot\epsilon \left(1+\mu_{S_2}^2\right)^{\tilde{I}} - \left(1-\mu_{S_1}^2\right)^{\tilde{I}} \cdot\epsilon \left(1-\mu_{S_2}^2\right)^{\tilde{I}}}{\left(1+\mu_{S_1}^2\right)^{\tilde{I}} \cdot\epsilon \left(1+\mu_{S_2}^2\right)^{\tilde{I}} + \left(1-\mu_{S_1}^2\right)^{\tilde{I}} \cdot\epsilon \left(1-\mu_{S_2}^2\right)^{\tilde{I}}}}, \\ \sqrt{\dfrac{\left(1+\mu_{T_1}^2\right)^{\tilde{I}} \cdot\epsilon \left(1+\mu_{T_2}^2\right)^{\tilde{I}} - \left(1-\mu_{T_1}^2\right)^{\tilde{I}} \cdot\epsilon \left(1-\mu_{T_2}^2\right)^{\tilde{I}}}{\left(1+\mu_{T_1}^2\right)^{\tilde{I}} \cdot\epsilon \left(1+\mu_{S_{T_2}}^2\right)^{\tilde{I}} + \left(1-\mu_{T_1}^2\right)^{\tilde{I}} \cdot\epsilon \left(1-\mu_{T_2}^2\right)^{\tilde{I}}}},\end{array}\right), \\ \left(\begin{array}{c}\dfrac{\sqrt{2}\left(\nu_{S_1}^{\tilde{I}} . \in \nu_{S_2}^{\tilde{I}}\right)}{\sqrt{\left(2-\nu_{S_1}^2\right)^{\tilde{I}} . \in \left(2-\nu_{S_2}^2\right)^{\tilde{I}} + \left(\nu_{S_1}^2\right)^{\tilde{I}} . \in \left(\nu_{S_2}^2\right)^{\tilde{I}}}}, \\ \dfrac{\sqrt{2}\left(\nu_{T_1}^{\tilde{I}} . \in \nu_{T_2}^{\tilde{I}}\right)}{\sqrt{\left(2-\nu_{T_1}^2\right)^{\tilde{I}} . \in \left(2-\nu_{T_2}^2\right)^{\tilde{I}} + \left(\nu_{T_1}^2\right)^{\tilde{I}} . \in \left(\nu_{T_2}^2\right)^{\tilde{I}}}}\end{array}\right)\end{bmatrix}$$

On the other hand,

$$\tilde{I}_{.\epsilon}Gz_1 = \left[\begin{array}{c}\left(\sqrt{\frac{(1+\mu_{S_1}^2)^{\tilde{I}}-(1-\mu_{S_1}^2)^{\tilde{I}}}{(1+\mu_{S_1}^2)^{\tilde{I}}+(1-\mu_{S_1}^2)^{\tilde{I}}}},\sqrt{\frac{\left(1+\mu_{T_1}^2\right)^{\tilde{I}}.-\left(1-\mu_{T_1}^2\right)^{\tilde{I}}}{\left(1+\mu_{T_1}^2\right)^{\tilde{I}}+\left(1-\mu_{T_1}^2\right)^{\tilde{I}}}},\right),\\ \left(\frac{\sqrt{2}\left(\nu_{S_1}^{\tilde{I}}\right)}{\sqrt{(2-\nu_{S_1}^2)^{\tilde{I}}+(\nu_{S_1}^2)^{\tilde{I}}}},\frac{\sqrt{2}\left(\nu_{T_1}^{\tilde{I}}\right)}{\sqrt{\left(2-\nu_{T_1}^2\right)^{\tilde{I}}+\left(\nu_{T_1}^2\right)^{\tilde{I}}}}\right)\end{array}\right]$$

$$= \left[\left(\sqrt{\frac{a_3-b_3}{a_3+b_3}},\sqrt{\frac{a_4-b_4}{a_4+b_4}}\right),\left(\frac{\sqrt{2}c_3}{\sqrt{d_3+c_3}},\frac{\sqrt{2}c_4}{\sqrt{d_4+c_4}}\right)\right]$$

and

$$\tilde{I}_{.\epsilon}Gz_2 = \left[\begin{array}{c}\left(\sqrt{\frac{(1+\mu_{S_2}^2)^{\tilde{I}}-(1-\mu_{S_2}^2)^{\tilde{I}}}{(1+\mu_{S_2}^2)^{\tilde{I}}+(1-\mu_{S_2}^2)^{\tilde{I}}}},\sqrt{\frac{\left(1+\mu_{T_2}^2\right)^{\tilde{I}}.-\left(1-\mu_{T_2}^2\right)^{\tilde{I}}}{\left(1+\mu_{T_2}^2\right)^{\tilde{I}}+\left(1-\mu_{T_2}^2\right)^{\tilde{I}}}},\right),\\ \left(\frac{\sqrt{2}\left(\nu_{S_2}^{\tilde{I}}\right)}{\sqrt{(2-\nu_{S_2}^2)^{\tilde{I}}+(\nu_{S_2}^2)^{\tilde{I}}}},\frac{\sqrt{2}\left(\nu_{T_2}^{\tilde{I}}\right)}{\sqrt{\left(2-\nu_{T_2}^2\right)^{\tilde{I}}+\left(\nu_{T_2}^2\right)^{\tilde{I}}}}\right)\end{array}\right]$$

$$= \left[\left(\sqrt{\frac{a_5-b_5}{a_5+b_5}},\sqrt{\frac{a_6-b_6}{a_6+b_6}}\right),\left(\frac{\sqrt{2}c_5}{\sqrt{d_5+c_5}},\frac{\sqrt{2}c_6}{\sqrt{d_6+c_6}}\right)\right],$$

where $a_3 = \left(1+\mu_{S_1}^2\right)^{\tilde{I}}$, $b_3 = \left(1-\mu_{S_1}^2\right)^{\tilde{I}}$, $a_4 = \left(1+\mu_{T_1}^2\right)^{\tilde{I}}$, $b_4 = \left(1-\mu_{T_1}^2\right)^{\tilde{I}}$ $c_3 = \left(\nu_{S_1}\right)^{\tilde{I}}$, $d_3 = \left(2-\nu_{S_1}^2\right)^{\tilde{I}}$, $c_4 = \left(\nu_{T_1}^{\tilde{I}}\right)$, $d_4 = \left(2-\nu_{T_1}^2\right)^{\tilde{I}}$, $a_5 = \left(1+\mu_{S_2}^2\right)^{\tilde{I}}$, $b_5 = \left(1-\mu_{S_2}^2\right)^{\tilde{I}}$, $a_6 = \left(1+\mu_{T_2}^2\right)^{\tilde{I}}$, $b_6 = \left(1-\mu_{T_2}^2\right)^{\tilde{I}}$ $c_5 = \left(\nu_{S_2}^{\tilde{I}}\right)$, $d_5 = \left(2-\nu_{S_2}^2\right)^{\tilde{I}}$, $c_6 = \left(\nu_{T_2}^{\tilde{I}}\right)$, $d_6 = \left(2-\nu_{T_2}^2\right)^{\tilde{I}}$. Therefore, in accordance with the operational rule of Einstein's addition, we get

$$\left(\tilde{I}_{.\epsilon}Gz_1\right) \oplus_\epsilon \left(\tilde{I}_{.\epsilon}Gz_2\right) = \left[\left(\sqrt{\frac{a_3-b_3}{a_3+b_3}},\sqrt{\frac{a_4-b_4}{a_4+b_4}}\right),\left(\frac{\sqrt{2}c_3}{\sqrt{d_3+c_3}},\frac{\sqrt{2}c_4}{\sqrt{d_4+c_4}}\right)\right] \oplus_\epsilon$$

$$\left[\left(\sqrt{\frac{a_5-b_5}{a_5+b_5}},\sqrt{\frac{a_6-b_6}{a_6+b_6}}\right),\left(\frac{\sqrt{2}c_5}{\sqrt{d_5+c_5}},\frac{\sqrt{2}c_6}{\sqrt{d_6+c_6}}\right)\right]$$

$$= \left[\begin{array}{c}\left(\sqrt{\dfrac{\frac{a_3-b_3}{a_3+b_3}+\frac{a_5-b_5}{a_5+b_5}}{1+\frac{a_3-b_3}{a_3+b_3}.\epsilon\frac{a_5-b_5}{a_5+b_5}}},\sqrt{\dfrac{\frac{a_4-b_4}{a_4+b_4}+\frac{a_6-b_6}{a_6+b_6}}{1+\frac{a_4-b_4}{a_4+b_4}.\epsilon\frac{a_6-b_6}{a_6+b_6}}}\right), \\ \left(\begin{array}{c}\dfrac{2\sqrt{\frac{c_3\cdot\epsilon c_5}{(d_3+c_3)\cdot\epsilon\,(d_5+c_5)}}}{\sqrt{1+\left(1-\frac{2c_3}{d_3+c_3}\right)\cdot\epsilon\left(1-\frac{2c_5}{d_5+c_5}\right)}}, \\ \dfrac{2\sqrt{\frac{c_4\cdot\epsilon c_6}{(d_4+c_6)\,.\epsilon\,(d_4+c_6)}}}{\sqrt{1+\left(1-\frac{2c_4}{d_4+c_4}\right)\cdot\epsilon\left(1-\frac{2c_6}{d_6+c_6}\right)}}\end{array}\right)\end{array}\right]$$

$$= \left[\begin{array}{c}\left(\sqrt{\dfrac{a_3.\epsilon a_5-b_3\cdot\epsilon b_5}{a_3\cdot\epsilon a_5+b_3\cdot\epsilon b_5}},\sqrt{\dfrac{a_4\cdot\epsilon a_6-b_4\cdot\epsilon b_6}{a_4\cdot\epsilon a_6+b_4\cdot\epsilon b_6}}\right), \\ \left(\sqrt{\dfrac{2c_3\cdot\epsilon c_5}{d_3\cdot\epsilon d_5+c_3\cdot\epsilon c_5}},\sqrt{\dfrac{2c_4\cdot\epsilon c_6}{d_4\cdot\epsilon d_6+c_4\cdot\epsilon c_6}}\right)\end{array}\right]$$

$$= \left[\begin{array}{c}\left(\begin{array}{c}\sqrt{\dfrac{\left(1+\mu_{S_1}^2\right)^{\tilde{I}}\cdot\epsilon\left(1+\mu_{S_2}^2\right)^{\tilde{I}}-\left(1-\mu_{S_1}^2\right)^{\tilde{I}}\cdot\epsilon\left(1-\mu_{S_2}^2\right)^{\tilde{I}}}{\left(1+\mu_{S_1}^2\right)^{\tilde{I}}\cdot\epsilon\left(1+\mu_{S_2}^2\right)^{\tilde{I}}+\left(1-\mu_{S_1}^2\right)^{\tilde{I}}\cdot\epsilon\left(1-\mu_{S_2}^2\right)^{\tilde{I}}}}, \\ \sqrt{\dfrac{\left(1+\mu_{T_1}^2\right)^{\tilde{I}}\cdot\epsilon\left(1+\mu_{T_2}^2\right)^{\tilde{I}}-\left(1-\mu_{T_1}^2\right)^{\tilde{I}}\cdot\epsilon\left(1-\mu_{T_2}^2\right)^{\tilde{I}}}{\left(1+\mu_{T_1}^2\right)^{\tilde{I}}\cdot\epsilon\left(1+\mu_{S_{T_2}}^2\right)^{\tilde{I}}+\left(1-\mu_{T_1}^2\right)^{\tilde{I}}\cdot\epsilon\left(1-\mu_{T_2}^2\right)^{\tilde{I}}}},\end{array}\right), \\ \left(\begin{array}{c}\dfrac{\sqrt{2}\left(\nu_{S_1}^{\tilde{I}}\cdot\in\nu_{S_2}^{\tilde{I}}\right)}{\sqrt{\left(2-\nu_{S_1}^2\right)^{\tilde{I}}\cdot\in\left(2-\nu_{S_2}^2\right)^{\tilde{I}}+\left(\nu_{S_1}^2\right)^{\tilde{I}}\cdot\in\left(\nu_{S_2}^2\right)^{\tilde{I}}}}, \\ \dfrac{\sqrt{2}\left(\nu_{T_1}^{\tilde{I}}\cdot\in\nu_{T_2}^{\tilde{I}}\right)}{\sqrt{\left(2-\nu_{T_1}^2\right)^{\tilde{I}}\cdot\in\left(2-\nu_{T_2}^2\right)^{\tilde{I}}+\left(\nu_{T_1}^2\right)^{\tilde{I}}\cdot\in\left(\nu_{T_2}^2\right)^{\tilde{I}}}}\end{array}\right)\end{array}\right]$$

Hence, $\tilde{I}_{.\epsilon}\left(Gz_1 \oplus_\epsilon Gz_2\right) = \left(\tilde{I}_{.\epsilon}Gz_1\right) \oplus_\epsilon \left(\tilde{I}_{.\epsilon}Gz_2\right)$

(5) For $\tilde{I}_1 > 0, \tilde{I}_2 > 0$

$$\tilde{I}_{1.\epsilon} Gz_1 = \left[\begin{array}{c} \left(\sqrt{\frac{\left(1+\mu_{S_1}^2\right)^{\tilde{I}_1} - \left(1-\mu_{S_1}^2\right)^{\tilde{I}_1}}{\left(1+\mu_{S_1}^2\right)^{\tilde{I}_1} + \left(1-\mu_{S_1}^2\right)^{\tilde{I}_1}}}, \sqrt{\frac{\left(1+\mu_{T_1}^2\right)^{\tilde{I}_1} .- \left(1-\mu_{T_1}^2\right)^{\tilde{I}_1}}{\left(1+\mu_{T_1}^2\right)^{\tilde{I}_1} + \left(1-\mu_{T_1}^2\right)^{\tilde{I}_1}}}, \right), \\ \left(\frac{\sqrt{2}\left(\nu_{S_1}^{\tilde{I}_1}\right)}{\sqrt{\left(2-\nu_{S_1}^2\right)^{\tilde{I}_1} + \left(\nu_{S_1}^2\right)^{\tilde{I}}}}, \frac{\sqrt{2}\left(\nu_{T_1}^{\tilde{I}_1}\right)}{\sqrt{\left(2-\nu_{T_1}^2\right)^{\tilde{I}_1} + \left(\nu_{T_1}^2\right)^{\tilde{I}_1}}} \right) \end{array} \right]$$

$$= \left[\left(\sqrt{\frac{a_3 - b_3}{a_3 + b_3}}, \sqrt{\frac{a_4 - b_4}{a_4 + b_4}} \right), \left(\frac{\sqrt{2} c_3}{\sqrt{d_3 + c_3}}, \frac{\sqrt{2} c_4}{\sqrt{d_4 + c_4}} \right) \right]$$

$$\tilde{I}_{2.\epsilon} Gz_1 = \left[\begin{array}{c} \left(\sqrt{\frac{\left(1+\mu_{S_1}^2\right)^{\tilde{I}_2} - \left(1-\mu_{S_1}^2\right)^{\tilde{I}_2}}{\left(1+\mu_{S_1}^2\right)^{\tilde{I}_2} + \left(1-\mu_{S_1}^2\right)^{\tilde{I}_2}}}, \sqrt{\frac{\left(1+\mu_{T_1}^2\right)^{\tilde{I}_2} .- \left(1-\mu_{T_1}^2\right)^{\tilde{I}_2}}{\left(1+\mu_{T_1}^2\right)^{\tilde{I}_2} + \left(1-\mu_{T_1}^2\right)^{\tilde{I}_2}}}, \right), \\ \left(\frac{\sqrt{2}\left(\nu_{S_1}^{\tilde{I}_2}\right)}{\sqrt{\left(2-\nu_{S_1}^2\right)^{\tilde{I}_2} + \left(\nu_{S_1}^2\right)^{\tilde{I}}}}, \frac{\sqrt{2}\left(\nu_{T_1}^{\tilde{I}_2}\right)}{\sqrt{\left(2-\nu_{T_1}^2\right)^{\tilde{I}_2} + \left(\nu_{T_1}^2\right)^{\tilde{I}_2}}} \right) \end{array} \right]$$

$$= \left[\left(\sqrt{\frac{a_5 - b_5}{a_5 + b_5}}, \sqrt{\frac{a_6 - b_6}{a_6 + b_6}} \right), \left(\frac{\sqrt{2} c_5}{\sqrt{d_5 + c_5}}, \frac{\sqrt{2} c_6}{\sqrt{d_6 + c_6}} \right) \right]$$

where $a_{\grave{\imath}} = \left(1 + \mu_{S_{\grave{\imath}}}^2\right)^{\tilde{I}_{\grave{\imath}}}$, $b_{\grave{\imath}} = \left(1 - \mu_{S_{\grave{\imath}}}^2\right)^{\tilde{I}_{\grave{\imath}}}$, $c_{\grave{\imath}} = \left(\nu_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}$, $d_{\grave{\imath}} = \left(2 - \nu_{S_{\grave{\imath}}}^2\right)^{\tilde{I}_{\grave{\imath}}}$ for $\grave{\imath} = 3, 5$ and $a_{\grave{\imath}} = \left(1 + \mu_{T_{\grave{\imath}}}^2\right)^{\tilde{I}_{\grave{\imath}}}$, $b_{\grave{\imath}} = \left(1 - \mu_{T_{\grave{\imath}}}^2\right)^{\tilde{I}_{\grave{\imath}}}$, $c_{\grave{\imath}} = \left(\nu_{T_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}$, $d_{\grave{\imath}} = \left(2 - \nu_{T_{\grave{\imath}}}^2\right)^{\tilde{I}_{\grave{\imath}}}$ for $\grave{\imath} = 4, 6$.

$$\left(\tilde{I}_{1.\epsilon} Gz_1\right) \oplus_\epsilon \left(\tilde{I}_{2.\epsilon} Gz_1\right) = \left[\left(\sqrt{\frac{a_3 - b_3}{a_3 + b_3}}, \sqrt{\frac{a_4 - b_4}{a_4 + b_4}} \right), \left(\frac{\sqrt{2} c_3}{\sqrt{d_3 + c_3}}, \frac{\sqrt{2} c_4}{\sqrt{d_4 + c_4}} \right) \right] \oplus_\epsilon \left[\left(\sqrt{\frac{a_5 - b_5}{a_5 + b_5}}, \sqrt{\frac{a_6 - b_6}{a_6 + b_6}} \right), \left(\frac{\sqrt{2} c_5}{\sqrt{d_5 + c_5}}, \frac{\sqrt{2} c_6}{\sqrt{d_6 + c_6}} \right) \right]$$

$$= \left[\begin{array}{c} \left(\sqrt{\dfrac{\dfrac{a_3-b_3}{a_3+b_3}+\dfrac{a_5-b_5}{a_5+b_5}}{1+\dfrac{a_3-b_3}{a_3+b_3}\cdot\epsilon\dfrac{a_5-b_5}{a_5+b_5}}}, \sqrt{\dfrac{\dfrac{a_4-b_4}{a_4+b_4}+\dfrac{a_6-b_6}{a_6+b_6}}{1+\dfrac{a_4-b_4}{a_4+b_4}\cdot\epsilon\dfrac{a_6-b_6}{a_6+b_6}}} \right), \\ \left(\begin{array}{c} \dfrac{2\sqrt{\dfrac{c_3\cdot\epsilon c_5}{(d_3+c_3)\cdot\epsilon\,(d_5+c_5)}}}{\sqrt{1+\left(1-\dfrac{2c_3}{d_3+c_3}\right)\cdot\epsilon\left(1-\dfrac{2c_5}{d_5+c_5}\right)}}, \\ \dfrac{2\sqrt{\dfrac{c_4.\epsilon c_6}{(d_4+c_6)\cdot\epsilon\,(d_4+c_6)}}}{\sqrt{1+\left(1-\dfrac{2c_4}{d_4+c_4}\right)\cdot\epsilon\left(1-\dfrac{2c_6}{d_6+c_6}\right)}} \end{array} \right) \end{array} \right]$$

$$= \left[\begin{array}{c} \left(\sqrt{\dfrac{a_3\cdot\epsilon a_5-b_3\cdot\epsilon b_5}{a_3\cdot\epsilon a_5+b_3\cdot\epsilon b_5}}, \sqrt{\dfrac{a_4\cdot\epsilon a_6-b_4\cdot\epsilon b_6}{a_4\cdot\epsilon a_6+b_4\cdot\epsilon b_6}} \right), \\ \left(\sqrt{\dfrac{2c_3\cdot\epsilon c_5}{d_3\cdot\epsilon d_5+c_3\cdot\epsilon c_5}}, \sqrt{\dfrac{2c_4\cdot\epsilon c_6}{d_4\cdot\epsilon d_6+c_4\cdot\epsilon c_6}} \right) \end{array} \right]$$

$$= \left[\begin{array}{c} \left(\begin{array}{c} \sqrt{\dfrac{\left(1+\mu_{S_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(1+\mu_{S_1}^2\right)^{\tilde{I}_2}-\left(1-\mu_{S_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(1-\mu_{S_1}^2\right)^{\tilde{I}_2}}{\left(1+\mu_{S_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(1+\mu_{S_1}^2\right)^{\tilde{I}_2}+\left(1-\mu_{S_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(1-\mu_{S_1}^2\right)^{\tilde{I}_2}}}, \\ \sqrt{\dfrac{\left(1+\mu_{T_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(1+\mu_{T_1}^2\right)^{\tilde{I}_2}-\left(1-\mu_{T_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(1-\mu_{T_1}^2\right)^{\tilde{I}_2}}{\left(1+\mu_{T_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(1+\mu_{S_{T_1}}^2\right)^{\tilde{I}_2}+\left(1-\mu_{T_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(1-\mu_{T_1}^2\right)^{\tilde{I}_2}}}, \end{array} \right), \\ \left(\begin{array}{c} \dfrac{\sqrt{2}\left(\nu_{S_1}^{\tilde{I}_1}\cdot\in\nu_{S_1}^{\tilde{I}_2}\right)}{\sqrt{\left(2-\nu_{S_1}^2\right)^{\tilde{I}_1}\cdot\in\left(2-\nu_{S_1}^2\right)^{\tilde{I}_2}+\left(\nu_{S_1}^2\right)^{\tilde{I}_1}\cdot\in\left(\nu_{S_1}^2\right)^{\tilde{I}_2}}}, \\ \dfrac{\sqrt{2}\left(\nu_{T_1}^{\tilde{I}_1}\cdot\in\nu_{T_1}^{\tilde{I}_2}\right)}{\sqrt{\left(2-\nu_{T_1}^2\right)^{\tilde{I}_1}\cdot\in\left(2-\nu_{T_1}^2\right)^{\tilde{I}_2}+\left(\nu_{T_1}^2\right)^{\tilde{I}_1}\cdot\in\left(\nu_{T_1}^2\right)^{\tilde{I}_2}}} \end{array} \right) \end{array} \right]$$

$$= \left[\begin{array}{c} \left(\begin{array}{c} \sqrt{\frac{\left(1+\mu_{S_1}^2\right)^{\tilde{I}_1+\tilde{I}_2}-\left(1-\mu_{S_1}^2\right)^{\tilde{I}_1+\tilde{I}_2}}{\left(1+\mu_{S_1}^2\right)^{\tilde{I}_1+\tilde{I}_2}+\left(1-\mu_{S_1}^2\right)^{\tilde{I}_1+\tilde{I}_2}}}, \\ \sqrt{\frac{\left(1+\mu_{T_1}^2\right)^{\tilde{I}_1+\tilde{I}_2}-\left(1-\mu_{T_1}^2\right)^{\tilde{I}_1+\tilde{I}_2}}{\left(1+\mu_{T_1}^2\right)^{\tilde{I}_1+\tilde{I}_2}+\left(1-\mu_{T_1}^2\right)^{\tilde{I}_1+\tilde{I}_2}}}, \end{array} \right), \\ \left(\begin{array}{c} \frac{\sqrt{2}\left(\nu_{S_1}\right)^{\tilde{I}_1+\tilde{I}_2}}{\sqrt{\left(2-\nu_{S_1}^2\right)^{\tilde{I}_1+\tilde{I}_2}+\left(\nu_{S_1}^2\right)^{\tilde{I}_1+\tilde{I}_2}}}, \\ \frac{\sqrt{2}\left(\nu_{T_1}\right)^{\tilde{I}_1+\tilde{I}_2}}{\sqrt{\left(2-\nu_{T_1}^2\right)^{\tilde{I}_1+\tilde{I}_2}+\left(\nu_{T_1}^2\right)^{\tilde{I}_1+\tilde{I}_2}}} \end{array} \right) \end{array} \right]$$

$$= \left(\tilde{I}_1 + \tilde{I}_2\right) \cdot_\epsilon Gz_1$$

Hence $\left(\tilde{I}_{1\cdot\epsilon} Gz_1\right) \oplus_\epsilon \left(\tilde{I}_{2\cdot\epsilon} Gz_1\right) = \left(\tilde{I}_1 + \tilde{I}_2\right) \cdot_\epsilon Gz_1$

Theorem 4.3. Let $Gz_1 = \left\{\left(\mu_{S_1}, \mu_{T_1}\right), \left(\nu_{S_1}, \nu_{T_1}\right)\right\}$, and $Gz_2 = \left\{\left(\mu_{S_2}, \mu_{T_2}\right), \left(\nu_{S_2}, \nu_{T_2}\right)\right\}$, be two PFZNs, then

(1) $G_{z_1}^c \wedge_\epsilon G_{z_2}^c = \left(G_{z_1} \vee_\epsilon G_{z_2}\right)^c$

(2) $G_{z_1}^c \vee_\epsilon G_{z_2}^c = \left(G_{z_1} \wedge_\epsilon G_{z_2}\right)^c$

(3) $G_{z_1}^c \oplus_\epsilon G_{z_2}^c = \left(G_{z_1} \otimes_\epsilon G_{z_2}\right)^c$

(4) $G_{z_1}^c \otimes_\epsilon G_{z_2}^c = \left(G_{z_1} \oplus_\epsilon G_{z_2}\right)^c$

(5) $\left(G_{z_1} \vee_\epsilon G_{z_2}\right) \oplus_\epsilon \left(G_{z_1} \wedge_\epsilon G_{z_2}\right) = G_{z_1} \oplus_\epsilon G_{z_2}$

(6) $\left(G_{z_1} \vee_\epsilon G_{z_2}\right) \otimes_\epsilon \left(G_{z_1} \wedge_\epsilon G_{z_2}\right) = G_{z_1} \otimes_\epsilon G_{z_2}$

Proof. It is omitted here because the proof is trivial.

Theorem 4.4. Let $Gz_1 = \left\{\left(\mu_{S_1}, \mu_{T_1}\right), \left(\nu_{S_1}, \nu_{T_1}\right)\right\}, Gz_2 = \left\{\left(\mu_{S_2}, \mu_{T_2}\right), \left(\nu_{S_2}, \nu_{T_2}\right)\right\}$,
and $Gz_3 = \left\{\left(\mu_{S_3}, \mu_{T_3}\right), \left(\nu_{S_3}, \nu_{T_3}\right)\right\}$ be three Pythagorean fuzzy ZNs, then

(1) $\left(G_{z_1} \vee_\epsilon G_{z_2}\right) \wedge_\epsilon G_{z_3} = \left(G_{z_1} \wedge_\epsilon G_{z_3}\right) \vee_\epsilon \left(G_{z_2} \wedge_\epsilon G_{z_3}\right)$

(2) $\left(G_{z_1} \wedge_\epsilon G_{z_2}\right) \vee_\epsilon G_{z_3} = \left(G_{z_1} \vee_\epsilon G_{z_3}\right) \wedge_\epsilon \left(G_{z_2} \vee_\epsilon G_{z_3}\right)$

(3) $\left(G_{z_1} \vee_\epsilon G_{z_2}\right) \oplus_\epsilon G_{z_3} = \left(G_{z_1} \oplus_\epsilon G_{z_3}\right) \vee_\epsilon \left(G_{z_2} \oplus_\epsilon G_{z_3}\right)$

(4) $\left(G_{z_1} \wedge_\epsilon G_{z_2}\right) \oplus_\epsilon G_{z_3} = \left(G_{z_1} \oplus_\epsilon G_{z_3}\right) \wedge_\epsilon \left(G_{z_2} \oplus_\epsilon G_{z_3}\right)$

(5) $\left(G_{z_1} \vee_\epsilon G_{z_2}\right) \otimes_\epsilon G_{z_3} = \left(G_{z_1} \otimes_\epsilon G_{z_3}\right) \vee_\epsilon \left(G_{z_2} \otimes_\epsilon G_{z_3}\right)$

(6) $\left(G_{z_1} \wedge_\epsilon G_{z_2}\right) \otimes_\epsilon G_{z_3} = \left(G_{z_1} \otimes_\epsilon G_{z_3}\right) \wedge_\epsilon \left(G_{z_2} \otimes_\epsilon G_{z_3}\right)$

Proof. It is omitted here because the proof is trivial.

5 Pythagorean Fuzzy Z-Numbers Einstein Weighted Aggregation Operators

The PFZNs weighted aggregating operators will be looked at in this section using Einstein operations.

Lemma 5.1. Let $Gz_{\grave{\imath}} = \left\{\left(\mu_{S_{\grave{\imath}}}, \mu_{T_{\grave{\imath}}}\right), \left(\nu_{S_{\grave{\imath}}}, \nu_{T_{\grave{\imath}}}\right)\right\}, \tilde{I}_{\grave{\imath}} > 0$ for ($\grave{\imath} = 1, 2, ..., \tilde{n}$) and $\sum_{\grave{\imath}=1}^{\tilde{n}} \tilde{I}_{\grave{\imath}} = 1$, then

$$\Pi_{\grave{\imath}=1}^{\tilde{n}} \left(Gz_{\grave{\imath}}\right)^{\tilde{I}_{\grave{\imath}}} \leq \sum_{\grave{\imath}=1}^{\tilde{n}} \tilde{I}_{\grave{\imath}} Gz_{\grave{\imath}}$$

with equality is true $\iff$ $Gz_1 = Gz_2 = ... = Gz_{\tilde{n}}$.

Theorem 5.1. Let $Gz_{\grave{\imath}} = \left\{\left(\mu_{S_{\grave{\imath}}}, \mu_{T_{\grave{\imath}}}\right), \left(\nu_{S_{\grave{\imath}}}, \nu_{T_{\grave{\imath}}}\right)\right\}, (\grave{\imath} = 1, 2, ..., \tilde{n})$ be a collection of *PFZNs*, then the aggregated value is a *PFZN*,

$$PFZNEWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \left[\begin{array}{c} \left(\begin{array}{c} \sqrt{\dfrac{\left(\Pi_{\grave{\imath}=1}^{\tilde{n}} 1 + \mu_{S_{\grave{\imath}}}^2\right)^{\tilde{I}_{\grave{\imath}}} - \Pi_{\grave{\imath}=1}^{\tilde{n}} \left(1 - \mu_{S_{\grave{\imath}}}^2\right)^{\tilde{I}_{\grave{\imath}}}}{\Pi_{\grave{\imath}=1}^{\tilde{n}} \left(1 + \mu_{S_{\grave{\imath}}}^2\right)^{\tilde{I}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}} \left(1 - \mu_{S_{\grave{\imath}}}^2\right)^{\tilde{I}_{\grave{\imath}}}}}, \\ \sqrt{\dfrac{\Pi_{\grave{\imath}=1}^{\tilde{n}} \left(1 + \mu_{T_{\grave{\imath}}}^2\right)^{\tilde{I}_{\grave{\imath}}} . - \Pi_{\grave{\imath}=1}^{\tilde{n}} \left(1 - \mu_{T_{\grave{\imath}}}^2\right)^{\tilde{I}_{\grave{\imath}}}}{\Pi_{\grave{\imath}=1}^{\tilde{n}} \left(1 + \mu_{T_{\grave{\imath}}}^2\right)^{\tilde{I}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}} \left(1 - \mu_{T_{\grave{\imath}}}^2\right)^{\tilde{I}_{\grave{\imath}}}}}, \end{array}\right), \\ \left(\begin{array}{c} \dfrac{\sqrt{2}\Pi_{\grave{\imath}=1}^{\tilde{n}} \left(\nu_{S_{\grave{\imath}}}^{\tilde{I}_{\grave{\imath}}}\right)}{\sqrt{\Pi_{\grave{\imath}=1}^{\tilde{n}} \left(2 - \nu_{S_{\grave{\imath}}}^2\right)^{\tilde{I}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}} \left(\nu_{S_{\grave{\imath}}}^2\right)^{\tilde{I}_{\grave{\imath}}}}}, \\ \dfrac{\sqrt{2}\Pi_{\grave{\imath}=1}^{\tilde{n}} \left(\nu_{T_{\grave{\imath}}}^{\tilde{I}_{\grave{\imath}}}\right)}{\sqrt{\Pi_{\grave{\imath}=1}^{\tilde{n}} \left(2 - \nu_{T_{\grave{\imath}}}^2\right)^{\tilde{I}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}} \left(\nu_{T_{\grave{\imath}}}^2\right)^{\tilde{I}_{\grave{\imath}}}}} \end{array}\right) \end{array}\right] \quad (3)$$

where $\tilde{I}_{\grave{\imath}}$ is the weight of $Gz_{\grave{\imath}} (\grave{\imath} = 1, 2, ..., \tilde{n})$ such that $\tilde{I}_{\grave{\imath}} \in [0, 1]$ and $\sum_{\grave{\imath}=1}^{\tilde{n}} \tilde{I}_{\grave{\imath}} = 1$.

Proof. By applying mathematical induction on n, we conclude Eq. (3), for this take the value $n = 2$, $PFZNEWA(Gz_1, Gz_2) = \tilde{I}_{1\cdot\epsilon} Gz_1 \oplus_\epsilon \tilde{I}_{2\cdot\epsilon} Gz_2$.

According to Theorem 4.2-operational law (5), we can see that both $\tilde{I}_{1\cdot\epsilon} Gz_1$ and $\tilde{I}_{2\cdot\epsilon} Gz_2$ are *PFZNs*, and the value of $\tilde{I}_{1\cdot\epsilon} Gz_1 \oplus_\epsilon \tilde{I}_{2\cdot\epsilon} Gz_2$ is a *PFZN*.

$$\tilde{I}_{1\cdot\epsilon} Gz_1 = \left[\begin{array}{c} \left(\sqrt{\dfrac{\left(1 + \mu_{S_1}^2\right)^{\tilde{I}_1} - \left(1 - \mu_{S_1}^2\right)^{\tilde{I}_1}}{\left(1 + \mu_{S_1}^2\right)^{\tilde{I}_1} + \left(1 - \mu_{S_1}^2\right)^{\tilde{I}_1}}}, \sqrt{\dfrac{\left(1 + \mu_{T_1}^2\right)^{\tilde{I}_1} . - \left(1 - \mu_{T_1}^2\right)^{\tilde{I}_1}}{\left(1 + \mu_{T_1}^2\right)^{\tilde{I}_1} + \left(1 - \mu_{T_1}^2\right)^{\tilde{I}_1}}},\right), \\ \left(\dfrac{\sqrt{2}\left(\nu_{S_1}^{\tilde{I}_1}\right)}{\sqrt{\left(2 - \nu_{S_1}^2\right)^{\tilde{I}_1} + \left(\nu_{S_1}^2\right)^{\tilde{I}}}}, \dfrac{\sqrt{2}\left(\nu_{T_1}^{\tilde{I}_1}\right)}{\sqrt{\left(2 - \nu_{T_1}^2\right)^{\tilde{I}_1} + \left(\nu_{T_1}^2\right)^{\tilde{I}_1}}}\right) \end{array}\right]$$

$$\tilde{I}_{2.\epsilon}Gz_2 = \left[\begin{array}{c}\left(\sqrt{\frac{\left(1+\mu_{S_2}^2\right)^{\tilde{I}_2}-\left(1-\mu_{S_2}^2\right)^{\tilde{I}_2}}{\left(1+\mu_{S_2}^2\right)^{\tilde{I}_2}+\left(1-\mu_{S_2}^2\right)^{\tilde{I}_2}}}, \sqrt{\frac{\left(1+\mu_{T_2}^2\right)^{\tilde{I}_2}.-\left(1-\mu_{T_2}^2\right)^{\tilde{I}_2}}{\left(1+\mu_{T_2}^2\right)^{\tilde{I}_2}+\left(1-\mu_{T_2}^2\right)\tilde{I}_2}}\right), \\ \left(\frac{\sqrt{2}\left(\nu_{S_2}^{\tilde{I}_2}\right)}{\sqrt{\left(2-\nu_{S_2}^2\right)^{\tilde{I}_2}+\left(\nu_{S_2}^2\right)^{\tilde{I}_2}}}, \frac{\sqrt{2}\left(\nu_{T_2}^{\tilde{I}_2}\right)}{\sqrt{\left(2-\nu_{T_2}^2\right)^{\tilde{I}_2}+\left(\nu_{T_2}^2\right)^{\tilde{I}_2}}}\right)\end{array}\right]$$

$$\Rightarrow\sim PFZNEWA(Gz_1, Gz_2) = \tilde{I}_{1.\epsilon}Gz_1 \oplus_\epsilon \tilde{I}_{2.\epsilon}Gz_2$$

$$= \left[\begin{array}{c}\left(\begin{array}{c}\sqrt{\frac{\frac{\left(1+\mu_{S_1}^2\right)^{\tilde{I}_1}-\left(1-\mu_{S_1}^2\right)^{\tilde{I}_1}}{\left(1+\mu_{S_1}^2\right)^{\tilde{I}_1}+\left(1-\mu_{S_1}^2\right)^{\tilde{I}_1}}+\frac{\left(1+\mu_{S_2}^2\right)^{\tilde{I}_2}-\left(1-\mu_{S_2}^2\right)^{\tilde{I}_2}}{\left(1+\mu_{S_2}^2\right)^{\tilde{I}_2}+\left(1-\mu_{S_2}^2\right)^{\tilde{I}_2}}}{1+\left(\frac{\left(1+\mu_{S_1}^2\right)^{\tilde{I}_1}-\left(1-\mu_{S_1}^2\right)^{\tilde{I}_1}}{\left(1+\mu_{S_1}^2\right)^{\tilde{I}_1}+\left(1-\mu_{S_1}^2\right)^{\tilde{I}_1}}\right)\cdot_\epsilon\frac{\left(1+\mu_{S_2}^2\right)^{\tilde{I}_2}-\left(1-\mu_{S_2}^2\right)^{\tilde{I}_2}}{\left(1+\mu_{S_2}^2\right)^{\tilde{I}_2}+\left(1-\mu_{S_2}^2\right)^{\tilde{I}_2}}}}, \\ \sqrt{\frac{\frac{\left(1+\mu_{T_1}^2\right)^{\tilde{I}_1}.-\left(1-\mu_{T_1}^2\right)^{\tilde{I}_1}}{\left(1+\mu_{T_1}^2\right)^{\tilde{I}_1}+\left(1-\mu_{T_1}^2\right)^{\tilde{I}_1}}+\frac{\left(1+\mu_{T_2}^2\right)^{\tilde{I}_2}.-\left(1-\mu_{T_2}^2\right)^{\tilde{I}_2}}{\left(1+\mu_{T_2}^2\right)^{\tilde{I}_2}+\left(1-\mu_{T_2}^2\right)\tilde{I}_2}}{1+\left(\frac{\left(1+\mu_{T_1}^2\right)^{\tilde{I}_1}.-\left(1-\mu_{T_1}^2\right)^{\tilde{I}_1}}{\left(1+\mu_{T_1}^2\right)^{\tilde{I}_1}+\left(1-\mu_{T_1}^2\right)^{\tilde{I}_1}}\right)\cdot_\epsilon\frac{\left(1+\mu_{T_2}^2\right)^{\tilde{I}_2}.-\left(1-\mu_{T_2}^2\right)^{\tilde{I}_2}}{\left(1+\mu_{T_2}^2\right)^{\tilde{I}_2}+\left(1-\mu_{T_2}^2\right)\tilde{I}_2}}}\end{array}\right), \\ \left(\begin{array}{c}\frac{\left(\frac{\sqrt{2}\left(\nu_{S_1}^{\tilde{I}_1}\right)}{\sqrt{\left(2-\nu_{S_1}^2\right)^{\tilde{I}_1}+\left(\nu_{S_1}^2\right)^{\tilde{I}}}}\right)\cdot_\epsilon\left(\frac{\sqrt{2}\left(\nu_{S_2}^{\tilde{I}_2}\right)}{\sqrt{\left(2-\nu_{S_2}^2\right)^{\tilde{I}_2}+\left(\nu_{S_2}^2\right)^{\tilde{I}_2}}}\right)}{\sqrt{1+\left(\frac{\sqrt{2}\left(\nu_{S_1}^{\tilde{I}_1}\right)}{\sqrt{\left(2-\nu_{S_1}^2\right)^{\tilde{I}_1}+\left(\nu_{S_1}^2\right)^{\tilde{I}_1}}}\right)\cdot_\epsilon\left(\frac{\sqrt{2}\left(\nu_{S_2}^{\tilde{I}_2}\right)}{\sqrt{\left(2-\nu_{S_2}^2\right)^{\tilde{I}_2}+\left(\nu_{S_2}^2\right)^{\tilde{I}_2}}}\right)}}, \\ \frac{\left(\frac{\sqrt{2}\left(\nu_{T_1}^{\tilde{I}_1}\right)}{\sqrt{\left(2-\nu_{T_1}^2\right)^{\tilde{I}_1}+\left(\nu_{T_1}^2\right)^{\tilde{I}_1}}}\right)\cdot_\epsilon\left(\frac{\sqrt{2}\left(\nu_{T_2}^{\tilde{I}_2}\right)}{\sqrt{\left(2-\nu_{T_2}^2\right)^{\tilde{I}_2}+\left(\nu_{T_2}^2\right)^{\tilde{I}_2}}}\right)}{\sqrt{1+\left(\frac{\sqrt{2}\left(\nu_{T_1}^{\tilde{I}_1}\right)}{\sqrt{\left(2-\nu_{T_1}^2\right)^{\tilde{I}_1}+\left(\nu_{T_1}^2\right)^{\tilde{I}_1}}}\right)\cdot_\epsilon\left(\frac{\sqrt{2}\left(\nu_{T_2}^{\tilde{I}_2}\right)}{\sqrt{\left(2-\nu_{T_2}^2\right)^{\tilde{I}_2}+\left(\nu_{T_2}^2\right)^{\tilde{I}_2}}}\right)}}\end{array}\right)\end{array}\right]$$

$$= \left[\begin{array}{c} \left(\sqrt{\frac{\left(1+\mu_{S_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(1+\mu_{S_2}^2\right)^{\tilde{I}_2}-\left(1-\mu_{S_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(1-\mu_{S_2}^2\right)^{\tilde{I}_2}}{\left(1+\mu_{S_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(1+\mu_{S_2}^2\right)^{\tilde{I}_2}+\left(1-\mu_{S_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(1-\mu_{S_2}^2\right)^{\tilde{I}_2}}}, \sqrt{\frac{\left(1+\mu_{T_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(1+\mu_{T_2}^2\right)^{\tilde{I}_2}-\left(1-\mu_{T_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(1-\mu_{T_2}^2\right)^{\tilde{I}_2}}{\left(1+\mu_{T_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(1+\mu_{T_2}^2\right)^{\tilde{I}_2}+\left(1-\mu_{T_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(1-\mu_{T_2}^2\right)^{\tilde{I}_2}}}\right), \\ \left(\frac{\sqrt{2}\left(\left(\nu_{S_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(\nu_{S_2}^2\right)^{\tilde{I}_2}\right)}{\sqrt{\left(2-\nu_{S_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(2-\nu_{S_2}^2\right)^{\tilde{I}_2}+\left(\nu_{S_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(\nu_{S_2}^2\right)^{\tilde{I}_2}}}, \frac{\sqrt{2}\left(\left(\nu_{T_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(\nu_{T_2}^2\right)^{\tilde{I}_2}\right)}{\sqrt{\left(2-\nu_{T_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(2-\nu_{T_2}^2\right)^{\tilde{I}_2}+\left(\nu_{T_1}^2\right)^{\tilde{I}_1}\cdot\epsilon\left(\nu_{T_2}^2\right)^{\tilde{I}_2}}}\right) \end{array}\right]$$

Therefore, the result is convincing when n is equal to 2.

Taking into account the fact that the result is accurate for $n = k$,

$$PFZNEWA(Gz_1, Gz_2, ..., Gz_k) = \left[\begin{array}{c} \left(\sqrt{\frac{\left(\Pi_{i=1}^k 1+\mu_{S_i}^2\right)^{\tilde{I}_i}-\Pi_{i=1}^k\left(1-\mu_{S_i}^2\right)^{\tilde{I}_i}}{\Pi_{i=1}^k\left(1+\mu_{S_i}^2\right)^{\tilde{I}_i}+\Pi_{i=1}^k\left(1-\mu_{S_i}^2\right)^{\tilde{I}_i}}}, \sqrt{\frac{\Pi_{i=1}^k\left(1+\mu_{T_i}^2\right)^{\tilde{I}_i}.-\Pi_{i=1}^k\left(1-\mu_{T_i}^2\right)^{\tilde{I}_i}}{\Pi_{i=1}^k\left(1+\mu_{T_i}^2\right)^{\tilde{I}_i}+\Pi_{i=1}^k\left(1-\mu_{T_i}^2\right)^{\tilde{I}_i}}},\right), \\ \left(\frac{\sqrt{2}\Pi_{i=1}^k\left(\nu_{S_i}^{\tilde{I}_i}\right)}{\sqrt{\Pi_{i=1}^k\left(2-\nu_{S_i}^2\right)^{\tilde{I}_i}+\Pi_{i=1}^k\left(\nu_{S_i}^2\right)^{\tilde{I}_i}}}, \frac{\sqrt{2}\Pi_{i=1}^k\left(\nu_{T_i}^{\tilde{I}_i}\right)}{\sqrt{\Pi_{i=1}^k\left(2-\nu_{T_i}^2\right)^{\tilde{I}_i}+\Pi_{i=1}^k\left(\nu_{T_i}^2\right)^{\tilde{I}_i}}}\right) \end{array}\right]$$

Hence, because $\tilde{n} = k + 1$, we yield

$$PFZNEWA(Gz_1, Gz_2, ..., Gz_{k+1}) = PFZNEWA(Gz_1, Gz_2, ..., Gz_k) \oplus \tilde{I}_{k+1} \cdot \epsilon Gz_{k+1}$$

$$= \left[\begin{pmatrix} \sqrt{\frac{\left(\Pi_{i=1}^{k} 1 + \mu_{S_i}^2\right)^{\tilde{I}_i} - \Pi_{i=1}^{k}\left(1 - \mu_{S_i}^2\right)^{\tilde{I}_i}}{\Pi_{i=1}^{k}\left(1 + \mu_{S_i}^2\right)^{\tilde{I}_i} + \Pi_{i=1}^{k}\left(1 - \mu_{S_i}^2\right)^{\tilde{I}_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^{k}\left(1 + \mu_{T_i}^2\right)^{\tilde{I}_i} . - \Pi_{i=1}^{k}\left(1 - \mu_{T_i}^2\right)^{\tilde{I}_i}}{\Pi_{i=1}^{k}\left(1 + \mu_{T_i}^2\right)^{\tilde{I}_i} + \Pi_{i=1}^{k}\left(1 - \mu_{T_i}^2\right)^{\tilde{I}_i}}}, \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}\Pi_{i=1}^{k}\left(\nu_{S_i}^{\tilde{I}_i}\right)}{\sqrt{\Pi_{i=1}^{k}\left(2 - \nu_{S_i}^2\right)^{\tilde{I}_i} + \Pi_{i=1}^{k}\left(\nu_{S_i}^2\right)^{\tilde{I}_i}}}, \\ \frac{\sqrt{2}\Pi_{i=1}^{k}\left(\nu_{T_i}^{\tilde{I}_i}\right)}{\sqrt{\Pi_{i=1}^{k}\left(2 - \nu_{T_i}^2\right)^{\tilde{I}_i} + \Pi_{i=1}^{k}\left(\nu_{T_i}^2\right)^{\tilde{I}_i}}} \end{pmatrix} \right]$$

$$\oplus \left[\begin{pmatrix} \sqrt{\frac{\left(1 + \mu_{S_{k+1}}^2\right)^{\tilde{I}_{k+1}} - \left(1 - \mu_{S_{k+1}}^2\right)^{\tilde{I}_{k+1}}}{\left(1 + \mu_{S_{k+1}}^2\right)^{\tilde{I}_{k+1}} + \left(1 - \mu_{S_{k+1}}^2\right)^{\tilde{I}_{k+1}}}}, \\ \sqrt{\frac{\left(1 + \mu_{T_{k+1}}^2\right)^{\tilde{I}_{k+1}} . - \left(1 - \mu_{T_{k+1}}^2\right)^{\tilde{I}_{k+1}}}{\left(1 + \mu_{T_{k+1}}^2\right)^{\tilde{I}_{k+1}} + \left(1 - \mu_{T_{k+1}}^2\right)^{\tilde{I}_{k+1}}}}, \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}\left(\nu_{S_{k+1}}^{\tilde{I}_{k+1}}\right)}{\sqrt{\left(2 - \nu_{S_{k+1}}^2\right)^{\tilde{I}_{k+1}} + \left(\nu_{S_{k+1}}^2\right)^{\tilde{I}_{k+1}}}}, \\ \frac{\sqrt{2}\left(\nu_{T_{k+1}}^{\tilde{I}_{k+1}}\right)}{\sqrt{\left(2 - \nu_{T_{k+1}}^2\right)^{\tilde{I}_i} + \left(\nu_{T_{k+1}}^2\right)^{\tilde{I}_i}}} \end{pmatrix} \right]$$

$$= \left[\begin{pmatrix} \sqrt{\frac{\left(\Pi_{i=1}^{k+1} 1 + \mu_{S_{k+1}}^2\right)^{\tilde{I}_{k+1}} - \Pi_{i=1}^{k+1}\left(1 - \mu_{S_{k+1}}^2\right)^{\tilde{I}_{k+1}}}{\Pi_{i=1}^{k+1}\left(1 + \mu_{S_{k+1}}^2\right)^{\tilde{I}_{k+1}} + \Pi_{i=1}^{k+1}\left(1 - \mu_{S_{k+1}}^2\right)^{\tilde{I}_{k+1}}}}, \\ \sqrt{\frac{\Pi_{i=1}^{k+1}\left(1 + \mu_{T_{k+1}}^2\right)^{\tilde{I}_{k+1}} . - \Pi_{i=1}^{k+1}\left(1 - \mu_{T_{k+1}}^2\right)^{\tilde{I}_{k+1}}}{\Pi_{i=1}^{k+1}\left(1 + \mu_{T_{k+1}}^2\right)^{\tilde{I}_{k+1}} + \Pi_{i=1}^{k+1}\left(1 - \mu_{T_{k+1}}^2\right)^{\tilde{I}_{k+1}}}}, \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}\Pi_{i=1}^{k+1}\left(\nu_{S_{k+1}}^{\tilde{I}_{k+1}}\right)}{\sqrt{\Pi_{i=1}^{k+1}\left(2 - \nu_{S_{k+1}}^2\right)^{\tilde{I}_{k+1}} + \Pi_{i=1}^{k+1}\left(\nu_{S_{k+1}}^2\right)^{\tilde{I}_{k+1}}}}, \\ \frac{\sqrt{2}\Pi_{i=1}^{k+1}\left(\nu_{T_{k+1}}^{\tilde{I}_{k+1}}\right)}{\sqrt{\Pi_{i=1}^{k+1}\left(2 - \nu_{T_{k+1}}^2\right)^{\tilde{I}_i} + \Pi_{i=1}^{k}\left(\nu_{T_{k+1}}^2\right)^{\tilde{I}_i}}} \end{pmatrix} \right]$$

This means that $\tilde{n} = k + 1$ also satisfies Eq. (3). This implies that Eq. (3) is valid for all values of $\tilde{n}$. The proof is completed.

Theorem 5.2. If $Gz_{\grave{\imath}} = \left\{\left(\mu_{S_{\grave{\imath}}}, \mu_{T_{\grave{\imath}}}\right), \left(\nu_{S_{\grave{\imath}}}, \nu_{T_{\grave{\imath}}}\right)\right\} \in PFZNs, \grave{\imath} = 1, 2, ..., \tilde{n}$, then the aggregated value by using the $PFZNEWA$ operator is again a $PFZN$, that is, $PFZNEWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) \in PFZN$.

Proof. Since $Gz_{\grave{\imath}} = \left\{\left(\mu_{S_{\grave{\imath}}}, \mu_{T_{\grave{\imath}}}\right), \left(\nu_{S_{\grave{\imath}}}, \nu_{T_{\grave{\imath}}}\right)\right\} \in PFZNs, \grave{\imath} = 1, 2, ..., \tilde{n}$, by definition of $PFZNs$, we have

$$0 \leq (\mu_{S_{\grave{\imath}}}(\wp))^2 + (\nu_{S_{\grave{\imath}}}(\wp))^2 \leq 1, \qquad 0 \leq (\mu_{T_{\grave{\imath}}}(\wp))^2 + (\nu_{T_{\grave{\imath}}}(\wp))^2 \leq 1.$$

Therefore,

$$\frac{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1+\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}} - \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}}}{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1+\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}}} = 1 - \frac{2\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}}}{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1+\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}}}$$

$$\leq 1 - \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}} \leq 1$$

Also $\left(1+\mu_{S_{\grave{\imath}}}^2\right) \geq \left(1-\mu_{S_{\grave{\imath}}}^2\right) \Rightarrow \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1+\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}} - \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}} \geq 0 \Rightarrow$

$$\frac{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1+\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}} - \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}}}{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1+\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}}} \geq 0.$$

$$0 \leq \mu_{PFZNEWA} \leq 1$$

On the other hand,

$$\frac{2\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\nu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}}}{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1+\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}}} = \frac{2\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}}}{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1+\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}}}$$

$$\leq \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}} \leq 1$$

Also,

$$\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\nu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}} \geq 0 \Leftrightarrow \frac{2\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\nu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}}}{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1+\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}}} \geq 0$$

Thus,

$$0 \leq \nu_{PFZNEWA} \leq 1$$

Moreover,

$$\mu_{PFZNEWA}^2 + \nu_{PFZNEWA}^2 = \frac{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1+\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}} - \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}}}{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1+\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{\imath}}}^2\right)^{\tilde{l}_{\grave{\imath}}}}$$

$$+\frac{2\Pi_{\grave{i}=1}^{\tilde{n}}\left(\nu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}}{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\mu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}+\Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}}$$

$$1-\frac{2\Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}}{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\mu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}+\Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}}$$

$$+\frac{2\Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}}{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\mu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}+\Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}}$$

$$\leq 1$$

Similarly, $0 \leq (\mu_{T_{\grave{i}}}(\wp))^2 + (\nu_{T_{\grave{i}}}(\wp))^2 \leq 1$. *PFZNEWA* $\in [0,1]$ follows, as a result, the *PFZNs* that the *PFZNEWA* operator aggregated are once again *PFZNs*.

Corollary 1. *The PFZNEWA and PFZNWA operators are related to one another as shown in:*

$$PFZNEWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) \leq PFZNWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}})$$

where $Gz_{\grave{i}}(\grave{i} = 1, 2, ..., \tilde{n})$ *be a collections of PFZNs and* $\tilde{I} = \left(\tilde{I}_1, \tilde{I}_2, ..., \tilde{I}_{\tilde{n}}\right)^T$ *is the weight vector of* $Gz_{\grave{i}}$ *such that* $\tilde{I}_{\grave{i}} \in [0, 1], (\grave{i} = 1, 2, ..., \tilde{n})$ *and* $\sum_{\grave{i}=1}^{\tilde{n}} \tilde{I}_{\grave{i}} = 1$.

Proof. Let $PFZNEWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \left\langle \left(\mu_{S_{\grave{i}}}^P, \mu_{T_{\grave{i}}}^P\right), \left(\nu_{S_{\grave{i}}}^P, \nu_{T_{\grave{i}}}^P\right) \right\rangle = G^P z$ and

$PFZNWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \left\langle \left(\mu_{S_{\grave{i}}}, \mu_{T_{\grave{i}}}\right), \left(\nu_{S_{\grave{i}}}, \nu_{T_{\grave{i}}}\right) \right\rangle = Gz$. Since $\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\mu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}} \leq \sum_{\grave{i}=1}^{\tilde{n}} \tilde{I}_{\grave{i}}\left(1+\mu_{S_{\grave{i}}}^{2}\right) + \sum_{\grave{i}=1}^{\tilde{n}} \tilde{I}_{\grave{i}}\left(1-\mu_{S_{\grave{i}}}^{2}\right) = 2$, we get

$$\sqrt{\frac{\left(\Pi_{\grave{i}=1}^{\tilde{n}}1+\mu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}-\Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}}{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\mu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}+\Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}}} \leq \sqrt{1-\Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}}$$

$$\Leftrightarrow \mu_{Gz}^{p} \leq \mu_{Gz}$$

where equality holds if and only if $\mu_{S_1} = \mu_{S_2} = ... = \mu_{S_{\tilde{n}}}$.

$$\frac{2\left(\Pi_{\grave{i}=1}^{\tilde{n}}\nu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}}{\Pi_{\grave{i}=1}^{\tilde{n}}\left(2-\nu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}+\left(\Pi_{\grave{i}=1}^{\tilde{n}}\nu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}} \geq \frac{2\Pi_{\grave{i}=1}^{\tilde{n}}(\nu_{S_{\grave{i}}}^{2})^{\tilde{I}_{\grave{i}}}}{\sum_{\grave{i}=1}^{\tilde{n}}\tilde{I}_{\grave{i}}\left(2-\nu_{S_{\grave{i}}}^{2}\right)+\left(\sum_{\grave{i}=1}^{\tilde{n}}\tilde{I}_{\grave{i}}\nu_{S_{\grave{i}}}^{2}\right)} \geq \Pi_{\grave{i}=1}^{\tilde{n}}(\nu_{S_{\grave{i}}}^{2})^{\tilde{I}_{\grave{i}}}$$

$$\Rightarrow \sqrt{\frac{2\left(\Pi_{\grave{i}=1}^{\tilde{n}}\nu_{S_{\grave{i}}}\right)^{\tilde{I}_{\grave{i}}}}{\Pi_{\grave{i}=1}^{\tilde{n}}\left(2-\nu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}+\left(\Pi_{\grave{i}=1}^{\tilde{n}}\nu_{S_{\grave{i}}}^{2}\right)^{\tilde{I}_{\grave{i}}}}} \geq \Pi_{\grave{i}=1}^{\tilde{n}}(\nu_{S_{\grave{i}}})^{\tilde{I}_{\grave{i}}}$$

$$\Rightarrow \nu_{Gz}^{p} \geq \nu_{Gz}$$

where equality holds if and only if $\nu_{S_1} = \nu_{S_2} = ... = \nu_{S_{\tilde{n}}}$. Thus,

$S\left(G^p z\right) = \left(\mu^p_{Gz}\right)^2 - \left(\nu^p_{Gz}\right)^2 \leq (\mu_{Gz})^2 - (\nu_{Gz})^2 = S\,(Gz)$

If $S\left(G^p z\right) < S\,(Gz)$, then by the score definition, for every $\tilde{I}$, we have

$PFZEWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) \leq PFZNWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}})$

If $S\left(G^p z\right) = S\,(Gz)$, that is, $\left(\mu^p_{Gz}\right)^2 - \left(\nu^p_{Gz}\right)^2 = (\mu_{Gz})^2 - (\nu_{Gz})^2$, then by the condition $\left(\mu^p_{Gz}\right) \leq (\mu_{Gz})$, and $\left(\nu^p_{Gz}\right) \geq (\nu_{Gz})$, we have $\left(\mu^p_{Gz}\right) = (\mu_{Gz})$, and $\left(\nu^p_{Gz}\right) \geq (\nu_{Gz})$; thus the accuracy function $H(G^p z) = \mu^p_{Gz} + \nu^p_{Gz} = \mu_{Gz} + \nu_{Gz} = H(Gz)$. Relies on the score definition in this instance, we demonstrate that:

$PFZNEWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = PFZNWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}})$

Hence,

$PFZNEWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) \leq PFZNWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}})$

At which the equality is valid if and only if $Gz_1 = Gz_2 = ... = Gz_{\tilde{n}}$. The proposed *PFZNEWA* operator illustrates a more optimistic decision maker behavior than the *PFZNWA* operator proposed by Yager and Abbasov [9].

Example 2. *Let* $Gz_1 = \left(\mu_{S_1}, \mu_{T_1}\right), \left(\nu_{S_1}, \nu_{T_1}\right) = \{(06, 0.4), (0.5, 0.3)\}$, $Gz_2 = \left(\mu_{S_2}, \mu_{T_2}\right), \left(\nu_{S_2}, \nu_{T_2}\right) = \{(0.7, 0.1), (0.3, 0.5)\}$, *and* $Gz_3 = \left(\mu_{S_3}, \mu_{T_3}\right), \left(\nu_{S_3}, \nu_{T_3}\right) = \{(0.4, 0.1), (0.8, 0.2)\}$ *be three PFZNs and* $\tilde{I} = (0.2, 0.5, 0.3)^T$ *be the weight vector of* $Gz_{\grave{\imath}}\,(\grave{\imath} = 1, 2, 3)$

$$PFZNEWA(Gz_1, Gz_2, Gz_3) = \begin{bmatrix} \left(\sqrt{\dfrac{\left(\Pi^3_{\grave{\imath}=1} 1+\mu^2_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}} - \Pi^3_{\grave{\imath}=1}\left(1-\mu^2_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}}{\Pi^3_{\grave{\imath}=1}\left(1+\mu^2_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}} + \Pi^3_{\grave{\imath}=1}\left(1-\mu^2_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}}}, \sqrt{\dfrac{\Pi^3_{\grave{\imath}=1}\left(1+\mu^2_{T_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}} .- \Pi^3_{\grave{\imath}=1}\left(1-\mu^2_{T_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}}{\Pi^3_{\grave{\imath}=1}\left(1+\mu^2_{T_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}} + \Pi^3_{\grave{\imath}=1}\left(1-\mu^2_{T_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}}}, \right), \\ \left(\dfrac{\sqrt{2}\Pi^3_{\grave{\imath}=1}\left(\nu^{\tilde{I}_{\grave{\imath}}}_{S_{\grave{\imath}}}\right)}{\sqrt{\Pi^3_{\grave{\imath}=1}\left(2-\nu^2_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}} + \Pi^3_{\grave{\imath}=1}\left(\nu^2_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}}}, \dfrac{\sqrt{2}\Pi^3_{\grave{\imath}=1}\left(\nu^{\tilde{I}_{\grave{\imath}}}_{T_{\grave{\imath}}}\right)}{\sqrt{\Pi^3_{\grave{\imath}=1}\left(2-\nu^2_{T_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}} + \Pi^3_{\grave{\imath}=1}\left(\nu^2_{T_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}}} \right) \end{bmatrix}$$

$$= [(0.619876, 0.200639), (0.8294563, 0.221921)]$$

If we aggregate the data employing the Yager et al. [9] *developed PFZNWA operator, PFZN* $Gz_{\grave{\imath}}\,(\grave{\imath} = 1, 2, 3)$, *then we have the following results*:

$$PFZNWA(Gz_1, Gz_2, Gz_3) = \begin{bmatrix} \left(\sqrt{1 - \Pi^3_{\grave{\imath}=1}\left(1 - \mu^2_{S_1}\right)^{\tilde{I}\grave{\imath}}}, \sqrt{1 - \Pi^3_{\grave{\imath}=1}\left(1 - \mu^2_{T_1}\right)^{\tilde{I}\grave{\imath}}} \right), \\ \left(\Pi^3_{\grave{\imath}=1} \nu^{\tilde{I}\grave{\imath}}_{S_1}, \Pi^3_{\grave{\imath}=1} \nu^{\tilde{I}\grave{\imath}}_{T_1} \right) \end{bmatrix}$$

$$= [(0.266926, 0.09527), (0.445945, 0.34294)]$$

Here, we have a few PFZNEWA operator properties based on Theorem 5.1.

PROPERTY: Let $Gz_{\grave{\imath}} = \left(\mu_{S_{\grave{\imath}}}, \mu_{T_{\grave{\imath}}}\right), \left(\nu_{S_{\grave{\imath}}}, \nu_{T_{\grave{\imath}}}\right)$ $(\grave{\imath} = 1, 2, ..., \tilde{n})$ be a collections of $PFZNs$ and $\tilde{I} = \left(\tilde{I}_1, \tilde{I}_2, ..., \tilde{I}_{\tilde{n}}\right)^T$ is the linked weighted vector of $Gz_{\grave{\imath}}$ such that $\tilde{I}_{\grave{\imath}} \in [0, 1]$, $(\grave{\imath} = 1, 2, ..., \tilde{n})$ and $\sum_{\grave{\imath}=1}^{\tilde{n}} \tilde{I}_{\grave{\imath}} = 1$; then, we have the following:

(1) Idempotency: If $Gz_{\grave{\imath}} = Gz_0 = \left\langle\left(\mu_{S_0}, \mu_{T_0}\right), \left(\nu_{S_0}, \nu_{T_0}\right)\right\rangle$ for all i, then
$PFZNEWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = Gz_0$.

(2) Boundedness: Let $G^{-}z_{\grave{\imath}} = \left\langle \min_{\grave{\imath}} \left(\mu_{S_{\grave{\imath}}}, \mu_{T_{\grave{\imath}}}\right), \max_{\grave{\imath}} \left(\nu_{S_{\grave{\imath}}}, \nu_{T_{\grave{\imath}}}\right)\right\rangle$, $G^{+}z_{\grave{\imath}} = \left\langle \max_{\grave{\imath}} \left(\mu_{S_{\grave{\imath}}}, \mu_{T_{\grave{\imath}}}\right) \min_{\grave{\imath}} \left(\nu_{S_{\grave{\imath}}}, \nu_{T_{\grave{\imath}}}\right)\right\rangle$ then
$G^{-}z_{\grave{\imath}} \leq PFZNEWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) \leq G^{+}z_{\grave{\imath}}$.

(3) Monotonicity: $Gz_{\grave{\imath}} \leq Dz_{\grave{\imath}}$, for all i, then
$PFZNEWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) \leq PFZNEWA(Dz_1, Dz_2, ..., Dz_{\tilde{n}})$.

Proof. (1) AS $Gz_{\grave{\imath}} = \left\langle\left(\mu_{S_0}, \mu_{T_0}\right), \left(\nu_{S_0}, \nu_{T_0}\right)\right\rangle \in PFZNs$ for all i, then

$$PFZNEWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \left[\begin{array}{c}\left(\begin{array}{c}\sqrt{\frac{\left(\Pi_{\grave{\imath}=1}^{\tilde{n}} 1+\mu_{S_0}^2\right)^{\tilde{I}_{\grave{\imath}}} - \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{S_0}^2\right)^{\tilde{I}_{\grave{\imath}}}}{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1+\mu_{S_0}^2\right)^{\tilde{I}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{S_0}^2\right)^{\tilde{I}_{\grave{\imath}}}}}, \\ \sqrt{\frac{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1+\mu_{T_0}^2\right)^{\tilde{I}_{\grave{\imath}}} . - \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{T_0}^2\right)^{\tilde{I}_{\grave{\imath}}}}{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1+\mu_{T_0}^2\right)^{\tilde{I}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu_{T_0}^2\right)^{\tilde{I}_{\grave{\imath}}}}},\end{array}\right), \\ \left(\begin{array}{c}\frac{\sqrt{2}\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\nu_{S_0}^{\tilde{I}_{\grave{\imath}}}\right)}{\sqrt{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(2-\nu_{S_0}^2\right)^{\tilde{I}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\nu_{S_0}^2\right)^{\tilde{I}_{\grave{\imath}}}}}, \\ \frac{\sqrt{2}\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\nu_{T_{\grave{\imath}}}^{\tilde{I}_{\grave{\imath}}}\right)}{\sqrt{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(2-\nu_{T_0}^2\right)^{\tilde{I}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\nu_{T_0}^2\right)^{\tilde{I}_{\grave{\imath}}}}}\end{array}\right)\end{array}\right]$$

$$= \left[\begin{array}{c}\left(\begin{array}{c}\sqrt{\frac{\left(1+\mu_{S_0}^2\right)^{\sum_{\grave{\imath}=1}^{\tilde{n}}\tilde{I}_{\grave{\imath}}} - \left(1-\mu_{S_0}^2\right)^{\sum_{\grave{\imath}=1}^{\tilde{n}}\tilde{I}_{\grave{\imath}}}}{\left(1+\mu_{S_0}^2\right)^{\sum_{\grave{\imath}=1}^{\tilde{n}}\tilde{I}_{\grave{\imath}}} + \left(1-\mu_{S_0}^2\right)^{\sum_{\grave{\imath}=1}^{\tilde{n}}\tilde{I}_{\grave{\imath}}}}}, \\ \sqrt{\frac{\left(1+\mu_{T_0}^2\right)^{\sum_{\grave{\imath}=1}^{\tilde{n}}\tilde{I}_{\grave{\imath}}} . - \left(1-\mu_{T_0}^2\right)^{\sum_{\grave{\imath}=1}^{\tilde{n}}\tilde{I}_{\grave{\imath}}}}{\left(1+\mu_{T_0}^2\right)^{\sum_{\grave{\imath}=1}^{\tilde{n}}\tilde{I}_{\grave{\imath}}} + \left(1-\mu_{T_0}^2\right)^{\sum_{\grave{\imath}=1}^{\tilde{n}}\tilde{I}_{\grave{\imath}}}}},\end{array}\right), \\ \left(\begin{array}{c}\frac{\sqrt{2}\left(\nu_{S_0}\right)^{\sum_{\grave{\imath}=1}^{\tilde{n}}\tilde{I}_{\grave{\imath}}}}{\sqrt{\left(2-\nu_{S_0}^2\right)^{\sum_{\grave{\imath}=1}^{\tilde{n}}\tilde{I}_{\grave{\imath}}} + \left(\nu_{S_0}^2\right)^{\sum_{\grave{\imath}=1}^{\tilde{n}}\tilde{I}_{\grave{\imath}}}}}, \\ \frac{\sqrt{2}\left(\nu_{T_{\grave{\imath}}}\right)^{\sum_{\grave{\imath}=1}^{\tilde{n}}\tilde{I}_{\grave{\imath}}}}{\sqrt{\left(2-\nu_{T_0}^2\right)^{\sum_{\grave{\imath}=1}^{\tilde{n}}\tilde{I}_{\grave{\imath}}} + \left(\nu_{T_0}^2\right)^{\sum_{\grave{\imath}=1}^{\tilde{n}}\tilde{I}_{\grave{\imath}}}}}\end{array}\right)\end{array}\right]$$

$$= \left\langle\left(\mu_{S_0}, \mu_{T_0}\right), \left(\nu_{S_0}, \nu_{T_0}\right)\right\rangle$$

$$= Gz_0$$

(2) Let $f(\wp) = \dfrac{1-\wp}{1+\wp}, \wp \in [0,1]$, then $f'(c) = \dfrac{-2}{(1+\wp)^2} < 0$; $f(\wp)$ is decreasing function. Since $\mu^2_{S_{\grave{\imath},\min}} \le \mu^2_{S_{\grave{\imath}}} \le \mu^2_{S_{\grave{\imath},\max}}$, for all $\grave{\imath} = 1, 2, ..., \tilde{n}$, then $f\left(\mu^2_{S_{\grave{\imath},\max}}\right) \le f\left(\mu^2_{S_{\grave{\imath}}}\right) \le f\left(\mu^2_{S_{\grave{\imath},\min}}\right)$ for all i, that is, $\dfrac{1-\mu^2_{S_{\grave{\imath},\max}}}{1+\mu^2_{S_{\grave{\imath},\max}}} \le \dfrac{1-\mu^2_{S_{\grave{\imath}}}}{1+\mu^2_{S_{\grave{\imath}}}} \le \dfrac{1-\mu^2_{S_{\grave{\imath},\min}}}{1+\mu^2_{S_{\grave{\imath},\min}}}$, for all i. Let $\tilde{I} = \left(\tilde{I}_1, \tilde{I}_2, ..., \tilde{I}_{\tilde{n}}\right)^T$ is the associated weighted vector of $Gz_{\grave{\imath}}$ such that $\tilde{I}_{\grave{\imath}} \in [0,1], (\grave{\imath} = 1, 2, ..., \tilde{n})$ and $\sum_{\grave{\imath}=1}^{\tilde{n}} \tilde{I}_{\grave{\imath}} = 1$; then for all i, we have

$$\left(\frac{1-\mu^2_{S_{\grave{\imath},\max}}}{1+\mu^2_{S_{\grave{\imath},\max}}}\right)^{\tilde{I}_{\grave{\imath}}} \le \left(\frac{1-\mu^2_{S_{\grave{\imath}}}}{1+\mu^2_{S_{\grave{\imath}}}}\right)^{\tilde{I}_{\grave{\imath}}} \le \left(\frac{1-\mu^2_{S_{\grave{\imath},\min}}}{1+\mu^2_{S_{\grave{\imath},\min}}}\right)^{\tilde{I}_{\grave{\imath}}}$$

Thus

$$\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\frac{1-\mu^2_{S_{\grave{\imath},\max}}}{1+\mu^2_{S_{\grave{\imath},\max}}}\right)^{\tilde{I}_{\grave{\imath}}} \le \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\frac{1-\mu^2_{S_{\grave{\imath}}}}{1+\mu^2_{S_{\grave{\imath}}}}\right)^{\tilde{I}_{\grave{\imath}}} \le \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\frac{1-\mu^2_{S_{\grave{\imath},\min}}}{1+\mu^2_{S_{\grave{\imath},\min}}}\right)^{\tilde{I}_{\grave{\imath}}}$$

$$\Leftrightarrow \left(\frac{1-\mu^2_{S_{\grave{\imath},\max}}}{1+\mu^2_{S_{\grave{\imath},\max}}}\right)^{\sum_{\grave{\imath}=1}^{\tilde{n}}\tilde{I}_{\grave{\imath}}} \le \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\frac{1-\mu^2_{S_{\grave{\imath}}}}{1+\mu^2_{S_{\grave{\imath}}}}\right)^{\tilde{I}_{\grave{\imath}}}$$

$$\le \left(\frac{1-\mu^2_{S_{\grave{\imath},\min}}}{1+\mu^2_{S_{\grave{\imath},\min}}}\right)^{\sum_{\grave{\imath}=1}^{\tilde{n}}\tilde{I}_{\grave{\imath}}}$$

$$\Leftrightarrow \left(\frac{1-\mu^2_{S_{\grave{\imath},\max}}}{1+\mu^2_{S_{\grave{\imath},\max}}}\right) \le \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\frac{1-\mu^2_{S_{\grave{\imath}}}}{1+\mu^2_{S_{\grave{\imath}}}}\right)^{\tilde{I}_{\grave{\imath}}} \le \left(\frac{1-\mu^2_{S_{\grave{\imath},\min}}}{1+\mu^2_{S_{\grave{\imath},\min}}}\right)$$

$$\Leftrightarrow \left(\frac{2}{1+\mu^2_{S_{\grave{\imath},\max}}}\right) \le 1 + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\frac{1-\mu^2_{S_{\grave{\imath}}}}{1+\mu^2_{S_{\grave{\imath}}}}\right)^{\tilde{I}_{\grave{\imath}}} \le \left(\frac{2}{1+\mu^2_{S_{\grave{\imath},\min}}}\right)$$

$$\Leftrightarrow \left(\frac{1+\mu^2_{S_{\grave{\imath},\min}}}{2}\right) \le \frac{1}{1 + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\frac{1-\mu^2_{S_{\grave{\imath}}}}{1+\mu^2_{S_{\grave{\imath}}}}\right)^{\tilde{I}_{\grave{\imath}}}} \le \left(\frac{1+\mu^2_{S_{\grave{\imath},\max}}}{2}\right)$$

$$\Leftrightarrow 1+\mu^2_{S_{\grave{\imath},\min}} \le \frac{2}{1 + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\frac{1-\mu^2_{S_{\grave{\imath}}}}{1+\mu^2_{S_{\grave{\imath}}}}\right)^{\tilde{I}_{\grave{\imath}}}} \le 1+\mu^2_{S_{\grave{\imath},\max}}$$

$$\Leftrightarrow \mu^2_{S_{\grave{\imath},\min}} \le \frac{2}{1 + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\frac{1-\mu^2_{S_{\grave{\imath}}}}{1+\mu^2_{S_{\grave{\imath}}}}\right)^{\tilde{I}_{\grave{\imath}}}} - 1 \le \mu^2_{S_{\grave{\imath},\max}}$$

$$\Leftrightarrow \mu^2_{S_{\grave{\imath},\min}} \le \frac{\left(\Pi_{\grave{\imath}=1}^{\tilde{n}} 1+\mu^2_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}} - \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu^2_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}}{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1+\mu^2_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu^2_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}} - 1 \le \mu^2_{S_{\grave{\imath},\max}}$$

$$\Leftrightarrow \mu^2_{S_{\grave{\imath}},\min} \leq \sqrt{\frac{\left(\Pi_{\grave{\imath}=1}^{\tilde{n}} 1+\mu^2_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}} - \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu^2_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}}{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1+\mu^2_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu^2_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}}} - 1 \leq \mu^2_{S_{\grave{\imath}},\max}$$

Similarly,

$$\mu^2_{T_{\grave{\imath}},\min} \leq \sqrt{\frac{\left(\Pi_{\grave{\imath}=1}^{\tilde{n}} 1+\mu^2_{T_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}} - \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu^2_{T_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}}{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1+\mu^2_{T_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(1-\mu^2_{T_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}}} - 1 \leq \mu^2_{T_{\grave{\imath}},\max}$$

On the other hand, let $g(y) = \frac{2-y}{y}$, $y \in (0,1]$; then $g'(y) = \frac{-2}{y^2} < 0$, i.e., $g(y)$ is a decreasing function on (0, 1]. Since $\nu^2_{S_{\grave{\imath}},\min} \leq \nu^2_{S_{\grave{\imath}}} \leq \nu^2_{S_{\grave{\imath}},\max}$, for all $\grave{\imath} = 1, 2, ..., \tilde{n}$, then $g\left(\nu^2_{S_{\grave{\imath}},\max}\right) \leq g\left(\nu^2_{S_{\grave{\imath}}}\right) \leq g\left(\nu^2_{S_{\grave{\imath}},\min}\right)$ for all i, that is, $\frac{2-\nu^2_{S_{\grave{\imath}},\max}}{\nu^2_{S_{\grave{\imath}},\max}} \leq \frac{2-\nu^2_{S_{\grave{\imath}}}}{\nu^2_{S_{\grave{\imath}}}} \leq \frac{2-\nu^2_{S_{\grave{\imath}},\min}}{\nu^2_{S_{\grave{\imath}},\min}}$. Then, we have

$$\left(\frac{2-\nu^2_{S_{\grave{\imath}},\max}}{\nu^2_{S_{\grave{\imath}},\max}}\right)^{\tilde{I}_{\grave{\imath}}} \leq \left(\frac{2-\nu^2_{S_{\grave{\imath}}}}{\nu^2_{S_{\grave{\imath}}}}\right)^{\tilde{I}_{\grave{\imath}}} \leq \left(\frac{2-\nu^2_{S_{\grave{\imath}},\min}}{\nu^2_{S_{\grave{\imath}},\min}}\right)^{\tilde{I}_{\grave{\imath}}}$$

Thus,

$$\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\frac{2-\nu^2_{S_{\grave{\imath}},\max}}{\nu^2_{S_{\grave{\imath}},\max}}\right)^{\tilde{I}_{\grave{\imath}}} \leq \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\frac{2-\nu^2_{S_{\grave{\imath}}}}{\nu^2_{S_{\grave{\imath}}}}\right)^{\tilde{I}_{\grave{\imath}}} \leq \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\frac{2-\nu^2_{S_{\grave{\imath}},\min}}{\nu^2_{S_{\grave{\imath}},\min}}\right)^{\tilde{I}_{\grave{\imath}}}$$

$$\Longrightarrow \frac{2-\nu^2_{S_{\grave{\imath}},\max}}{\nu^2_{S_{\grave{\imath}},\max}} \leq \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\frac{2-\nu^2_{S_{\grave{\imath}}}}{\nu^2_{S_{\grave{\imath}}}}\right)^{\tilde{I}_{\grave{\imath}}} \leq \frac{2-\nu^2_{S_{\grave{\imath}},\min}}{\nu^2_{S_{\grave{\imath}},\min}}$$

$$\Longrightarrow \frac{2}{\nu^2_{S_{\grave{\imath}},\max}} \leq \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\frac{2-\nu^2_{S_{\grave{\imath}}}}{\nu^2_{S_{\grave{\imath}}}}\right)^{\tilde{I}_{\grave{\imath}}} + 1 \leq \frac{2}{\nu^2_{S_{\grave{\imath}},\min}}$$

$$\Longrightarrow \frac{\nu^2_{S_{\grave{\imath}},\min}}{2} \leq \frac{1}{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\frac{2-\nu^2_{S_{\grave{\imath}}}}{\nu^2_{S_{\grave{\imath}}}}\right)^{\tilde{I}_{\grave{\imath}}} + 1} \leq \frac{\nu^2_{S_{\grave{\imath}},\max}}{2}$$

$$\Longrightarrow \nu^2_{S_{\grave{\imath}},\min} \leq \frac{2}{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\frac{2-\nu^2_{S_{\grave{\imath}}}}{\nu^2_{S_{\grave{\imath}}}}\right)^{\tilde{I}_{\grave{\imath}}} + 1} \leq \nu^2_{S_{\grave{\imath}},\max}$$

$$\Longrightarrow \nu^2_{S_{\grave{\imath}},\min} \leq \frac{2\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\nu_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}}{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(2-\nu^2_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\nu^2_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}} \leq \nu^2_{S_{\grave{\imath}},\max}$$

i.e,

$$\Longrightarrow \nu_{S_{\grave{\imath}},\min} \leq \frac{\sqrt{2}\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\nu_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}}{\sqrt{\Pi_{\grave{\imath}=1}^{\tilde{n}}\left(2-\nu^2_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}} + \Pi_{\grave{\imath}=1}^{\tilde{n}}\left(\nu^2_{S_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}}} \leq \nu_{S_{\grave{\imath}},\max}$$

Similarly,

$$\nu_{T_{\grave{\imath}},\min} \leq \frac{\sqrt{2}\Pi_{i=1}^{\tilde{n}}\left(\nu_{T_{\grave{\imath}}}\right)^{\tilde{I}_{\grave{\imath}}}}{\sqrt{\Pi_{i=1}^{\tilde{n}}\left(2-\nu_{T_{\grave{\imath}}}^{2}\right)^{\tilde{I}_{\grave{\imath}}}+\Pi_{i=1}^{\tilde{n}}\left(\nu_{T_{\grave{\imath}}}^{2}\right)^{\tilde{I}_{\grave{\imath}}}}} \leq \nu_{S_{\grave{\imath}},\max}$$

Let $PFZNEWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \langle(\mu_{G_Z}, \mu_{G_Z}), (\nu_{G_Z}, \nu_{G_Z})\rangle$, then we have the following results, respectively;

$\mu^2_{S_{\grave{\imath}},\min} \leq \mu_{G_Z} \leq \mu^2_{S_{\grave{\imath}},\max}$, $\mu^2_{T_{\grave{\imath}},\min} \leq \mu_{G_Z} \leq \mu^2_{T_{\grave{\imath}},\max}$,

and

$\nu^2_{S_{\grave{\imath}},\min} \leq \nu_{G_Z} \leq \nu^2_{S_{\grave{\imath}},\max}$, $\nu^2_{T_{\grave{\imath}},\min} \leq \nu_{G_Z} \leq \nu^2_{T_{\grave{\imath}},\max}$.

where $\mu_{S_{\grave{\imath}},\min} = \min_{\grave{\imath}}(\mu_{S_{\grave{\imath}}}), \mu_{S_{\grave{\imath}},\max} = \max_{\grave{\imath}}(\mu_{S_{\grave{\imath}}}), \mu_{T_{\grave{\imath}},\min} = \min_{\grave{\imath}}(\mu_{T_{\grave{\imath}}}), \mu_{T_{\grave{\imath}},\max} = \max_{\grave{\imath}}(\mu_{T_{\grave{\imath}}}), \nu_{S_{\grave{\imath}},\min} = \min_{\grave{\imath}}(\nu_{S_{\grave{\imath}}}), \nu_{S_{\grave{\imath}},\max} = \max_{\grave{\imath}}(\nu_{S_{\grave{\imath}}})$, and $\nu_{S_{\grave{\imath}},\min} = \min_{\grave{\imath}}(\nu_{S_{\grave{\imath}}}), \nu_{S_{\grave{\imath}},\max} = \max_{\grave{\imath}}(\nu_{S_{\grave{\imath}}})$. So, $S(G_Z) = \mu^2_{G_Z} - \nu^2_{G_Z} = \mu^2_{\max} - \nu^2_{\min} = S(G^+_Z)$ and $S(G_Z) = \mu^2_{G_Z} - \nu^2_{G_Z} = \mu^2_{\min} - \nu^2_{\max} = S(G^-_Z)$. If $S(G_Z) < S(G^+_Z)$ and $S(G_Z) > S(G^-_Z)$, then by order relation between two *PFZNs*, we have

$G^-_Z \leq PFZNEWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) \leq G^+_Z$

the proof (3) is similar as (2).

6 Einstein Ordered Weighted Averaging Operator, Einstein Weighted Geometric Averaging, and Order Weighted Geometric Averaging Aggregating Operators under Pythagorean Fuzzy Z-Numbers

This part is divided into two subsections: in Subsection 6.1 we offered the PFZNEOWA operator, and in Subsection 6.2 we established the PFZNEWGA and PFZNEOWGA.

6.1 Pythagorean Fuzzy Z-Number Einstein Ordered Weighted Averaging Operator

In this part, we present the PFZNEOWA operator, which combines the OWA concept with the PFZNEWA operator. The OWA concept is not taken into consideration by the PFZNEWA operator throughout the information fusion process. The PFZNEOWA operator is discussed in the sections that follow, first with a brief introduction and then with a numerical example.

Definition 6.1. Let $Gz_{\grave{\imath}} = \left\{(\mu_{S_{\alpha_{\grave{\imath}}}}, \mu_{T_{\alpha_{\grave{\imath}}}})(\nu_{S_{\alpha_{\grave{\imath}}}}, \nu_{T_{\alpha_{\grave{\imath}}}})\right\}$ $(\grave{\imath} = 1, 2, ..., \tilde{n})$ be a collection of *PFZNs*, and $\tilde{I} = (\grave{\imath} = 1, 2, ..., \tilde{n})^T$ is the associated weighted vector of $Gz_{\grave{\imath}}(\grave{\imath} = 1, 2, ..., \tilde{n})$ with $0 \leq \tilde{I}_{\grave{\imath}} \leq 1$ and $\sum_{i=1}^{\tilde{n}} \tilde{I}_{\grave{\imath}} = 1$; then the PFZN Einstein order weighted averaging aggregation operator of dimension $\tilde{n}$ is a mapping $PFZNEOWA : \varpi^{\tilde{n}} \longrightarrow \varpi$, and

$$PFZNEOWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \tilde{I}_{1\cdot\epsilon}Gz_{\sigma(1)} \oplus_\epsilon \tilde{I}_{2\cdot\epsilon}Gz_{\sigma(2)} \oplus_\epsilon ... \oplus_\epsilon \tilde{I}_{\tilde{n}\cdot\epsilon}Gz_{\sigma(\tilde{n})}$$

where ϖ is the set of all *PFZNs*, $(\sigma(1), \sigma(2), ..., \sigma(\tilde{n}))$ is a permutation of $(\grave{\imath} = 1, 2, ..., \tilde{n})$ such that $Gz_{\sigma(\grave{\imath}-1)} \geq Gz_{\sigma(\grave{\imath})}$ for all i.

The *PFZNEOWA* operator can be changed into the following form employing the *PFZNs*' Einstein operational rules.

Theorem 6.1. Let $Gz_{\grave{\imath}} = \left\{(\mu_{S_{\alpha_{\grave{\imath}}}}, \mu_{T_{\alpha_{\grave{\imath}}}})(\nu_{S_{\alpha_{\grave{\imath}}}}, \nu_{T_{\alpha_{\grave{\imath}}}})\right\}$ $(\grave{\imath} = 1, 2, ..., \tilde{n})$ be a collection of *PFZNs*, then their aggregated value by employing the *PFZNEOWA* operator is also a *PFZN* and

$$PFZNEOWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \left[\begin{array}{c} \left(\begin{array}{c} \sqrt{\frac{\left(\Pi_{\grave{i}=1}^{\tilde{n}} 1+\mu_{S_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}} - \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{S_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}}}{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\mu_{S_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{S_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}}}}, \\ \sqrt{\frac{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\mu_{T_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}} . - \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{T_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}}}{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\mu_{T_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{T_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}}}}, \end{array} \right), \\ \left(\begin{array}{c} \frac{\sqrt{2}\Pi_{\grave{i}=1}^{\tilde{n}}\left(\nu_{S_{\sigma(\grave{i})}}^{\tilde{I}_{\grave{i}}}\right)}{\sqrt{\Pi_{\grave{i}=1}^{\tilde{n}}\left(2-\nu_{S_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(\nu_{S_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}}}}, \\ \frac{\sqrt{2}\Pi_{\grave{i}=1}^{\tilde{n}}\left(\nu_{T_{\sigma(\grave{i})}}^{\tilde{I}_{\grave{i}}}\right)}{\sqrt{\Pi_{\grave{i}=1}^{\tilde{n}}\left(2-\nu_{T_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(\nu_{T_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}}}} \end{array} \right) \end{array} \right] \quad (4)$$

$(\sigma(1), \sigma(2), ..., \sigma(\tilde{n}))$ is a permutation of $(\grave{i} = 1, 2, ..., \tilde{n})$ such that $Gz_{\sigma(\grave{i}-1)} \geq Gz_{\sigma(\grave{i})}$ for all i, $\tilde{I} = (\grave{i} = 1, 2, ..., \tilde{n})^T$ is the associated weighted vector of $Gz_{\grave{i}}(\grave{i} = 1, 2, ..., \tilde{n})$ with $0 \leq \tilde{I}_{\grave{i}} \leq 1$ and $\sum_{\grave{i}=1}^{\tilde{n}} \tilde{I}_{\grave{i}} = 1$.

Proof. This theorem is included in this discussion because the evidence for it is quite similar to the proof for Theorem 5.1.

Corollary 2. *The PFZNEOWA operator and PFZNOWA operator has the underlying links*:

$PFZNEOWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) \leq PFZNOWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}})$

where $Gz_{\grave{i}} = \left\{(\mu_{S_{\alpha_{\grave{i}}}}, \mu_{T_{\alpha_{\grave{i}}}})(\nu_{S_{\alpha_{\grave{i}}}}, \nu_{T_{\alpha_{\grave{i}}}})\right\}$ $(\grave{i} = 1, 2, ..., \tilde{n})$ *be a collection of PFZNs, and* $\tilde{I} = (\grave{i} = 1, 2, ..., \tilde{n})^T$ *is the associated weighted vector of* $Gz_{\grave{i}}(\grave{i} = 1, 2, ..., \tilde{n})$ *with* $0 \leq \tilde{I}_{\grave{i}} \leq 1$ *and* $\sum_{\grave{i}=1}^{\tilde{n}} \tilde{I}_{\grave{i}} = 1$.

Proof. The concept of verification is quite similar to that of Corollary 1.

Example 3. *Let* $Gz_1 = (\mu_{S_1}, \mu_{T_1}), (\nu_{S_1}, \nu_{T_1}) = \{(06, 0.4), (0.5, 0.3)\}$, $Gz_2 = (\mu_{S_2}, \mu_{T_2}), (\nu_{S_2}, \nu_{T_2}) = \{(0.7, 0.1), (0.3, 0.5)\}$, *and* $Gz_3 = (\mu_{S_3}, \mu_{T_3}), (\nu_{S_3}, \nu_{T_3}) = \{(0.4, 0.1), (0.8, 0.2)\}$ *be three PFZNs and* $\tilde{I} = (0.2, 0.5, 0.3)^T$ *be the weight vector of* $Gz_{\grave{i}}(\grave{i} = 1, 2, 3)$. *Then*

$$PFZNEOWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \left[\begin{array}{c} \left(\begin{array}{c} \sqrt{\frac{\left(\Pi_{\grave{i}=1}^{\tilde{n}} 1+\mu_{S_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}} - \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{S_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}}}{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\mu_{S_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{S_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}}}}, \\ \sqrt{\frac{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\mu_{T_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}} . - \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{T_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}}}{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\mu_{T_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{T_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}}}}, \end{array} \right), \\ \left(\begin{array}{c} \frac{\sqrt{2}\Pi_{\grave{i}=1}^{\tilde{n}}\left(\nu_{S_{\sigma(\grave{i})}}^{\tilde{I}_{\grave{i}}}\right)}{\sqrt{\Pi_{\grave{i}=1}^{\tilde{n}}\left(2-\nu_{S_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(\nu_{S_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}}}}, \\ \frac{\sqrt{2}\Pi_{\grave{i}=1}^{\tilde{n}}\left(\nu_{T_{\sigma(\grave{i})}}^{\tilde{I}_{\grave{i}}}\right)}{\sqrt{\Pi_{\grave{i}=1}^{\tilde{n}}\left(2-\nu_{T_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(\nu_{T_{\sigma(\grave{i})}}^2\right)^{\tilde{I}_{\grave{i}}}}} \end{array} \right) \end{array} \right]$$

$$= [(0.610681, 0.200639), (0.294563, 0.221921)]$$

If we employ the PFZNOWA operator to aggregate the PFZNs $Gz_{\grave{\imath}}s$, then we obtain

$$PFZNOWA(\alpha_1, \alpha_2, ...\alpha_{\tilde{n}}) = \left\{ \begin{array}{c} \left(\sqrt{1 - \Pi_{i=1}^{\tilde{n}} \left(1 - \mu^2_{S\alpha_{\sigma(i)}}\right)^{\tilde{I}\grave{\imath}}}, \right. \\ \left. \sqrt{1 - \Pi_{i=1}^{\tilde{n}} \left(1 - \mu^2_{T_{\alpha_{\sigma(i)}}}\right)^{\tilde{I}\grave{\imath}}} \right), \\ \left(\Pi_{i=1}^{\tilde{n}} \nu^{\tilde{I}\grave{\imath}}_{S\alpha_{\sigma(i)}}, \Pi_{i=1}^{\tilde{n}} \nu^{\tilde{I}\grave{\imath}}_{T\alpha_{\sigma(i)}} \right) \end{array} \right\}$$

$$= \{(0.266926, 0.09527), (0.445945, 0.34294)\}$$

Similarly the PFZNEWA operator, the PFZNEOWA operator has some properties as follows.

Proposition 6.1. Let $Gz_{\grave{\imath}} = \left(\mu_{S_{\grave{\imath}}}, \mu_{T_{\grave{\imath}}}\right), \left(\nu_{S_{\grave{\imath}}}, \nu_{T_{\grave{\imath}}}\right) (\grave{\imath} = 1, 2, ..., \tilde{n})$ be a collections of *PFZNs* and $\tilde{I} = \left(\tilde{I}_1, \tilde{I}_2, ..., \tilde{I}_{\tilde{n}}\right)^T$ is the associated weighted vector of $Gz_{\grave{\imath}}$ such that $\tilde{I}_{\grave{\imath}} \in [0, 1], (\grave{\imath} = 1, 2, ..., \tilde{n})$ and $\sum_{\grave{\imath}=1}^{\tilde{n}} \tilde{I}_{\grave{\imath}} = 1$; then, we have the following:

(1) Idempotency: If all $Gz_{\grave{\imath}} (\grave{\imath} = 1, 2, ..., \tilde{n})$ are equal that is $Gz_{\grave{\imath}} = Gz$ for all i, then

$PFZNEOWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = Gz$

(2) Boundedness: Let $G^-z_{\grave{\imath}} = \langle \min_{\grave{\imath}} \left(\mu_{S_{\grave{\imath}}}, \mu_{T_{\grave{\imath}}}\right), \max_{\grave{\imath}} \left(\nu_{S_{\grave{\imath}}}, \nu_{T_{\grave{\imath}}}\right)\rangle$, $G^+z_{\grave{\imath}} = \langle \max_{\grave{\imath}} \left(\mu_{S_{\grave{\imath}}}, \mu_{T_{\grave{\imath}}}\right) \min_{\grave{\imath}} \left(\nu_{S_{\grave{\imath}}}, \nu_{T_{\grave{\imath}}}\right)\rangle$ then

$G^-z_{\grave{\imath}} \leq PFZNEOWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) \leq G^+z_{\grave{\imath}}$

(3) Monotonicity: Let $Gz_{\grave{\imath}} = \left(\mu_{S_{G\grave{\imath}}}, \mu_{T_{G\grave{\imath}}}\right), \left(\nu_{S_{G\grave{\imath}}}, \nu_{T_{G\grave{\imath}}}\right)$ and $Dz_{\grave{\imath}} = \left(\mu_{S_{D\grave{\imath}}}, \mu_{T_{D\grave{\imath}}}\right), \left(\nu_{S_{D\grave{\imath}}}, \nu_{T_{D\grave{\imath}}}\right)$ be collection of *PFZNs* and $Gz_{\grave{\imath}} \leq Dz_{\grave{\imath}}$, for all i, then

$PFZNEWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) \leq PFZNEWA(Dz_1, Dz_2, ..., Dz_{\tilde{n}})$

6.2 Pythagorean Fuzzy Z-Numbers Einstein Weighted Geometric and Order Weighted Geometric Aggregating Operators

With the use of Einstein operations, we will explore the PFZNs geometric and order geometric aggregating operators in this section.

Definition 6.2. Let $Gz_{\grave{\imath}} = \left\{\left(\mu_{S_{\grave{\imath}}}, \mu_{T_{\grave{\imath}}}\right), \left(\nu_{S_{\grave{\imath}}}, \nu_{T_{\grave{\imath}}}\right)\right\}, (\grave{\imath} = 1, 2, ..., \tilde{n})$ be a catalogue of *PFZNs* and $\tilde{I}_{\grave{\imath}}$ is the weight of $\tilde{I}_{\grave{\imath}} (\grave{\imath} = 1, 2, ..., \tilde{n})$ such that $\tilde{I}_{\grave{\imath}} \in [0, 1]$ and $\sum_{\grave{\imath}=1}^{\tilde{n}} \tilde{I}_{\grave{\imath}} = 1$ then, a *PFZNEGA* operator is a mapping of dimension n $PFZNEGA : \varpi^{\tilde{n}} \rightarrow \varpi$, and

$PFZNEGA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \Pi_{\grave{\imath}=1}^{\tilde{n}} G^{\tilde{I}_{\grave{\imath}}} z_{\grave{\imath}}$

where ϖ is the collection of all *PFZNs*.

Theorem 6.2. Let $Gz_{\grave{\imath}} = \left\{\left(\mu_{S_{\grave{\imath}}}, \mu_{T_{\grave{\imath}}}\right), \left(\nu_{S_{\grave{\imath}}}, \nu_{T_{\grave{\imath}}}\right)\right\}, (\grave{\imath} = 1, 2, ..., \tilde{n})$ be a catalogue of *PFZNs*, then the aggregated value are determined by

$$PFZNEGA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \begin{bmatrix} \left(\dfrac{\sqrt{2}\Pi_{\imath=1}^{\tilde{n}}\left(\mu_{S_\imath}^{\tilde{I}_\imath}\right)}{\sqrt{\Pi_{\imath=1}^{\tilde{n}}\left(2-\mu_{S_\imath}^2\right)^{\tilde{I}_\imath}+\Pi_{\imath=1}^{\tilde{n}}\left(\mu_{S_\imath}^2\right)^{\tilde{I}_\imath}}}, \dfrac{\sqrt{2}\Pi_{\imath=1}^{\tilde{n}}\left(\mu_{T_\imath}^{\tilde{I}_\imath}\right)}{\sqrt{\Pi_{\imath=1}^{\tilde{n}}\left(2-\mu_{T_\imath}^2\right)^{\tilde{I}_\imath}+\Pi_{\imath=1}^{\tilde{n}}\left(\mu_{T_\imath}^2\right)^{\tilde{I}_\imath}}} \right), \\ \left(\sqrt{\dfrac{\left(\Pi_{\imath=1}^{\tilde{n}}1+\nu_{S_\imath}^2\right)^{\tilde{I}_\imath}-\Pi_{\imath=1}^{\tilde{n}}\left(1-\nu_{S_\imath}^2\right)^{\tilde{I}_\imath}}{\Pi_{\imath=1}^{\tilde{n}}\left(1+\nu_{S_\imath}^2\right)^{\tilde{I}_\imath}+\Pi_{\imath=1}^{\tilde{n}}\left(1-\nu_{S_\imath}^2\right)^{\tilde{I}_\imath}}}, \sqrt{\dfrac{\Pi_{\imath=1}^{\tilde{n}}\left(1+\nu_{T_\imath}^2\right)^{\tilde{I}_\imath}.-\Pi_{\imath=1}^{\tilde{n}}\left(1-\nu_{T_\imath}^2\right)^{\tilde{I}_\imath}}{\Pi_{\imath=1}^{\tilde{n}}\left(1+\nu_{T_\imath}^2\right)^{\tilde{I}_\imath}+\Pi_{\imath=1}^{\tilde{n}}\left(1-\nu_{T_\imath}^2\right)^{\tilde{I}_\imath}}}, \right) \end{bmatrix} \quad (5)$$

where $\tilde{I}_\imath$ is the weight of $Gz_\imath(\imath = 1, 2, ..., \tilde{n})$ such that $\tilde{I}_\imath \in [0, 1]$ and $\sum_{\imath=1}^{\tilde{n}} \tilde{I}_\imath = 1$.

Proof. Verification is trivial, therefore it is valid here.

Theorem 6.3. Let $Gz_\imath = \left\{\left(\mu_{S_\imath}, \mu_{T_\imath}\right), \left(\nu_{S_\imath}, \nu_{T_\imath}\right)\right\}, (\imath = 1, 2, ..., \tilde{n})$ be a catalogue of *PFZNs*, then the aggregated value are determined by

$$PFZNEOGA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \begin{bmatrix} \left(\dfrac{\sqrt{2}\Pi_{\imath=1}^{\tilde{n}}\left(\mu_{S_{\sigma(\imath)}}^{\tilde{I}_\imath}\right)}{\sqrt{\Pi_{\imath=1}^{\tilde{n}}\left(2-\mu_{S_{\sigma(\imath)}}^2\right)^{\tilde{I}_\imath}+\Pi_{\imath=1}^{\tilde{n}}\left(\mu_{S_{\sigma(\imath)}}^2\right)^{\tilde{I}_\imath}}}, \dfrac{\sqrt{2}\Pi_{\imath=1}^{\tilde{n}}\left(\mu_{T_{\sigma(\imath)}}^{\tilde{I}_\imath}\right)}{\sqrt{\Pi_{\imath=1}^{\tilde{n}}\left(2-\mu_{T_{\sigma(\imath)}}^2\right)^{\tilde{I}_\imath}+\Pi_{\imath=1}^{\tilde{n}}\left(\mu_{T_{\sigma(\imath)}}^2\right)^{\tilde{I}_\imath}}} \right), \\ \left(\sqrt{\dfrac{\left(\Pi_{\imath=1}^{\tilde{n}}1+\nu_{S_{\sigma(\imath)}}^2\right)^{\tilde{I}_\imath}-\Pi_{\imath=1}^{\tilde{n}}\left(1-\nu_{S_{\sigma(\imath)}}^2\right)^{\tilde{I}_\imath}}{\Pi_{\imath=1}^{\tilde{n}}\left(1+\nu_{S_{\sigma(\imath)}}^2\right)^{\tilde{I}_\imath}+\Pi_{\imath=1}^{\tilde{n}}\left(1-\nu_{S_{\sigma(\imath)}}^2\right)^{\tilde{I}_\imath}}}, \sqrt{\dfrac{\Pi_{\imath=1}^{\tilde{n}}\left(1+\nu_{T_{\sigma(\imath)}}^2\right)^{\tilde{I}_\imath}.-\Pi_{\imath=1}^{\tilde{n}}\left(1-\nu_{T_{\sigma(\imath)}}^2\right)^{\tilde{I}_\imath}}{\Pi_{\imath=1}^{\tilde{n}}\left(1+\nu_{T_{\sigma(\imath)}}^2\right)^{\tilde{I}_\imath}+\Pi_{\imath=1}^{\tilde{n}}\left(1-\nu_{T_{\sigma(\imath)}}^2\right)^{\tilde{I}_\imath}}}, \right) \end{bmatrix} \quad (6)$$

where $\tilde{I}_\imath$ is the weight of $Gz_\imath(\imath = 1, 2, ..., \tilde{n})$ such that $\tilde{I}_\imath \in [0, 1]$ and $\sum_{\imath=1}^{\tilde{n}} \tilde{I}_\imath = 1$.

Proof. It is omitted here because the proof is trivial.

7 Multiple Aggregated Operators with Example

We developed MCDM for the PFZNEWA, PFZNEOWA, PFZNEWGA, and PFZNEOWGA aggregating operators. An example is given to demonstrate the usefulness of these aggregating operators, and a comparison of the these aggregating operators is included as well.

7.1 Multi-Criteria Decision-Making Approach Using the PFZNEWA, PFZNEOWA, PFZNEWGA, and PFZNEOWGA Operators

MCDM is a method used to make decisions in situations where there are multiple conflicting criteria. The PFZNEWA, PFZNEOWA, PFZNEWGA, and PFZNEOWGA operators are four types of fuzzy aggregation operators that can be used in MCDM. To use these operators in MCDM, the decision maker needs to first define the set of criteria, which can be represented as a set of Pythagorean fuzzy Z-numbers. The PFZNEWA, PFZNEOWA, PFZNEWGA, and PFZNEOWGA operators can then be used to aggregate the criteria and produce a final decision. The choice of which operator to use depends on the decision-maker's preferences and the characteristics of the criteria. In order to handle MCDM difficulties, this part develops an MCDM methodology using assessment data for both Pythagorean values and Pythagorean reliability measures. This approach relates to PFZNEWA, PFZNEOWA, PFZNEWGA, and PFZNEOWGA operators and the score function. The weight $\tilde{I}_i$, are define as the weight vector $\tilde{I}_i = (\tilde{I}_1, \tilde{I}_2, ..., \tilde{I}_{\tilde{n}})$. Consider $PFZNs$ $Gz_{jd} = \left\{\left(\mu_{S_{jd}}, \mu_{T_{jd}}\right), \left(\nu_{S_{jd}}, \nu_{T_{jd}}\right)\right\}$ $(jd = 1, 2, ..., \tilde{n})$, where $\mu_{S_{jd}}, \mu_{T_{jd}} \in [0, 1]$ and $\nu_{S_{jd}}, \nu_{T_{jd}} \in [0, 1]$. Now the decision matrix of $PFZN$ is determined as $Gz_{jd} = \left(Gz_{jd}\right)_{r\times m}$. The decision process is defined in MCDM problem as following:

Step 1: Using PFZNEWA, PFZNEOWA, PFZNEWGA, and PFZNEOWGA operators, we defined the PFZNs in Equations, i.e., Eqs. (3)–(6), respectively.

Step 2: Using Eq. (1), the score values of $J(Gz_i)(i = 1, 2, ..., n)$ are calculated.

Step 3: The best option among the rated options is chosen based on the score values.

Step 4: End.

7.2 Numerical Illustration

To illustrate the relevance and efficacy of the suggested MCDM technique with PZN information, this section gives an example concerning the challenge of agriculture fields such as productivity profitability and energy use efficiency of Wheat crop in different tillage system. Energy inputs estimations were relies on the human labor requirement, use of different types of machinery and quantity of materials, energy calculation was completed using different input and output energy equivalent. Energy efficiency in agricultural production is becoming more popular due to rising fuel prices. The agriculture sector, like other sectors, relied on energy sources like electricity and fossil fuels to produce more food than the growing population demanded. The expert panel provides a set of 5 alternatives $Q = \{Q_1, Q_2, Q_3, Q_4, Q_5\}$ from potential agriculture partner. The three criteria's weight vector is written as $(0.2, 0.5, 0.3)$ to denote their relative importance.

Seed energy: was calculated by the multiplication of quantity (Qs) of seed (kg ha^{-1}) used at the time of sowing with the amount of energy stored (SE) in each seed unit.

Total fuel energy: was quantify using volumetric method such as tillage, sowing, harvesting and threshing operations it was calculated by the multiplication of quantity of diesel (Qd) with the energy present (TE) in each liter.

Human labor energy: is required for individually in each tillage, fertilizer, irrigation, chemicals, harvesting and threshing operations during crop growth period. It was calculated by multiplication of total hours (Hd) per day needed with the human energy used (HE) per hour.

Irrigation energy: which applied to field during crop growth period measured with water flow meter. It was calculated by multiplication of total quantity of water (Qw) to field with the water energy (IE) use during irrigations As a result, $PFZN$ decision matrix can be used to create all $PFZNs$ in Table 1.

Table 1: $Gz = (Gz_{\grave{\imath}d})_{5\times 3}$

Tillage methods	(SE, TF, HE, IE)	(SE, TF, HE, IE)	(SE, TF, HE, IE)
RTB (Q_1)	$\langle(0.6, 0.4), (0.5, 0.3)\rangle$	$\langle(0.7, 0.1), (0.3, 0.5)\rangle$	$\langle(0.4, 0.1), (0.8, 0.2)\rangle$
ZTB (Q_2)	$\langle(0.3, 0.1), (0.4, 0.5)\rangle$	$\langle(0.4, 0.3), (0.6, 0.1)\rangle$	$\langle(0.2, 0.7), (0.1, 0.3)\rangle$
RT (Q_3)	$\langle(0.2, 0.8), (0.4, 0.1)\rangle$	$\langle(0.6, 0.1), (0.6, 0.2)\rangle$	$\langle(0.3, 0.5), (0.6, 0.2)\rangle$
CT (Q_4)	$\langle(0.5, 0.6), (0.7, 0.2)\rangle$	$\langle(0.3, 0.6), (0.4, 0.1)\rangle$	$\langle(0.6, 0.3), (0.1, 0.4)\rangle$
CTB (Q_5)	$\langle(0.3, 0.4), (0.1, 0.5)\rangle$	$\langle(0.6, 0.1), (0.7, 0.2)\rangle$	$\langle(0.5, 0.4), (0.6, 0.1)\rangle$

By using PFZNEWA operator: Now, we can apply *PFZNEWA* on Table 1.

Step 1: To find PFZNs $G_{Z_{\grave{\imath}}}$ $(\grave{\imath} = 1, 2, 3, 4, 5)$ using Eq. (3) is defined as

$$Gz_1 = \{(0.61068082, 0.200638579), (0.294562982, 0.221921072)\},$$
$$Gz_2 = \{(0.332111451, 0.452762799), (0.217490322, 0.12872272)\},$$
$$Gz_3 = \{(0.468903966, 0.478663074), (0.350640262, 0.115056262)\},$$
$$Gz_4 = \{(0.45416745, 0.531936912), (0.198209476, 0.116587122)\},$$
$$Gz_5 = \{(0.525153511, 0.292377101), (0.306654337, 0.130166267)\}.$$

The calculation step for Table 1 PFZNEWA operator for the first step Gz_1 can be calculated as i.e.,

$$\Pi_{\grave{\imath}=1}^{3}(1+\mu_{S_{\grave{\imath}}}^{2})^{\tilde{I}_{\grave{\imath}}} = (1.36)^{.2} \times (1.49)^{.5} \times (1.16)^{.3} = 1.357182692$$

$$\Pi_{\grave{\imath}=1}^{3}(1-\mu_{S_{\grave{\imath}}}^{2})^{\tilde{I}_{\grave{\imath}}} = (0.64)^{.2} \times (0.51)^{.5} \times (0.84)^{.3} = 0.619876067$$

$$\Pi_{\grave{\imath}=1}^{3}\left(1+\mu_{T_{\grave{\imath}}}^{2}\right)^{\tilde{I}_{\grave{\imath}}} = (1.16)^{.2} \times (1.01)^{.5} \times (1.01)^{.3} = 1.038361786$$

$$\Pi_{\grave{\imath}=1}^{3}\left(1+\mu_{T_{\grave{\imath}}}^{2}\right)^{\tilde{I}_{\grave{\imath}}} = (0.84)^{.2} \times (0.99)^{.5} \times (0.99)^{.3} = 0.9579967$$

$$\Pi_{\grave{\imath}=1}^{3}\left(\nu_{S_{\grave{\imath}}}^{2}\right)^{\tilde{I}_{\grave{\imath}}} = (0.5)^{.2} \times (0.3)^{.5} \times (0.8)^{.3} = 0.445945334$$

$$\Pi_{\grave{\imath}=1}^{3}\left(2-\nu_{S_{\grave{\imath}}}^{2}\right)^{\tilde{I}_{\grave{\imath}}} = (1.75)^{.2} \times (1.91)^{.5} \times (1.36)^{.3} = 1.695063475.$$

$$\Pi_{\grave{\imath}=1}^{3}\left(\nu_{T_{\grave{\imath}}}^{2}\right)^{\tilde{I}_{\grave{\imath}}} = (0.3)^{.2} \times (0.5)^{.5} \times (0.2)^{.3} = 0.342940086$$

$$\Pi_{\grave{\imath}=1}^{3}\left(2-\nu_{T_{\grave{\imath}}}^{2}\right)^{\tilde{I}_{\grave{\imath}}} = (1.91)^{.2} \times (1.75)^{.5} \times (1.96)^{.3} = 1.842478884$$

Thus

$$PFZNEWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \left[\begin{array}{c} \left(\sqrt{\frac{\left(\Pi_{\grave{i}=1}^{\tilde{n}} 1+\mu_{S_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}} - \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}}}{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\mu_{S_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{S_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}}}}, \sqrt{\frac{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\mu_{T_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}} .- \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{T_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}}}{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\mu_{T_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{T_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}}}}, \right), \\ \left(\frac{\sqrt{2}\Pi_{\grave{i}=1}^{\tilde{n}}\left(\nu_{S_{\grave{i}}}^{\tilde{I}_{\grave{i}}}\right)}{\sqrt{\Pi_{\grave{i}=1}^{\tilde{n}}\left(2-\nu_{S_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(\nu_{S_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}}}}, \frac{\sqrt{2}\Pi_{\grave{i}=1}^{\tilde{n}}\left(\nu_{T_{\grave{i}}}^{\tilde{I}_{\grave{i}}}\right)}{\sqrt{\Pi_{\grave{i}=1}^{\tilde{n}}\left(2-\nu_{T_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(\nu_{T_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}}}} \right) \end{array} \right],$$

$$= \{(0.61068082, 0.200638579), (0.294562982, 0.221921072)\}.$$

Step 2: The score values $J(Gz_d)$ of $PFZNEWA$ for the alternatives $Q_j = \{1, 2, 3, 4, 5\}$ are given below:

$$J(Gz_1) = 0.5285782,\ J(Gz_2) = 0.561185882,\ J(Gz_3) = 0.592051828,$$
$$J(Gz_4) = 0.609239879,\ J(Gz_5) = 0.556813405.$$

Step 3: According to the score values $J(Gz_4) \geq J(Gz_3) \geq J(Gz_2) \geq J(Gz_5) \geq J(Gz_1)$, the five alternatives are ranked as $Q_4 \geq Q_3 \geq Q_2 \geq Q_5 \geq Q_1$. Hence the best supplier is Q_4.

Here we first order the given matrix with the help of score function then the original matrix becomes: **By using PFZNEOWA operator:** Now, we can apply $PFZNEOWA$ on Table 1.

Step 1: To find $PFZNs$ $G_{Z\grave{i}}$ $(\grave{i} = 1, 2, 3, 4, 5)$ using Eq. (3) is defined as:

$$Gz_1 = \{(0.61068082, 0.200638579), (0.294562982, 0.221921072)\},$$
$$Gz_2 = \{(0.339494347, 0.392551733), (0.245832943, 0.135896766)\},$$
$$Gz_3 = \{(0.405806341, 0.524471713), (0.350640262, 0.115056262)\},$$
$$Gz_4 = \{(0.45416745, 0.531936912), (0.198209476, 0.116587122)\},$$
$$Gz_5 = \{(0.45416745, 0.339784731), (0.179706884, 0.18252723)\}.$$

Step 2: The score values $J(Gz_d)$ of $PFZNEOWA$ for the alternatives $Q_j = \{1, 2, 3, 4, 5\}$ are given below:

$$J(Gz_1) = 0.5285782, J(Gz_2) = 0.549930596, J(Gz_3) = 0.586245294,$$
$$J(Gz_4) = 0.609239879, J(Gz_5) = 0.560758883.$$

Step 3: According to the score values $J(Gz_4) \geq J(Gz_3) \geq J(Gz_5) \geq J(Gz_2) \geq J(Gz_1)$, the five alternatives are ranked as $Q_4 \geq Q_3 \geq Q_5 \geq Q_2 \geq Q_1$. Hence the best supplier is Q_4. Now we can apply $PFZNEWGA$, in MCDM problem can be solved using the invented MCDM strategy using the $PFZNEWGA$ operator, which is illustrated by using the accompanying decision-making procedure:

By using PFZNEWGA operator: Now, we can apply *PFZNEWGA* on Table 1.

Step 1: The total number of $PFZNGz_j (j = 1, 2, 3, 4, 5)$ are obtained as follows:

$$Gz_1 = \{(0.368418396, 0.089242623), (0.559293884, 0.39555748)\},$$

$$Gz_2 = \{(0.197536032, 0.208601424), (0.469106778, 0.288036181)\},$$

$$Gz_3 = \{(0.254931725, 0.171976027), (0.567378715, 0.184401829)\},$$

$$Gz_4 = \{(0.262715535, 0.312701971), (0.434303643, 0.247675851)\}.$$

$$Gz_5 = \{(0.31523819, 0.133824743), (0.604477583, 00.271952294)\}.$$

Step 2: The score values $J(Gz_d)$ of *PFZNEWGA* for the corresponding $Q_i = \{1, 2, 3, 4, 5\}$ are given below:

$$J(Gz_1) = 0.405822872, J(Gz_2) = 0.453043286, J(Gz_3) = 0.469608236,$$

$$J(Gz_4) = 0.487292571, J(Gz_5) = 0.438898802.$$

Step 3: According to the score values $J(Gz_4) \geq J(Gz_3) \geq J(Gz_2) \geq J(Gz_5) \geq J(Gz_1)$, the five alternatives are ranked as $Q_4 \geq Q_3 \geq Q_2 \geq Q_5 \geq Q_1$. Hence the best supplier is Q_4. Now we can apply *PFZNEOWGA*, in MCDM problem can be solved using the invented MCDM strategy using the *PFZNEOWGA* operator, which is illustrated by using the accompanying decision-making procedure:

By using PFZNEOWGA operator: Now, we can apply *PFZNEOWGA* on Table 1.

Step 1: The total number of $PFZN\ Gz_j (j = 1, 2, 3, 4, 5)$ are obtained as follows:

$$Gz_1 = \{(0.368418396, 0.089242623), (0.499927467, 0.39555748)\},$$

$$Gz_2 = \{(0.204980379, 0.17209745), (0.494956732, 0.315188962)\},$$

$$Gz_3 = \{(0.221479946, 0.231744777), (0.534224391, 0.184401829)\},$$

$$Gz_4 = \{(0.262715535, 0.312701971), (0.456694776, 0.247675851)\}.$$

$$Gz_5 = \{(0.262715535, 0.173775546), (0.430236559, 0.375195664)\}.$$

Step 2: The score values $J(Gz_d)$ of *PFZNEOWGA* for the alternatives $Q_j = \{1, 2, 3, 4, 5\}$ are given below:

$$J(Gz_1) = 0.417564287, J(Gz_2) = 0.439635851, J(Gz_3) = 0.476407433,$$

$$J(Gz_4) = 0.484519699, J(Gz_5) = 0.442115322.$$

Step 3: According to the score values $J(Gz_4) \geq J(Gz_3) \geq J(Gz_5) \geq J(Gz_2) \geq J(Gz_1)$, the five alternatives are ranked as $Q_4 \geq Q_3 \geq Q_5 \geq Q_2 \geq Q_1$. Hence the best supplier is Q_4. The four sorts of ranking orders indicated above for the five possibilities and the best option are the same, according to the developed MCDM technique, which uses the PFZNEWA, PFZNEOWA, PFZNEWGA, and PFZNEOWGA operators as well as the score function. The established MCDM technique thus functions.

8 Improved EDAS Method Based on Pythagorean Fuzzy Z-Number Einstein Aggregation Operators

MCDM is beneficial in locating solutions to a wide range of decision-making challenges that arise in the real world. Evaluation based on the distance from the average solution (EDAS) is an innovative and viable tool for MCDM tactic. The needed concerns, such as the average response being determined using weighted mean in this manner, may be effectively addressed by the EDAS technique. For the purpose of assessing the efficacy of the PFZN weighted geometric AOs, an original extended EDAS strategy is created to handle the complicated uncertain data in real-world DS scenarios. Assume there are a number of "alternatives" $\{Q_1, Q_2, ..., Q_l\}$, and a satisfactory rating $\{R_1, R_2, ..., R_m\}$ for each. Then, $\tilde{I}_d = (\tilde{I}_1, \tilde{I}_2, ..., \tilde{I}_m)^T$ specifies the usefulness of various characteristics $R_d(d = 1, 2, ..., m)$, such that $\tilde{I}_d > 0$ and $\Sigma_{d=1}^{m} \tilde{I}_d = 1$. Let $Gz_{jd} = \{(U_{Sjd}, U_{Tjd}), \sim (V_{Sjd}, V_{Tjd})\}$ where $0 \leq (U_{Sjd})^2 + \left(U_{Tjd}\right)^2 \leq 1$, $0 \leq (V_{Sjd})^2 + \left(V_{Tjd}\right)^2 \leq 1$ be the permissible rating for each attribute for each option.

Step 1: Select a set of qualities that can be used to evaluate the problem: Prospective assessment characteristics are acquired by a review of the literature, and an expert decision-making committee is established to filter the characteristics for the purpose to generate a legitimate set of evaluation criteria. $R_d(d = 1, 2, ..., m)$.

$$Gz_{jd} = \begin{pmatrix} R_1 & R_2 & \cdots & R_m \\ ((U_{S11}, U_{T11}), (V_{S11}, (V_{T11})) & ((U_{S12}, U_{T12}), (V_{S12}, V_{T12})) & \cdots & ((U_{S1m}, U_{T1m}), (V_{S1m}, V_{T1m})) \\ ((U_{S21}, U_{T21}), (V_{S21}, (V_{T21})) & ((U_{S22}, U_{T22}), (V_{S22}, V_{T22})) & \cdots & ((U_{S2m}, U_{T2m}), (V_{S2m}, V_{T2m})) \\ \cdots & \cdots & \cdots & \cdots \\ ((U_{Sr1}, U_{Tr1}), (V_{Sr1}, (V_{Tr1})) & ((U_{Sr2}, U_{Tr2}), (V_{Sr2}, V_{Tr2})) & \cdots & ((U_{Srm}, U_{Trm}), (V_{Srm}, V_{Trm})) \end{pmatrix}$$

Step 2: Using normalization, the following steps are taken to construct the normalized decision matrix:

$$Gz_{jd} = \begin{cases} ((U_{Sjd}, U_{Tjd}),\ (V_{Sjd}, V_{Tjd})), & \text{if } C_I, \\ ((V_{Sjd}, V_{Tjd}), (U_{Sjd}, U_{Tjd})), & \text{if } C_{II}. \end{cases}$$

if $R_d(d = 1, 2, ..., m)$ is a benefit criterion, the statement use C_I, if $R_d(d = 1, 2, ..., m)$ is a cost criterion, the statement C_{II} use.

Step 3: Aggregated Data: The skilled uncertain data of required situations are aggregated using established *PFZNWGA* operators.

$$PFZEGA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \Pi_{\grave{i}=1}^{\tilde{n}} G^{\tilde{I}_{\grave{i}}} z_{\grave{i}}$$

$$= \left[\begin{array}{c} \left(\dfrac{\sqrt{2}\Pi_{\grave{i}=1}^{\tilde{n}}\left(\mu_{S_{\grave{i}}}^{\tilde{I}_{\grave{i}}}\right)}{\sqrt{\Pi_{\grave{i}=1}^{\tilde{n}}\left(2-\mu_{S_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(\mu_{S_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}}}}, \dfrac{\sqrt{2}\Pi_{\grave{i}=1}^{\tilde{n}}\left(\mu_{T_{\grave{i}}}^{\tilde{I}_{\grave{i}}}\right)}{\sqrt{\Pi_{\grave{i}=1}^{\tilde{n}}\left(2-\mu_{T_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(\mu_{T_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}}}} \right), \\ \left(\sqrt{\dfrac{\left(\Pi_{\grave{i}=1}^{\tilde{n}} 1+\nu_{S_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}} - \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\nu_{S_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}}}{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\nu_{S_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\nu_{S_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}}}}, \sqrt{\dfrac{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\nu_{T_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}} .- \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\nu_{T_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}}}{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\nu_{T_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\nu_{T_{\grave{i}}}^2\right)^{\tilde{I}_{\grave{i}}}}} \right). \end{array} \right]$$

Step 4: Verify the average solution (A_VS), which is relies on all the criteria given. $A_VS = [A_VS_d]_{1\times m} = \left\{\frac{\Sigma_{j=1}^{\acute{n}} Gz_{jd}}{\acute{n}}\right\}_{1\times m}$ Using Definition 3, we obtain $A_VS = [A_VS_d]_{1\times m} = \left\{\frac{\Sigma_{j=1}^{\acute{n}} Gz_{jd}}{\acute{n}}\right\}_{1\times m}$

$$PFZNEWA(Gz_1, Gz_2, ..., Gz_{\tilde{n}}) = \left[\begin{array}{c}\left(\begin{array}{c}\sqrt{\frac{\left(\Pi_{\grave{i}=1}^{\tilde{n}} 1+\mu_{S_0}^2\right)^{\tilde{I}_{\grave{i}}} - \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{S_0}^2\right)^{\tilde{I}_{\grave{i}}}}{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\mu_{S_0}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{S_0}^2\right)^{\tilde{I}_{\grave{i}}}}},\\ \sqrt{\frac{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\mu_{T_0}^2\right)^{\tilde{I}_{\grave{i}}} .- \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{T_0}^2\right)^{\tilde{I}_{\grave{i}}}}{\Pi_{\grave{i}=1}^{\tilde{n}}\left(1+\mu_{T_0}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(1-\mu_{T_0}^2\right)^{\tilde{I}_{\grave{i}}}}},\end{array}\right),\\ \left(\begin{array}{c}\frac{\sqrt{2}\Pi_{\grave{i}=1}^{\tilde{n}}\left(\nu_{S_0}^{\tilde{I}_{\grave{i}}}\right)}{\sqrt{\Pi_{\grave{i}=1}^{\tilde{n}}\left(2-\nu_{S_0}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(\nu_{S_0}^2\right)^{\tilde{I}_{\grave{i}}}}},\\ \frac{\sqrt{2}\Pi_{\grave{i}=1}^{\tilde{n}}\left(\nu_{T_{\grave{i}}}^{\tilde{I}_{\grave{i}}}\right)}{\sqrt{\Pi_{\grave{i}=1}^{\tilde{n}}\left(2-\nu_{T_0}^2\right)^{\tilde{I}_{\grave{i}}} + \Pi_{\grave{i}=1}^{\tilde{n}}\left(\nu_{T_0}^2\right)^{\tilde{I}_{\grave{i}}}}}\end{array}\right)\end{array}\right]$$

Step 5: The positive (PDA_v) and negative (NDA_v) distances from average must be determined from the A_VS values:

$$PDA_v = \frac{\max(0, (Gz_{jd} - A_VS))}{A_VS}, \text{ and } NDA_v = \frac{\max(0, (A_VS - Gz_{jd}))}{A_VS}.$$

The score function of *PFZNs* specified in Definition 3 can be employed as follows to assess the *PDA* and *NDA*:

$$PDA_v = \frac{\max(0, (J(Gz_{jd}) - J(A_VS))}{J(A_VS)}, \text{ and } \acute{n}DA_v = \frac{\max(0, (J(A_VS) - J\left(Gz_{jd}\right))}{J(A_VS)}.$$

where W shows the score value.

Step 6: Calculate *SPDA* and *SNDA*, which represent for *PDA* and *NDA′s* weighted average, respectively:

$$SPDA = \sum_{d=1}^{m} \tilde{I}_d PDA_d,\ SNDA = \sum_{d=1}^{m} \tilde{I}_d NDA_d,\ \tilde{I}_d \in [0, 1] \text{ and } \Sigma_{d=1}^{m} \tilde{I}_d = 1.$$

Step 7: Normalize weighted sum of *PDA* and *NDA* is defined as, respectively

$$NSPDA = \frac{SPDA}{max(SPDA)}, \text{ and } NSPDA = \frac{SNDA}{max(SNDA)}$$

Step 8: Compute the values of ASC depends on each alternative's as

$$ASC = \frac{1}{2}((NSPDA + 1 - NSNDA).$$

Step 9: Depending on the *ASC* calculations, alternatives are sorted in decreasing order, and the higher the ASC number, the better options will be.

The flow chart of EDAS method is given in Fig. 1.

Figure 1: Flow chart of EDAS method

8.1 Numerical Illustration by Using EDAS Method

Step 1: The expert decision information is given in Table 2.

Table 2: $Gz = \left(Gz_{jd}\right)_{5\times 3}$

Tillage methods	(SE, TF, HE, IE)	(SE, TF, HE, IE)	(SE, TF, HE, IE)
RTB	$\langle(0.6, 0.4), (0.5, 0.3)\rangle$	$\langle(0.7, 0.1), (0.3, 0.5)\rangle$	$\langle(0.4, 0.1), (0.8, 0.2)\rangle$
ZTB	$\langle(0.3, 0.1), (0.4, 0.5)\rangle$	$\langle(0.4, 0.3), (0.6, 0.1)\rangle$	$\langle(0.2, 0.7), (0.1, 0.3)\rangle$
RT	$\langle(0.2, 0.8), (0.4, 0.1)\rangle$	$\langle(0.6, 0.1), (0.6, 0.2)\rangle$	$\langle(0.3, 0.5), (0.6, 0.2)\rangle$
CT	$\langle(0.5, 0.6), (0.7, 0.2)\rangle$	$\langle(0.3, 0.6), (0.4, 0.1)\rangle$	$\langle(0.6, 0.3), (0.1, 0.4)\rangle$
CTB	$\langle(0.3, 0.4), (0.1, 0.5)\rangle$	$\langle(0.6, 0.1), (0.7, 0.2)\rangle$	$\langle(0.5, 0.4), (0.6, 0.1)\rangle$

Step 2: The normalized decision matrix is created using normalization as follows:

$$Gz_{jd} = \begin{cases} ((U_{Sjd}, U_{Tjd}), (V_{Sjd}, V_{Tjd})), & \text{if } C_I, \\ ((V_{Sjd}, V_{Tjd}), (U_{Sjd}, U_{Tjd})), & \text{if } C_{II}. \end{cases}$$

if $R_d(d = 1, 2, ..., m)$ is a benefit criterion, the statement use C_I, if $R_d(d = 1, 2, ..., m)$ is a cost criterion, the statement C_{II} use. Here the given system is benefit type, so we not need to normalized. Step 3: Now that $PFZNWGA$ is available, we can use it to address $MCDM$ problems using the newly created MCDM method, as shown by the following decision-making procedure. The overall collected $PFZN$ $Gz_j(j = 1, 2, 3, 4, 5)$ are obtained as follows:

$$Gz_1 = \{(0.353445079, 0.107280055), (0.593185747, 0.357392481)\},$$
$$Gz_2 = \{(0.186477202, 0.189131059), (0.424938759, 0.343341288)\},$$
$$Gz_3 = \{(0.216371437, 0.23958899), (0.5437384, 0.173133908)\},$$
$$Gz_4 = \{(0.286909561, 0.306199937), (0.480953768, 0.265109265)\},$$
$$Gz_5 = \{(0.286909561, 0.166765999), (0.545908753, 0.318371496)\}.$$

Step 4: The score values $J(Gz_d)$ of $PFZNEWGA$ for the alternatives $Q_j = \{1, 2, 3, 4, 5\}$ are given below:

$$J(Gz_1) = 0.412958741, J(Gz_2) = 0.444684805, J(Gz_3) = 0.47885033,$$
$$J(Gz_4) = 0.480173195, J(Gz_5) = 0.437022487.$$

And verify the average solution $(A_V S)$ as

$$A_V(Gz_1) = 0.266022568, A_V(Gz_2) = 0.201793208, A_V(Gz_3) = 0.517745085, A_V(Gz_4) = 0.291469687.$$

Score function of average solution $(A_V S)$ we have

$$J[A_V(Gz)] = 0.451387275$$

Step 5: Using the $A_V S$ values, the positive distance from average (PDA_v) and the negative distance from average (NDA_v) is calculated in Tables 3 and 4, respectively.

Table 3: Positive distance from average

0.328626016	0	0.145710237	0.22617244
0	0	0	0.177964415
0	0.187736358	0.050205024	0
0.078514118	0.517952881	0	0
0.078514118	0	0.054396958	0.092295933

Table 4: Negative distance from average

0	0.468170799	0	0
0.2990185	0.06240335	0.179250868	0
0.186643871	0	0	0.405997503
0	0	0.071060527	0.090440647
0	0.173275702	0	0

By using the score function of *PFZNs* mentioned in Definition 3, the *PDA* and *NDA* is calculated in Tables 5 and 6, respectively:

Table 5: Find PDA using average of PFZNWGA

0	0	0.060842094	0.06377276	0

Table 6: Find NDA using average of PFZNWGA

0.085133731	0.014848002	0	0	0.031823055

Step 6: Determined *SPDA* and *SNDA* in Tables 7 and 8, respectively, which reflect the weighted average for *PDA* and *NDA's*, and attributes weighting vector $\tilde{I} = (0.333, 0.333, 0.333, 0.333, 0.333)$, we may acquire the results as follows:

Table 7: Find SPDA using weight vector

0	0	0.020260417	0.021236329	0

Table 8: Find NPDA using weight vector

0.028349532	0.004944385	0	0	0.010597077

Step 7: The total *PDA* and *NDA* is presented in Tables 9 and 10, respectively, normalized by weight are characterized as:

Table 9: NSPDA

0	0	0.954059955	1.000015503	0

Table 10: NSNDA

0.999983502	0.174405105	0	0	0.373794614

Step 8: Values of the ASC are determined in Table 11:

Table 11: Appraisal score (ASC)

0.000008	0.412797447	0.977029977	1.000007751	0.313102693

Step 9: Ranking of *EDAS* method is given in Table 12 by using weighted geometric aggregation operator:

Table 12: Ranking of EDAS method based on PFZNWGA

0.000008	4
0.412797447	5
0.977029977	3
1.000007751	1
0.313102693	2

$\Rightarrow \quad Q_4 \geq Q_3 \geq Q_5 \geq Q_2 \geq Q_1$. Hence the best supplier is Q_4.

Validation and Sensitivity Analysis

This section provides an example concerning the challenges that are faced in the agricultural fields, such as the productivity, profitability, and energy use efficiency of wheat crops grown under a variety of tillage systems. The purpose of this section is to illustrate the relevance and efficacy of the developed MCDM technique with PFZN information. Energy input estimates were based on the amount of human labor needed, the use of various kinds of equipment, and the amount of materials. Energy calculations were carried out using various input and output energy equivalents. Since the price of fuel continues to rise, there is a growing interest in improving agricultural production's energy efficiency. To meet the needs of a more populous world, other industries, like agriculture, had to rely on non-renewable energy sources like electricity and fossil fuels to manufacture more food than was required.

Here, sensitivity analysis is performed to adjust the behavior of the offered methodologies. In Table 2, we provide five sets of possibilities. One criterion has a higher weight than the others for each set, as shown in the table. By following this technique, a large enough space of criterion weights has been constructed to test the method's sensitivity to changes in the weighting of criteria. In Table 13 and Fig. 2, we provide the results of a sensitivity study that ranks different operators based on an extended EDAS method amenity alternative and a variety of criterion weight.

The graphical representation of ranking result is shown with the help of a Fig. 2.

Table 13: Comparison by all proposed operators

Existing operator	Ranking	Best
PFZEWA	$Q_4 \geq Q_3 \geq Q_2 \geq Q_5 \geq Q_1$	Q_4
PFZNEWGA	$Q_4 \geq Q_3 \geq Q_5 \geq Q_2 \geq Q_1$	Q_4
PFZNEOWA	$Q_4 \geq Q_3 \geq Q_5 \geq Q_2 \geq Q_1$	Q_4
PFZNOWGA	$Q_4 \geq Q_3 \geq Q_5 \geq Q_2 \geq Q_1$	Q_4
EDAS for *PFZNEWGA*	$Q_4 \geq Q_3 \geq Q_2 \geq Q_5 \geq Q_1$	Q_4

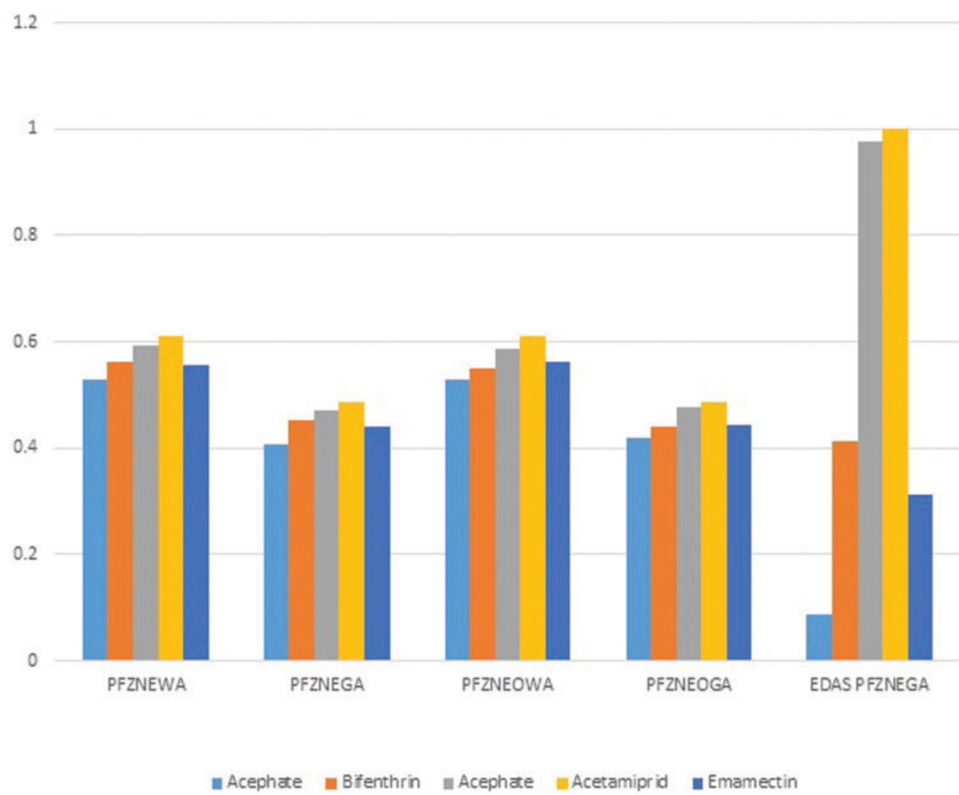

Figure 2: Ranking results

8.2 Comparison Analysis

In this section we discussed the ranking results with the existing studies in the literature. The Tables 14 and 15 are given as follows:

Table 14: Comparison of extended EDAS method with other differents methods

Operators	Ranking	Best alternative
EDAS for PFZNEWGA	$Q_4 \geq Q_3 \geq Q_2 \geq Q_5 \geq Q_1$	Q_4
TOPSIS	$Q_4 \geq Q_1 \geq Q_3 \geq Q_2 \geq Q_5$	Q_4
PFZN Yager weighted average	$Q_4 \geq Q_3 \geq Q_2 \geq Q_5 \geq Q_1$	Q_4
PFZN ordered weighted average	$Q_4 \geq Q_3 \geq Q_2 \geq Q_5 \geq Q_1$	Q_4
PFZN geometric average	$Q_4 \geq Q_3 \geq Q_2 \geq Q_5 \geq Q_1$	Q_4
PFZN ordered geometric average	$Q_4 \geq Q_3 \geq Q_2 \geq Q_5 \geq Q_1$	Q_4
Sine trigonometry	$Q_4 \geq Q_2 \geq Q_5 \geq Q_3 \geq Q_1$	Q_4
Dombi	$Q_4 \geq Q_3 \geq Q_1 \geq Q_2 \geq Q_5$	Q_4

Table 15: Comparison between proposed technique and some existing approaches

	Set	MG	Non-MG	Truthness of MG	Truthness of non-MG
Zadeh [36]	Fuzzy set	✓	×	×	×
Atanassov [13]	Intuitionistic fuzzy set	✓	✓	×	×
Yager et al. [9]	Pythagorean fuzzy set	✓	✓	×	×
Zadeh [37]	Fuzzy Z-number	✓	×	✓	×
Proposed technique	Pythagorean Z-number	✓	✓	✓	✓

9 Conclusion

The wheat crop had a distinct set of energy inputs compared to the other crops. The primary reason for the disparity in wheat energy inputs was the use of different fuels. There is a need for human labor, irrigation, and seed energy. Different types of tillage required different types of energy expenditure, one of which was higher in CTB. CT, RTB, RT, and ZTB all help to reduce fuel energy use when applied to wheat crops. The performance values of alternatives in MCDM circumstances are prone to being distributed arbitrarily, despite the fact that these values are essential in a wide variety of scientific, technical, and managerial disciplines. The optimistic assessment score and the pessimistic assessment score were used together to make the enhanced evaluation score, which was then used to make the final evaluation of the different options. We have defined favorable and unfavorable values for a number of components of the recommended method in order to account for the unpredictability of the data across the entire analytic strategy. This is done to work through numerical examples and demonstrate how the proposed method functions, as well as showcase the situations where it may be useful. We use the second example to compare the ranking results of the recommended technique with a few other ways, and the first example gives a step-by-step discussion of the EDAS strategy. In addition, the decisive rating outcomes of the competitions that were recommended were better aligned with the real decision-making techniques that were used.

Future Scope and Direction

There are several future research directions that can be explored based on the proposed method of handling Pythagorean fuzzy Z-numbers with Einstein aggregation operators and extended EDAS method. Some of these directions include:

1. Developing hybrid methods: The proposed method can be combined with other existing methods, such as TOPSIS, AHP, or PROMETHEE, to enhance its performance and accuracy.

2. Generalizing the proposed method: The proposed method can be generalized to handle other types of fuzzy numbers, such as interval-valued fuzzy numbers, type-2 fuzzy sets, and hesitant fuzzy sets.

3. Applications in diverse fields: The proposed method can be applied to various real-life problems, such as supplier selection, project evaluation, medical diagnosis, financial analysis, and environmental management.

4. Extending the EDAS method: The EDAS method can be further extended by incorporating additional distance metrics or modifying the weights calculation to improve its performance.

5. Development of software tools: Software tools can be developed to facilitate the implementation of the proposed method in decision-making problems.

6. The proposed method can be applicable to several real-life problems that involve decision-making under uncertainty and imprecision. For instance, in supplier selection, the proposed method can be used to evaluate and rank potential suppliers based on multiple criteria, such as quality, price, and delivery time. In financial analysis, the proposed method can be used to evaluate investment opportunities based on factors such as risk, return, and liquidity. In medical diagnosis, the proposed method can be used to analyze patient data and provide a diagnosis based on multiple symptoms and test results. Overall, the proposed method has the potential to be widely applied in various fields and can contribute to making more accurate and informed decisions in uncertain and imprecise situations.

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