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ARTICLE

On Riemann-Type Weighted Fractional Operators and Solutions to Cauchy Problems

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ABSTRACT

In this paper, we establish the new forms of Riemann-type fractional integral and derivative operators. The novel fractional integral operator is proved to be bounded in Lebesgue space and some classical fractional integral and differential operators are obtained as special cases. The properties of new operators like semi-group, inverse and certain others are discussed and its weighted Laplace transform is evaluated. Fractional integro-differential free-electron laser (FEL) and kinetic equations are established. The solutions to these new equations are obtained by using the modified weighted Laplace transform. The Cauchy problem and a growth model are designed as applications along with graphical representation. Finally, the conclusion section indicates future directions to the readers.

KEYWORDS

Weighted fractional operators; weighted laplace transform; integro-differential free-electron laser equation; kinetic differ-integral equation

1 Introduction

The analysis and applications of non-integer order derivatives and integrals are known as fractional calculus. Fractional calculus theory has developed rapidly in recent years and has played a number of pivotal roles in science and engineering, helping as a strong and efficient resource for numerous physical phenomena. Over the last two decades, it has been extensively studied by several mathematicians [1–6].

The literature suggests that the Riemann-Liouville fractional (RLF) derivative plays a crucial part in fractional calculus. Researchers are encouraged to broaden the meanings of fractional derivatives due to the variety of applications. Some of the applications are available in [7–12]. Akgül [13] and Atangana et al. [14] investigated the fractional derivative with non-local and non-singular kernel. In

[15] Caputo et al. examined the non-local fractional derivative which can work more efficiently with Fourier transformation. Some applications of fractional order operators are available in [16,17]. The existence of solution of Riemann-Liouville fractional integro-differential equations with fractional non-local multi-point boundary conditions and system of Riemann-Liouville fractional boundary value problems with ρ-Laplacian operators are briefly discussed in [18,19]. Currently, Jarad et al. [20] defined the weighted fractional derivatives and fractional integrals. To study fractional calculus and its applications, we refer to the readers [21–27].

Motivated by the recent studies presented in [20] and by combining this idea to extend the RLF operators, we will introduce the generalized weighted *(k, s)*-RLF operators and study their properties. The weighted Laplace transform to such fractional operators and some applications in mathematical physics will be discussed. Finally, we will finish with some closing remarks.

In the beginning, we recall some related definitions and notions. The integral form of the k-gamma and k-beta functions given in [28] are defined as follows:

Definition 1.1. The k-gamma function is defined by

$$\Gamma_k(\zeta) = \int_0^\infty \alpha^{\zeta-1} e^{\frac{-\alpha^k}{k}} d\alpha, \Re(\zeta) > 0.$$

Note that: $\Gamma(\zeta) = \lim\limits_{k\to1} \Gamma_k(\zeta)$ and $\Gamma_k(\zeta) = k^{\frac{\zeta}{k}-1}\Gamma\left(\frac{\zeta}{k}\right)$.

Definition 1.2. For $\Re(\zeta), \Re(\eta) > 0$ and $k > 0$, the k-beta function is defined as

$$B_k(\zeta,\eta) = \frac{1}{k}\int_0^1 \tau^{\frac{\zeta}{k}-1}(1-\tau)^{\frac{\eta}{k}-1} d\tau,$$

where the Γ_k and B_k functions are related with an identity $B_k(\zeta,\eta) = \dfrac{\Gamma_k(\zeta)\Gamma_k(\eta)}{\Gamma_k(\zeta+\eta)}$,

Definition 1.3. [29] Suppose that the Ω be a continuous function on interval [*a,b*]. Then weighted *(k, s)*-RLF integral of order ζ is given by

$$\left({}^s_k\mathfrak{J}^\zeta_{a^+,\rho}\Omega\right)(\alpha) = \frac{(s+1)^{1-\frac{\zeta}{k}}\rho^{-1}(\alpha)}{k\Gamma_k(\zeta)}\int_a^\alpha \left(\alpha^{s+1}-t^{s+1}\right)^{\frac{\zeta}{k}-1} t^s\rho(t)\,\Omega(t)\,dt,\ \ \alpha\in[a,b],$$

where $\zeta, k > 0$, $\rho(\alpha) \neq 0$ and $s \in \mathbb{R}\backslash\{-1\}$.

Definition 1.4. [29] Let Ω be a continuous function on $[0,\infty)$ and $s \in \mathbb{R}\backslash\{-1\}$, with $n = [\zeta]+1$, ζ, $\rho(\alpha) \neq 0$, and $k > 0$. Then for all $0 < t < \alpha < \infty$

$$\begin{aligned}\left({}^s_k\mathfrak{D}^\zeta_{a^+,\rho}\Omega\right)(\alpha) &= \rho^{-1}(\alpha)\left(k\alpha^{-s}\frac{d}{d\alpha}\right)^n \rho(\alpha)\left({}^s_k\mathfrak{J}^{nk-\zeta}_{a^+,\rho}\Omega\right)(t)\\ &= \frac{k^{n-1}(s+1)^{\frac{\zeta-nk+k}{k}}\rho^{-1}(\alpha)}{\Gamma_k(nk-\zeta)}\left(\alpha^{-s}\frac{d}{d\alpha}\right)^n\\ &\quad\times\int_a^\alpha\left(\alpha^{s+1}-t^{s+1}\right)^{\frac{nk-\zeta}{k}-1} t^s\rho(t)\,\Omega(t)\,dt.\end{aligned}$$

where ${}^s_k\mathfrak{J}^{nk-\zeta}_{a^+,\rho}$ is a weighted *(k, s)*-RLF integral.

Jarad et al. [20] defined the generalized weighted Laplace transform as follows:

Definition 1.5. Let ρ, Υ be functions with values in $\mathbb{R}$. Furthermore, $\Upsilon(\alpha)$ is continuous and $\Upsilon'(\alpha) > 0$ on $[a, \infty)$. The weighted generalized Laplace transform of Ω is given by

$$L_\Upsilon^\rho\{\Omega(t)\}(u) = \int_a^\infty e^{-u(\Upsilon(t)-\Upsilon(a))}\rho(t)\Omega(t)\Upsilon'(t)dt, \tag{1}$$

and is true for all values of u for which (1) exists.

Theorem 1.1. [20] If $\Omega \in AC_\rho[a,\alpha)$ and of weighted Υ-exponential order. Suppose that the $\mathfrak{D}_\rho\Omega$ be a piecewise continuous function on every interval $[a, T]$, then the weighted generalized Laplace transform of $\mathfrak{D}_\rho\Omega$ exists and

$$\mathfrak{L}_\Upsilon^\rho\{\mathfrak{D}_\rho\Omega\}(u) = u\mathfrak{L}_\Upsilon^\rho\{\Omega(\alpha)\}(u) - \rho(a)\Omega(a).$$

The generalized form of Theorem 1.1 is stated in the next result.

Theorem 1.2. Let $\Omega \in AC_\rho^{n-1}[a,\alpha)$, such that $\mathfrak{D}_\rho^k\Omega$, $k = 0, 1, 2, \ldots,$ n−1 are of weighted Υ-exponential order. If $\mathfrak{D}_\rho^n\Omega$ is a continuous function on all intervals $[a, T]$, the weighted generalized Laplace transform of $\mathfrak{D}_\rho^n\Omega$ exists and

$$\mathfrak{L}_\Upsilon^\rho\left\{\mathfrak{D}_\rho^n\Omega\right\}(u) = u^n\mathfrak{L}_\Upsilon^\rho\{\Omega(\alpha)\}(u) - \sum_{k=0}^{n-1}u^{n-k-1}\Omega_k(a).$$

Definition 1.6. [20] The generalization of the weighted convolution of Ω and Υ is defined by

$$(\Omega *_\Upsilon^\rho h)(s) = \rho^{-1}(s)\int_\alpha^s \rho(\Upsilon^{-1}(\Upsilon(s)+\Upsilon(\alpha)-\Upsilon(t)))$$
$$\times\, \Omega(\Upsilon^{-1}(\Upsilon(s)+\Upsilon(\alpha)-\Upsilon(t)))\rho(t)h(t)\Upsilon'(t)dt.$$

2 Generalized Weighted *(k, s)*-Riemann-Liouville Fractional Operators

In this section, we introduce the generalized weighted *(k, s)*-RLF operators and describe some of their features.

Definition 2.1. Suppose that the Ω be a continuous function on the finite real interval $[a,b]$ and Υ is strictly increasing function. Then the generalized weighted *(k, s)*-RLF integral of order ζ is defined by

$$({}_{\Upsilon,k}^{s}\mathfrak{J}_{a^+,\rho}^{\zeta}\Omega)(\alpha) = \frac{(s+1)^{1-\frac{\zeta}{k}}\rho^{-1}(\alpha)}{k\Gamma_k(\zeta)}\int_a^\alpha (\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(t))^{\frac{\zeta}{k}-1}\Upsilon^s(t)\,\Upsilon'(t)\,\rho(t)\,\Omega(t)\,dt,\ \ \alpha > a, \tag{2}$$

where $\zeta, k > 0$, $\rho(\alpha) \neq 0$, $s \in \mathbb{R}\backslash\{-1\}$ and $\Upsilon^{s+1}(x) = (\Upsilon(x))^{s+1}$.

The integral operator defined in 2 cover many fractional integral operators. For instance,

I. if we set $s = 0$ and $k = 1$ in (2), we get the generalized weighted-RLF integral given in [20].

II. If we set $\Upsilon(\alpha) = \alpha$ in (2), we get the weighted *(k, s)*-RLF integral presented in [29].

III. If we set $\rho(\alpha) = 1$ and $\Upsilon(\alpha) = \alpha$ in (2), we get the weighted *(k, s)*-RLF integral [29].

IV. If we set $s = 0$, $\Upsilon(\alpha) = \alpha$ and $\rho(\alpha) = 1$ in (2), k-RLF integral is obtained [30].

V. If we set $k = 1$, $s = 0$, $\Upsilon(\alpha) = \alpha$ and $\rho(\alpha) = 1$ in (2), it gives RLF integral [3].

VI. For $s \to -1^+$, $\Upsilon(\alpha) = \alpha$ and $\rho(\alpha) = 1$ in (2), we obtain k-Hadamard fractional integral [31].

The corresponding weighted generalized fractional derivative is defined by the following definition.

Definition 2.2. Let Ω be continuous function on $[0,\infty)$ and $s \in \mathbb{R}\backslash\{-1\}$, $n = [\zeta]+1$, $\zeta, k > 0$, and $\rho(\alpha) \neq 0$. Then for all $0 < t < \alpha < \infty$, the inverse derivative operator of integral operator 2 is defined by

$$\begin{aligned}
&\left({}_{\Upsilon,k}^{\;s}\mathfrak{D}^{\zeta}_{a+,\rho}\Omega\right)(\alpha)\\
&= \rho^{-1}(\alpha)\left(\frac{k}{\Upsilon^{s}(\alpha)\Upsilon'(\alpha)}\frac{d}{d\alpha}\right)^{n}\rho(\alpha)\left({}_{\Upsilon,k}^{\;s}\mathfrak{J}^{nk-\zeta}_{a+,\rho}\Omega\right)(t)\\
&= \frac{k^{n-1}(s+1)^{1-\frac{nk-\zeta}{k}}\rho^{-1}(\alpha)}{\Gamma_{k}(nk-\zeta)}\left(\frac{1}{\Upsilon^{s}(\alpha)\Upsilon'(\alpha)}\frac{d}{d\alpha}\right)^{n}\\
&\quad\times\int_{a}^{\alpha}\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(t)\right)^{\frac{nk-\zeta}{k}-1}\Upsilon^{s}(t)\Upsilon'(t)\rho(t)\Omega(t)dt. \qquad (3)
\end{aligned}$$

where ${}_{\Upsilon,k}^{\;s}\mathfrak{J}^{nk-\zeta}_{a+,\rho}$ is a generalized weighted *(k, s)*-RLF integral.

There are many other fractional derivative operators as special cases of the operator (3).

I. If we choose $s = 0$ and $k = 1$ in (3), we get the weighted *(k, s)*-RLF derivative presented in [20]

II. If we choose $\Upsilon(\alpha) = \alpha$ in (3), we get weighted *(k, s)*-RLF derivative presented in [29].

III. If we choose $\rho(\alpha) = 1$ and $\Upsilon(\alpha) = \alpha$ in (3), we get *(k, s)*-RLF derivative [32].

IV. If we set $s = 0$, $\Upsilon(\alpha) = \alpha$ and $\rho(\alpha) = 1$ in (3) it gives to k-RLF derivatives [33].

V. If we set $k = 1$, $s = 0$, $\Upsilon(\alpha) = \alpha$ and $\rho(\alpha) = 1$ in (3), it reduces to RLF derivative [34].

VI. (3) reduces to k-Hadamard fractional derivative for $s \to -1^{+}$, $\Upsilon(\alpha) = \alpha$ and $\rho(\alpha) = 1$ [31].

In the following definition, we define the space where the generalized weighted *(k, s)*-RLF integral is bounded.

Definition 2.3. Let f be defined on $[a,b]$ and $X^{p}_{\rho}(a,b)$, $1 \leq p \leq \infty$ be the space of all Lebesgue measurable functions for which $\parallel \Omega \parallel_{X^{p}_{\rho}} < \infty$, where

$$\parallel \Omega \parallel_{X^{p}_{\rho}} = \left[(s+1)\int_{a}^{b} \mid \rho(\alpha)\,\Omega(\alpha) \mid^{p} \Upsilon^{s}(\alpha)\,\Upsilon'(\alpha)\,d\alpha\right]^{\frac{1}{p}}, 1 \leq p < \infty,$$

$\rho(\alpha) \neq 0, s \in \mathbb{R}\backslash\{-1\}$ and

$$\parallel \Omega \parallel_{X^{\infty}_{\rho}} = ess\, sup_{a\leq\alpha\leq b}\left|\rho(\alpha)\Omega(\alpha)\right| < \infty.$$

Note that $\Omega \in X^{p}_{\rho}(a,b) \Leftrightarrow \rho(\alpha)\,\Omega(\alpha)\,(\Upsilon^{s}(\alpha)\,\Upsilon'(\alpha))^{\frac{1}{p}} \in L_{p}(a,b)$ for $1 \leq p < \infty$ and $\Omega \in X^{\infty}_{\rho}(a,b)$ $\Leftrightarrow \rho(\alpha)\Omega(\alpha) \in L_{\infty}(a,b)$.

Theorem 2.1. Let $\zeta > 0$, $k > 0$, $1 \leq p \leq \infty$ and $\Omega \in X^{p}_{\rho}(a,b)$. Then ${}_{\Upsilon,k}^{\;s}\mathfrak{J}^{\zeta}_{a+,\rho}\Omega$ is bounded in $X^{p}_{\rho}(a,b)$ and

$$\parallel {}_{\Upsilon,k}^{\;s}\mathfrak{J}^{\zeta}_{a+,\rho}\Omega \parallel_{X^{p}_{\rho}} \leq \frac{(s+1)^{-\frac{\zeta}{k}}\left(\Upsilon^{s+1}(b)-\Upsilon^{s+1}(a)\right)^{\frac{\zeta}{k}}}{\Gamma_{k}(\zeta+1)} \parallel \Omega \parallel_{X^{p}_{\rho}}.$$

Proof. For $1 \leq p < \infty$, we have

$$\|_{\Upsilon,k}^{s}\mathfrak{J}_{a^+,\rho}^{\zeta}\Omega\|_{X_\rho^p}$$

$$= \left[(s+1)\int_a^b \left|\rho(\alpha)\frac{(s+1)^{1-\frac{\zeta}{k}}\rho^{-1}(\alpha)}{k\Gamma_k(\zeta)}\right.\right.$$

$$\left.\left.\times\int_a^\alpha (\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(t))^{\frac{\zeta}{k}-1}\Upsilon^s(t)\Upsilon'(t)\rho(t)\Omega(t)dt\right|^p\Upsilon^s(\alpha)\Upsilon'(\alpha)d\alpha\right]^{\frac{1}{p}}$$

$$= \frac{(s+1)^{-\frac{\zeta}{k}}}{k\Gamma_k(\zeta)}\left[\int_a^b\left|\int_a^\alpha (\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(t))^{\frac{\zeta}{k}-1}(s+1)\Upsilon^s(t)\Upsilon'(t)\rho(t)\Omega(t)dt\right|^p\right.$$

$$\left.\times(s+1)\Upsilon^s(\alpha)\Upsilon'(\alpha)d\alpha\right]^{\frac{1}{p}}. \tag{4}$$

Substituting $\Upsilon^{s+1}(\alpha) = v$ and $\Upsilon^{s+1}(t) = u$ on the right side of (4), we obtain

$$\|_{\Upsilon,k}^{s}\mathfrak{J}_{a^+,\rho}^{\zeta}\Omega\|_{X_\rho^p}$$

$$= \frac{(s+1)^{-\frac{\zeta}{k}}}{k\Gamma_k(\zeta)}\left[\int_{\Upsilon^{s+1}(a)}^{\Upsilon^{s+1}(b)}\left|\int_{\Upsilon^{s+1}(a)}^{\Upsilon^{s+1}(\alpha)}(v-u)^{\frac{\zeta}{k}-1}\rho\left(\Upsilon^{-1}\left(u^{\frac{1}{s+1}}\right)\right)\Omega\left(\Upsilon^{-1}\left(u^{\frac{1}{s+1}}\right)\right)du\right|^p dv\right]^{\frac{1}{p}}.$$

By using Minkowski's inequality, we have

$$\|_{\Upsilon,k}^{s}\mathfrak{J}_{a^+,\rho}^{\zeta}\Omega\|_{X_\rho^p}$$

$$\leq \frac{(s+1)^{-\frac{\zeta}{k}}}{k\Gamma_k(\zeta)}\int_{\Upsilon^{s+1}(a)}^{\Upsilon^{s+1}(b)}\left|\rho\left(\Upsilon^{-1}\left(u^{\frac{1}{s+1}}\right)\right)\Omega\left(\Upsilon^{-1}\left(u^{\frac{1}{s+1}}\right)\right)\right|\left[\int_u^{\Upsilon^{s+1}(b)}(v-u)^{\left(\frac{\zeta}{k}-1\right)p}dv\right]^{\frac{1}{p}}du$$

$$\leq \frac{(s+1)^{-\frac{\zeta}{k}}}{k\Gamma_k(\zeta)}\int_{\Upsilon^{s+1}(a)}^{\Upsilon^{s+1}(b)}\left|\rho\left(\Upsilon^{-1}\left(u^{\frac{1}{s+1}}\right)\right)\Omega\left(\Upsilon^{-1}\left(u^{\frac{1}{s+1}}\right)\right)\right|\left[\frac{\left(\Upsilon^{s+1}(b)-u\right)^{\left(\frac{\zeta}{k}-1\right)p+1}}{\left(\frac{\zeta}{k}-1\right)p+1}\right]^{\frac{1}{p}}du.$$

Applying Hölder's inequality, we get

$$\|_{\Upsilon,k}^{s}\mathfrak{J}_{a^+,\rho}^{\zeta}\Omega\|_{X_\rho^p} \leq \frac{(s+1)^{-\frac{\zeta}{k}}}{k\Gamma_k(\zeta)}\left[\int_{\Upsilon^{s+1}(a)}^{\Upsilon^{s+1}(b)}\left|\rho\left(\Upsilon^{-1}\left(u^{\frac{1}{s+1}}\right)\right)\Omega\left(\Upsilon^{-1}\left(u^{\frac{1}{s+1}}\right)\right)\right|^p du\right]^{\frac{1}{p}}$$

$$\times\left[\int_{\Upsilon^{s+1}(a)}^{\Upsilon^{s+1}(b)}\left(\frac{\left(\Upsilon^{s+1}(b)-u\right)^{\left(\frac{\zeta}{k}-1\right)p+1}}{\left(\frac{\zeta}{k}-1\right)p+1}\right)^{\frac{q}{p}}du\right]^{\frac{1}{q}},$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

$$\|{}_{\Upsilon}{}^{s}_{k}\mathfrak{J}^{\zeta}_{a^{+},\rho}\Omega\|_{X^{p}_{\rho}} \leq \frac{(s+1)^{-\frac{\zeta}{k}}}{k\Gamma_{k}(\zeta)}\left[\int_{a}^{b}\left|\rho(t)\,\Omega(t)\right|^{p}(s+1)\,\Upsilon^{s}(t)\,\Upsilon'(t)\,dt\right]^{\frac{1}{p}}$$

$$\times\left[\int_{\Upsilon^{s+1}(a)}^{\Upsilon^{s+1}(b)}\left(\frac{\left(\Upsilon^{s+1}(b)-u\right)^{\left(\frac{\zeta}{k}-1\right)p+1}}{\left(\frac{\zeta}{k}-1\right)p+1}\right)^{\frac{q}{p}}du\right]^{\frac{1}{q}}$$

$$\leq \frac{(s+1)^{-\frac{\zeta}{k}}\left(\Upsilon^{s+1}(b)-\Upsilon^{s+1}(a)\right)^{\frac{\zeta}{k}}}{\Gamma_{k}(\zeta+1)}\,\|\,\Omega\,\|_{X^{p}_{\rho}}.$$

For $p = \infty$, we obtain

$$\left|\rho(\alpha)\,{}_{\Upsilon}{}^{s}_{k}\mathfrak{J}^{\zeta}_{a^{+},\rho}\Omega(\alpha)\right| = \frac{(s+1)^{-\frac{\zeta}{k}}\left(\Upsilon^{s+1}(b)-\Upsilon^{s+1}(a)\right)^{\frac{\zeta}{k}}}{\Gamma_{k}(\zeta+1)}\,\|\,\Omega\,\|_{X^{\infty}_{\rho}}.$$

Hence the proof is done.

Theorem 2.2. Let Ω be continuous on $[0, \infty)$ and $s \in \mathbb{R}\backslash\{-1\}$ and $\rho(\alpha) \neq 0$, $n = [\zeta] + 1$. Then for all $0 < a < \alpha$.

$${}_{\Upsilon}{}^{s}_{k}\mathfrak{D}^{\zeta}_{a+,\rho}({}_{\Upsilon}{}^{s}_{k}\mathfrak{J}^{\zeta}_{a+,\rho}\Omega)(\alpha) = \Omega(\alpha),$$

where $\zeta, k > 0$ and $nk - \zeta > 0$.

Proof. Consider

$${}_{\Upsilon}{}^{s}_{k}\,\mathfrak{D}^{\zeta}_{a^{+},\rho}({}_{\Upsilon}{}^{s}_{k}\,\mathfrak{J}^{\zeta}_{a^{+},\rho}\Omega)(\alpha)$$

$$= \frac{(s+1)^{\frac{\zeta-nk+k}{k}}\rho^{-1}(\alpha)}{k\Gamma_{k}(nk-\zeta)}\left(\frac{1}{\Upsilon^{s}(\alpha)\Upsilon'(\alpha)}\frac{d}{d\alpha}\right)^{n}k^{n}$$

$$\times\int_{a}^{\alpha}(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(y))^{\frac{nk-\zeta}{k}-1}\Upsilon^{s}(y)\Upsilon'(y)\rho(y)({}_{\Upsilon}{}^{s}_{k}\,\mathfrak{J}^{\zeta}_{a^{+},\rho}\Omega)(y)dy$$

$$= \frac{(s+1)^{\frac{\zeta-nk+k}{k}}\rho^{-1}(\alpha)}{k\Gamma_{k}(nk-\zeta)}\left(\frac{1}{\Upsilon^{s}(\alpha)\Upsilon'(\alpha)}\frac{d}{d\alpha}\right)^{n}k^{n}\int_{a}^{\alpha}\Bigg[(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(y))^{\frac{nk-\zeta}{k}-1}$$

$$\times\Upsilon^{s}(y)\Upsilon'(y)\rho(y)\frac{(s+1)^{1-\frac{\zeta}{k}}\rho^{-1}(\alpha)}{k\Gamma_{k}(\zeta)}\int_{a}^{y}(\Upsilon^{s+1}(y)-\Upsilon^{s+1}(t))^{\frac{\zeta}{k}-1}\Upsilon^{s}(t)\Upsilon'(t)\rho(t)\Omega(t)dt\Bigg]dy$$

$$= \frac{(s+1)^{2-n}\rho^{-1}(\alpha)}{k^{2}\Gamma_{k}(\zeta)\Gamma_{k}(nk-\zeta)}\left(\frac{1}{\Upsilon^{s}(\alpha)\Upsilon'(\alpha)}\frac{d}{d\alpha}\right)^{n}k^{n}\int_{a}^{\alpha}\Upsilon^{s}(t)\Upsilon'(t)\rho(t)\Omega(t)$$

$$\times\left[\int_{t}^{\alpha}(\Upsilon^{s+1}(y)-\Upsilon^{s+1}(t))^{\frac{\zeta}{k}-1}(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(y))^{\frac{nk-\zeta}{k}-1}\Upsilon^{s}(y)\Upsilon'(y)dy\right]dt. \tag{5}$$

Substitute $z = \dfrac{\Upsilon^{s+1}(y)-\Upsilon^{s+1}(t)}{\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(t)}$ on the right side of (5), we get

$$
\begin{aligned}
&{}_{\Upsilon,k}^{\;s}\mathfrak{D}^{\zeta}_{a^+,\rho}({}_{\Upsilon,k}^{\;s}\mathfrak{J}^{\zeta}_{a^+,\rho}\Omega)(\alpha)\\
&=\frac{(s+1)^{1-n}\rho^{-1}(\alpha)}{k^2\Gamma_k(\zeta)\Gamma_k(nk-\zeta)}\left(\frac{1}{\Upsilon^s(\alpha)\Upsilon'(\alpha)}\frac{d}{d\alpha}\right)^n k^n\\
&\quad\times\int_a^\alpha \Upsilon^s(t)\,\Upsilon'(t)\,\rho(t)\,\Omega(t)\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(t)\right)^{n-1}\left[\int_t^\alpha(1-z)^{\frac{\zeta}{k}-1}(z)^{\frac{nk-\zeta}{k}-1}dz\right]dt\\
&=\frac{(s+1)^{1-n}\rho^{-1}(\alpha)}{k^2\Gamma_k(\zeta)\Gamma_k(nk-\zeta)}\left(\frac{1}{\Upsilon^s(\alpha)\Upsilon'(\alpha)}\frac{d}{d\alpha}\right)^n k^n\\
&\quad\times\int_a^\alpha \Upsilon^s(t)\Upsilon'(t)\rho(t)\Omega(t)(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(t))^{n-1}[kB_k(\zeta,nk-\zeta)]dt\\
&=\frac{(s+1)^{1-n}\rho^{-1}(\alpha)}{k\Gamma_k(nk)}\left(\frac{1}{\Upsilon^s(\alpha)\Upsilon'(\alpha)}\frac{d}{d\alpha}\right)^n k^n\\
&\quad\times\int_a^\alpha \Upsilon^s(t)\Upsilon'(t)\rho(t)\Omega(t)(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(t))^{n-1}dt\\
&=\frac{(s+1)^{1-n}\rho^{-1}(\alpha)}{k^n\Gamma(n)}\left(\frac{1}{\Upsilon^s(\alpha)\Upsilon'(\alpha)}\frac{d}{d\alpha}\right)^n k^n\\
&\quad\times\int_a^\alpha \Upsilon^s(t)\Upsilon'(t)\rho(t)\Omega(t)(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(t))^{n-1}dt,
\end{aligned}
$$

which gives

$$
{}_{\Upsilon,k}^{\;s}\mathfrak{D}^{\zeta}_{a^+,\rho}({}_{\Upsilon,k}^{\;s}\mathfrak{J}^{\zeta}_{a^+,\rho}\Omega)(\alpha)=\Omega(\alpha).
$$

This proved the inverse property.

Corollary 2.1. Let the function Ω be continuous on $[0,\infty)$ and $s\in\mathbb{R}\backslash\{-1\}$ and $\rho(\alpha)\neq 0$, $m=[\eta]+1$, $n=[\zeta]+1$. Then for all $0<a<\alpha$

$$
{}_{\Upsilon,k}^{\;s}\mathfrak{D}^{\zeta}_{a^+,\rho}({}_{\Upsilon,k}^{\;s}\mathfrak{J}^{\eta}_{a^+,\rho}\Omega)(\alpha)=({}_{\Upsilon,k}^{\;s}\mathfrak{D}^{\zeta-\eta}_{a^+,\rho}\Omega)(\alpha),
$$

where $\zeta,\eta,k>0$.

Corollary 2.2. Let the function Ω be continuous on $[0,\infty)$ and $s\in\mathbb{R}\backslash\{-1\}$, $\rho(\alpha)\neq 0$, $n=[\zeta]+1$, $m=[\eta]+1$ and $\zeta+\eta<nk$. Then for all $0<a<\alpha$

$$
{}_{\Upsilon,k}^{\;s}\mathfrak{D}^{\zeta}_{a^+,\rho}({}_{\Upsilon,k}^{\;s}\mathfrak{D}^{\eta}_{a^+,\rho}\Omega)(\alpha)=({}_{\Upsilon,k}^{\;s}\mathfrak{D}^{\zeta+\eta}_{a^+,\rho}\Omega)(\alpha),
$$

where $\zeta,\eta,k>0$.

Proof. By using Definition 2.2, we have

$$
\begin{aligned}
&{}_{\Upsilon,k}^{\;s}\mathfrak{D}^{\zeta}_{a^+,\rho}({}_{\Upsilon,k}^{\;s}\mathfrak{D}^{\eta}_{a^+,\rho}\Omega)(\alpha)\\
&=\rho^{-1}(\alpha)\left(\frac{k}{\Upsilon^s(\alpha)\Upsilon'(\alpha)}\frac{d}{d\alpha}\right)^n\rho(\alpha)\left({}_{\Upsilon,k}^{\;s}\mathfrak{J}^{nk-\zeta}_{a^+,\rho}\right)\left({}_{\Upsilon,k}^{\;s}\mathfrak{D}^{\eta}_{a^+,\rho}\Omega\right)(\alpha)\\
&=\rho^{-1}(\alpha)\left(\frac{k}{\Upsilon^s(\alpha)\Upsilon'(\alpha)}\frac{d}{d\alpha}\right)^n\rho(\alpha)\left({}_{\Upsilon,k}^{\;s}\mathfrak{J}^{nk-\zeta}_{a^+,\rho}\right)({}_{\Upsilon,k}^{\;s}\mathfrak{D}^{\eta}_{a^+,\rho}\left({}_{\Upsilon,k}^{\;s}\mathfrak{J}^{\eta}_{a^+,\rho}\,{}_{\Upsilon,k}^{\;s}\mathfrak{J}^{-\eta}_{a^+,\rho}\Omega\right)(\alpha)\,.
\end{aligned}
$$

By using Theorem 2.2, we have

$$\Upsilon,{}^{s}_{k}\mathfrak{D}^{\zeta}_{a^+,\rho}({}_{\Upsilon},{}^{s}_{k}\mathfrak{D}^{\eta}_{a^+,\rho}\Omega)(\alpha)$$

$$= \rho^{-1}(\alpha)\left(\frac{k}{\Upsilon^{s}(\alpha)\Upsilon'(\alpha)}\frac{d}{d\alpha}\right)^{n}\rho(\alpha)\left({}_{\Upsilon},{}^{s}_{k}\mathfrak{J}^{nk-\zeta}_{a^+,\rho}\right)\left({}_{\Upsilon},{}^{s}_{k}\mathfrak{J}^{-\eta}_{a^+,\rho}\Omega\right)(\alpha)$$

$$= \rho^{-1}(\alpha)\left(\frac{k}{\Upsilon^{s}(\alpha)\Upsilon'(\alpha)}\frac{d}{d\alpha}\right)^{n}\rho(\alpha)\left({}_{\Upsilon},{}^{s}_{k}\mathfrak{J}^{nk-(\zeta+\eta)}_{a^+,\rho}\Omega\right)(\alpha),$$

which implies

$${}_{\Upsilon},{}^{s}_{k}\mathfrak{D}^{\zeta}_{a^+,\rho}({}_{\Upsilon},{}^{s}_{k}\mathfrak{D}^{\eta}_{a^+,\rho}\Omega)(\alpha) = ({}_{\Upsilon},{}^{s}_{k}\mathfrak{D}^{\zeta+\eta}_{a^+,\rho}\Omega)(\alpha).$$

Hence the semi-group property of new derivative operator is proved.

Corollary 2.3. Suppose that the Ω be a continuous function on $[0,\infty)$ and $\zeta,\eta \in \mathbb{R}^+$, $\rho(\alpha) \neq 0$ and $s \in \mathbb{R}\backslash\{-1\}$. Then for all $0 < a < \alpha$

$${}_{\Upsilon},{}^{s}_{k}\mathfrak{D}^{\zeta}_{a^+,\rho}({}_{\Upsilon},{}^{s}_{k}\mathfrak{D}^{\eta}_{a^+,\rho}\Omega)(\alpha) ={}_{\Upsilon},{}^{s}_{k}\mathfrak{D}^{\eta}_{a^+,\rho}({}_{\Upsilon},{}^{s}_{k}\mathfrak{D}^{\zeta}_{a^+,\rho}\Omega)(\alpha),$$

where $n = [\zeta] + 1$, $m = [\eta] + 1$ and $\zeta + \eta < nk$.

Theorem 2.3. Let the function Ω be continuous on $[a,b]$ and $k > 0$, $\rho(\alpha) \neq 0$ and $s \in \mathbb{R}\backslash\{-1\}$

$${}_{\Upsilon},{}^{s}_{k}\mathfrak{J}^{\eta}_{a^+,\rho}[{}_{\Upsilon},{}^{s}_{k}\mathfrak{J}^{\zeta}_{a^+,\rho}\Omega(\alpha)] ={}_{\Upsilon},{}^{s}_{k}\mathfrak{J}^{\zeta}_{a^+,\rho}[{}_{\Upsilon},{}^{s}_{k}\mathfrak{J}^{\eta}_{a^+,\rho}\Omega(\alpha)] ={}_{\Upsilon},{}^{s}_{k}\mathfrak{J}^{\zeta+\eta}_{a^+,\rho}\Omega(\alpha),$$

for all ζ, $\eta > 0$ and $\alpha \in [a,b]$.

Proof. By utilizing the Definition 2.1 and Dirichlet's formula, we get

$$\Upsilon,{}^{s}_{k}\mathfrak{J}^{\zeta}_{a^+,\rho}\left[{}_{\Upsilon},{}^{s}_{k}\mathfrak{J}^{\eta}_{a^+,\rho}\Omega(\alpha)\right]$$

$$= \frac{(s+1)^{1-\frac{\zeta}{k}}\rho^{-1}(\alpha)}{k\Gamma_k(\zeta)}\int_a^{\alpha}(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(t))^{\frac{\zeta}{k}-1}\Upsilon^{s}(t)\Upsilon'(t)\rho(t)\left({}_{\Upsilon},{}^{s}_{k}\mathfrak{J}^{\eta}_{a^+,\rho}\Omega\right)(t)\,dt$$

$$= \frac{(s+1)^{1-\frac{\zeta}{k}}\rho^{-1}(\alpha)}{k\Gamma_k(\zeta)}\int_a^{\alpha}(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(t))^{\frac{\zeta}{k}-1}\Upsilon^{s}(t)\Upsilon'(t)\rho(t)$$

$$\times\left[\frac{(s+1)^{1-\frac{\eta}{k}}\rho^{-1}(t)}{k\Gamma_k(\eta)}\int_a^{t}(\Upsilon^{s+1}(t)-\Upsilon^{s+1}(\tau))^{\frac{\eta}{k}-1}\Upsilon^{s}(\tau)\Upsilon'(\tau)\rho(\tau)\Omega(\tau)\,d\tau\right]dt$$

$$= \frac{(s+1)^{2-\frac{\zeta+\eta}{k}}\rho^{-1}(\alpha)}{k^2\Gamma_k(\zeta)\Gamma_k(\eta)}\int_a^{\alpha}\Upsilon^{s}(\tau)\Upsilon'(\tau)\rho(\tau)\Omega(\tau)$$

$$\times\int_{\tau}^{\alpha}(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(t))^{\frac{\zeta}{k}-1}\left(\Upsilon^{s+1}(t)-\Upsilon^{s+1}(\tau)\right)^{\frac{\eta}{k}-1}\Upsilon^{s}(t)\Upsilon'(t)\,dtd\tau. \tag{6}$$

Substitute $y = \dfrac{\Upsilon^{s+1}(t)-\Upsilon^{s+1}(\tau)}{\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(\tau)}$ on the right side of (6), we obtain

$$
\begin{aligned}
&{}_{\Upsilon,k}^{\;s}\mathfrak{J}_{a^+,\rho}^{\zeta}\left[{}_{\Upsilon,k}^{\;s}\mathfrak{J}_{a^+,\rho}^{\eta}\Omega(\alpha)\right]\\
&=\frac{(s+1)^{2-\frac{\zeta+\eta}{k}}\rho^{-1}(\alpha)}{k^2\Gamma_k(\zeta)\Gamma_k(\eta)}\\
&\quad\times\int_a^\alpha\frac{\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(\tau)\right)^{\frac{\zeta+\eta}{k}-1}}{(s+1)}\Upsilon^s(\tau)\Upsilon'(\tau)\rho(\tau)\Omega(\tau)kB_k(\zeta,\eta)\,d\tau\\
&=\frac{(s+1)^{1-\frac{\zeta+\eta}{k}}\rho^{-1}(\alpha)}{k\Gamma_k(\zeta+\eta)}\int_a^\alpha\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(\tau)\right)^{\frac{\zeta+\eta}{k}-1}\Upsilon^s(\tau)\Upsilon'(\tau)\rho(\tau)\Omega(\tau)\,d\tau\\
&={}_{\Upsilon,k}^{\;s}\mathfrak{J}_{a^+,\rho}^{\zeta+\eta}\Omega(\alpha).
\end{aligned}
$$

This completes the proof.

Theorem 2.4. Let $\zeta,\eta,k>0$, $\rho(\alpha)\neq 0$ and $s\in\mathbb{R}\backslash\{-1\}$. Then we have

$$
{}_{\Upsilon,k}^{\;s}\mathfrak{J}_{a^+,\rho}^{\eta}\left[\rho^{-1}(\alpha)\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a)\right)^{\frac{\eta}{k}-1}\right]=\frac{\Gamma_k(\eta)\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a)\right)^{\frac{\zeta+\eta}{k}-1}\rho^{-1}(\alpha)}{(s+1)^{\frac{\zeta}{k}}\Gamma_k(\zeta+\eta)},
$$

where $\Gamma_k(.)$ represents the k-Gamma function.

Proof. By Definition 2.1, we get

$$
\begin{aligned}
&{}_{\Upsilon,k}^{\;s}\mathfrak{J}_{a^+,\rho}^{\eta}\left[\rho^{-1}(\alpha)\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a)\right)^{\frac{\eta}{k}-1}\right]\\
&=\frac{(s+1)^{1-\frac{\zeta}{k}}\rho^{-1}(\alpha)}{k\Gamma_k(\zeta)}\int_a^\alpha(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(t))^{\frac{\zeta}{k}-1}\Upsilon^s(t)\Upsilon'(t)\,dt\\
&\quad\times\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a)\right)^{\frac{\eta}{k}-1}\Omega(t)\,dt.
\end{aligned}\tag{7}
$$

Substitute $y=\dfrac{\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(t)}{\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a)}$ on the right side of (7), we get

$$
\begin{aligned}
&{}_{\Upsilon,k}^{\;s}\mathfrak{J}_{a^+,\rho}^{\eta}\left[\rho^{-1}(\alpha)\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a)\right)^{\frac{\eta}{k}-1}\right]\\
&=\frac{(s+1)^{\frac{-\zeta}{k}}\rho^{-1}(\alpha)\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a)\right)^{\frac{\zeta+\eta}{k}-1}}{k\Gamma_k(\zeta)}\\
&\quad\times\int_0^1(1-y)^{\frac{\zeta}{k}-1}(y)^{\frac{\eta}{k}-1}\,dy\\
&=\frac{(s+1)^{\frac{-\zeta}{k}}\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a)\right)^{\frac{\zeta+\eta}{k}-1}\rho^{-1}(\alpha)}{k\Gamma_k(\zeta)}kB_k(\zeta,\eta)\\
&=\frac{\Gamma_k(\eta)\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a)\right)^{\frac{\zeta+\eta}{k}-1}\rho^{-1}(\alpha)}{(s+1)^{\frac{\zeta}{k}}\Gamma_k(\zeta+\eta)}.
\end{aligned}
$$

The proof is done.

Example 2.1. Corresponding to the choice of the parameters $s = 0, k = 1, \eta = 3, a = 0$ and $\rho(t) = 1$, we get the following graphs with different choices of the function $\Upsilon(t)$.

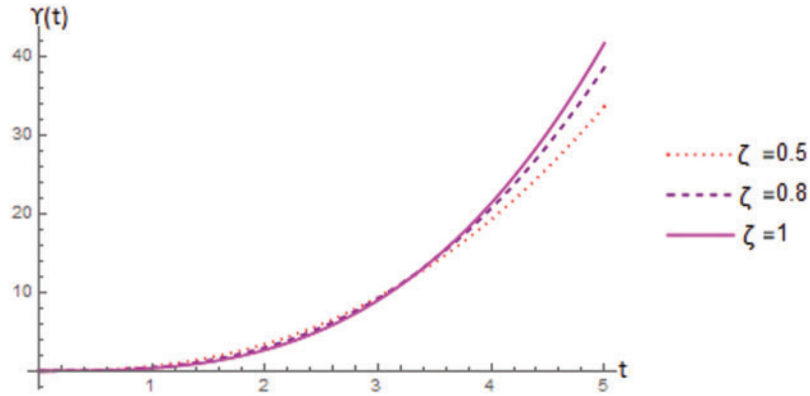

Figure 1: For $\Upsilon(t)=t$ the graph in Fig. 1 shows the increasing behaviour with $0^+ \leq t \leq 5$

3 The Generalized Weighted Laplace Transform

In the following section, we use the weighted Laplace transformation to the new fractional operators. Firstly, we present the following definition which is a modified form of the Definition 1.5.

Definition 3.1. Suppose that the Ω be a real valued function defined on $\Omega \in [a, \infty)$ and $s \in \mathbb{R}\backslash\{-1\}$. The weighted generalized Laplace transform of Ω is given by

$$\mathfrak{L}_{\Upsilon}^{\rho}\Omega(u) = (s+1)\int_a^{\infty} e^{-u(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a))}\rho(\alpha)\Upsilon^s(\alpha)\Upsilon'(\alpha)\Omega(\alpha)d\alpha$$

holds for all values of u.

Proposition 3.1.

$$\mathfrak{L}_{\Upsilon}^{\rho}\left\{\rho^{-1}(\alpha)\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a)\right)^{\frac{\zeta}{k}-1}\right\}(u) = \frac{\Gamma\left(\dfrac{\zeta}{k}\right)}{u^{\frac{\zeta}{k}}}, \quad u > 0.$$

Proof. By the Definition 3.1, we have

$$\mathfrak{L}_{\Upsilon}^{\rho}\left\{\rho^{-1}(\alpha)\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a)\right)^{\frac{\zeta}{k}-1}\right\}(u)$$

$$= (s+1)\int_a^{\infty} e^{-u(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a))}\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a)\right)^{\frac{\zeta}{k}-1}\Upsilon^s(\alpha)\Upsilon'(\alpha)\,d\alpha. \tag{8}$$

Substitute $t = (\Upsilon^{s+1}(\alpha) - \Upsilon^{s+1}(a))$ on the right side of (8), we get

$$\mathfrak{L}_{\Upsilon}^{\rho}\left\{\rho^{-1}(\alpha)\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a)\right)^{\frac{\zeta}{k}-1}\right\}(u)$$
$$=\int_{0}^{\infty}e^{-ut}t^{\frac{\zeta}{k}-1}dt$$
$$=\int_{0}^{\infty}e^{-ut}\frac{(ut)^{\frac{\zeta}{k}-1}}{(u)^{\frac{\zeta}{k}-1}}\frac{u}{u}dt$$
$$=\frac{1}{u^{\frac{\zeta}{k}}}\int_{0}^{\infty}e^{-ut}(ut)^{\frac{\zeta}{k}-1}udt,$$

the proof is done.

Theorem 3.1. Let the function Ω be continuous on each interval a, α and of weighted Υ^{s+1}-exponential order. Then

$$\mathfrak{L}_{\Upsilon}^{\rho}\left(\left({}_{\Upsilon,k}^{\;\;s}\mathfrak{J}_{a^{+},\rho}^{\zeta}\Omega\right)(\alpha)\right)(u)=((s+1)\,uk)^{\frac{-\zeta}{k}}\,\mathfrak{L}_{\Upsilon}^{\rho}\{\Omega(\alpha)\}(u),$$

where $k>0$, $\rho(\alpha)\neq 0$, $s\in\mathbb{R}\backslash\{-1\}$.

Proof. By the Definitions 2.1, 1.6 and Proposition 3.1, we have

$$\mathfrak{L}_{\Upsilon}^{\rho}\{({}_{\Upsilon,k}^{\;\;s}\mathfrak{J}_{a^{+},\rho}^{\zeta}\Omega)(\alpha)\}(u)$$
$$=\mathfrak{L}_{\Upsilon}^{\rho}\left\{\frac{(s+1)^{1-\frac{\zeta}{k}}\rho^{-1}(\alpha)}{k\Gamma_{k}(\zeta)}\int_{a}^{\alpha}(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(t))^{\frac{\zeta}{k}-1}\Upsilon^{s}(t)\,\Upsilon'(t)\,\rho(t)\,\Omega(t)\,dt\right\}(u)$$
$$=\frac{(s+1)^{-\frac{\zeta}{k}}}{k\Gamma_{k}(\zeta)}\mathfrak{L}_{\Upsilon}^{\rho}\left\{\rho^{-1}(\alpha)\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a)\right)^{\frac{\zeta}{k}-1}*_{\Upsilon^{s+1}}^{\rho}\Omega(\alpha)\right\}(u)$$
$$=\frac{(s+1)^{-\frac{\zeta}{k}}}{k\Gamma_{k}(\zeta)}\mathfrak{L}_{\Upsilon}^{\rho}\left\{\rho^{-1}(\alpha)\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a)\right)^{\frac{\zeta}{k}-1}\right\}(u)\,\mathfrak{L}_{\Upsilon}^{\rho}\{\Omega(\alpha)\}(u)$$
$$=\frac{(s+1)^{-\frac{\zeta}{k}}}{k\Gamma_{k}(\zeta)}\frac{\Gamma\left(\dfrac{\zeta}{k}\right)}{u^{\frac{\zeta}{k}}}\mathfrak{L}_{\Upsilon}^{\rho}\{\Omega(\alpha)\}(u)$$
$$=((s+1)\,uk)^{-\frac{\zeta}{k}}\,\mathfrak{L}_{\Upsilon}^{\rho}\{\Omega(\alpha)\}(u).$$

This completes the proof.

Theorem 3.2. The generalized weighted Laplace transform of the novel derivative is

$$\mathfrak{L}_{\Upsilon}^{\rho}\left\{({}_{\Upsilon,k}^{\;\;s}\mathfrak{D}_{a^{+},\rho}^{\zeta}\Omega)(\alpha)\right\}(u)=(s+1)^{-\frac{nk-\zeta}{k}}(ku)^{\frac{\zeta}{k}}\,\mathfrak{L}_{\Upsilon}^{\rho}\{\Omega(\alpha)\}(u)$$
$$-k^{n}\sum_{m=0}^{n-1}u^{n-m-1}\left({}_{\Upsilon,k}^{\;\;s}\mathfrak{J}_{a^{+},\rho}^{nk-\zeta}\Omega\right)_{m}\left(a^{+}\right).$$

Proof. By the Definition 2.2, Theorem 1.2 and Theorem 3.1, we get

$$\begin{aligned}\mathfrak{L}^{\rho}_{\Upsilon}\left\{\left({}_{\Upsilon,k}^{\;s}\mathfrak{D}^{\zeta}_{a^+,\rho}\Omega\right)(\alpha)\right\}(u) &= \mathfrak{L}^{\rho}_{\Upsilon}\left\{\rho^{-1}(\alpha)\left(\frac{k}{\Upsilon^{s}(\alpha)\Upsilon'(\alpha)}\frac{d}{d\alpha}\right)^{n}\rho(\alpha)\left({}_{\Upsilon,k}^{\;s}\mathfrak{J}^{nk-\zeta}_{a^+,\rho}\Omega\right)(t)\right\}(u)\\
&= k^{n}u^{n}\mathfrak{L}^{\rho}_{\Upsilon}\{({}_{\Upsilon,k}^{\;s}\mathfrak{J}^{nk-\zeta}_{a^+,\rho}\Omega)(t)\}(u)\\
&\quad - k^{n}\sum_{m=0}^{n-1}u^{n-m-1}\left({}_{\Upsilon,k}^{\;s}\mathfrak{J}^{nk-\zeta}_{a^+,\rho}\Omega\right)_{m}\left(a^{+}\right)\\
&= (uk)^{n}\left((s+1)\,uk\right)^{-\frac{nk-\zeta}{k}}\mathfrak{L}^{\rho}_{\Upsilon}\{\Omega(\alpha)\}(u)\\
&\quad - k^{n}\sum_{m=0}^{n-1}u^{n-m-1}\left({}_{\Upsilon,k}^{\;s}\mathfrak{J}^{nk-\zeta}_{a^+,\rho}\Omega\right)_{m}\left(a^{+}\right)\\
&= (s+1)^{-\frac{nk-\zeta}{k}}(ku)^{\frac{\zeta}{k}}\mathfrak{L}^{\rho}_{\Upsilon}\{\Omega(\alpha)\}(u)\\
&\quad - k^{n}\sum_{m=0}^{n-1}u^{n-m-1}\left({}_{\Upsilon,k}^{\;s}\mathfrak{J}^{nk-\zeta}_{a^+,\rho}\Omega\right)_{m}\left(a^{+}\right).\end{aligned}$$

The proof is completed.

4 Fractional Free Electron Laser Equation with Solution

In this section, we investigate the fractional generalization FEL by using the introduced fractional integral given in (2) and the fractional derivative presented in (3). The series form solution is obtained by employing the weighted generalized Laplace transform introduced by Jarad et al. [20].

Theorem 4.1. The solution of the cauchy problem

$${}_{\Upsilon,k}^{\;s}\mathfrak{D}^{\zeta}_{a^+,\rho}\Omega(\alpha) = \lambda\,{}_{\Upsilon,k}^{\;s}\mathfrak{J}^{\eta}_{a^+,\rho}\Omega(\alpha) + f(\alpha), \tag{9}$$

$${}_{\Upsilon,k}^{\;s}\mathfrak{J}^{nk-\zeta}_{a^+,\rho}\Omega(a^{+}) = d, \quad d \geq 0, \tag{10}$$

where $\alpha \in (0,\infty), f \in L^{1}[a,\infty)$, $a \geq 0$, $\rho \neq 0$and $\lambda \in \mathbb{R}$ is given by

$$\begin{aligned}\Omega(\alpha) &= d\rho^{-1}(\alpha)\sum_{m=0}^{\infty}\lambda^{m}\frac{(s+1)^{-\frac{(\zeta+\eta-k)m+\zeta-k}{k}}}{\Gamma_{k}\left((\zeta+\eta)m+\zeta\right)}\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a)\right)^{\frac{(\zeta+\eta)m+\zeta}{k}-1}\\
&\quad + \sum_{m=0}^{\infty}\lambda^{m}(s+1)^{m+1}\left({}_{\Upsilon,k}^{\;s}\mathfrak{J}^{(\zeta+\eta)m+\zeta}_{a^+,\rho}f\right)(\alpha).\end{aligned} \tag{11}$$

Proof. Applying generalized weighted Laplace transform on (9) and using Theorems 3.1 and 3.2, we get

$$\mathfrak{L}^{\rho}_{\Upsilon}\{{}_{\Upsilon,k}^{\;s}\mathfrak{D}^{\zeta}_{a^+\rho}\Omega(\alpha)\}(u) = \lambda\mathfrak{L}^{\rho}_{\Upsilon}\{{}_{\Upsilon,k}^{\;s}\mathfrak{J}^{\eta}_{a^+\rho}\Omega(\alpha)\}(u) + \mathfrak{L}^{\rho}_{\Upsilon}\{f(\alpha)\}(u).$$

The above equation implies that

$$\mathfrak{L}_{\Upsilon}^{\rho}\{\Omega(\alpha)\}(u)$$

$$= dk\left[\frac{(s+1)^{\frac{k-\zeta}{k}}(ku)^{\frac{-\zeta}{k}}}{1-\lambda(s+1)^{\frac{k-\zeta-\eta}{k}}(ku)^{\frac{-(\zeta+\eta)}{k}}}\right]$$

$$+\left[\frac{(s+1)^{\frac{k-\zeta}{k}}(ku)^{\frac{-\zeta}{k}}}{1-\lambda(s+1)^{\frac{k-\zeta-\eta}{k}}(ku)^{\frac{-(\zeta+\eta)}{k}}}\right]\mathfrak{L}_{\Upsilon}^{\rho}\{f(\alpha)\}(u).$$

Taking $\left|\lambda(s+1)^{-\frac{\zeta+\eta-k}{k}}(ku)^{-\frac{\zeta+\eta}{k}}\right| \le 1$ and using the binomial expansion, we get

$$\mathfrak{L}_{\Upsilon}^{\rho}\{\Omega(\alpha)\}(u)$$

$$=\left\{dk(s+1)^{\frac{k-\zeta}{k}}(ku)^{\frac{-\zeta}{k}}(s+1)^{\frac{k-\zeta}{k}}(ku)^{\frac{-\zeta}{k}}\mathfrak{L}_{\Upsilon}^{\rho}\{f(\alpha)\}(u)\right\}$$

$$\times\sum_{m=0}^{\infty}\lambda^m(s+1)^{-\frac{(\zeta+\eta-k)m}{k}}(ku)^{-\frac{(\zeta+\eta)m}{k}}$$

$$= dk\sum_{m=0}^{\infty}\lambda^m(s+1)^{-\frac{(\zeta+\eta-k)m+\zeta-k}{k}}(ku)^{-\frac{(\zeta+\eta)m+\zeta}{k}}$$

$$+\sum_{m=0}^{\infty}\lambda^m(s+1)^{-\frac{(\zeta+\eta-k)m+\zeta-k}{k}}(ku)^{-\frac{(\zeta+\eta)m+\zeta}{k}}\left\{\mathfrak{L}_{\Upsilon}^{\rho}\{f(\alpha)\}(u)\right\}.$$

By using the inverse Laplace transform, we obtain

$$\Omega(\alpha) = d\rho^{-1}(\alpha)\sum_{m=0}^{\infty}\lambda^m\frac{(s+1)^{-\frac{(\zeta+\eta-k)m+\zeta-k}{k}}}{\Gamma_k((\zeta+\eta)m+\zeta)}\left(\Upsilon^{s+1}(\alpha)-\Upsilon^{s+1}(a)\right)^{\frac{(\zeta+\eta)m+\zeta}{k}-1}$$

$$+\sum_{m=0}^{\infty}\lambda^m(s+1)^{m+1}\left({}_{\Upsilon,k}^{\ s}\mathfrak{J}_{a^+,\rho}^{(\zeta+\eta)m+\zeta}f\right)(\alpha),$$

the result is completed.

Remark 4.1. If we set $s = 0$, $k = 1$, $\zeta = \eta = 1$, $f(\alpha) = 0$, $\rho = ir$, $\lambda = -i\Pi p$, $(r, p \in \mathbb{R})$ and $\Upsilon^{s+1}(\alpha) = \alpha$, in 9 and 10, then the original free electron laser equation given in [35] is obtained.

The following is the cauchy problem based on Theorem 4.1.

Example 4.1. The solution of the cauchy problem

$$\Upsilon,{}_{k}^{s}\mathfrak{D}_{a^+,\rho}^{\zeta}\Omega(\alpha) = \lambda\,{}_{\Upsilon,k}^{\ s}\mathfrak{J}_{a^+,\rho}^{\eta}\Omega(\alpha) + f(\alpha),$$

where

$$f(\alpha) = \rho^{-1}(\alpha)\left(\Omega^{(s+1)}(\alpha) - \Omega^{(s+1)}(a^+)\right) \tag{12}$$

subject to the condition

$${}_{\Upsilon,k}^{\ s}\mathfrak{J}_{a^+,\rho}^{nk-\zeta}\Omega(a^+) = 0 \tag{13}$$

with $\alpha \in (0,\infty)$, $a \geq 0$, $\rho \neq 0$ and $\lambda \in \mathbb{R}$ is given by

$$\Omega(\alpha) = \sum_{m=0}^{\infty} \lambda^m (s+1)^{(m+1)-\frac{\zeta}{k}} \frac{\Gamma_k((\zeta+\eta)m+\zeta)\left(\Upsilon^{s+1}(\alpha) - \Upsilon^{s+1}(a)\right)^{\frac{(\zeta+\eta)m+2\zeta}{k}-1} \rho^{-1}(\alpha)}{\Gamma_k((\zeta+\eta)m+2\zeta)} \tag{14}$$

Solution 4.1. For the function given by (12) subjected to the condition presented in (13) the Eq. (11) becomes

$$\Omega(\alpha) = \sum_{m=0}^{\infty} \lambda^m (s+1)^{m+1} \left({}_{\Upsilon}{}^{s}_{k}\mathfrak{J}^{(\zeta+\eta)m+\zeta}_{a^+,\rho} f\right)(\alpha). \tag{15}$$

Consider

$$({}_{\Upsilon}{}^{s}_{k}\mathfrak{J}^{(\zeta+\eta)m+\zeta}_{a^+,\rho} f)(\alpha) = {}_{\Upsilon}{}^{s}_{k}\mathfrak{J}^{(\zeta+\eta)m+\zeta}_{a^+,\rho} \left(\rho^{-1}(\alpha)\left(\Omega^{(s+1)}(\alpha) - \Omega^{(s+1)}(a^+)\right)\right)$$

$$= \frac{\Gamma_k((\zeta+\eta)m+\zeta)\left(\Omega^{s+1}(\alpha) - \Omega^{s+1}(a)\right)^{\frac{(\zeta+\eta)m+2\zeta}{k}-1} \rho^{-1}(\alpha)}{(s+1)^{\frac{\zeta}{k}}\Gamma_k((\zeta+\eta)m+2\zeta)} \tag{16}$$

Using (16) in (15), we obtain (14).

5 Fractional Kinetic Differ-Integral Equation with Solution

In the last decade, fractional calculus has opened up new vistas of research and brought a revolution in the study of fractional PDE's and ODE's [36–38]. Fractional kinetic equation has been successfully used to predict physical phenomena such as diffusion in permeable media, reactions and unwinding forms in complicated framework. The fractional form of the kinetic equation has gained attention due to the its relationship with the CTRW-theory [39]. This section is dedicated to investigating a new weighted fractional kinetic equations to explain the continuity of the motion of the material and the fundamental equations of natural sciences. The series solution of this new fractional kinetic equation by applying weighted generalized fractional laplace is also part of this section. The fractional kinetic equation is

$$b\left({}_{\Upsilon}{}^{s}_{k}\mathfrak{D}^{\zeta}_{a^+,\rho} N\right)(t) - N_0\Omega(t) \quad = c\left({}_{\Upsilon}{}^{s}_{k}\mathfrak{J}^{\eta}_{a^+,\rho} N\right)(t), \quad \Omega \in L^1[a,\infty) \tag{17}$$

subject to

$$\left({}_{\Upsilon}{}^{s}_{k}\mathfrak{J}^{nk-\zeta}_{a^+,\rho} N\right)(a^+) \quad = d, \quad d \geq 0, \tag{18}$$

where $a, \zeta \geq 0$, $b, c \in R (b \neq 0)$, $k > 0$, $n = \left[\frac{\zeta}{k}\right] = 1$.

Theorem 5.1. The solution of (17) with initial condition (18) is

$$N(t) = d\rho^{-1}(t) \sum_{m=0}^{\infty} \left(\frac{c}{b}\right)^m \frac{(s+1)^{-\frac{(\zeta-k)(m+1)+m\eta}{k}}}{\Gamma_k((\zeta+\eta)m+\zeta)} \left(\Upsilon^{s+1}(\alpha) - \Upsilon^{s+1}(a)\right)^{\frac{(\zeta+\eta)m+\zeta}{k}-1}$$

$$+ \frac{N_0}{b} \sum_{m=0}^{\infty} \left(\frac{c}{b}\right)^m (s+1)^{(m+1)} \left({}_{\Upsilon}{}^{s}_{k}\mathfrak{J}^{(\zeta+\eta)m+\zeta}_{a^+,\rho} \Omega\right)(t).$$

Proof. By applying the modified weighted Laplace transform on both side of (17), we get

$$b\mathfrak{L}^{\rho}_{\Upsilon}\{({}_{\Upsilon}{}^{s}_{k}\mathfrak{D}^{\zeta}_{a^+,\rho} N)(t)\}(u) - \mathfrak{L}^{\rho}_{\Upsilon}\{N_0\Omega(t)\}(u) \quad = c\mathfrak{L}^{\rho}_{\Upsilon}\{({}_{\Upsilon}{}^{s}_{k}\mathfrak{J}^{\eta}_{a^+,\rho} N)(t)\}(u).$$

Using Theorems 3.1 and 3.2, we get

$$b\,(s+1)^{-\frac{k-\zeta}{k}}\,(ku)^{\frac{\zeta}{k}}\,\mathfrak{L}_{\Upsilon}^{\rho}\{N(t)\}(u) - k\left({}_{\Upsilon,k}^{\;s}\mathfrak{J}_{a^+,\rho}^{k-\zeta}N\right)(a) - N_0\mathfrak{L}_{\Upsilon}^{\rho}\{\Omega(t)\}(u) = c\,(s+1)^{\frac{-\zeta}{k}}\,(uk)^{\frac{-\zeta}{k}}\,\mathfrak{L}_{\Upsilon}^{\rho}\{N(t)\}(u)$$

$$\left[\frac{b - c\,(s+1)^{-\frac{\zeta-k+\eta}{k}}\,(ku)^{-\frac{\zeta+\eta}{k}}}{(s+1)^{-\frac{\zeta-k}{k}}\,(ku)^{-\frac{\zeta}{k}}}\right]\mathfrak{L}_{\Upsilon}^{\rho}\{N(t)\} = bkd + N_0\mathfrak{L}_{\Upsilon}^{\rho}\{\Omega(t)\}(u)$$

$$\mathfrak{L}_{\Upsilon}^{\rho}\{N(t)\} = bkd\left[\frac{(s+1)^{-\frac{\zeta-k}{k}}\,(ku)^{-\frac{\zeta}{k}}}{b - c\,(s+1)^{-\frac{\zeta-k+\eta}{k}}\,(ku)^{-\frac{\zeta+\eta}{k}}}\right] + \left[\frac{(s+1)^{-\frac{\zeta-k}{k}}\,(ku)^{-\frac{\zeta}{k}}}{b - c\,(s+1)^{-\frac{\zeta-k+\eta}{k}}\,(ku)^{-\frac{\zeta+\eta}{k}}}\right] \times N_0\mathfrak{L}_{\Upsilon}^{\rho}\{\Omega(t)\}(u).$$

Taking $\left|\frac{c}{b}\,(s+1)^{-\frac{\zeta-k+\eta}{k}}\,(ku)^{-\frac{\zeta+\eta}{k}}\right| \le 1$, we get

$$\begin{aligned}
\mathfrak{L}_{\Upsilon}^{\rho}\{N(t)\} &= \left[kd\left[(s+1)^{-\frac{\zeta-k}{k}}\,(ku)^{-\frac{\zeta}{k}}\right] + a^{-1}N_0\left[(s+1)^{-\frac{\zeta-k}{k}}\,(ku)^{-\frac{\zeta}{k}}\right]\right] \\
&\quad \times \sum_{m=0}^{\infty}\left(\frac{c}{b}\right)^m (s+1)^{-\frac{(\zeta-k+\eta)m}{k}}\,(ku)^{-\frac{(\zeta+\eta)m}{k}}\,\mathfrak{L}_{\Upsilon}^{\rho}\{\Omega(t)\}(u) \\
&= kd\left[(s+1)^{-\frac{\zeta-k}{k}}\,(ku)^{-\frac{\zeta}{k}}\right]\sum_{m=0}^{\infty}\left(\frac{c}{b}\right)^m (s+1)^{-\frac{(\zeta-k+\eta)m}{k}}\,(ku)^{-\frac{(\zeta+\eta)m}{k}} \\
&\quad + b^{-1}N_0\left[(s+1)^{-\frac{\zeta-k}{k}}\,(ku)^{-\frac{\zeta}{k}}\right] \\
&\quad \times \sum_{m=0}^{\infty}\left(\frac{c}{b}\right)^m (s+1)^{-\frac{(\zeta-k+\eta)m}{k}}\,(ku)^{-\frac{(\zeta+\eta)m}{k}}\,\mathfrak{L}_{\Upsilon}^{\rho}\{\Omega(t)\}(u) \\
&= kd\sum_{m=0}^{\infty}\left(\frac{c}{b}\right)^m (s+1)^{-\frac{(\zeta-k)(m+1)+m\eta}{k}}\,(ku)^{-\frac{(\zeta+\eta)m+\zeta}{k}} \\
&\quad + \frac{N_0}{b}\sum_{m=0}^{\infty}\left(\frac{c}{b}\right)^m (s+1)^{-\frac{(\zeta-k)(m+1)+m\eta}{k}}\,(ku)^{-\frac{(\zeta+\eta)m+\zeta}{k}}\,\mathfrak{L}_{\Upsilon}^{\rho}\{\Omega(t)\}(u) \\
&= kd\sum_{m=0}^{\infty}\left(\frac{c}{b}\right)^m (s+1)^{-\frac{(\zeta-k)(m+1)+m\eta}{k}}\,(ku)^{-\frac{(\zeta+\eta)m+\zeta}{k}} \\
&\quad + \frac{N_0}{b}\sum_{m=0}^{\infty}\left(\frac{c}{b}\right)^m (s+1)^{-\frac{(\zeta+\eta)m+\zeta}{k}}\,(s+1)^{(m+1)}\,(ku)^{-\frac{(\zeta+\eta)m+\zeta}{k}}\,\mathfrak{L}_{\Upsilon}^{\rho}\{\Omega(t)\}(u).
\end{aligned}$$

By applying the inverse Laplace transform, we get

$$N(t) = d\rho^{-1}(t) \sum_{m=0}^{\infty} \left(\frac{c}{b}\right)^{m} \frac{(s+1)^{-\frac{(\zeta-k)(m+1)+m\eta}{k}}}{\Gamma_k\left((\zeta+\eta)m+\zeta\right)} \left(\Upsilon^{s+1}(\alpha) - \Upsilon^{s+1}(a)\right)^{\frac{(\zeta+\eta)m+\zeta}{k}-1}$$

$$+ \frac{N_0}{b} \sum_{m=0}^{\infty} \left(\frac{c}{b}\right)^{m} (s+1)^{(m+1)} \left({}_{\Upsilon,k}^{\;s}\mathfrak{J}_{a^+,\rho}^{(\zeta+\eta)m+\zeta}\Omega\right)(t).$$

Next, we include an example in the field of engineering using our defined operators.

Example 5.1. Consider a famous growth model given by

$$\Upsilon,{}_{k}^{s}\mathfrak{D}_{a^+,\rho}^{\zeta} N(t) - N(t) = 0 \tag{19}$$

subject to the condition

$${}_{\Upsilon,k}^{\;s}\mathfrak{J}_{a^+,\rho}^{nk-\zeta} N(0) = d_0, \tag{20}$$

where $\zeta \geq 0, k > 0, n = \left[\frac{\zeta}{k}\right] = 1$. The solution to the growth model (19) is

$$N(t) \;= d_0\rho^{-1}(t) \sum_{m=0}^{\infty} \frac{(s+1)^{-\frac{(\zeta-k)(m+1)+m}{k}}}{\Gamma_k\left((\zeta)m+\zeta\right)} \left(\Upsilon^{s+1}(\alpha) - \Upsilon^{s+1}(a)\right)^{\frac{(m+1)\zeta}{k}-1}. \tag{21}$$

Solution 5.1. By choosing $b = c = 1 N_0 = 0, \eta = 0$ in (17) and $a = 0, d = d_0$ in (18), we obtain the growth model with solution (21). Further with the choice of parameters $k = 1, s = 0, \zeta = 1.5$, $d_0 = 1, \rho(t) = 1$ and $\Upsilon(\alpha) = \alpha$, we get

$$N(\alpha) \;= \sum_{m=0}^{\infty} \frac{(\alpha)^{1.5m+0.5}}{\Gamma((1+m)1.5)}.$$

The graph of the function $N(\alpha)$ is presented as follows:

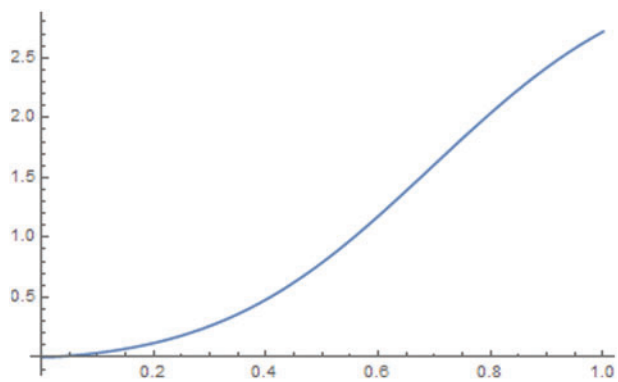

Figure 2: For $0 < \alpha < 1$, the graph in Fig. 2 indicates the increasing and convergent behaviour of the infinite series

6 Conclusion

In this paper, the weighted generalized fractional integral and derivative operators of Riemann-type are investigated. We discuss some properties of the fractional operators in certain spaces.

Specifically, the semi-group and inverse properties are proved for the introduced operators. The modified weighted Laplace transform of novel operators is also examined which is compatible with the introduced operators. It is worth mentioning that many established operators unify some operators that exist in literature. Finally, the solutions of the weighted generalized fractional free electron laser and kinetic equations are obtained by utilizing the skillful technique of the weighted Laplace transform, which has been applied in many mathematical and physical problems. Furthermore, a Cauchy problem and a growth model for a specific choice of parameters involved are designed and sketched in their graphs to check the validity.

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