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ARTICLE

Analysis and Numerical Computations of the Multi-Dimensional, Time-Fractional Model of Navier-Stokes Equation with a New Integral Transformation

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ABSTRACT

The analytical solution of the multi-dimensional, time-fractional model of Navier-Stokes equation using the triple and quadruple Elzaki transform decomposition method is presented in this article. The aforesaid model is analyzed by employing Caputo fractional derivative. We deliberated three stimulating examples that correspond to the triple and quadruple Elzaki transform decomposition methods, respectively. The findings illustrate that the established approaches are extremely helpful in obtaining exact and approximate solutions to the problems. The exact and estimated solutions are delineated via numerical simulation. The proposed analysis indicates that the projected configuration is extremely meticulous, highly efficient, and precise in understanding the behavior of complex evolutionary problems of both fractional and integer order that classify affiliated scientific fields and technology.

KEYWORDS

Caputo derivative; Elzaki transform; time-fractional Navier-Stokes equation; decomposition method

1 Introduction

Physical and technical workflows are recognized by fractional calculus (*FC*) and are succinctly explained by fractional differential equations (*FDEs*) [1–4]. Admittedly, classical simulation results of integer-order derivatives, which include empirical systems, do not underlie well in numerous situations [5–9]. *FC* has contributed substantially to quantum physics, nuclear magnetic resonance, image recognition, circuit theory, transport phenomena, chaos, and epidemiology. Diverse aspects of *FC* have been researched by Kilbas et al. [10–14].

Fractional differential equations (*FDEs*) are viewed as the most vital and effective mechanism for describing and modeling complex anomalies in general, such as seismic nonlinear vibration. Because of the revelation of fractal frameworks in finance, many researchers have examined fractional configurations in recent years. *FDEs* are also employed to simulate computational anatomy, biochemical mechanisms, and a variety of other natural or physical structures [15–19]. Nonlinear problems are interesting to architects, astronomers, and cosmologists since most physical processes in nature are nonlinear. On the other hand, nonlinear equations are complicated to fix and can yield fascinating results [20–26]. In the analysis of high-order nonlinear equations, the actual solutions of transition models contribute significantly.

Even so, humanity is constantly seeking improvements in correctness or accuracy, computation complexity, trustworthiness and appropriateness. Recently, Tarig Elzaki envisioned the Elazki transform ($\mathbb{E}$-transform) by merging well-noted transforms, the Laplace and Sumudu transformations. It can indubitably strengthen the quantitative expression of differential equations in the same way that Laplace and Sumudu transformations have. The $\mathbb{E}$-transform is developed from the traditional Fourier integral, with emphasis on its computational adaptability and promising effects. The $\mathbb{E}$-transform is aimed at resolving ordinary and *PDEs* in the time realm. The Fourier, Laplace, and Sumudu transforms, in particular, are impacting, pragmatic computational methodologies for addressing *FDEs*. It is worth noting that the $\mathbb{E}$-transformation was formerly recommended over an ancient, more intricate methodology, including the Sumudu technique. Nonetheless, we need to show that the Elzaki transformation can resolve issues that Laplace cannot [27]. In this article, the $\mathbb{E}$-transform is used to reconfigure the iterative algorithm, and the novel paradigm is known as the "Elzaki Adomian Decomposition Method ($EADM$)."

Mathematicians have progressively focused their consideration on approximate and analytical solutions to *PDEs* for developing advanced mathematical strategies address *PDEs*. Some well acquainted strategies concerning the solution of *PDEs* are the homotopy analysis method (HAM), new iteration method (NIM), Laplace transform algorithm (LTA), the Haar wavelet technique (HWT) and many more.

In 1982, the Navier-Stokes equation (*NSe*) was first devised by Navier [28]. The equation is a confluence of the momentum equation, the continuity equation, and the energy equation, and it can be considered Newton's second law of motion for fluid substances. The *NS*-model is significant because it explains a variety of interesting physical phenomena that occur in various areas of applied sciences, such as ocean circulation, atmosphere, air circulation across awing, and hydrological cycle in tubes, see [29,30].

Several researchers have attempted to solve the *NSe* problem using various approaches. To efficiently strengthen the algorithmic structure, Eltayeb et al. [31] proposed unifying the multi-Laplace transform decomposition method for *NSe*. Shah et al. [32] presented the analytical solution of *NSe* utilizing the natural transform decomposition method. Singh et al. [33] have expounded numerical simulation by considering fractional reduced differential transformation method for discovering an analytical solution of the time-fractional model of *NSe*. Khan et al. [34] introduced the Elzaki transform decomposition approach for finding the analytical solution of *NSe*.

Inspired by the work of [31,34], we aim to investigate the multi-dimensional time fractional model of *NSe* with the aid of triple Elzaki Adomian decomposition method ($TEADM$) and quadruple Elzaki Adomian decomposition method ($QEADM$). Elzaki transformation [35,36] and Adomian decomposition method (ADM) [37,38] are incorporated in the suggested framework. For the formulations of quantitative and qualitative analysis, the Elzaki transformation [39–41] and ADM [37,38] have been utilized independently and supply the exact solutions in terms of convergent series. The exact and approximated solution of nonlinear *NSe* are addressed using advanced technique. With the assistance of graphs, the findings are illustrated and validated. The new approach is demonstrated to solve fractional-order equations in a quite clear and concise way. Other modulation problems in different fields of mathematical and physical sciences can be addressed using the current approach.

The structure of the article is as follows: Section 2 represents the basic definitions related to fractional calculus and the proposed Elzaki transform. Section 3 illustrates the road map for the analytical approximations of triple and quadruple Elzaki Adomian decomposition methods. Numerical experiments and their graphical illustrations concerning the TEADM and QEADM are conducted in Section 4. Theoretical findings are elaborated via comparison analysis with the previous findings demonstrated in Section 5. In a nutshell, we summarized the concluding remarks with open problems.

2 Prelude

In this unit, we have addressed several of the key aspects of *FC* and triple Elzaki transform. For more details, see [10,12].

The Elzaki transform [35] is a novel integral operator described for functions of exponential order. We shall look at mappings in the set G that are specified by

$$G = \left\{\hbar(\xi) : \exists \mathcal{M}, \kappa_1, \kappa_2 > 0; |\hbar(\xi)| < \mathcal{M}e^{\frac{|\xi|}{\kappa}}; \xi \in (-1)^{\jmath} \times [0, +\infty)\right\}.$$

Definition 2.1. ([35]) Elzaki transform for function $\hbar(\xi)$ is defined as

$$\mathbb{E}[\hbar(\xi)] = \mathcal{H}(p) = p\int_0^{+\infty} \hbar(\xi)e^{-\frac{\xi}{p}}d\xi, \quad \xi > 0, \kappa_1 < p < \kappa_2. \tag{1}$$

This transform has a stronger resemblance to the Laplace transform such as $\mathcal{L}(\mathfrak{s}) = \eta\mathbb{E}\left(\frac{1}{\eta}\right)$. The Elzaki transform is an efficient tool for providing the solution of integral equations, *FDEs* and *PDEs*.

Now we define the triple Elzaki transform as follows:

Definition 2.2. For $\varpi_1, \varpi_2, \xi > 0$ and let $\hbar$ be a mapping of three variables ϖ_1, ϖ_2 and ξ. Then the triple Elzaki transform of $\hbar$ is stated as

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\hbar(\varpi_1, \varpi_2, \xi)\right] = \mathcal{H}(\eta, \zeta, \mathfrak{s}) = \eta\zeta\mathfrak{s}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \hbar(\varpi_1, \varpi_2, \xi)e^{-\frac{\varpi_1}{\eta}-\frac{\varpi_2}{\zeta}-\frac{\xi}{\mathfrak{s}}}d\xi d\varpi_2 d\varpi_1, \tag{2}$$

where $\eta, \zeta, \mathfrak{s} \in (\kappa_1, \kappa_2)$. Also, the triple Elzaki transforms of the first and second order partial derivatives are presented by

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\hbar_{\varpi_1}(\varpi_1, \varpi_2, \xi)\right] = \frac{1}{\eta}\mathcal{H}(\eta, \zeta, \mathfrak{s}) - \mathcal{H}(0, \zeta, \mathfrak{s}),$$

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\hbar_{\varpi_2}(\varpi_1, \varpi_2, \xi)\right] = \frac{1}{\zeta}\mathcal{H}(\eta, \zeta, \mathfrak{s}) - \mathcal{H}(\eta, 0, \mathfrak{s}),$$

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\hbar_{\xi}(\varpi_1,\varpi_2,\xi)\right]=\frac{1}{\mathfrak{s}}\mathcal{H}(\eta,\zeta,\mathfrak{s})-\mathcal{H}(\eta,\zeta,0). \tag{3}$$

Analogously, we have

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\hbar_{\varpi_1\varpi_1}(\varpi_1,\varpi_2,\xi)\right]=\frac{1}{\eta^2}\mathcal{H}(\eta,\zeta,\mathfrak{s})-\mathcal{H}(0,\zeta,\mathfrak{s})-\eta\hbar_{\varpi_1}(0,\zeta,\mathfrak{s}),$$

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\hbar_{\varpi_2\varpi_2}(\varpi_1,\varpi_2,\xi)\right]=\frac{1}{\zeta^2}\mathcal{H}(\eta,\zeta,\mathfrak{s})-\mathcal{H}(\eta,0,\mathfrak{s})-\zeta\hbar_{\varpi_1}(\eta,0,\mathfrak{s}),$$

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\hbar_{\xi\xi}(\varpi_1,\varpi_2,\xi)\right]=\frac{1}{\mathfrak{s}^2}\mathcal{H}(\eta,\zeta,\mathfrak{s})-\mathcal{H}(\eta,\zeta,0)-\mathfrak{s}\hbar_{\varpi_1}(\eta,\zeta,0). \tag{4}$$

The inverse triple Elzaki transform $\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathcal{H}(\eta,\zeta,\mathfrak{s})\right]=\hbar(\varpi_1,\varpi_2,\xi)$ is stated as follows:

$$\hbar(\varpi_1,\varpi_2,\xi)=\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathcal{H}(\eta,\zeta,\mathfrak{s})\right]=\eta\zeta\mathfrak{s}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}e^{\eta\varpi_1+\zeta\varpi_2+\mathfrak{s}\xi}\mathbb{E}\left(\frac{1}{\eta}\right)\mathbb{E}\left(\frac{1}{\zeta}\right)\mathbb{E}\left(\frac{1}{\mathfrak{s}}\right)d\eta\, d\zeta\, d\mathfrak{s}. \tag{5}$$

Definition 2.3. ([13]) The Caputo fractional operator of a function $\hbar$ of order $\phi>0$ can be stated as follows:

$${}^{c}\mathcal{D}^{\phi}\hbar(\varpi_1)=\begin{cases}\dfrac{1}{\Gamma(\ell-\phi)}\displaystyle\int_0^{\varpi_1}(\varpi_1-\xi)^{\ell-\phi-1}\hbar(\xi)d\xi, & \ell-1<\phi<\ell,\\ \dfrac{d^{\ell}}{d\varpi_1^{\ell}}\hbar(\varpi_1) & \ell=\phi.\end{cases}$$

Our next result is important for further investigation of this article.

Theorem 2.1. For $\phi_1,\phi_2,\phi_3>0,\ell-1<\phi_1\leq\ell,n-1<\phi_2\leq n,\ q-1<\phi_3\leq q$ and $\ell,n,q\in\mathbb{N}$, so that $f_1\in\mathbb{C}^{i}(\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+)$, $i=\max\{\ell,n,q\},f_1^{(i)}\in L_1[(0,a_1)\times(0,b_1)\times(0,c_1)]$ for any $a_1,b_1,c_1>0,|f_1(\varpi_1,\varpi_2,q)|\leq\mathcal{M}e^{\frac{\varpi_1}{\xi_1}+\frac{\varpi_2}{\xi_2}+\frac{\varpi_3}{\xi_3}},0<a_1<\varpi_1,0<b_1<\varpi_2,0<c_1<\xi$, then the triple Elzaki transforms of Caputo's fractional derivatives $\mathcal{D}_{\xi}^{\phi_1}\hbar(\varpi_1,\varpi_2,\xi),\mathcal{D}_{\xi}^{\phi_2}\hbar(\varpi_1,\varpi_2,\xi)$ and $\mathcal{D}_{\xi}^{\phi_3}\hbar(\varpi_1,\varpi_2,\xi)$ are described by

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\mathcal{D}_{\xi}^{\phi_1}\hbar(\varpi_1,\varpi_2,\xi)\right]=\mathfrak{s}^{-\phi_1}\mathcal{H}(\eta,\zeta,\mathfrak{s})-\sum_{i=0}^{\ell-1}\mathfrak{s}^{2-\phi_1+i}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_1}\left[\mathcal{D}_{\xi}^{i}\hbar(\varpi_1,\varpi_2,0)\right],\quad \ell-1<\phi_1<\ell,$$

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\mathcal{D}_{\varpi_2}^{\phi_2}\hbar(\varpi_1,\varpi_2,\xi)\right]=\zeta^{-\phi_2}\mathcal{H}(\eta,\zeta,\mathfrak{s})-\sum_{j=0}^{n-1}\zeta^{2-\phi_2+j}\mathbb{E}_{\varpi_1}\mathbb{E}_{\xi}\left[\mathcal{D}_{\varpi_2}^{j}\hbar(\varpi_1,0,\xi)\right],\quad n-1<\phi_2<n$$

and

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\mathcal{D}_{\varpi_1}^{\phi_3}\hbar(\varpi_1,\varpi_2,\xi)\right]=\eta^{-\phi_3}\mathcal{H}(\eta,\zeta,\mathfrak{s})-\sum_{\kappa=0}^{q-1}\eta^{2-\phi_3+\kappa}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\mathcal{D}_{\varpi_1}^{\kappa}\hbar(0,\varpi_2,\xi)\right],\quad q-1<\phi_3<q.$$

3 Description of the Method for Triple Elzaki Transform Decomposition Method

The constructive approach of the $TEADM$ for multi-dimensional time-fractional NSe is presented in this section. Assume the following framework of two-dimensional time-fractional NSe to demonstrate the underlying strategy of the $TEADM$:

$$\begin{aligned}
&\frac{\partial^{\phi}\Phi}{\partial\xi^{\phi}}+\Phi\frac{\partial\Phi}{\partial\varpi_1}+\Psi\frac{\partial\Phi}{\partial\varpi_2}=\rho_0\left(\frac{\partial^2\Phi}{\partial\varpi_1^2}+\frac{\partial^2\Phi}{\partial\varpi_2^2}\right)+\frac{1}{\rho}\frac{\partial\sigma}{\partial\varpi_1}, \quad \varpi_1,\varpi_2,\xi>0,\\
&\frac{\partial^{\phi}\Psi}{\partial\xi^{\phi}}+\Phi\frac{\partial\Psi}{\partial\varpi_1}+\Psi\frac{\partial\Psi}{\partial\varpi_2}=\rho_0\left(\frac{\partial^2\Psi}{\partial\varpi_1^2}+\frac{\partial^2\Psi}{\partial\varpi_2^2}\right)-\frac{1}{\rho}\frac{\partial\sigma}{\partial\varpi_2}, \quad \varpi_1,\varpi_2,\xi>0,
\end{aligned} \tag{6}$$

subject to

$$\Phi(\varpi_1,\varpi_2,0)=f_1(\varpi_1,\varpi_2), \qquad \Psi(\varpi_1,\varpi_2,0)=g_1(\varpi_1,\varpi_2),$$

where $\frac{\partial^{\phi}}{\partial\xi^{\phi}}$ is the Caputo fractional derivative and σ is the pressure. Also, if σ is known, then $\hbar_1=\frac{1}{\rho}\frac{\partial\sigma}{\partial\varpi_1}$ and $\hbar_2=\frac{1}{\rho}\frac{\partial\sigma}{\partial\varpi_2}$. To employ the $TEADM$, multiply (6) by ϖ_1, which leads us

$$\begin{aligned}
\frac{1}{\mathfrak{s}^{\phi}}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\Phi(\varpi_1,\varpi_2,\xi)\right]-\sum_{\kappa=0}^{\ell-1}\Phi_{(\kappa)}(\eta,\zeta,0)\mathfrak{s}^{2-\phi+\kappa}&=-\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\Phi\frac{\partial\Phi}{\partial\varpi_1}+\Psi\frac{\partial\Phi}{\partial\varpi_2}\right)\\
&\quad+\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\rho_0\left(\frac{\partial^2\Phi}{\partial\varpi_1^2}+\frac{\partial^2\Phi}{\partial\varpi_2^2}\right)+\hbar_1\right),\\
\frac{1}{\mathfrak{s}^{\phi}}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\Psi(\varpi_1,\varpi_2,\xi)\right]-\sum_{\kappa=0}^{\ell-1}\Psi_{(\kappa)}(\eta,\zeta,0)\mathfrak{s}^{2-\phi+\kappa}&=-\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\Phi\frac{\partial\Psi}{\partial\varpi_1}+\Psi\frac{\partial\Psi}{\partial\varpi_2}\right)\\
&\quad+\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\rho_0\left(\frac{\partial^2\Psi}{\partial\varpi_1^2}+\frac{\partial^2\Psi}{\partial\varpi_2^2}\right)\right)-\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\hbar_2\right).
\end{aligned} \tag{7}$$

By the virtue of Theorem 2.1, we obtain

$$\begin{aligned}
\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\Phi(\varpi_1,\varpi_2,\xi)\right]&=\mathfrak{s}^2\mathcal{F}_1(\eta,\zeta)-\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\Phi\frac{\partial\Phi}{\partial\varpi_1}+\Psi\frac{\partial\Phi}{\partial\varpi_2}\right)\\
&\quad+\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\rho_0\left(\frac{\partial^2\Phi}{\partial\varpi_1^2}+\frac{\partial^2\Phi}{\partial\varpi_2^2}\right)\right)+\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\hbar_1\right),\\
\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\Psi(\varpi_1,\varpi_2,\xi)\right]&=\mathfrak{s}^2\mathcal{G}_1(\eta,\zeta)-\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\Phi\frac{\partial\Psi}{\partial\varpi_1}+\Psi\frac{\partial\Psi}{\partial\varpi_2}\right)\\
&\quad+\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\rho_0\left(\frac{\partial^2\Psi}{\partial\varpi_1^2}+\frac{\partial^2\Psi}{\partial\varpi_2^2}\right)\right)-\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\hbar_2\right).
\end{aligned} \tag{8}$$

By implementing the triple inverse Elzaki transformation for (8), we obtain

$$\Phi(\varpi_1,\varpi_2,\xi)=\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Bigg[\mathfrak{s}^2\mathcal{F}_1(\eta,\zeta)-\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left(\Phi\frac{\partial\Phi}{\partial\varpi_1}+\Psi\frac{\partial\Phi}{\partial\varpi_2}\right)$$
$$+\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left(\rho_0\left(\frac{\partial^2\Phi}{\partial\varpi_1^2}+\frac{\partial^2\Phi}{\partial\varpi_2^2}\right)\right)+\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left(\hbar_1\right)\Bigg],$$
$$\Psi(\varpi_1,\varpi_2,\xi)=\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Bigg[\mathfrak{s}^2\mathcal{G}_1(\eta,\zeta)-\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left(\Phi\frac{\partial\Psi}{\partial\varpi_1}+\Psi\frac{\partial\Psi}{\partial\varpi_2}\right)$$
$$+\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left(\rho_0\left(\frac{\partial^2\Psi}{\partial\varpi_1^2}+\frac{\partial^2\Psi}{\partial\varpi_2^2}\right)\right)-\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left(\hbar_2\right)\Bigg].\tag{9}$$

The infinite series solutions $\Phi(\varpi_1,\varpi_2,\xi)$ and $\Psi(\varpi_1,\varpi_2,\xi)$ are presented as follows:

$$\Phi(\varpi_1,\varpi_2,\xi)=\sum_{\ell=0}^{+\infty}\Phi_\ell(\varpi_1,\varpi_2,\xi),\qquad\Psi(\varpi_1,\varpi_2,\xi)=\sum_{\ell=0}^{+\infty}\Psi_\ell(\varpi_1,\varpi_2,\xi).\tag{10}$$

Meanwhile, the non-linear expressions $\mathcal{N}_1=\Phi\frac{\partial\Phi}{\partial\varpi_1}$, $\mathcal{N}_2=\Psi\frac{\partial\Phi}{\partial\varpi_2}$, $\mathcal{N}_3=\Phi\frac{\partial\Psi}{\partial\varpi_1}$ and $\mathcal{N}_4=\Psi\frac{\partial\Psi}{\partial\varpi_2}$ are denoted by

$$\mathcal{N}_1(\Phi,\Psi)=\sum_{\ell=0}^{+\infty}\mathcal{A}_\ell,\qquad\mathcal{N}_2(\Phi,\Psi)=\sum_{\ell=0}^{+\infty}\mathcal{B}_\ell,\qquad\mathcal{N}_3(\Phi,\Psi)=\sum_{\ell=0}^{+\infty}\mathcal{C}_\ell,\qquad\mathcal{N}_4(\Phi,\Psi)=\sum_{\ell=0}^{+\infty}\mathcal{D}_\ell\tag{11}$$

and are presented by the Adomian polynomials as

$$\mathcal{A}_\ell=\frac{1}{\ell!}\left[\frac{\partial^\ell}{\partial\theta^\ell}\left\{\mathcal{N}_1\left(\sum_{\kappa=0}^{+\infty}\theta^\kappa\Phi_\kappa,\sum_{\kappa=0}^{+\infty}\theta^\kappa\Psi_\kappa\right)\right\}\right]_{\theta=0},$$
$$\mathcal{B}_\ell=\frac{1}{\ell!}\left[\frac{\partial^\ell}{\partial\theta^\ell}\left\{\mathcal{N}_2\left(\sum_{\kappa=0}^{+\infty}\theta^\kappa\Phi_\kappa,\sum_{\kappa=0}^{+\infty}\theta^\kappa\Psi_\kappa\right)\right\}\right]_{\theta=0},$$
$$\mathcal{C}_\ell=\frac{1}{\ell!}\left[\frac{\partial^\ell}{\partial\theta^\ell}\left\{\mathcal{N}_3\left(\sum_{\kappa=0}^{+\infty}\theta^\kappa\Phi_\kappa,\sum_{\kappa=0}^{+\infty}\theta^\kappa\Psi_\kappa\right)\right\}\right]_{\theta=0},$$
$$\mathcal{D}_\ell=\frac{1}{\ell!}\left[\frac{\partial^\ell}{\partial\theta^\ell}\left\{\mathcal{N}_4\left(\sum_{\kappa=0}^{+\infty}\theta^\kappa\Phi_\kappa,\sum_{\kappa=0}^{+\infty}\theta^\kappa\Psi_\kappa\right)\right\}\right]_{\theta=0}.\tag{12}$$

By inserting (12) into (9), we attain

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left[\sum_{\ell=0}^{+\infty}\Phi_\ell(\varpi_1,\varpi_2,\xi)\right]=\mathfrak{s}^2\mathcal{F}_1(\eta,\zeta)-\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left(\sum_{\ell=0}^{+\infty}\left(\mathcal{A}_\ell+\mathcal{B}_\ell\right)\right)$$
$$+\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left(\rho_0\left(\sum_{\ell=0}^{+\infty}\frac{\partial^2\Phi_\ell}{\partial\varpi_1^2}+\sum_{\ell=0}^{+\infty}\frac{\partial^2\Phi_\ell}{\partial\varpi_2^2}\right)\right)+\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left(\hbar_1\right)\tag{13}$$

and

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\sum_{\ell=0}^{+\infty}\Psi_\ell(\varpi_1,\varpi_2,\xi)\right]=\mathfrak{s}^2\mathcal{G}_1(\eta,\zeta)-\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\sum_{\ell=0}^{+\infty}\left(\mathcal{C}_\ell+\mathcal{D}_\ell\right)\right)$$
$$+\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\rho_0\left(\sum_{\ell=0}^{+\infty}\frac{\partial^2\Psi_\ell}{\partial\varpi_1^2}+\sum_{\ell=0}^{+\infty}\frac{\partial^2\Psi_\ell}{\partial\varpi_2^2}\right)\right)-\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\hbar_2\right). \quad (14)$$

By implementing the inverse Elzaki transformation to (13) and (14) we have

$$\sum_{\ell=0}^{+\infty}\Phi_\ell(\varpi_1,\varpi_2,\xi)=\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^2\mathcal{F}_1(\eta,\zeta)\right]-\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\sum_{\ell=0}^{+\infty}\left(\mathcal{A}_\ell+\mathcal{B}_\ell\right)\right)\right]$$
$$+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\rho_0\left(\sum_{\ell=0}^{+\infty}\frac{\partial^2\Phi_\ell}{\partial\varpi_1^2}+\sum_{\ell=0}^{+\infty}\frac{\partial^2\Phi_\ell}{\partial\varpi_2^2}\right)\right)+\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\hbar_1\right)\right] \quad (15)$$

and

$$\sum_{\ell=0}^{+\infty}\Psi_\ell(\varpi_1,\varpi_2,\xi)=\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^2\mathcal{G}_1(\eta,\zeta)\right]-\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\sum_{\ell=0}^{+\infty}\left(\mathcal{C}_\ell+\mathcal{D}_\ell\right)\right)\right]$$
$$+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\rho_0\left(\sum_{\ell=0}^{+\infty}\frac{\partial^2\Psi_\ell}{\partial\varpi_1^2}+\sum_{\ell=0}^{+\infty}\frac{\partial^2\Psi_\ell}{\partial\varpi_2^2}\right)\right)-\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\hbar_2\right)\right]. \quad (16)$$

Taking into account the *TEADM*, we develop iterative connections as follows:

$$\Phi_0(\varpi_1,\varpi_2,\xi)=\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^2\mathcal{F}_1(\eta,\zeta)\right],$$
$$\Psi_0(\varpi_1,\varpi_2,\xi)=\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^2\mathcal{G}_1(\eta,\zeta)\right], \quad (17)$$

and the rest of the components $\Phi_{\ell+1}$, $\ell\geq 0$ are presented by

$$\Phi_{\ell+1}(\varpi_1,\varpi_2,\xi)=-\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\mathcal{A}_\ell+\mathcal{B}_\ell\right)\right]$$
$$+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\rho_0\left(\frac{\partial^2\Phi_\ell}{\partial\varpi_1^2}+\frac{\partial^2\Phi_\ell}{\partial\varpi_2^2}\right)\right)+\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\hbar_1\right)\right]. \quad (18)$$

and

$$\Psi_{\ell+1}(\varpi_1,\varpi_2,\xi)=-\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\mathcal{C}_\ell+\mathcal{D}_\ell\right)\right]$$
$$+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\rho_0\left(\frac{\partial^2\Psi_\ell}{\partial\varpi_1^2}+\frac{\partial^2\Psi_\ell}{\partial\varpi_2^2}\right)\right)-\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left(\hbar_2\right)\right]. \quad (19)$$

where $\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}$ is the triple $\mathbb{E}$-transform in respecting ϖ_1,ϖ_2,ξ_1 and $\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}$ is the triple inverse $\mathbb{E}$-transform in respecting $\eta,\zeta,\mathfrak{s}$. For (17)–(19), we assumed that the triple inverse $\mathbb{E}$-transform exists.

4 Application of the Proposed Method

In this section, we evaluate the efficacy of our current techniques by introducing the decomposition method in combination with the triple and quadruple $\mathbb{E}$-transform.

Problem 4.1. We assume the two dimensional time-fractional *NSe*:

$$\begin{cases} \dfrac{\partial^{\phi}\Phi}{\partial\xi^{\phi}} + \Phi\dfrac{\partial\Phi}{\partial\varpi_1} + \Psi\dfrac{\partial\Phi}{\partial\varpi_2} = \rho\Big(\dfrac{\partial^2\Phi}{\partial\varpi_1^2} + \dfrac{\partial^2\Phi}{\partial\varpi_2^2}\Big) + \hbar, \quad \varpi_1, \varpi_2, \xi > 0, \\ \dfrac{\partial^{\phi}\Psi}{\partial\xi^{\phi}} + \Phi\dfrac{\partial\Psi}{\partial\varpi_1} + \Psi\dfrac{\partial\Psi}{\partial\varpi_2} = \rho\Big(\dfrac{\partial^2\Psi}{\partial\varpi_1^2} + \dfrac{\partial^2\Psi}{\partial\varpi_2^2}\Big) - \hbar, \quad \varpi_1, \varpi_2, \xi > 0, \\ \ell - 1 < \phi < \ell; \end{cases} \tag{20}$$

subject to

$$\begin{cases} \Phi(\varpi_1, \varpi_2, 0) = -\sin(\varpi_1 + \varpi_2), \\ \Psi(\varpi_1, \varpi_2, 0) = \sin(\varpi_1 + \varpi_2). \end{cases} \tag{21}$$

Proof. By applying the triple Elzaki transform on (20), we have

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\frac{\partial^{\phi}\Phi}{\partial\xi^{\phi}} + \Phi\frac{\partial\Phi}{\partial\varpi_1} + \Psi\frac{\partial\Phi}{\partial\varpi_2}\Big] = \mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\rho\Big(\frac{\partial^2\Phi}{\partial\varpi_1^2} + \frac{\partial^2\Phi}{\partial\varpi_2^2}\Big) + \hbar\Big],$$

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\frac{\partial^{\phi}\Psi}{\partial\xi^{\phi}} + \Phi\frac{\partial\Psi}{\partial\varpi_1} + \Psi\frac{\partial\Psi}{\partial\varpi_2}\Big] = \mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\rho\Big(\frac{\partial^2\Psi}{\partial\varpi_1^2} + \frac{\partial^2\Psi}{\partial\varpi_2^2}\Big) - \hbar\Big].$$

On making the use of differentiation property of the Elzaki transform, we obtain

$$\frac{1}{\mathfrak{s}^{\phi}}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\big[\Phi(\varpi_1, \varpi_2, \xi)\big] - \sum_{\kappa=0}^{\ell-1}\Phi_{(\kappa)}(\eta, \zeta, 0)\mathfrak{s}^{2-\phi+\kappa}$$
$$= -\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\Phi\frac{\partial\Phi}{\partial\varpi_1} + \Psi\frac{\partial\Phi}{\partial\varpi_2}\Big] + \mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\rho\Big(\frac{\partial^2\Phi}{\partial\varpi_1^2} + \frac{\partial^2\Phi}{\partial\varpi_2^2}\Big) + \hbar\Big],$$
$$\frac{1}{\mathfrak{s}^{\phi}}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\big[\Psi(\varpi_1, \varpi_2, \xi)\big] - \sum_{\kappa=0}^{\ell-1}\Psi_{(\kappa)}(\eta, \zeta, 0)\mathfrak{s}^{2-\phi+\kappa}$$
$$= -\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\Phi\frac{\partial\Psi}{\partial\varpi_1} + \Psi\frac{\partial\Psi}{\partial\varpi_2}\Big] + \mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\rho\Big(\frac{\partial^2\Psi}{\partial\varpi_1^2} + \frac{\partial^2\Psi}{\partial\varpi_2^2}\Big) - \hbar\Big]. \tag{22}$$

According to initial conditions and simple computations yields

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\big[\Phi(\varpi_1, \varpi_2, \xi)\big] = -\mathfrak{s}^2\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\Big[\sin(\varpi_1 + \varpi_2)\Big] - \mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\Phi\frac{\partial\Phi}{\partial\varpi_1} + \Psi\frac{\partial\Phi}{\partial\varpi_2}\Big]$$
$$+ \mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\rho\Big(\frac{\partial^2\Phi}{\partial\varpi_1^2} + \frac{\partial^2\Phi}{\partial\varpi_2^2}\Big) + \hbar\Big],$$

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\big[\Psi(\varpi_1, \varpi_2, \xi)\big] = \mathfrak{s}^2\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\Big[\sin(\varpi_1 + \varpi_2)\Big] - \mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\Phi\frac{\partial\Psi}{\partial\varpi_1} + \Psi\frac{\partial\Psi}{\partial\varpi_2}\Big]$$
$$+ \mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\rho\Big(\frac{\partial^2\Psi}{\partial\varpi_1^2} + \frac{\partial^2\Psi}{\partial\varpi_2^2}\Big) - \hbar\Big].$$

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\Phi(\varpi_1,\varpi_2,\xi)\right] = -\mathfrak{s}^2\frac{\eta^2\zeta^2(\eta+\zeta)}{(\eta^2+1)(\zeta^2+1)} - \mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\Phi\frac{\partial\Phi}{\partial\varpi_1}+\Psi\frac{\partial\Phi}{\partial\varpi_2}\right]$$
$$+\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2\Phi}{\partial\varpi_1^2}+\frac{\partial^2\Phi}{\partial\varpi_2^2}\right)+\hbar\right],$$
$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\Psi(\varpi_1,\varpi_2,\xi)\right] = \mathfrak{s}^2\frac{\eta^2\zeta^2(\eta+\zeta)}{(\eta^2+1)(\zeta^2+1)} - \mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\Phi\frac{\partial\Psi}{\partial\varpi_1}+\Psi\frac{\partial\Psi}{\partial\varpi_2}\right]$$
$$+\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2\Psi}{\partial\varpi_1^2}+\frac{\partial^2\Psi}{\partial\varpi_2^2}\right)-\hbar\right]. \tag{23}$$

Employing the inverse triple Elzaki transform for (23)

$$\Phi(\varpi_1,\varpi_2,\xi) = -\sin(\varpi_1+\varpi_2) + \mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}(\hbar)\right]$$
$$-\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\Phi\frac{\partial\Phi}{\partial\varpi_1}+\Psi\frac{\partial\Phi}{\partial\varpi_2}\right]\right]$$
$$+\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2\Phi}{\partial\varpi_1^2}+\frac{\partial^2\Phi}{\partial\varpi_2^2}\right)\right]\right],$$
$$\Psi(\varpi_1,\varpi_2,\xi) = \sin(\varpi_1+\varpi_2) - \mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}(\hbar)\right]$$
$$-\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\Phi\frac{\partial\Psi}{\partial\varpi_1}+\Psi\frac{\partial\Psi}{\partial\varpi_2}\right]\right]$$
$$+\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2\Psi}{\partial\varpi_1^2}+\frac{\partial^2\Psi}{\partial\varpi_2^2}\right)\right]\right]. \tag{24}$$

It follows that

$$\Phi(\varpi_1,\varpi_2,\xi) = -\sin(\varpi_1+\varpi_2) + \mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\hbar\eta^2\varsigma^2\mathfrak{s}^{\phi+2}\right]$$
$$-\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\Phi\frac{\partial\Phi}{\partial\varpi_1}+\Psi\frac{\partial\Phi}{\partial\varpi_2}\right]\right]$$
$$+\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2\Phi}{\partial\varpi_1^2}+\frac{\partial^2\Phi}{\partial\varpi_2^2}\right)\right]\right],$$
$$\Psi(\varpi_1,\varpi_2,\xi) = \sin(\varpi_1+\varpi_2) - \mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\hbar\eta^2\varsigma^2\mathfrak{s}^{\phi+2}\right]$$
$$-\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\Phi\frac{\partial\Psi}{\partial\varpi_1}+\Psi\frac{\partial\Psi}{\partial\varpi_2}\right]\right]$$
$$+\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2\Psi}{\partial\varpi_1^2}+\frac{\partial^2\Psi}{\partial\varpi_2^2}\right)\right]\right]. \tag{25}$$

Consequently, we have

$$\Phi(\varpi_1,\varpi_2,\xi)=-\sin(\varpi_1+\varpi_2)+\frac{\hbar\xi^{\phi}}{\Gamma(\phi+1)}$$
$$-\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\Phi\frac{\partial\Phi}{\partial\varpi_1}+\Psi\frac{\partial\Phi}{\partial\varpi_2}\right]\right]$$
$$+\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2\Phi}{\partial\varpi_1^2}+\frac{\partial^2\Phi}{\partial\varpi_2^2}\right)\right]\right],$$
$$\Psi(\varpi_1,\varpi_2,\xi)=\sin(\varpi_1+\varpi_2)-\frac{\hbar\xi^{\phi}}{\Gamma(\phi+1)}$$
$$-\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\Phi\frac{\partial\Psi}{\partial\varpi_1}+\Psi\frac{\partial\Psi}{\partial\varpi_2}\right]\right]$$
$$+\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2\Psi}{\partial\varpi_1^2}+\frac{\partial^2\Psi}{\partial\varpi_2^2}\right)\right]\right]. \tag{26}$$

The infinite series solution for unknown function $\Phi(\varpi_1,\varpi_2,\xi)$ and $\Psi(\varpi_1,\varpi_2,\xi)$ have the following form:

$$\Phi(\varpi_1,\varpi_2,\xi)=\sum_{\ell=0}^{+\infty}\Phi_{\ell}(\varpi_1,\varpi_2,\xi),\quad \Psi(\varpi_1,\varpi_2,\xi)=\sum_{\ell=0}^{+\infty}\Psi_{\ell}(\varpi_1,\varpi_2,\xi). \tag{27}$$

The Adomian approach is required to determine the zeroth elements Φ_0 and Ψ_0. Therefore, it includes initial condition, all of which are considered to be identified. Consequently, we devised

$$\Phi_0=-\sin(\varpi_1+\varpi_2)+\hbar\frac{\xi^{\phi}}{\Gamma(\phi+1)},\quad \Psi_0=\sin(\varpi_1+\varpi_2)-\hbar\frac{\xi^{\phi}}{\Gamma(\phi+1)}.$$

Remember that $\Phi\frac{\partial\Phi(\varpi_1,\xi)}{\partial\varpi_1}=\sum_{\ell=0}^{+\infty}\mathcal{A}_{\ell}$, $\Psi\frac{\partial\Phi(\varpi_1,\xi)}{\partial\varpi_2}=\sum_{\ell=0}^{+\infty}\mathcal{B}_{\ell}$, $\Phi\frac{\partial\Psi(\varpi_1,\xi)}{\partial\varpi_1}=\sum_{\ell=0}^{+\infty}\mathcal{C}_{\ell}$ and $\Psi\frac{\partial\Psi(\varpi_1,\xi)}{\partial\varpi_2}=\sum_{\ell=0}^{+\infty}\mathcal{D}_{\ell}$ are the Adomian terms and nonlinear terms were characterized. Now, (26) can be expressed in an iterative way with the aid of (27), as follows:

$$\sum_{\ell=0}^{+\infty}\Phi_{\ell}(\varpi_1,\varpi_2,\xi)=-\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\sum_{\ell=0}^{+\infty}\mathcal{A}_{\ell}+\sum_{\ell=0}^{+\infty}\mathcal{B}_{\ell}\right]\right]$$
$$+\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2\Phi_{\ell}}{\partial\varpi_1^2}+\frac{\partial^2\Phi_{\ell}}{\partial\varpi_2^2}\right)\right]\right],$$
$$\sum_{\ell=0}^{+\infty}\Psi_{\ell}(\varpi_1,\varpi_2,\xi)=-\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\sum_{\ell=0}^{+\infty}\mathcal{C}_{\ell}+\sum_{\ell=0}^{+\infty}\mathcal{D}_{\ell}\right]\right]$$
$$+\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2\Psi_{\ell}}{\partial\varpi_1^2}+\frac{\partial^2\Psi_{\ell}}{\partial\varpi_2^2}\right)\right]\right].$$

Thanks to (11), the Adomian polynomials will express all forms of nonlinearity as

$$\mathcal{A}_0 = \Phi_0 \frac{\partial \Phi_0}{\partial \varpi_1}, \quad \mathcal{A}_1 = \Phi_0 \frac{\partial \Phi_1}{\partial \varpi_1} + \Phi_1 \frac{\partial \Phi_0}{\partial \varpi_1}, \quad \mathcal{A}_2 = \Phi_0 \frac{\partial \Phi_2}{\partial \varpi_1} + \Phi_1 \frac{\partial \Phi_1}{\partial \varpi_1} + \Phi_2 \frac{\partial \Phi_0}{\partial \varpi_1},$$
$$\mathcal{B}_0 = \Psi_0 \frac{\partial \Phi_0}{\partial \varpi_2}, \quad \mathcal{B}_1 = \Psi_0 \frac{\partial \Phi_1}{\partial \varpi_2} + \Psi_1 \frac{\partial \Phi_0}{\partial \varpi_2}, \quad \mathcal{B}_2 = \Psi_0 \frac{\partial \Phi_2}{\partial \varpi_2} + \Psi_1 \frac{\partial \Phi_1}{\partial \varpi_2} + \Psi_2 \frac{\partial \Phi_0}{\partial \varpi_2},$$
$$\mathcal{C}_0 = \Phi_0 \frac{\partial \Psi_0}{\partial \varpi_1}, \quad \mathcal{C}_1 = \Phi_0 \frac{\partial \Psi_1}{\partial \varpi_1} + \Phi_1 \frac{\partial \Psi_0}{\partial \varpi_1}, \quad \mathcal{C}_2 = \Phi_0 \frac{\partial \Psi_2}{\partial \varpi_1} + \Phi_1 \frac{\partial \Psi_1}{\partial \varpi_1} + \Phi_2 \frac{\partial \Psi_0}{\partial \varpi_1},$$
$$\mathcal{D}_0 = \Psi_0 \frac{\partial \Psi_0}{\partial \varpi_2}, \quad \mathcal{D}_1 = \Psi_0 \frac{\partial \Psi_1}{\partial \varpi_2} + \Psi_1 \frac{\partial \Psi_0}{\partial \varpi_2}, \quad \mathcal{D}_2 = \Psi_0 \frac{\partial \Psi_2}{\partial \varpi_2} + \Psi_1 \frac{\partial \Psi_1}{\partial \varpi_2} + \Psi_2 \frac{\partial \Psi_0}{\partial \varpi_2}.$$

For $\ell = 0$, we have

$$\begin{aligned}
\Phi_1(\varpi_1, \varpi_2, \xi) &= -\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi \mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Big[\mathcal{A}_0 + \mathcal{B}_0\Big]\Big] \\
&\quad + \mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi \mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Big[\rho\Big(\frac{\partial^2 \Phi_0}{\partial \varpi_1^2} + \frac{\partial^2 \Phi_0}{\partial \varpi_2^2}\Big)\Big]\Big] \\
&= \mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[2\rho \mathfrak{s}^{\phi+2} \frac{\eta^2\zeta^2(\eta+\zeta)}{(\eta^2+1)(\zeta^2+1)}\Big] \\
&= 2\frac{\rho \xi^\phi}{\Gamma(\phi+1)} \sin(\varpi_1 + \varpi_2).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\Psi_1(\varpi_1, \varpi_2, \xi) &= -\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi \mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Big[\mathcal{C}_0 + \mathcal{D}_0\Big]\Big] \\
&\quad + \mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi \mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Big[\rho\Big(\frac{\partial^2 \Psi_0}{\partial \varpi_1^2} + \frac{\partial^2 \Psi_0}{\partial \varpi_2^2}\Big)\Big]\Big] \\
&= -2\frac{\rho \xi^\phi}{\Gamma(\phi+1)} \sin(\varpi_1 + \varpi_2)
\end{aligned}$$

Analogously, for $\ell = 1$,

$$\begin{aligned}
\Phi_2(\varpi_1, \varpi_2, \xi) &= -\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi \mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Big[\mathcal{A}_1 + \mathcal{B}_1\Big]\Big] \\
&\quad + \mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi \mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Big[\rho\Big(\frac{\partial^2 \Phi_1}{\partial \varpi_1^2} + \frac{\partial^2 \Phi_1}{\partial \varpi_2^2}\Big)\Big]\Big] \\
&= \mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[-4\rho^2 \mathfrak{s}^{2\phi+2} \frac{\eta^2\zeta^2(\eta+\zeta)}{(\eta^2+1)(\zeta^2+1)}\Big] \\
&= -\frac{(2\rho)^2 \xi^{2\phi}}{\Gamma(2\phi+1)} \sin(\varpi_1 + \varpi_2)
\end{aligned}$$

and

$$\Psi_2(\varpi_1,\varpi_2,\xi) = -\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Big[\mathcal{C}_1+\mathcal{D}_1\Big]\Big]$$
$$+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Big[\rho\Big(\frac{\partial^2\Psi_1}{\partial\varpi_1^2}+\frac{\partial^2\Psi_1}{\partial\varpi_2^2}\Big)\Big]\Big]$$
$$=\frac{(2\rho)^2\xi^{2\phi}}{\Gamma(2\phi+1)}\sin(\varpi_1+\varpi_2).$$

For $\ell = 2$,

$$\Phi_3(\varpi_1,\varpi_2,\xi) = -\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Big[\mathcal{A}_2+\mathcal{B}_2\Big]\Big]$$
$$+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Big[\rho\Big(\frac{\partial^2\Phi_2}{\partial\varpi_1^2}+\frac{\partial^2\Phi_2}{\partial\varpi_2^2}\Big)\Big]\Big]$$
$$=\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[8\rho^3\mathfrak{s}^{3\phi+2}\frac{\eta^2\zeta^2(\eta+\zeta)}{(\eta^2+1)(\zeta^2+1)}\Big]$$
$$=\frac{(2\rho)^3\xi^{3\phi}}{\Gamma(3\phi+1)}\sin(\varpi_1+\varpi_2)$$

and

$$\Psi_3(\varpi_1,\varpi_2,\xi) = -\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Big[\mathcal{C}_2+\mathcal{D}_2\Big]\Big]$$
$$+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Big[\rho\Big(\frac{\partial^2\Psi_2}{\partial\varpi_1^2}+\frac{\partial^2\Psi_2}{\partial\varpi_2^2}\Big)\Big]\Big]$$
$$=-\frac{(2\rho)^3\xi^{3\phi}}{\Gamma(3\phi+1)}\sin(\varpi_1+\varpi_2).$$

Continuing the same way, the iterative terms of Φ_ℓ and Ψ_ℓ, $(\ell > 4)$ are presented as follows:

$$\Phi(\varpi_1,\varpi_2,\xi) = \sum_{\ell=0}^{+\infty}\Phi_\ell(\varpi_1,\varpi_2,\xi) = \Phi_0(\varpi_1,\varpi_2,\xi)+\Phi_1(\varpi_1,\varpi_2,\xi)+\Phi_2(\varpi_1,\varpi_2,\xi)$$
$$+\Phi_3(\varpi_1,\varpi_2,\xi)+...,$$
$$\Psi(\varpi_1,\varpi_2,\xi) = \sum_{\ell=0}^{+\infty}\Psi_\ell(\varpi_1,\varpi_2,\xi) = \Psi_0(\varpi_1,\varpi_2,\xi)+\Psi_1(\varpi_1,\varpi_2,\xi)+\Psi_2(\varpi_1,\varpi_2,\xi)$$
$$+\Psi_3(\varpi_1,\varpi_2,\xi)+...,$$
$$\Phi(\varpi_1,\varpi_2,\xi) = \frac{\hbar\xi^\phi}{\Gamma(\phi+1)} - \sin(\varpi_1+\varpi_2)\Big[1-2\frac{\rho\xi^\phi}{\Gamma(\phi+1)}+4\frac{\rho^2\xi^{2\phi}}{\Gamma(2\phi+1)}-8\frac{\rho^3\xi^{3\phi}}{\Gamma(3\phi+1)}+...\Big],$$
$$\Psi(\varpi_1,\varpi_2,\xi) = -\frac{\hbar\xi^\phi}{\Gamma(\phi+1)} + \sin(\varpi_1+\varpi_2)\Big[1-2\frac{\rho\xi^\phi}{\Gamma(\phi+1)}+4\frac{\rho^2\xi^{2\phi}}{\Gamma(2\phi+1)}-8\frac{\rho^3\xi^{3\phi}}{\Gamma(3\phi+1)}+...\Big].$$

At $\phi = 1$ and $\hbar = 0$, the actual solution of classical *NSe* is

$$\Phi(\varpi_1, \varpi_2, \xi) = -\exp(-2\rho\xi)\sin(\varpi_1 + \varpi_2), \quad \Psi(\varpi_1, \varpi_2, \xi) = \exp(-2\rho\xi)\sin(\varpi_1 + \varpi_2).$$

Problem 4.2. We assume the two dimensional time-fractional *NSe*:

$$\begin{cases} \dfrac{\partial^{\phi}\Phi(\varpi_1, \xi)}{\partial \xi^{\phi}} + \Phi\dfrac{\partial \Phi(\varpi_1, \xi)}{\partial \varpi_1} + \Psi\dfrac{\partial \Phi(\varpi_1, \xi)}{\partial \varpi_2} = \rho\Big(\dfrac{\partial^2 \Phi}{\partial \varpi_1^2} + \dfrac{\partial^2 \Phi}{\partial \varpi_2^2}\Big) + \hbar, \\ \dfrac{\partial^{\phi}\Psi(\varpi_1, \xi)}{\partial \xi^{\phi}} + \Phi\dfrac{\partial \Psi(\varpi_1, \xi)}{\partial \varpi_1} + \Psi\dfrac{\partial \Psi(\varpi_1, \xi)}{\partial \varpi_2} = \rho\Big(\dfrac{\partial^2 \Psi}{\partial \varpi_1^2} + \dfrac{\partial^2 \Psi}{\partial \varpi_2^2}\Big) - \hbar, \end{cases} \tag{28}$$

subject to

$$\begin{cases} \Phi(\varpi_1, \varpi_2, 0) = -\exp(\varpi_1 + \varpi_2), \\ \Psi(\varpi_1, \varpi_2, 0) = \exp(\varpi_1 + \varpi_2). \end{cases} \tag{29}$$

Proof. By means of (22) and according to initial conditions given in (29), we have

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\big[\Phi(\varpi_1, \varpi_2, \xi)\big] = -\mathfrak{s}^2\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\Big[\exp(\varpi_1 + \varpi_2)\Big] - \mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\Phi\frac{\partial \Phi(\varpi_1, \xi)}{\partial \varpi_1} + \Psi\frac{\partial \Phi(\varpi_1, \xi)}{\partial \varpi_2}\Big]$$
$$+ \mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\rho\Big(\frac{\partial^2 \Phi}{\partial \varpi_1^2} + \frac{\partial^2 \Phi}{\partial \varpi_2^2}\Big) + \hbar\Big],$$
$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\big[\Psi(\varpi_1, \varpi_2, \xi)\big] = \mathfrak{s}^2\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\Big[\exp(\varpi_1 + \varpi_2)\Big] - \mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\Phi\frac{\partial \Psi(\varpi_1, \xi)}{\partial \varpi_1} + \Psi\frac{\partial \Psi(\varpi_1, \xi)}{\partial \varpi_2}\Big]$$
$$+ \mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\rho\Big(\frac{\partial^2 \Psi}{\partial \varpi_1^2} + \frac{\partial^2 \Psi}{\partial \varpi_2^2}\Big) - \hbar\Big].$$
$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\big[\Phi(\varpi_1, \varpi_2, \xi)\big] = -\mathfrak{s}^2\frac{\eta^2\zeta^2}{(1-\eta)(1-\zeta)} - \mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\Phi\frac{\partial \Phi(\varpi_1, \xi)}{\partial \varpi_1} + \Psi\frac{\partial \Phi(\varpi_1, \xi)}{\partial \varpi_2}\Big]$$
$$+ \mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\rho\Big(\frac{\partial^2 \Phi}{\partial \varpi_1^2} + \frac{\partial^2 \Phi}{\partial \varpi_2^2}\Big) + \hbar\Big],$$
$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\big[\Psi(\varpi_1, \varpi_2, \xi)\big] = \mathfrak{s}^2\frac{\eta^2\zeta^2}{(1-\eta)(1-\zeta)} - \mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\Phi\frac{\partial \Psi(\varpi_1, \xi)}{\partial \varpi_1} + \Psi\frac{\partial \Psi(\varpi_1, \xi)}{\partial \varpi_2}\Big]$$
$$+ \mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\rho\Big(\frac{\partial^2 \Psi}{\partial \varpi_1^2} + \frac{\partial^2 \Psi}{\partial \varpi_2^2}\Big) - \hbar\Big]. \tag{30}$$

Employing the inverse triple Elzaki transform for (30)

$$\Phi(\varpi_1, \varpi_2, \xi) = -\exp(\varpi_1 + \varpi_2) + \mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\Big[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}(\hbar)\Big]$$
$$- \mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\Big[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\Phi\frac{\partial \Phi(\varpi_1, \xi)}{\partial \varpi_1} + \Psi\frac{\partial \Phi(\varpi_1, \xi)}{\partial \varpi_2}\Big]\Big]$$
$$+ \mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\Big[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\Big[\rho\Big(\frac{\partial^2 \Phi}{\partial \varpi_1^2} + \frac{\partial^2 \Phi}{\partial \varpi_2^2}\Big)\Big]\Big],$$

$$\Psi(\varpi_1,\varpi_2,\xi)=\exp(\varpi_1+\varpi_2)-\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi(\hbar)\Big]$$
$$-\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left[\Phi\frac{\partial\Psi(\varpi_1,\xi)}{\partial\varpi_1}+\Psi\frac{\partial\Psi(\varpi_1,\xi)}{\partial\varpi_2}\right]\right]$$
$$+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left[\rho\Big(\frac{\partial^2\Psi}{\partial\varpi_1^2}+\frac{\partial^2\Psi}{\partial\varpi_2^2}\Big)\right]\right]. \tag{31}$$

It follows that

$$\Phi(\varpi_1,\varpi_2,\xi)=-\exp(\varpi_1+\varpi_2)+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\hbar\eta^2\varsigma^2\mathfrak{s}^{\phi+2}\Big]$$
$$-\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left[\Phi\frac{\partial\Phi(\varpi_1,\xi)}{\partial\varpi_1}+\Psi\frac{\partial\Phi(\varpi_1,\xi)}{\partial\varpi_2}\right]\right]$$
$$+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left[\rho\Big(\frac{\partial^2\Phi}{\partial\varpi_1^2}+\frac{\partial^2\Phi}{\partial\varpi_2^2}\Big)\right]\right],$$
$$\Psi(\varpi_1,\varpi_2,\xi)=\exp(\varpi_1+\varpi_2)-\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\hbar\eta^2\varsigma^2\mathfrak{s}^{\phi+2}\Big]$$
$$-\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left[\Phi\frac{\partial\Psi(\varpi_1,\xi)}{\partial\varpi_1}+\Psi\frac{\partial\Psi(\varpi_1,\xi)}{\partial\varpi_2}\right]\right]$$
$$+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left[\rho\Big(\frac{\partial^2\Psi}{\partial\varpi_1^2}+\frac{\partial^2\Psi}{\partial\varpi_2^2}\Big)\right]\right]. \tag{32}$$

Consequently, we have

$$\Phi(\varpi_1,\varpi_2,\xi)=-\exp(\varpi_1+\varpi_2)+\frac{\hbar\xi^\phi}{\Gamma(\phi+1)}-\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\hbar\eta^2\varsigma^2\mathfrak{s}^{\phi+2}\Big]$$
$$-\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left[\Phi\frac{\partial\Phi(\varpi_1,\xi)}{\partial\varpi_1}+\Psi\frac{\partial\Phi(\varpi_1,\xi)}{\partial\varpi_2}\right]\right]$$
$$+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left[\rho\Big(\frac{\partial^2\Phi}{\partial\varpi_1^2}+\frac{\partial^2\Phi}{\partial\varpi_2^2}\Big)\right]\right],$$
$$\Psi(\varpi_1,\varpi_2,\xi)=\exp(\varpi_1+\varpi_2)-\frac{\hbar\xi^\phi}{\Gamma(\phi+1)}$$
$$+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left[\Phi\frac{\partial\Psi(\varpi_1,\xi)}{\partial\varpi_1}+\Psi\frac{\partial\Psi(\varpi_1,\xi)}{\partial\varpi_2}\right]\right]$$
$$+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\left[\rho\Big(\frac{\partial^2\Psi}{\partial\varpi_1^2}+\frac{\partial^2\Psi}{\partial\varpi_2^2}\Big)\right]\right]. \tag{33}$$

The infinite series solution for unknown function $\Phi(\varpi_1,\varpi_2,\xi)$ and $\Psi(\varpi_1,\varpi_2,\xi)$ have the following form:

$$\Phi(\varpi_1,\varpi_2,\xi)=\sum_{\ell=0}^{+\infty}\Phi_\ell(\varpi_1,\varpi_2,\xi),\quad \Psi(\varpi_1,\varpi_2,\xi)=\sum_{\ell=0}^{+\infty}\Psi_\ell(\varpi_1,\varpi_2,\xi). \tag{34}$$

The Adomian approach is required to determine the zeroth elements Φ_0 and Ψ_0. Therefore, it includes initial condition, all of which are considered to be identified. Consequently, we devised

$$\Phi_0 = -\exp(\varpi_1 + \varpi_2) + \frac{\hbar \xi^{\phi}}{\Gamma(\phi + 1)}, \quad \Psi_0 = \exp(\varpi_1 + \varpi_2) - \frac{\hbar \xi^{\phi}}{\Gamma(\phi + 1)}.$$

Remember that $\Phi \dfrac{\partial \Phi(\varpi_1, \xi)}{\partial \varpi_1} = \sum_{\ell=0}^{+\infty} \mathcal{A}_\ell$, $\Psi \dfrac{\partial \Phi(\varpi_1, \xi)}{\partial \varpi_2} = \sum_{\ell=0}^{+\infty} \mathcal{B}_\ell$, $\Phi \dfrac{\partial \Psi(\varpi_1, \xi)}{\partial \varpi_1} = \sum_{\ell=0}^{+\infty} \mathcal{C}_\ell$ and $\Psi \dfrac{\partial \Psi(\varpi_1, \xi)}{\partial \varpi_2} = \sum_{\ell=0}^{+\infty} \mathcal{D}_\ell$ are the Adomian terms and nonlinear terms were characterized. Now, (33) can be expressed in an iterative way with the aid of (34), as follows:

$$\begin{aligned}
\sum_{\ell=0}^{+\infty} \Phi_\ell(\varpi_1, \varpi_2, \xi) = &-\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\sum_{\ell=0}^{+\infty}\mathcal{A}_\ell + \sum_{\ell=0}^{+\infty}\mathcal{B}_\ell\right]\right] \\
&+ \mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2 \Phi_\ell}{\partial \varpi_1^2} + \frac{\partial^2 \Phi_\ell}{\partial \varpi_2^2}\right)\right]\right], \\
\sum_{\ell=0}^{+\infty} \Psi_\ell(\varpi_1, \varpi_2, \xi) = &-\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\sum_{\ell=0}^{+\infty}\mathcal{C}_\ell + \sum_{\ell=0}^{+\infty}\mathcal{D}_\ell\right]\right] \\
&+ \mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2 \Psi_\ell}{\partial \varpi_1^2} + \frac{\partial^2 \Psi_\ell}{\partial \varpi_2^2}\right)\right]\right].
\end{aligned}$$

Thanks to (11), the Adomian polynomials will express all forms of nonlinearity as

$$\mathcal{A}_0 = \Phi_0\frac{\partial \Phi_0}{\partial \varpi_1}, \quad \mathcal{A}_1 = \Phi_0\frac{\partial \Phi_1}{\partial \varpi_1} + \Phi_1\frac{\partial \Phi_0}{\partial \varpi_1}, \quad \mathcal{A}_2 = \Phi_0\frac{\partial \Phi_2}{\partial \varpi_1} + \Phi_1\frac{\partial \Phi_1}{\partial \varpi_1} + \Phi_2\frac{\partial \Phi_0}{\partial \varpi_1},$$

$$\mathcal{B}_0 = \Psi_0\frac{\partial \Phi_0}{\partial \varpi_2}, \quad \mathcal{B}_1 = \Psi_0\frac{\partial \Phi_1}{\partial \varpi_2} + \Psi_1\frac{\partial \Phi_0}{\partial \varpi_2}, \quad \mathcal{B}_2 = \Psi_0\frac{\partial \Phi_2}{\partial \varpi_2} + \Psi_1\frac{\partial \Phi_1}{\partial \varpi_2} + \Psi_2\frac{\partial \Phi_0}{\partial \varpi_2},$$

$$\mathcal{C}_0 = \Phi_0\frac{\partial \Psi_0}{\partial \varpi_1}, \quad \mathcal{C}_1 = \Phi_0\frac{\partial \Psi_1}{\partial \varpi_1} + \Phi_1\frac{\partial \Psi_0}{\partial \varpi_1}, \quad \mathcal{C}_2 = \Phi_0\frac{\partial \Psi_2}{\partial \varpi_1} + \Phi_1\frac{\partial \Psi_1}{\partial \varpi_1} + \Phi_2\frac{\partial \Psi_0}{\partial \varpi_1},$$

$$\mathcal{D}_0 = \Psi_0\frac{\partial \Psi_0}{\partial \varpi_2}, \quad \mathcal{D}_1 = \Psi_0\frac{\partial \Psi_1}{\partial \varpi_2} + \Psi_1\frac{\partial \Psi_0}{\partial \varpi_2}, \quad \mathcal{D}_2 = \Psi_0\frac{\partial \Psi_2}{\partial \varpi_2} + \Psi_1\frac{\partial \Psi_1}{\partial \varpi_2} + \Psi_2\frac{\partial \Psi_0}{\partial \varpi_2}.$$

For $\ell = 0$, we have

$$\begin{aligned}
\Phi_1(\varpi_1, \varpi_2, \xi) &= -\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\mathcal{A}_0 + \mathcal{B}_0\right]\right] \\
&\quad + \mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2 \Phi_0}{\partial \varpi_1^2} + \frac{\partial^2 \Phi_0}{\partial \varpi_2^2}\right)\right]\right] \\
&= \mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[-2\rho\mathfrak{s}^{\phi+2}\frac{\eta^2\zeta^2}{(1-\eta)(1-\zeta)}\right] \\
&= -2\frac{\rho\xi^{\phi}}{\Gamma(\phi+1)}\exp(\varpi_1 + \varpi_2).
\end{aligned}$$

In a similar manner, we obtain

$$\begin{aligned}\Psi_1(\varpi_1,\varpi_2,\xi) &= -\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Big[\mathcal{C}_0+\mathcal{D}_0\Big]\Big]\\ &\quad+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Bigg[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Bigg[\rho\Big(\frac{\partial^2\Psi_0}{\partial\varpi_1^2}+\frac{\partial^2\Psi_0}{\partial\varpi_2^2}\Big)\Bigg]\Bigg]\\ &= 2\frac{\rho\xi^\phi}{\Gamma(\phi+1)}\exp(\varpi_1+\varpi_2).\end{aligned}$$

Analogously, for $\ell=1$,

$$\begin{aligned}\Phi_2(\varpi_1,\varpi_2,\xi) &= -\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Big[\mathcal{A}_1+\mathcal{B}_1\Big]\Big]\\ &\quad+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Bigg[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Bigg[\rho\Big(\frac{\partial^2\Phi_1}{\partial\varpi_1^2}+\frac{\partial^2\Phi_1}{\partial\varpi_2^2}\Big)\Bigg]\Bigg]\\ &= \mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[-4\rho^2\mathfrak{s}^{2\phi+2}\frac{\eta^2\zeta^2}{(1-\eta)(1-\zeta)}\Big]\\ &= -\frac{(2\rho)^2\xi^{2\phi}}{\Gamma(2\phi+1)}\exp(\varpi_1+\varpi_2)\end{aligned}$$

and

$$\begin{aligned}\Psi_2(\varpi_1,\varpi_2,\xi) &= -\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Big[\mathcal{C}_1+\mathcal{D}_1\Big]\Big]\\ &\quad+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Bigg[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Bigg[\rho\Big(\frac{\partial^2\Psi_1}{\partial\varpi_1^2}+\frac{\partial^2\Psi_1}{\partial\varpi_2^2}\Big)\Bigg]\Bigg]\\ &= \frac{(2\rho)^2\xi^{2\phi}}{\Gamma(2\phi+1)}\exp(\varpi_1+\varpi_2).\end{aligned}$$

For $\ell=2$,

$$\begin{aligned}\Phi_3(\varpi_1,\varpi_2,\xi) &= -\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Big[\mathcal{A}_2+\mathcal{B}_2\Big]\Big]\\ &\quad+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Bigg[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Bigg[\rho\Big(\frac{\partial^2\Phi_2}{\partial\varpi_1^2}+\frac{\partial^2\Phi_2}{\partial\varpi_2^2}\Big)\Bigg]\Bigg]\\ &= \mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[-8\rho^3\mathfrak{s}^{3\phi+2}\frac{\eta^2\zeta^2}{(1-\eta)(1-\zeta)}\Big]\\ &= -\frac{(2\rho)^3\xi^{3\phi}}{\Gamma(3\phi+1)}\exp(\varpi_1+\varpi_2)\end{aligned}$$

and

$$\begin{aligned}\Psi_3(\varpi_1,\varpi_2,\xi) &= -\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Big[\mathcal{C}_2+\mathcal{D}_2\Big]\Big]\\ &\quad+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Bigg[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_\xi\Bigg[\rho\Big(\frac{\partial^2\Psi_2}{\partial\varpi_1^2}+\frac{\partial^2\Psi_2}{\partial\varpi_2^2}\Big)\Bigg]\Bigg]\end{aligned}$$

$$= \frac{(2\rho)^3 \xi^{3\phi}}{\Gamma(3\phi+1)} \exp(\varpi_1 + \varpi_2).$$

Continuing the same way, the iterative terms of Φ_ℓ and Ψ_ℓ, $(\ell > 4)$ are presented as follows:

$$\Phi(\varpi_1, \varpi_2, \xi) = \sum_{\ell=0}^{+\infty} \Phi_\ell(\varpi_1, \varpi_2, \xi) = \Phi_0(\varpi_1, \varpi_2, \xi) + \Phi_1(\varpi_1, \varpi_2, \xi) + \Phi_2(\varpi_1, \varpi_2, \xi) + \Phi_3(\varpi_1, \varpi_2, \xi) + ...,$$

$$\Psi(\varpi_1, \varpi_2, \xi) = \sum_{\ell=0}^{+\infty} \Psi_\ell(\varpi_1, \varpi_2, \xi) = \Psi_0(\varpi_1, \varpi_2, \xi) + \Psi_1(\varpi_1, \varpi_2, \xi) + \Psi_2(\varpi_1, \varpi_2, \xi) + \Psi_3(\varpi_1, \varpi_2, \xi) + ...,$$

$$\Phi(\varpi_1, \varpi_2, \xi) = \frac{\hbar \xi^\phi}{\Gamma(\phi+1)} - \exp(\varpi_1 + \varpi_2) \sum_{\ell=0}^{+\infty} \frac{(2\rho)^\ell \xi^{\ell\phi}}{\Gamma(\ell\phi+1)},$$

$$\Psi(\varpi_1, \varpi_2, \xi) = -\frac{\hbar \xi^\phi}{\Gamma(\phi+1)} + \exp(\varpi_1 + \varpi_2) \sum_{\ell=0}^{+\infty} \frac{(2\rho)^\ell \xi^{\ell\phi}}{\Gamma(\ell\phi+1)}.$$

At $\phi = 1$ and $\hbar = 0$, the actual solution of classical *NSe* is

$$\Phi(\varpi_1, \varpi_2, \xi) = -\exp(\varpi_1 + \varpi_2 + 2\rho\xi), \quad \Psi(\varpi_1, \varpi_2, \xi) = \exp(\varpi_1 + \varpi_2 + 2\rho\xi).$$

Problem 4.3. We assume the system of time-fractional *NSe* with $\hbar_1 = \hbar_1 = \hbar_3 = 0$

$$\begin{cases} \frac{\partial^\phi \Phi}{\partial \xi^\phi} + \Phi \frac{\partial \Phi}{\partial \varpi_1} + \Psi \frac{\partial \Phi}{\partial \varpi_2} + \Upsilon \frac{\partial \Phi}{\partial \varpi_3} = \rho\left(\frac{\partial^2 \Phi}{\partial \varpi_1^2} + \frac{\partial^2 \Phi}{\partial \varpi_2^2} + \frac{\partial^2 \Phi}{\partial \varpi_3^2}\right), \\ \frac{\partial^\phi \Psi}{\partial \xi^\phi} + \Phi \frac{\partial \Psi}{\partial \varpi_1} + \Psi \frac{\partial \Psi}{\partial \varpi_2} + \Upsilon \frac{\partial \Psi}{\partial \varpi_3} = \rho\left(\frac{\partial^2 \Psi}{\partial \varpi_1^2} + \frac{\partial^2 \Psi}{\partial \varpi_2^2} + \frac{\partial^2 \Psi}{\partial \varpi_3^2}\right), \\ \frac{\partial^\phi \Upsilon}{\partial \xi^\phi} + \Phi \frac{\partial \Upsilon}{\partial \varpi_1} + \Psi \frac{\partial \Upsilon}{\partial \varpi_2} + \Upsilon \frac{\partial \Upsilon}{\partial \varpi_3} = \rho\left(\frac{\partial^2 \Upsilon}{\partial \varpi_1^2} + \frac{\partial^2 \Upsilon}{\partial \varpi_2^2} + \frac{\partial^2 \Upsilon}{\partial \varpi_3^2}\right), \end{cases} \tag{35}$$

subject to

$$\begin{cases} \Phi(\varpi_1, \varpi_2, \varpi_3, 0) = -0.5\varpi_1 + \varpi_2 + \varpi_3, \\ \Psi(\varpi_1, \varpi_2, \varpi_3, 0) = \varpi_1 - 0.5\varpi_2 + \varpi_3, \\ \Upsilon(\varpi_1, \varpi_2, \varpi_3, 0) = \varpi_1 + \varpi_2 - 0.5\varpi_3. \end{cases} \tag{36}$$

Proof. By applying the quadruple Elzaki transform on (35), we have

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\frac{\partial^\phi \Phi}{\partial \xi^\phi} + \Phi \frac{\partial \Phi}{\partial \varpi_1} + \Psi \frac{\partial \Phi}{\partial \varpi_2} + \Upsilon \frac{\partial \Phi}{\partial \varpi_3}\right] = \mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2 \Phi}{\partial \varpi_1^2} + \frac{\partial^2 \Phi}{\partial \varpi_2^2} + \frac{\partial^2 \Phi}{\partial \varpi_3^2}\right)\right],$$

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\frac{\partial^\phi \Psi}{\partial \xi^\phi} + \Phi \frac{\partial \Psi}{\partial \varpi_1} + \Psi \frac{\partial \Psi}{\partial \varpi_2} + \Upsilon \frac{\partial \Psi}{\partial \varpi_3}\right] = \mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2 \Psi}{\partial \varpi_1^2} + \frac{\partial^2 \Psi}{\partial \varpi_2^2} + \frac{\partial^2 \Psi}{\partial \varpi_3^2}\right)\right],$$

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\frac{\partial^\phi \Upsilon}{\partial \xi^\phi} + \Phi \frac{\partial \Upsilon}{\partial \varpi_1} + \Psi \frac{\partial \Phi}{\partial \varpi_2} + \Upsilon \frac{\partial \Upsilon}{\partial \varpi_3}\right] = \mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2 \Upsilon}{\partial \varpi_1^2} + \frac{\partial^2 \Upsilon}{\partial \varpi_2^2} + \frac{\partial^2 \Upsilon}{\partial \varpi_3^2}\right)\right].$$

On making the use of differentiation property of the Elzaki transform, we obtain

$$\frac{1}{\mathfrak{s}^{\phi}}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\Phi(\varpi_1,\varpi_2,\varpi_3,\xi)\right]-\sum_{\kappa=0}^{\ell-1}\Phi_{(\kappa)}(\eta,\zeta,r_1,0)\mathfrak{s}^{2-\phi+\kappa}$$
$$=-\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\Phi\frac{\partial\Phi}{\partial\varpi_1}+\Psi\frac{\partial\Phi}{\partial\varpi_2}+\Upsilon\frac{\partial\Phi}{\partial\varpi_3}\right]+\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2\Phi}{\partial\varpi_1^2}+\frac{\partial^2\Phi}{\partial\varpi_2^2}+\frac{\partial^2\Phi}{\partial\varpi_3^2}\right)\right],$$
$$\frac{1}{\mathfrak{s}^{\phi}}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\Psi(\varpi_1,\varpi_2,\varpi_3,\xi)\right]-\sum_{\kappa=0}^{\ell-1}\Psi_{(\kappa)}(\eta,\zeta,r_1,0)\mathfrak{s}^{2-\phi+\kappa}$$
$$=-\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\Phi\frac{\partial\Psi}{\partial\varpi_1}+\Psi\frac{\partial\Psi}{\partial\varpi_2}+\Upsilon\frac{\partial\Psi}{\partial\varpi_3}\right]+\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2\Psi}{\partial\varpi_1^2}+\frac{\partial^2\Psi}{\partial\varpi_2^2}+\frac{\partial^2\Psi}{\partial\varpi_3^2}\right)\right],$$
$$\frac{1}{\mathfrak{s}^{\phi}}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\Upsilon(\varpi_1,\varpi_2,\varpi_3,\xi)\right]-\sum_{\kappa=0}^{\ell-1}\Upsilon_{(\kappa)}(\eta,\zeta,r_1,0)\mathfrak{s}^{2-\phi+\kappa}$$
$$=-\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\Phi\frac{\partial\Upsilon}{\partial\varpi_1}+\Psi\frac{\partial\Upsilon}{\partial\varpi_2}+\Upsilon\frac{\partial\Upsilon}{\partial\varpi_3}\right]+\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2\Upsilon}{\partial\varpi_1^2}+\frac{\partial^2\Upsilon}{\partial\varpi_2^2}+\frac{\partial^2\Upsilon}{\partial\varpi_3^2}\right)\right]. \quad (37)$$

According to initial conditions and simple computations yields

$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\Phi(\varpi_1,\varpi_2,\varpi_3,\xi)\right]$$
$$=-\mathfrak{s}^2\left[-0.5\eta^3\zeta^2r_1^2+\eta^2\zeta^3r_2^2+\eta^2\zeta^2r_1^3\right]-\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\Phi\frac{\partial\Phi}{\partial\varpi_1}+\Psi\frac{\partial\Phi}{\partial\varpi_2}+\Upsilon\frac{\partial\Phi}{\partial\varpi_3}\right]$$
$$+\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2\Phi}{\partial\varpi_1^2}+\frac{\partial^2\Phi}{\partial\varpi_2^2}+\frac{\partial^2\Phi}{\partial\varpi_3^2}\right)\right],$$
$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\Psi(\varpi_1,\varpi_2,\varpi_3,\xi)\right]$$
$$=-\mathfrak{s}^2\left[\eta^3\zeta^2r_1^2-0.5\eta^2\zeta^3r_2^2+\eta^2\zeta^2r_1^3\right]-\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\Phi\frac{\partial\Psi}{\partial\varpi_1}+\Psi\frac{\partial\Psi}{\partial\varpi_2}+\Upsilon\frac{\partial\Psi}{\partial\varpi_3}\right]$$
$$+\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2\Psi}{\partial\varpi_1^2}+\frac{\partial^2\Psi}{\partial\varpi_2^2}+\frac{\partial^2\Psi}{\partial\varpi_3^2}\right)\right],$$
$$\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\Upsilon(\varpi_1,\varpi_2,\varpi_3,\xi)\right]$$
$$=-\mathfrak{s}^2\left[\eta^3\zeta^2r_1^2+\eta^2\zeta^3r_2^2-0.5\eta^2\zeta^2r_1^3\right]-\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\Phi\frac{\partial\Upsilon}{\partial\varpi_1}+\Psi\frac{\partial\Upsilon}{\partial\varpi_2}+\Upsilon\frac{\partial\Upsilon}{\partial\varpi_3}\right]$$
$$+\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2\Upsilon}{\partial\varpi_1^2}+\frac{\partial^2\Upsilon}{\partial\varpi_2^2}+\frac{\partial^2\Upsilon}{\partial\varpi_3^2}\right)\right]. \quad (38)$$

Employing the inverse quadruple Elzaki transform for (38)

$$\Phi(\varpi_1,\varpi_2,\varpi_3,\xi)=-0.5\varpi_1+\varpi_2+\varpi_3-\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\Phi\frac{\partial\Phi}{\partial\varpi_1}+\Psi\frac{\partial\Phi}{\partial\varpi_2}+\Upsilon\frac{\partial\Phi}{\partial\varpi_3}\right]\right]$$
$$+\mathbb{E}_{\eta}^{-1}\mathbb{E}_{\zeta}^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_{\mathfrak{s}}^{-1}\left[\mathfrak{s}^{\phi}\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_{\xi}\left[\rho\left(\frac{\partial^2\Phi}{\partial\varpi_1^2}+\frac{\partial^2\Phi}{\partial\varpi_2^2}+\frac{\partial^2\Phi}{\partial\varpi_3^2}\right)\right]\right],$$

$$\Psi(\varpi_1,\varpi_2,\varpi_3,\xi)=\varpi_1-0.5\varpi_2+\varpi_3-\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_\xi\left[\Phi\frac{\partial\Psi}{\partial\varpi_1}+\Psi\frac{\partial\Psi}{\partial\varpi_2}+\Upsilon\frac{\partial\Psi}{\partial\varpi_3}\right]\right]$$
$$+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_\xi\left[\rho\left(\frac{\partial^2\Psi}{\partial\varpi_1^2}+\frac{\partial^2\Psi}{\partial\varpi_2^2}+\frac{\partial^2\Psi}{\partial\varpi_3^2}\right)\right]\right],$$
$$\Upsilon(\varpi_1,\varpi_2,\varpi_3,\xi)=\varpi_1+\varpi_2-0.5\varpi_3-\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_\xi\left[\Phi\frac{\partial\Upsilon}{\partial\varpi_1}+\Psi\frac{\partial\Upsilon}{\partial\varpi_2}+\Upsilon\frac{\partial\Upsilon}{\partial\varpi_3}\right]\right]$$
$$+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_\xi\left[\rho\left(\frac{\partial^2\Upsilon}{\partial\varpi_1^2}+\frac{\partial^2\Upsilon}{\partial\varpi_2^2}+\frac{\partial^2\Upsilon}{\partial\varpi_3^2}\right)\right]\right].$$
(39)

The infinite series solution for unknown function $\Phi(\varpi_1,\varpi_2,\varpi_3,\xi)$, $\Psi(\varpi_1,\varpi_2,\varpi_3,\xi)$ and $\Upsilon(\varpi_1,\varpi_2,\varpi_3,\xi)$ have the following form:

$$\Phi(\varpi_1,\varpi_2,\varpi_3,\xi)=\sum_{\ell=0}^{+\infty}\Phi_\ell(\varpi_1,\varpi_2,\varpi_3,\xi),\Psi(\varpi_1,\varpi_2,\varpi_3,\xi)=\sum_{\ell=0}^{+\infty}\Psi_\ell(\varpi_1,\varpi_2,\varpi_3,\xi),$$
$$\Upsilon(\varpi_1,\varpi_2,\varpi_3,\xi)=\sum_{\ell=0}^{+\infty}\Upsilon_\ell(\varpi_1,\varpi_2,\varpi_3,\xi). \tag{40}$$

The Adomian approach is required to determine the zeroth elements Φ_0, Ψ_0 and Υ_0. Therefore, it includes initial condition, all of which are considered to be identified. Consequently, we devised

$$\Phi_0=-0.5\varpi_1+\varpi_2+\varpi_3,\quad \Psi_0=\varpi_1-0.5\varpi_2+\varpi_3,\quad \Upsilon_0=\varpi_1+\varpi_2-0.5\varpi_3.$$

$$\sum_{\ell=0}^{+\infty}\Phi_\ell(\varpi_1,\varpi_2,\varpi_3,\xi)=-\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_\xi\left[\sum_{\ell=0}^{+\infty}\mathcal{A}_\ell+\sum_{\ell=0}^{+\infty}\mathcal{B}_\ell+\sum_{\ell=0}^{+\infty}\mathcal{C}_\ell\right]\right]$$
$$+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_\xi\left[\rho\left(\frac{\partial^2\Phi_\ell}{\partial\varpi_1^2}+\frac{\partial^2\Phi_\ell}{\partial\varpi_2^2}+\frac{\partial^2\Phi_\ell}{\partial\varpi_3^2}\right)\right]\right],$$
$$\sum_{\ell=0}^{+\infty}\Psi_\ell(\varpi_1,\varpi_2,\varpi_3,\xi)=-\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_\xi\left[\sum_{\ell=0}^{+\infty}\mathcal{D}_\ell+\sum_{\ell=0}^{+\infty}\mathcal{E}_\ell+\sum_{\ell=0}^{+\infty}\mathcal{F}_\ell\right]\right]$$
$$+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_\xi\left[\rho\left(\frac{\partial^2\Psi_\ell}{\partial\varpi_1^2}+\frac{\partial^2\Psi_\ell}{\partial\varpi_2^2}+\frac{\partial^2\Psi_\ell}{\partial\varpi_3^2}\right)\right]\right],$$
$$\sum_{\ell=0}^{+\infty}\Upsilon_\ell(\varpi_1,\varpi_2,\varpi_3,\xi)=-\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_\xi\left[\sum_{\ell=0}^{+\infty}\mathcal{G}_\ell+\sum_{\ell=0}^{+\infty}\mathcal{H}_\ell+\sum_{\ell=0}^{+\infty}\mathcal{I}_\ell\right]\right]$$
$$+\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\left[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_\xi\left[\rho\left(\frac{\partial^2\Upsilon_\ell}{\partial\varpi_1^2}+\frac{\partial^2\Upsilon_\ell}{\partial\varpi_2^2}+\frac{\partial^2\Upsilon_\ell}{\partial\varpi_3^2}\right)\right]\right].$$

Thanks to (11), the Adomian polynomials will express all forms of nonlinearity as

$$\mathcal{A}_0 = \Phi_0 \frac{\partial \Phi_0}{\partial \varpi_1}, \quad \mathcal{A}_1 = \Phi_0 \frac{\partial \Phi_1}{\partial \varpi_1} + \Phi_1 \frac{\partial \Phi_0}{\partial \varpi_1}, \quad \mathcal{A}_2 = \Phi_0 \frac{\partial \Phi_2}{\partial \varpi_1} + \Phi_1 \frac{\partial \Phi_1}{\partial \varpi_1} + \Phi_2 \frac{\partial \Phi_0}{\partial \varpi_1},$$
$$\mathcal{B}_0 = \Psi_0 \frac{\partial \Phi_0}{\partial \varpi_2}, \quad \mathcal{B}_1 = \Psi_0 \frac{\partial \Phi_1}{\partial \varpi_2} + \Psi_1 \frac{\partial \Phi_0}{\partial \varpi_2}, \quad \mathcal{B}_2 = \Psi_0 \frac{\partial \Phi_2}{\partial \varpi_2} + \Psi_1 \frac{\partial \Phi_1}{\partial \varpi_2} + \Psi_2 \frac{\partial \Phi_0}{\partial \varpi_2},$$
$$\mathcal{C}_0 = \Upsilon_0 \frac{\partial \Phi_0}{\partial \varpi_3}, \quad \mathcal{C}_1 = \Upsilon_0 \frac{\partial \Phi_1}{\partial \varpi_3} + \Upsilon_1 \frac{\partial \Phi_0}{\partial \varpi_3}, \quad \mathcal{C}_2 = \Upsilon_0 \frac{\partial \Phi_2}{\partial \varpi_3} + \Upsilon_1 \frac{\partial \Phi_1}{\partial \varpi_1} + \Upsilon_2 \frac{\partial \Phi_0}{\partial \varpi_3},$$
$$\mathcal{D}_0 = \Phi_0 \frac{\partial \Psi_0}{\partial \varpi_1}, \quad \mathcal{D}_1 = \Phi_0 \frac{\partial \Psi_1}{\partial \varpi_1} + \Phi_1 \frac{\partial \Psi_0}{\partial \varpi_1}, \quad \mathcal{D}_2 = \Phi_0 \frac{\partial \Psi_2}{\partial \varpi_1} + \Phi_1 \frac{\partial \Psi_1}{\partial \varpi_1} + \Phi_2 \frac{\partial \Psi_0}{\partial \varpi_1},$$
$$\mathcal{E}_0 = \Psi_0 \frac{\partial \Psi_0}{\partial \varpi_2}, \quad \mathcal{E}_1 = \Psi_0 \frac{\partial \Psi_1}{\partial \varpi_2} + \Psi_1 \frac{\partial \Psi_0}{\partial \varpi_2}, \quad \mathcal{E}_2 = \Psi_0 \frac{\partial \Psi_2}{\partial \varpi_2} + \Psi_1 \frac{\partial \Psi_1}{\partial \varpi_1} + \Psi_2 \frac{\partial \Psi_0}{\partial \varpi_2},$$
$$\mathcal{F}_0 = \Upsilon_0 \frac{\partial \Psi_0}{\partial \varpi_3}, \quad \mathcal{F}_1 = \Upsilon_0 \frac{\partial \Psi_1}{\partial \varpi_3} + \Upsilon_1 \frac{\partial \Psi_0}{\partial \varpi_3}, \quad \mathcal{F}_2 = \Upsilon_0 \frac{\partial \Psi_2}{\partial \varpi_3} + \Upsilon_1 \frac{\partial \Psi_1}{\partial \varpi_1} + \Upsilon_2 \frac{\partial \Psi_0}{\partial \varpi_3},$$
$$\mathcal{G}_0 = \Phi_0 \frac{\partial \Upsilon_0}{\partial \varpi_1}, \quad \mathcal{G}_1 = \Phi_0 \frac{\partial \Upsilon_1}{\partial \varpi_1} + \Phi_1 \frac{\partial \Upsilon_0}{\partial \varpi_1}, \quad \mathcal{G}_2 = \Phi_0 \frac{\partial \Upsilon_2}{\partial \varpi_1} + \Phi_1 \frac{\partial \Upsilon_1}{\partial \varpi_1} + \Phi_2 \frac{\partial \Upsilon_0}{\partial \varpi_1},$$
$$\mathcal{H}_0 = \Psi_0 \frac{\partial \Upsilon_0}{\partial \varpi_2}, \quad \mathcal{H}_1 = \Psi_0 \frac{\partial \Upsilon_1}{\partial \varpi_2} + \Psi_1 \frac{\partial \Upsilon_0}{\partial \varpi_2}, \quad \mathcal{H}_2 = \Psi_0 \frac{\partial \Upsilon_2}{\partial \varpi_2} + \Psi_1 \frac{\partial \Upsilon_1}{\partial \varpi_1} + \Psi_2 \frac{\partial \Upsilon_0}{\partial \varpi_2},$$
$$\mathcal{I}_0 = \Upsilon_0 \frac{\partial \Upsilon_0}{\partial \varpi_3}, \quad \mathcal{I}_1 = \Upsilon_0 \frac{\partial \Upsilon_1}{\partial \varpi_3} + \Upsilon_1 \frac{\partial \Upsilon_0}{\partial \varpi_3}, \quad \mathcal{I}_2 = \Upsilon_0 \frac{\partial \Upsilon_2}{\partial \varpi_3} + \Upsilon_1 \frac{\partial \Upsilon_1}{\partial \varpi_1} + \Upsilon_2 \frac{\partial \Upsilon_0}{\partial \varpi_3},$$

For $\ell = 0$, we have

$$\begin{aligned}
\Phi_1(\varpi_1, \varpi_2, \varpi_3, \xi) &= -\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi \mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_\xi\Big[\mathcal{A}_0 + \mathcal{B}_0 + \mathcal{C}_0\Big]\Big] \\
&\quad + \mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi \mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_\xi\Big[\rho\Big(\frac{\partial^2 \Phi_0}{\partial \varpi_1^2} + \frac{\partial^2 \Phi_0}{\partial \varpi_2^2} + \frac{\partial^2 \Phi_0}{\partial \varpi_3^2}\Big)\Big]\Big] \\
&= -\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[2.25\eta^3\zeta^2 r_1^2 \mathfrak{s}^{\phi+2}\Big] \\
&= \frac{-2.25\varpi_1\xi^\phi}{\Gamma(\phi+1)}.
\end{aligned}$$

In a similar way, we obtain

$$\Psi_1(\varpi_1, \varpi_2, \varpi_3, \xi) = \frac{-2.25\varpi_2\xi^\phi}{\Gamma(\phi+1)},$$
$$\Upsilon_1(\varpi_1, \varpi_2, \varpi_3, \xi) = \frac{-2.25\varpi_3\xi^\phi}{\Gamma(\phi+1)}.$$

Analogously, for $\ell = 1$,

$$
\begin{aligned}
\Phi_2(\varpi_1,\varpi_2,\varpi_3,\xi) &= -\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_\xi\Big[\mathcal{A}_1+\mathcal{B}_1+\mathcal{C}_1\Big]\Big] \\
&\quad + \mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_\xi\Big[\rho\Big(\frac{\partial^2\Phi_1}{\partial\varpi_1^2}+\frac{\partial^2\Phi_1}{\partial\varpi_2^2}+\frac{\partial^2\Phi_1}{\partial\varpi_3^2}\Big)\Big]\Big] \\
&= -\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[2.25\eta^3\zeta^2 r_1^2\mathfrak{s}^{2\phi+2} - 4.5\eta^2\zeta^3 r_1^2\mathfrak{s}^{2\phi+2} - 4.5\eta^2\zeta^2 r_1^3\mathfrak{s}^{2\phi+2}\Big] \\
&= \frac{2(2.25)\xi^{2\phi}}{\Gamma(2\phi+1)}(-0.5\varpi_1+\varpi_2+\varpi_3).
\end{aligned}
$$

In a similar manner, we obtain

$$
\Psi_2(\varpi_1,\varpi_2,\varpi_3,\xi) = \frac{2(2.25)\xi^{2\phi}}{\Gamma(2\phi+1)}(\varpi_1-0.5\varpi_2+\varpi_3),
$$

$$
\Upsilon_2(\varpi_1,\varpi_2,\varpi_3,\xi) = \frac{2(2.25)\xi^{2\phi}}{\Gamma(2\phi+1)}(\varpi_1+\varpi_2-0.5\varpi_3).
$$

For $\ell = 2$,

$$
\begin{aligned}
\Phi_3(\varpi_1,\varpi_2,\varpi_3,\xi) &= -\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_\xi\Big[\mathcal{A}_2+\mathcal{B}_2+\mathcal{C}_2\Big]\Big] \\
&\quad + \mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\mathfrak{s}^\phi\mathbb{E}_{\varpi_1}\mathbb{E}_{\varpi_2}\mathbb{E}_{\varpi_3}\mathbb{E}_\xi\Big[\rho\Big(\frac{\partial^2\Phi_2}{\partial\varpi_1^2}+\frac{\partial^2\Phi_2}{\partial\varpi_2^2}+\frac{\partial^2\Phi_2}{\partial\varpi_3^2}\Big)\Big]\Big] \\
&= -\mathbb{E}_\eta^{-1}\mathbb{E}_\zeta^{-1}\mathbb{E}_{r_1}^{-1}\mathbb{E}_\mathfrak{s}^{-1}\Big[\frac{(2.25)^2\big(4(\Gamma(\phi+1))^2+\Gamma(2\phi+1)\big)}{(\Gamma(\phi+1))^2}\mathfrak{s}^{3\phi+2}\eta^3\zeta^2 r_1^2\Big] \\
&= -\frac{(2.25)^2\big(4(\Gamma(\phi+1))^2+\Gamma(2\phi+1)\big)}{(\Gamma(\phi+1))^2}\frac{\varpi_1\xi^{3\phi}}{\Gamma(3\phi+1)},
\end{aligned}
$$

$$
\Psi_3(\varpi_1,\varpi_2,\varpi_3,\xi) = -\frac{(2.25)^2\big(4(\Gamma(\phi+1))^2+\Gamma(2\phi+1)\big)}{(\Gamma(\phi+1))^2}\frac{\varpi_2\xi^{3\phi}}{\Gamma(3\phi+1)},
$$

and

$$
\Upsilon_1(\varpi_1,\varpi_2,\varpi_3,\xi) = -\frac{(2.25)^2\big(4(\Gamma(\phi+1))^2+\Gamma(2\phi+1)\big)}{(\Gamma(\phi+1))^2}\frac{\varpi_3\xi^{3\phi}}{\Gamma(3\phi+1)}.
$$

Continuing the same way, the iterative terms of Φ_ℓ, Ψ_ℓ and Υ_ℓ $(\ell > 4)$ are presented as follows:

$$
\begin{aligned}
\Phi(\varpi_1,\varpi_2,\varpi_3,\xi) &= \sum_{\ell=0}^{+\infty}\Phi_\ell(\varpi_1,\varpi_2,\varpi_3,\xi) = \Phi_0(\varpi_1,\varpi_2,\varpi_3,\xi)+\Phi_1(\varpi_1,\varpi_2,\varpi_3,\xi)+\Phi_2(\varpi_1,\varpi_2,\varpi_3,\xi) \\
&\quad + \Phi_3(\varpi_1,\varpi_2,\varpi_3,\xi)+\ldots, \\
\Psi(\varpi_1,\varpi_2,\varpi_3,\xi) &= \sum_{\ell=0}^{+\infty}\Psi_\ell(\varpi_1,\varpi_2,\varpi_3,\xi) = \Psi_0(\varpi_1,\varpi_2,\varpi_3,\xi)+\Psi_1(\varpi_1,\varpi_2,\varpi_3,\xi)+\Psi_2(\varpi_1,\varpi_2,\varpi_3,\xi) \\
&\quad + \Psi_2(\varpi_1,\varpi_2,\varpi_3,\xi)+\ldots,
\end{aligned}
$$

$$\Upsilon(\varpi_1,\varpi_2,\varpi_3,\xi)=\sum_{\ell=0}^{+\infty}\Upsilon_\ell(\varpi_1,\varpi_2,\varpi_3,\xi)=\Upsilon_0(\varpi_1,\varpi_2,\varpi_3,\xi)+\Upsilon_1(\varpi_1,\varpi_2,\varpi_3,\xi)+\Upsilon_2(\varpi_1,\varpi_2,\varpi_3,\xi)$$
$$+\Upsilon_3(\varpi_1,\varpi_2,\varpi_3,\xi)+\ldots,$$

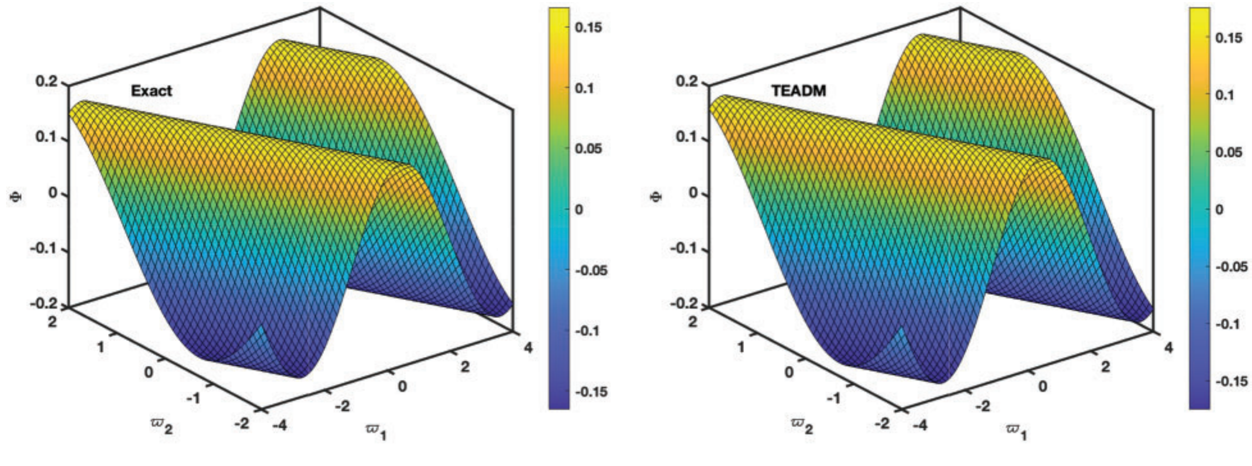
$$\Phi(\varpi_1,\varpi_2,\varpi_3,\xi)=-0.5\varpi_1+\varpi_2+\varpi_3-\frac{2.25\varpi_1\xi^\phi}{\Gamma(\phi+1)}+\frac{2(2.25)\xi^{2\phi}}{\Gamma(2\phi+1)}(-0.5\varpi_1+\varpi_2+\varpi_3)$$
$$-\frac{(2.25)^2\big(4(\Gamma(\phi+1))^2+\Gamma(2\phi+1)\big)}{(\Gamma(\phi+1))^2}\frac{\varpi_1\xi^{3\phi}}{\Gamma(3\phi+1)}+\ldots,$$
$$\Psi(\varpi_1,\varpi_2,\varpi_3,\xi)=\varpi_1-0.5\varpi_2+\varpi_3-\frac{2.25\varpi_2\xi^\phi}{\Gamma(\phi+1)}+\frac{2(2.25)\xi^{2\phi}}{\Gamma(2\phi+1)}(\varpi_1-0.5\varpi_2+\varpi_3)$$
$$-\frac{(2.25)^2\big(4(\Gamma(\phi+1))^2+\Gamma(2\phi+1)\big)}{(\Gamma(\phi+1))^2}\frac{\varpi_2\xi^{3\phi}}{\Gamma(3\phi+1)}+\ldots,$$
$$\Upsilon(\varpi_1,\varpi_2,\varpi_3,\xi)=\varpi_1+\varpi_2-0.5\varpi_3-\frac{2.25\varpi_3\xi^\phi}{\Gamma(\phi+1)}+\frac{2(2.25)\xi^{2\phi}}{\Gamma(2\phi+1)}(\varpi_1+\varpi_2-0.5\varpi_3)$$
$$-\frac{(2.25)^2\big(4(\Gamma(\phi+1))^2+\Gamma(2\phi+1)\big)}{(\Gamma(\phi+1))^2}\frac{\varpi_3\xi^{3\phi}}{\Gamma(3\phi+1)}+\ldots.$$

At $\phi=1$, the actual solution of system of *NSe* is

$$\Phi(\varpi_1,\varpi_2,\varpi_3,\xi)=(-0.5\varpi_1+\varpi_2+\varpi_3-2.25\varpi_1\xi)(1-2.25\xi^2)^{-1},$$
$$\Psi(\varpi_1,\varpi_2,\varpi_3,\xi)=(\varpi_1-0.5\varpi_2+\varpi_3-2.25\varpi_2\xi)(1-2.25\xi^2)^{-1},$$
$$\Upsilon(\varpi_1,\varpi_2,\varpi_3,\xi)=(\varpi_1+\varpi_2-0.5\varpi_3-2.25\varpi_3\xi)(1-2.25\xi^2)^{-1}.$$

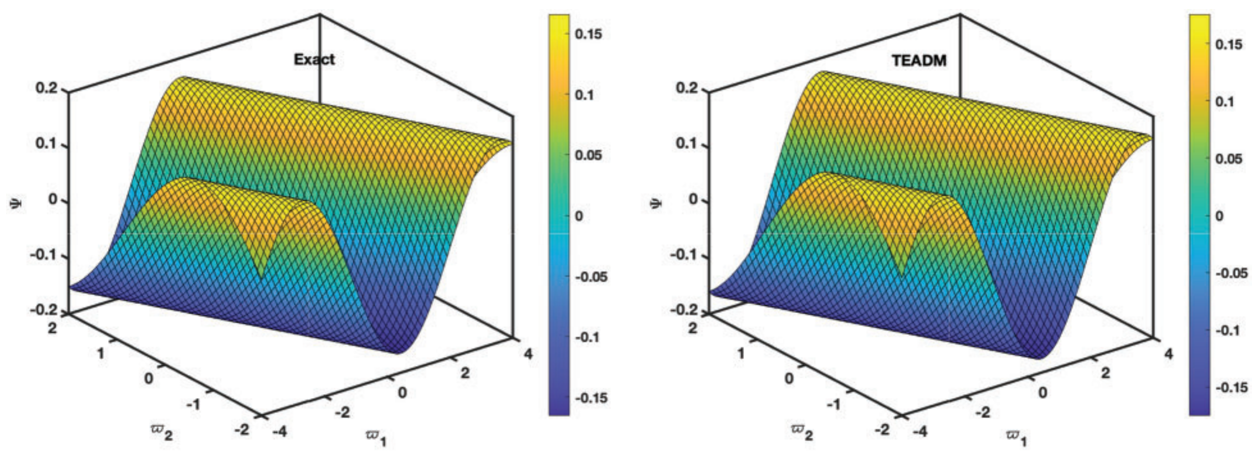
5 Results and Discussion

• In Figs. 1 and 2, the 3D surfaces with respect to the Caputo fractional derivative operator at $\phi = 1, \rho = 0.3$ and $\hbar = 0$ are presented. Furthermore, we have demonstrated the graphs of the approximate solutions of functional values $\Phi(\varpi_1,\varpi_2,\xi)$ and $\Psi(\varpi_1,\varpi_2,\xi)$ of the *NSe* for Caputo fractional derivative operator, respectively. The Figs. 1a and 2b depicted the approximate solutions, because the *TEADM* proved to be relatively compelling for solving nonlinear *PDEs* without linearization, perturbation or discretization. In Figs. 3 and 4, we have described the approximate solution of the problem with respect to the proposed method and exact solution, respectively. One can deduce from the figures that approximate solution approaches the exact solution as the fractional parameter changes from integer-order $\phi = 1$.



(a) (b)

Figure 1: Simulation of the (a) exact and (b) $TEADM$ solutions of $\Phi(\varpi_1, \varpi_2, \xi)$ at $\xi = 3$ of Problem 4.1 when $\phi = 1$, $\rho = 0.3$, and $\hbar = 0$



(a) (b)

Figure 2: Simulation of the (a) exact and (b) $TEADM$ solutions of $\Psi(\varpi_1, \varpi_2, \xi)$ of Problem 4.1 with the parameters when $\phi = 1$, $\rho = 0.3$, and $\hbar = 0$

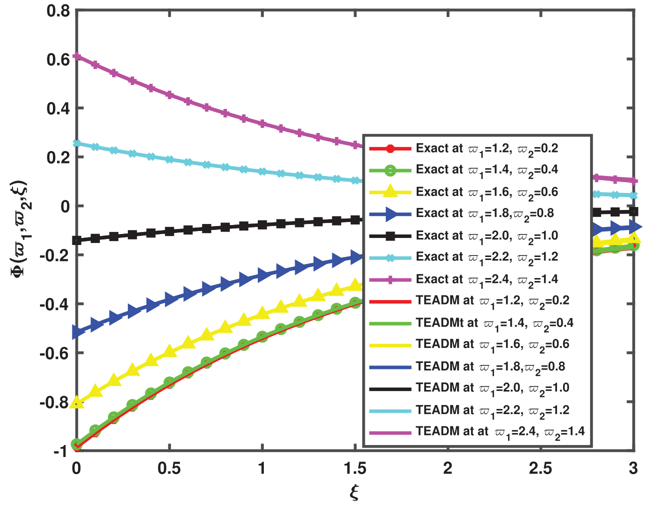


Figure 3: Simulation of the exact and $TEADM$ solutions of $\Phi(\varpi_1, \varpi_2, \xi)$ of Problem 4.1 when ϖ_1 =1.2 to 2.4, ϖ_2 =0.2 to 1.4, at $\phi = 1$, $\rho = 0.3$, and $\hbar = 0$, for various values of ξ

- In Figs. 5–8, the 3D surfaces with respect to the Caputo fractional derivative operator at $\phi = 1, \rho = 0.3$ and $\hbar = 0$ are presented. Furthermore, we have demonstrated the graphs of the approximate solutions of functional values $\Phi(\varpi_1, \varpi_2, \xi)$ and $\Psi(\varpi_1, \varpi_2, \xi)$ of the NSe for Caputo fractional derivative operator, respectively. The Figs. 5b and 6b depicted the approximate solutions, because the $TEADM$ proved to be relatively promising for solving nonlinear $PDEs$ without linearization, perturbation or discretization. Also, in Fig. 7, the comparison simulation for both Φ and Ψ are also presented to demonstrate the strong correlation between the exact and approximate solution of the proposed technique.

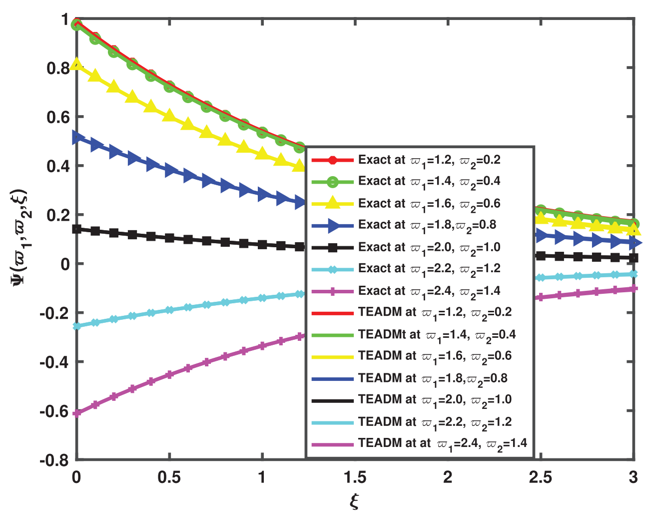
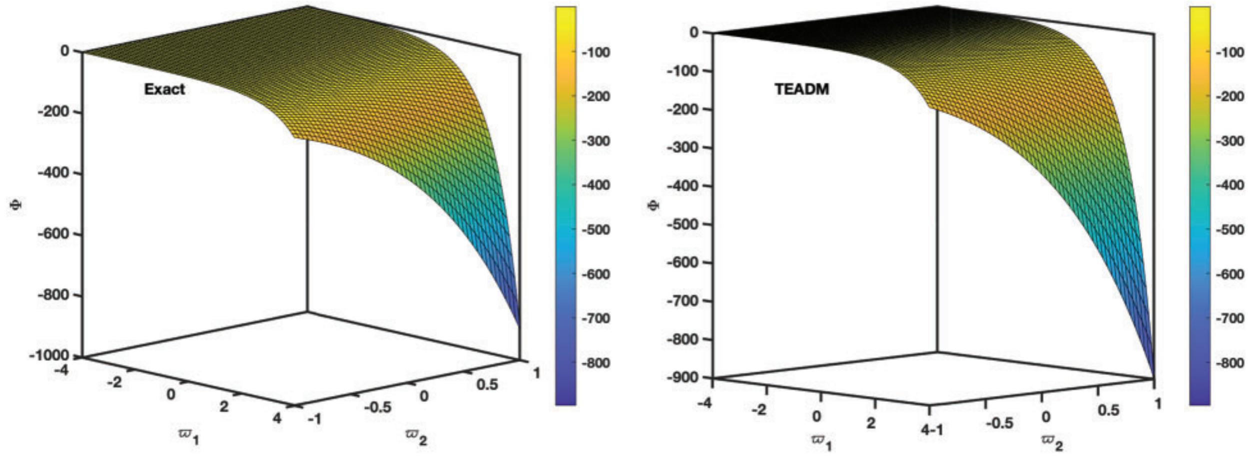
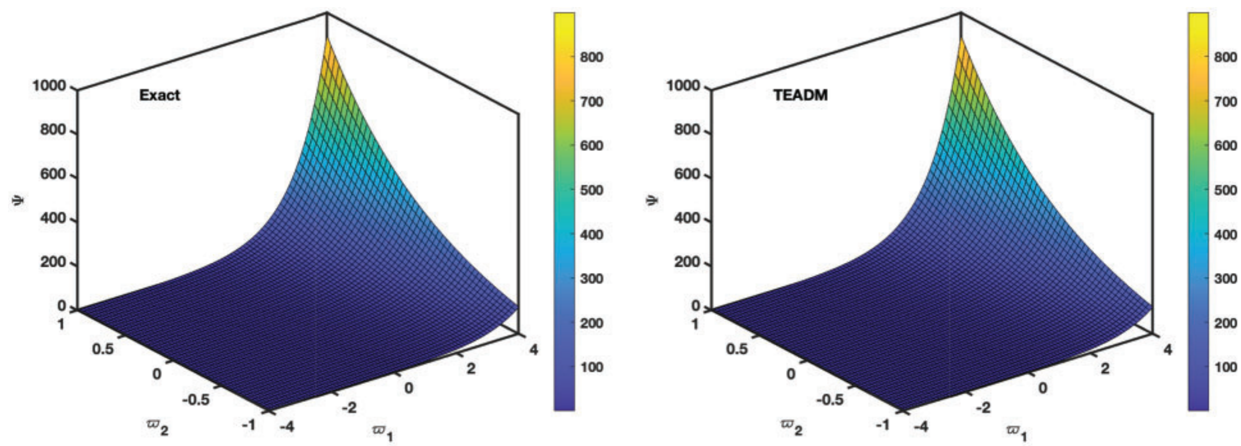


Figure 4: Simulation of the exact and $TEADM$ solutions of $\Psi(\varpi_1, \varpi_2, \xi)$ of Problem 4.1 when ϖ_1 =1.2 to 2.4, ϖ_2 =0.2 to 1.4, at $\phi = 1$, $\rho = 0.3$, and $\hbar = 0$, for various values of ξ



(a) (b)

Figure 5: Simulation of the (a) exact and (b) $TEADM$ solutions of $\Phi(\varpi_1, \varpi_2, \xi)$ at $\xi = 3$ of Problem 4.2 with the parameters when $\phi = 1$, $\rho = 0.3$, and $\hbar = 0$



(a) (b)

Figure 6: Simulation of the (a) exact and (b) $TEADM$ solutions of $\Psi(\varpi_1, \varpi_2, \xi)$ at $\xi = 3$ of Problem 4.2 with the parameters $\phi = 1$, $\rho = 0.3$, and $\hbar = 0$

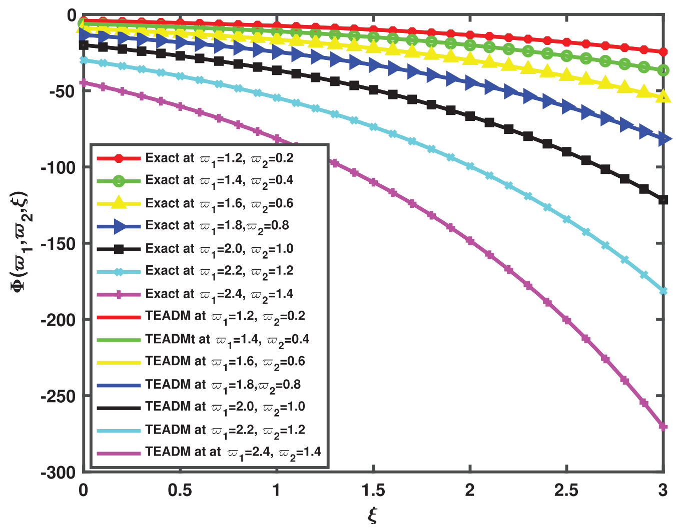


Figure 7: Simulation of the exact and $TEADM$ solutions of $\Phi(\varpi_1, \varpi_2, \xi)$ of Problem 4.2 for ϖ_1 =1.2 to 2.4, ϖ_2 =0.2 to 1.4, at $\phi = 1$, $\rho = 0.3$, and $\hbar = 0$, for various values of ξ

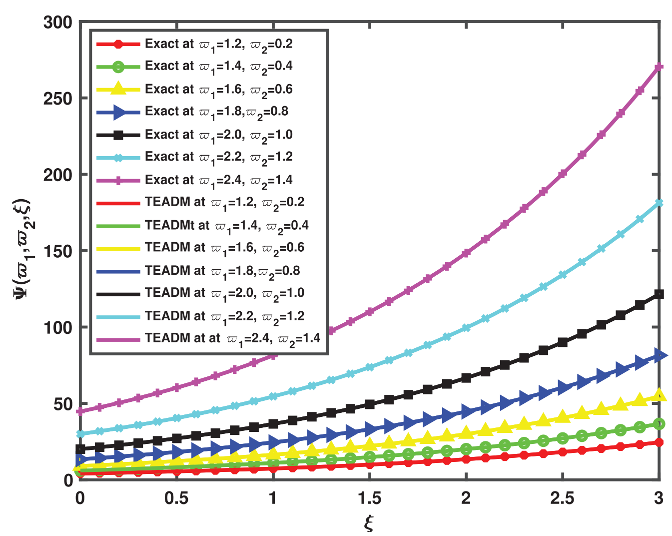


Figure 8: Simulation of the exact and $TEADM$ solutions of $\Psi(\varpi_1, \varpi_2, \xi)$ of Problem 4.2 when ϖ_1 =1.2 to 2.4, ϖ_2 =0.2 to 1.4, at $\phi = 1$, $\rho = 0.3$, and $\hbar = 0$, for various values of ξ

• In Figs. 9–11, we have given the comparison of the exact, $\varpi_1\varpi_2$-slice of exact solution given in (35) and in Figs. 12–14, it can be seen that the solution to the problem which is given by (35) with respect to integer-order parameter in the framework of Caputo fractional derivative. Taking into consideration the results of the article, we can view that only a few components of the series derived by the Adomian decomposition method coupled with the Elzaki transform provide the exact solution. Additionally, the comparison simulation for Φ, Ψ and Υ are also presented to demonstrate the strong correlation between the exact and approximate solution by implementing $QEADM$.

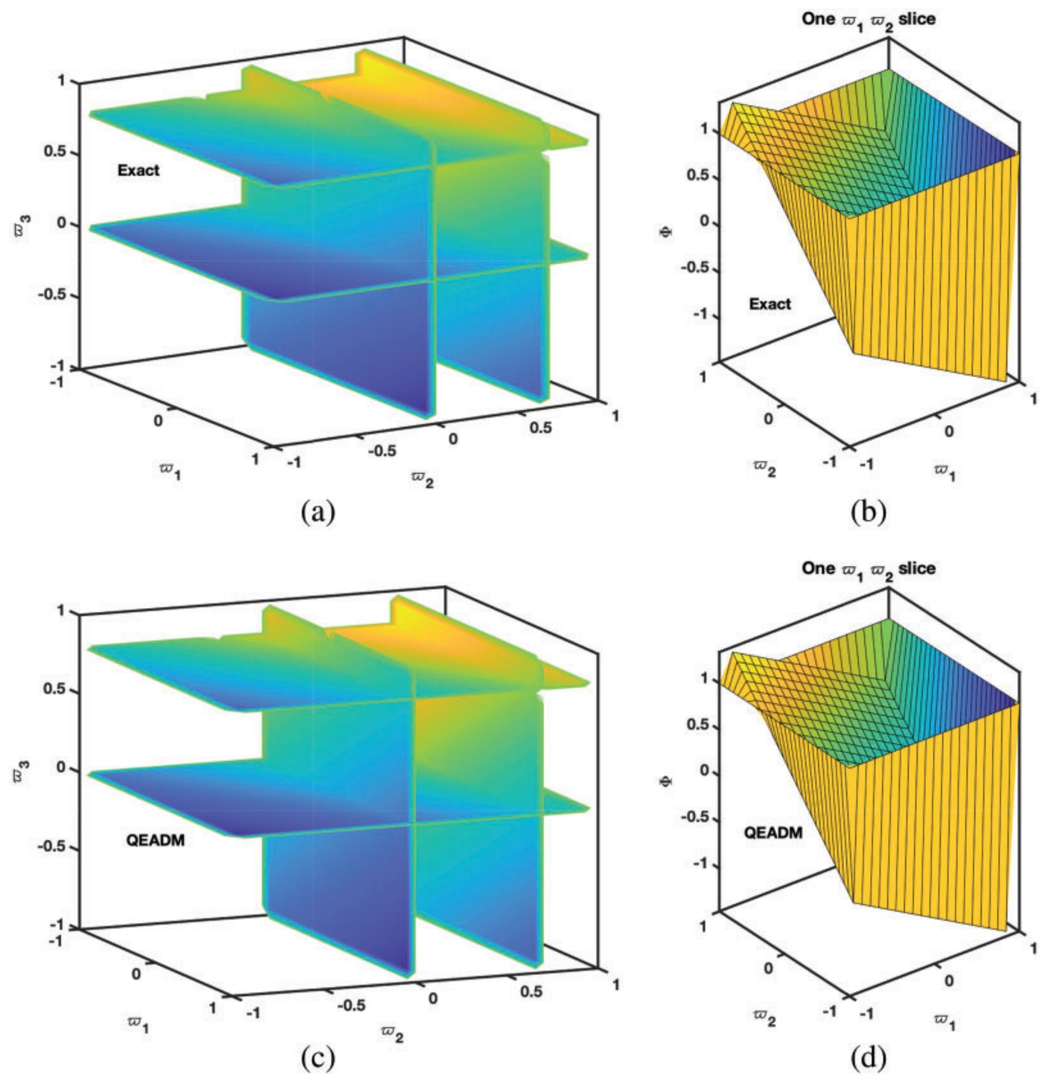


Figure 9: Simulation of the (a) exact solution of Problem 4.3 (b) $\varpi_1\varpi_2$-slice of exact solution (c) $QEADM$ solutions of $\Phi(\varpi_1, \varpi_2, \varpi_3, \xi)$ of Problem 4.3 when $\phi = 1$, $\rho = 0.3$, $\xi = 0.01$ and $\hbar = 0$, (d) $\varpi_1\varpi_2$-slice of solution of (c)

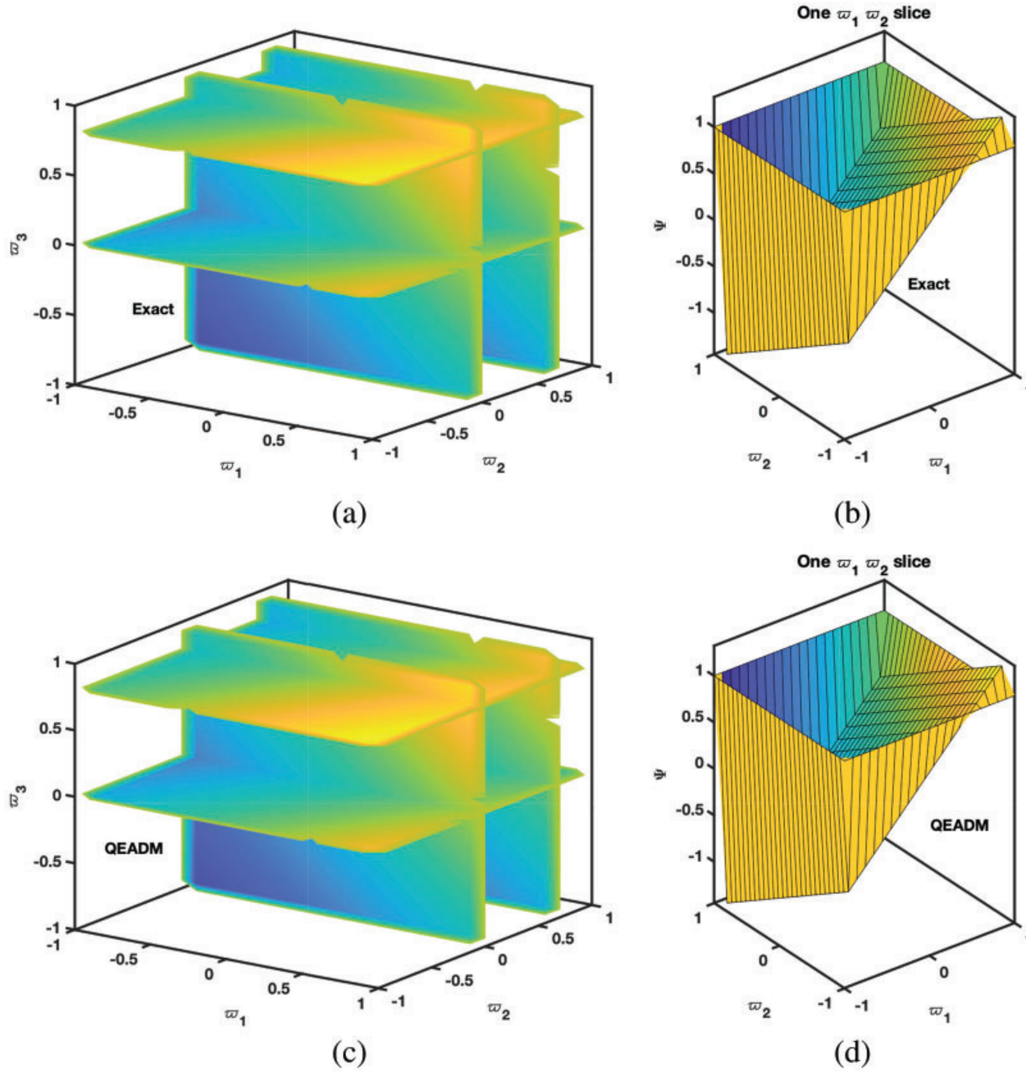


Figure 10: Simulation of the (a) exact solution of Problem 4.3 (b) $\varpi_1\varpi_2$-slice of exact solution (c) $QEADM$ solutions of $\Psi(\varpi_1, \varpi_2, \varpi_3, \xi)$ of problem 4.3, at $\phi = 1$, $\rho = 0.3$, $\xi = 0.01$ and $\hbar = 0$, (d) $\varpi_1\varpi_2$-slice of solution of (c)

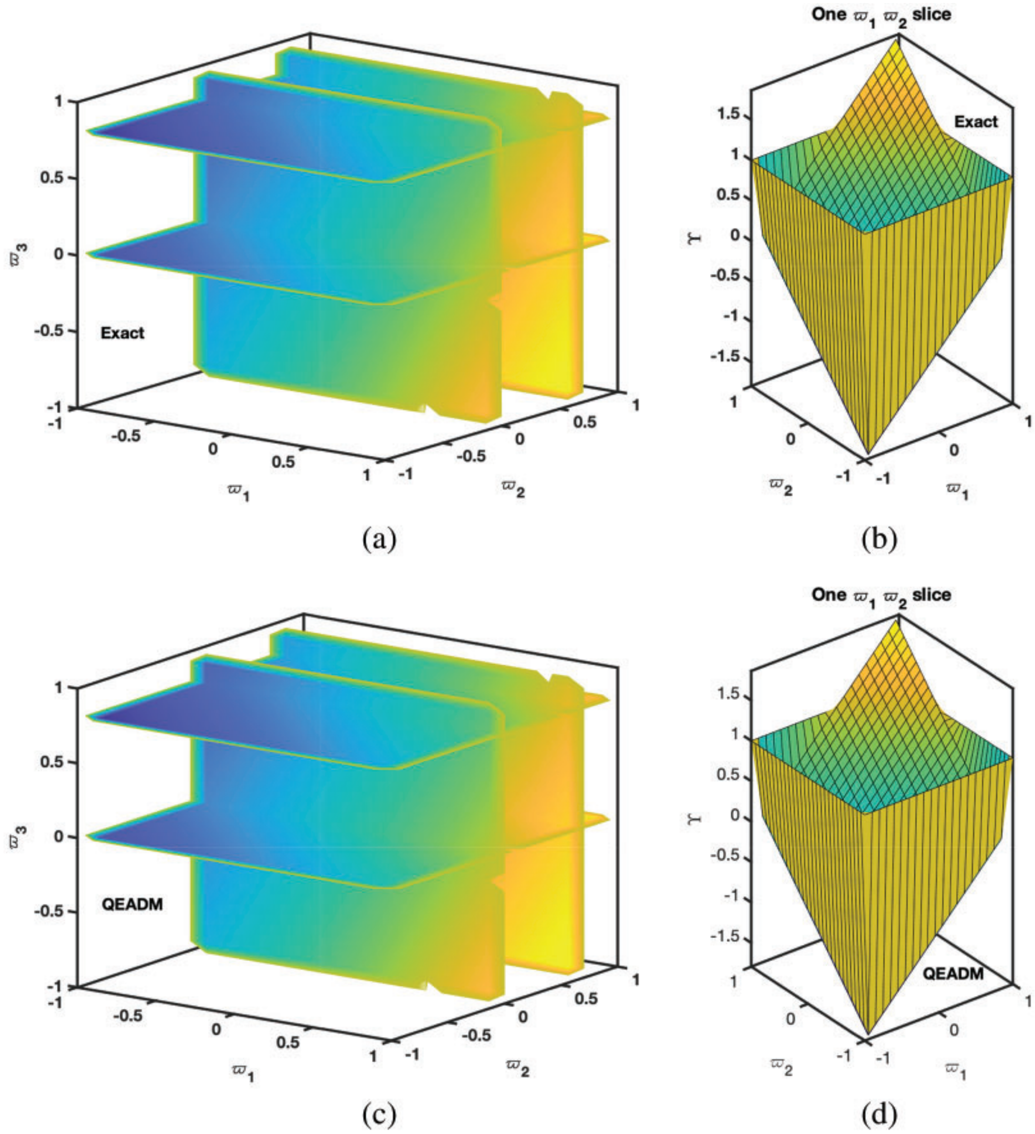


Figure 11: Simulation of the (a) exact solution of Problem 4.3 (b) $\varpi_1\varpi_2$-slice of exact solution (c) $QEADM$ solutions of $\Upsilon(\varpi_1, \varpi_2, \varpi_3, \xi)$ of Problem 4.3 when $\phi = 1$, $\rho = 0.3$, $\xi = 0.01$ and $\hbar = 0$, (d) $\varpi_1\varpi_2$-slice of solution of (c)

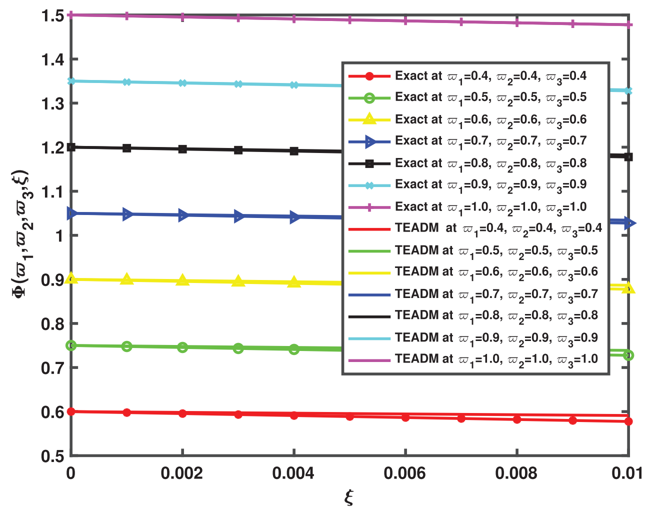


Figure 12: Comparison between the exact and $QEADM$ solutions of $\Phi(\varpi_1, \varpi_2, \varpi_3, \xi)$ of Problem 4.3 for $\varpi_1, \varpi_2, \varpi_3$= 0.4 to 1.0 at $\phi = 1$, $\rho = 0.3$, and $\hbar = 0$, for various values of ξ

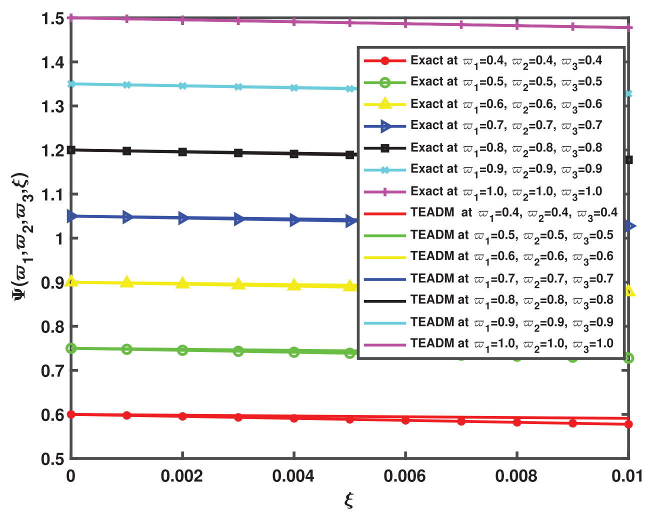


Figure 13: Comparison between the exact and $QEADM$ solutions of $\Psi(\varpi_1, \varpi_2, \varpi_3, \xi)$ of Problem 4.3 for $\varpi_1, \varpi_2, \varpi_3$= 0.4 to 1.0 at $\phi = 1$, $\rho = 0.3$, and $\hbar = 0$, for various values of ξ

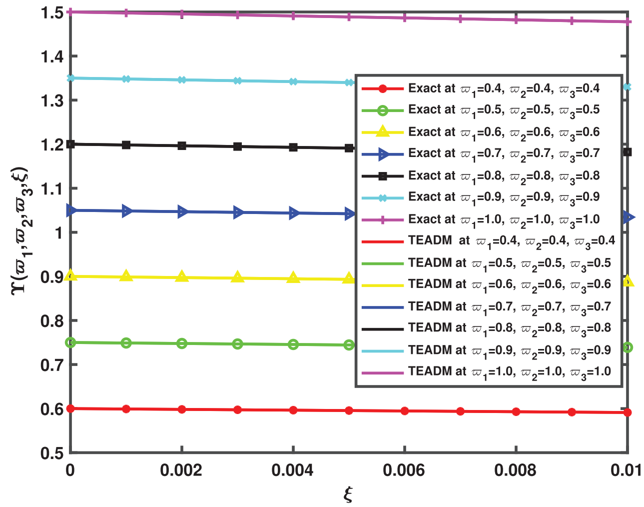


Figure 14: Comparison between the exact and $QEADM$ solutions of $\Upsilon(\varpi_1, \varpi_2, \varpi_3, \xi)$ of Problem 4.3 for $\varpi_1, \varpi_2, \varpi_3 =$ 0.4 to 1.0 at $\phi = 1$, $\rho = 0.3$, and $\hbar = 0$, for various values of ξ

• Tables 1–7 presented a comparison analysis of the findings shown in [31]. The analysis predicts that our projected scheme is close to harmony with the exact solutions.

Table 1: Comparison analysis of approximate and exact solutions of $\Phi(\varpi_1, \varpi_2, \xi)$ of Problem 4.1 with absolute error when $\phi = 1$, $\xi = 0.1$, for various values of ϖ_1 and ϖ_2 with multi-Laplace transform decomposition method (LTDM) proposed by [31]

ϖ_1	ϖ_2	Exact solution	Approximate solution	Absolute error	LTDM [31]
1.2	0.2	−0.928061605333306	−0.928061605333849	$5.432321259490891\times10^{-13}$	−0.928061605333849
1.4	0.4	−0.917135159876129	−0.917135159876666	$5.369038547087257\times10^{-13}$	−0.917135159876666
1.6	0.6	−0.761413238647699	−0.761413238648144	$4.457545443870004\times10^{-13}$	−0.761413238648144
1.8	0.8	−0.485480908995482	−0.485480908995766	$2.842170943040401\times10^{-13}$	−0.485480908995766
2	1	−0.132901818569906	−0.132901818569984	$7.779887845060784\times10^{-14}$	−0.132901818569984
2.2	1.2	0.240659546761904	0.240659546762045	$1.408873018249324\times10^{-13}$	0.240659546762045
2.4	1.4	0.576226061283512	0.576226061283849	$3.372857548811226\times10^{-13}$	0.576226061283849

Table 2: Comparison analysis of approximate and exact solutions of $\Psi(\varpi_1, \varpi_2, \xi)$ of Problem 4.1 with absolute error when $\phi = 1$, $\xi = 0.1$, for various values of ϖ_1 and ϖ_2 with multi-Laplace transform decomposition method (LTDM) proposed by [31]

ϖ_1	ϖ_2	Exact solution	Approximate solution	Absolute error	LTDM [31]
1.2	0.2	0.928061605333306	0.928061605333849	$5.432321259490891\times10^{-13}$	0.928061605333849
1.4	0.4	0.917135159876129	0.917135159876666	$5.369038547087257\times10^{-13}$	0.917135159876666
1.6	0.6	0.761413238647699	0.761413238648144	$4.457545443870004\times10^{-13}$	0.761413238648144
1.8	0.8	0.485480908995482	0.485480908995766	$2.842170943040401\times10^{-13}$	0.485480908995766
2	1	0.132901818569906	0.132901818569984	$7.779887845060784\times10^{-14}$	0.132901818569984
2.2	1.2	−0.240659546761904	−0.240659546762045	$1.408873018249324\times10^{-13}$	−0.240659546762045
2.4	1.4	−0.576226061283512	−0.576226061283849	$3.372857548811226\times10^{-13}$	−0.576226061283849

Table 3: Comparison analysis of approximate and exact solutions of $\Phi(\varpi_1, \varpi_2, \xi)$ of Problem 4.2 with absolute error when $\phi = 1$, $\xi = 0.1$, for various values of ϖ_1 and ϖ_2 with multi-Laplace transform decomposition method (LTDM) proposed by [31]

ϖ_1	ϖ_2	Exact solution	Approximate solution	Absolute error	LTDM [31]
1.2	0.2	−4.305959528345206	−4.305959528345206	0	−4.305959528345206
1.4	0.4	−6.423736771429134	−6.423736771429133	$8.881784197001252\times10^{-16}$	−6.423736771429133
1.6	0.6	−9.583089166764379	−9.583089166764379	0	−9.583089166764379
1.8	0.8	−14.296289098677603	−14.296289098677603	0	−14.296289098677603
2	1	−21.327557162026903	−21.327557162026903	0	−21.327557162026903
2.2	1.2	−31.816976514667704	−31.816976514667704	0	−31.816976514667704
2.4	1.4	−47.465351368853518	−47.465351368853518	0	−47.465351368853518

Table 4: Comparison analysis of approximate and exact solutions of $\Psi(\varpi_1, \varpi_2, \xi)$ of Problem 4.2 with absolute error when $\phi = 1$, $\xi = 0.1$, for various values of ϖ_1 and ϖ_2 with multi-Laplace transform decomposition method (LTDM) proposed by [31]

ϖ_1	ϖ_2	Exact solution	Approximate solution	Absolute error	LTDM [31]
1.2	0.2	4.305959528345206	4.305959528345206	0	4.305959528345206
1.4	0.4	6.423736771429134	6.423736771429133	$8.881784197001252\times10^{-16}$	6.423736771429133
1.6	0.6	9.583089166764379	9.583089166764379	0	9.583089166764379
1.8	0.8	14.296289098677603	14.296289098677603	0	14.296289098677603
2	1	21.327557162026903	21.327557162026903	0	21.327557162026903
2.2	1.2	31.816976514667704	31.816976514667704	0	31.816976514667704
2.4	1.4	47.465351368853518	47.465351368853518	0	47.465351368853518

Table 5: Comparison analysis of approximate and exact solutions of $\Phi(\varpi_1, \varpi_2, \varpi_3, \xi)$ of Problem 4.3 with absolute error when $\phi = 1$, $\xi = 0.1$, for various values of ϖ_1, φ_2 and ϖ_3 with multi-Laplace transform decomposition method (LTDM) proposed by [31]

ϖ_1	ϖ_2	ϖ_3	Exact solution	Approximate solution	Absolute error	LTDM [31]
0.1	0.1	0.1	0.130434782608696	0.130368750000000	$6.603260869569860\times10^{-5}$	0.130368750000000
0.2	0.2	0.2	0.260869565217391	0.260737500000000	$1.320652173913972\times10^{-4}$	0.260737500000000
0.3	0.3	0.3	0.391304347826087	0.391106250000000	$1.980978260869848\times10^{-4}$	0.391106250000000
0.4	0.4	0.4	0.521739130434783	0.521475000000000	$2.641304347827944\times10^{-4}$	0.521475000000000
0.5	0.5	0.5	0.652173913043478	0.651843750000000	$3.301630434783265\times10^{-4}$	0.651843750000000
0.6	0.6	0.6	0.782608695652174	0.782212500000000	$3.961956521739696\times10^{-4}$	0.782212500000000
0.7	0.7	0.7	0.913043478260870	0.912581250000000	$4.622282608696127\times10^{-4}$	0.912581250000000
0.8	0.8	0.8	1.043478260869566	1.042950000000000	$5.282608695655888\times10^{-4}$	1.042950000000000
0.9	0.9	0.9	1.173913043478261	1.173318750000000	$5.942934782610099\times10^{-4}$	1.173318750000000
1	1	1	1.304347826086957	1.303687500000000	$6.603260869566530\times10^{-4}$	1.303687500000000

Table 6: Comparison analysis of approximate and exact solutions of $\Psi(\varpi_1, \varpi_2, \varpi_3, \xi)$ of Problem 4.3 with absolute error when $\phi = 1$, $\xi = 0.1$, for various values of ϖ_1, φ_2 and ϖ_3 with multi-Laplace transform decomposition method (LTDM) proposed by [31]

ϖ_1	ϖ_2	ϖ_3	Exact solution	Approximate solution	Absolute error	LTDM [31]
0.1	0.1	0.1	0.130434782608696	0.130368750000000	$6.603260869569860\times10^{-5}$	0.130368750000000
0.2	0.2	0.2	0.260869565217391	0.260737500000000	$1.320652173913972\times10^{-4}$	0.260737500000000
0.3	0.3	0.3	0.391304347826087	0.391106250000000	$1.980978260869848\times10^{-4}$	0.391106250000000
0.4	0.4	0.4	0.521739130434783	0.521475000000000	$2.641304347827944\times10^{-4}$	0.521475000000000
0.5	0.5	0.5	0.652173913043478	0.651843750000000	$3.301630434783265\times10^{-4}$	0.651843750000000
0.6	0.6	0.6	0.782608695652174	0.782212500000000	$3.961956521739696\times10^{-4}$	0.782212500000000
0.7	0.7	0.7	0.913043478260870	0.912581250000000	$4.622282608696127\times10^{-4}$	0.912581250000000
0.8	0.8	0.8	1.043478260869566	1.042950000000000	$5.282608695655888\times10^{-4}$	1.042950000000000
0.9	0.9	0.9	1.173913043478261	1.173318750000000	$5.942934782610099\times10^{-4}$	1.173318750000000
1	1	1	1.304347826086957	1.303687500000000	$6.603260869566530\times10^{-4}$	1.303687500000000

Table 7: Comparison analysis of approximate and exact solutions of $\Upsilon(\varpi_1, \varpi_2, \varpi_3, \xi)$ of Problem 4.3 with absolute error when $\phi = 1$, $\xi = 0.1$, for various values of ϖ_1, φ_2 and ϖ_3 with multi-Laplace transform decomposition method (LTDM) proposed by [31]

ϖ_1	ϖ_2	ϖ_3	Exact solution	Approximate solution	Absolute error	LTDM [31]
0.1	0.1	0.1	0.130434782608696	0.130368750000000	$6.603260869569860\times10^{-5}$	0.130368750000000
0.2	0.2	0.2	0.260869565217391	0.260737500000000	$1.320652173913972\times10^{-4}$	0.260737500000000
0.3	0.3	0.3	0.391304347826087	0.391106250000000	$1.980978260869848\times10^{-4}$	0.391106250000000
0.4	0.4	0.4	0.521739130434783	0.521475000000000	$2.641304347827944\times10^{-4}$	0.521475000000000
0.5	0.5	0.5	0.652173913043478	0.651843750000000	$3.301630434783265\times10^{-4}$	0.651843750000000
0.6	0.6	0.6	0.782608695652174	0.782212500000000	$3.961956521739696\times10^{-4}$	0.782212500000000
0.7	0.7	0.7	0.913043478260870	0.912581250000000	$4.622282608696127\times10^{-4}$	0.912581250000000
0.8	0.8	0.8	1.043478260869566	1.042950000000000	$5.282608695655888\times10^{-4}$	1.042950000000000
0.9	0.9	0.9	1.173913043478261	1.173318750000000	$5.942934782610099\times10^{-4}$	1.173318750000000
1	1	1	1.304347826086957	1.303687500000000	$6.603260869566530\times10^{-4}$	1.303687500000000

6 Conclusion

The exploration of theoretical models designed to highlight physical process manifestations has indeed drawn researchers' consideration to their potential to generate provocative results with adequate techniques. In the present investigation, we succeeded in finding analytical solutions to certain nonlinear fractional Navier-Stokes equations employing $TEADM$ and $QEADM$. More precisely, the hybrid method reported the series of solutions by considering a new iterative approach. The high accuracy of the obtained results with the anticipated technique is expounded in the context of numerical simulation, and the highly complicated behavior has been captured in terms of surface plots. These kinds of investigations can help us examine more intriguing system results, and they widen the window for advancements and reformation in the notion of examining and predicting the more dynamic behavior of the commensurate models that illustrate physical processes. For future research, we extend this study by using the time delays and white noise environments to incorporate the revolutionary techniques of fractional calculus.

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Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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