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ARTICLE

A Study on the Nonlinear Caputo-Type Snakebite Envenoming Model with Memory

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ABSTRACT

In this article, we introduce a nonlinear Caputo-type snakebite envenoming model with memory. The well-known Caputo fractional derivative is used to generalize the previously presented integer-order model into a fractional-order sense. The numerical solution of the model is derived from a novel implementation of a finite-difference predictor-corrector (L1-PC) scheme with error estimation and stability analysis. The proof of the existence and positivity of the solution is given by using the fixed point theory. From the necessary simulations, we justify that the first-time implementation of the proposed method on an epidemic model shows that the scheme is fully suitable and time-efficient for solving epidemic models. This work aims to show the novel application of the given scheme as well as to check how the proposed snakebite envenoming model behaves in the presence of the Caputo fractional derivative, including memory effects.

KEYWORDS

Mathematical model; Caputo fractional derivative; L1-predictor-corrector method; error estimation; stability; graphical simulations

1 Introduction

Nowadays, fractional calculus [1–3] is being applied to solve various real-world problems in terms of mathematical modeling. Different fractional derivatives [4] have been successfully used to model various problems. More specifically, several deadly epidemics have been modeled by using mathematical models in a fractional-order sense. It is a well-known fact that the fractional-order operators are non-local in nature and may be more effective for modeling history (memory)-dependent systems. Moreover, a fractional order can be fixed as any positive real number that better fits a real-data. So, by using such an operator, an accurate adjustment can be done in a model to fit with real data for better predicting the outbreaks of an epidemic. Recently, several applications of fractional derivatives have been recorded in epidemiology. In [5–7], the authors have studied the dynamics of the COVID-19 epidemic by using fractional-order mathematical models. In [8], the authors used non-singular Caputo-Fabrizio and Atangana-Baleanu fractional derivatives to study the dynamical nature of a malaria epidemic model. Kumar et al. in [9] have solved canine distemper virus and rabies epidemic models in the sense of generalized Caputo derivative. Kumar et al. [10] defined two different types of fractional-order models to study the dynamics of mosaic disease. A novel application of the generalized Caputo derivative in environmental infection dynamics can be seen in [11]. Sinan et al. [12] proposed the mathematical modeling of typhoid fever in terms of fractional derivatives. In [13], some novel analyses on the numerical modeling of biological systems using fractional derivatives have been given. Rihan et al. [14] studied fractional-order predator-prey models, including delay with Holling type-II functional response. In [15], some novel applications of delay differential equations were proposed. Work on the application of fractional derivatives in ecology and psychology can be understood from [16,17], respectively.

To analyze the various types of fractional-order systems, several computational schemes have been proposed by researchers. Odibat et al. [18] derived a new generalized form of the predictor-corrector (PC) scheme to investigate fractional initial value problems. Kumar et al. [19] introduced a new method to simulate fractional-order systems with various examples. In [20], the PC method was derived to simulate delay fractional differential equations. A modified form of the PC scheme in terms of the generalized Caputo derivative to solve delay-type systems has been introduced in reference [21]. Odibat et al. [22] have derived the generalized differential transform method for solving fractional impulsive differential equations. In [23], some computational schemes to solve delayed parabolic and time-fractional partial differential equations have been proposed. Shah et al. [24] simulated the dynamics of some important fractional order differential equations. In [25], some novel analyses of the Cauchy-type fractional-order dynamical system in terms of piecewise equations have been given. Some more applications of fractional derivatives can be seen in [26–28].

Jhinga et al. [29] introduced a novel finite-difference predictor-corrector (L1-PC) scheme to solve fractional-order systems in the sense of the Caputo derivative. That L1-PC scheme is not yet applied to any epidemic model to check how it will work and show the disease dynamics. In this paper, we fill this research gap by implementing the given L1-PC method on a Caputo-type snakebite envenoming (SBE) mathematical model, which was previously proposed in the integer-order sense in reference [30]. Currently, SBE is a deathly neglected disease, mainly in developing nations. The World Health Organization (WHO) recognized SBE as a fatal disease that generally results from the injection of a combination of various toxins (venom) following a venomous snakebite. Mainly poor, rural societies in tropical and subtropical nations all over the world are typically affected by SBE, which is a big threat to the health and well-being of about 5.8 billion population [31]. More information on SBE can be collected from the references [32–34]. To date, only a few research studies have gone into the mathematical modeling of SBE. In [35], the authors introduced a model using the law of mass action to analyze snakebite incidence. In [36], a mathematical model considering the socio-demographic components that influence the death risk from SBE in India was proposed.

The motivation of this study is to justify the application of the aforementioned L1-PC scheme in epidemiology by solving a mathematical model of SBE. Also, the aim is to check how the proposed SBE model will behave in the presence of the Caputo fractional derivative, which is a non-local differential operator that allows memory effects in the system. The given study is designed using the following systems: Section 2 recalls some preliminaries of fractional calculus. The description of the considered model of SBE in terms of the Caputo derivative is given in Section 3. The necessary numerical analysis, like the solution algorithm, error estimation, and method stability, is performed in

Section 4. A brief discussion of the proposed methodology, its novelty, and outputs is given in Section 5 with supporting conclusions.

2 Preliminaries

Here we recall some preliminaries of fractional calculus.

Definition 1. *A real function $f(s)$, $s > 0$ belongs to the space*

a) C_η, $\eta \in \mathbb{R}$ if there exists a real number $q > \eta$, such that $f(s) = s^q f_1(s), f_1 \in C[0,\infty)$. Clearly, $C_\eta \subset C_\alpha$ if $\alpha \leq \eta$.

b) C_η^m, $m \in \mathbb{N} \cup \{0\}$ if $f^m \in C_\eta$.

Definition 2. [2] *The Riemann-Liouville fractional integral of $f(t) \in C_\eta$ $(\eta \geq -1)$ is defined as follows:*

$$J^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s) ds,$$
$$J^0 f(t) = f(t).$$

Definition 3. [2] *The Caputo fractional derivative of $f \in C_{-1}^m$ is written by*

$$D_t^\gamma f(t) = \begin{cases} \dfrac{d^m f(t)}{dt^m}, & \text{if } \gamma = m \in \mathbb{N} \\ \dfrac{1}{\Gamma(m-\gamma)} \int_0^t (t-\vartheta)^{m-\gamma-1} f^{(m)}(\vartheta)\, d\vartheta, & \text{if } m-1 < \gamma < m,\ m \in \mathbb{N}. \end{cases} \tag{1}$$

Theorem 1. [37] *Consider the function f such that f and ${}^C D_{c+}^\gamma f$ are continuous for $\gamma \in (0,1]$. Then, $\forall\, t \in (c,d]$, there exists some $k \in (c,t)$ following the constraint*

$$f(t) = f(c) + \frac{1}{\Gamma(\gamma+1)}\, {}^C D_{c+}^\gamma f(k)\,(t-c)^\gamma.$$

Therefore, from Theorem 1, we say that if ${}^C D_{c+}^\gamma f(t) > 0$, for all $t \in [c,d]$, then f is strictly increasing, and if ${}^C D_{c+}^\gamma f(t) < 0$, for all $t \in [c,d]$, then the function f is strictly decreasing.

Lemma 1. [38] *Let $\gamma \in (0,1), n \in \mathbb{N}$ and define the vectors $X := (x_1, x_2, \ldots, x_n)$ and $Y := (y_1, y_2, \ldots, y_n)$. For each $i = 1, 2, \ldots, n$, let us consider $G_i : [c,d] \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function fulfils the Lipschitz condition with respect to the second variable such that*

$$|G_i(t,X) - G_i(t,Y)| \leq L_i \|X - Y\|,$$

where L_i is a constant. Let us write $G := (G_1, G_2, \ldots, G_n)$ and define the two fractional differential equations

$${}^C D_{c+}^\gamma X(t) = G(t,X) + \frac{1}{j} \text{ and } {}^C D_{c+}^\gamma X(t) = G(t,X), \tag{2}$$

with the same initial constraints, where j is a positive integer. If ${}_jX^* := ({}_jx_1^*, \ldots, {}_jx_n^*)$ and $X^* := (x_1^*, x_2^*, \ldots, x_n^*)$ are the solutions of (2), simultaneously, then ${}_jX^*(t) \to X^*(t)$ as $j \to \infty$ for all $t \in [c,d]$.

3 Model Description

Here we introduce the generalized form of the previously published integer-order model [30] into Caputo-type fractional-order sense ($^{C}D^{\gamma}$). In our model, the total human population $N_H(t)$ at time t is split out into the following mutual exclusive classes: unaware susceptible humans, ($S_U(t)$), aware susceptible humans, ($S_E(t)$), SBE population, ($I(t)$), humans getting early remedy with antivenom, ($T_E(t)$), peoples getting late remedy with antivenom, ($T_L(t)$), peoples suffering from early adverse reaction (EAR) at the time of early remedy, ($V_E(t)$), peoples facing EAR at the time of late remedy, ($V_L(t)$), recovered peoples with disabilities, ($R_D(t)$), and recovered peoples without disabilities, ($R_W(t)$). Therefore, the total population size is defined as

$$N(t) = S_U(t) + S_E(t) + I(t) + T_E(t) + T_L(t) + V_E(t) + V_L(t) + R_D(t) + R_W(t).$$

The total snake population is defined by ($N_S(t)$) and $D(t)$ shows the total deaths caused by snakebite. For taking the equal time-dimension $day^{-\gamma}$ on the both sides of the fractional-order model, the power γ is applied to the parameters in time unit day^{-1} in the classical case. Therefore, the fractional-order nonlinear model for SBE is given as follows:

$$\begin{aligned}
{}^{C}D^{\gamma} S_U &= \Lambda_H^{\gamma} - (\lambda + K_1) S_U, \\
{}^{C}D^{\gamma} S_E &= \epsilon^{\gamma} S_U + \phi_1^{\gamma} R_D + \phi_2^{\gamma} R_W - (\Pi_1 \lambda + K_2) S_E, \\
{}^{C}D^{\gamma} I &= (\Pi_1 S_E + S_U)\lambda - K_3 I, \\
{}^{C}D^{\gamma} T_E &= \tau^{\gamma} k I - K_4 T_E, \\
{}^{C}D^{\gamma} T_L &= \tau^{\gamma} \Pi_2 I - K_5 T_L, \\
{}^{C}D^{\gamma} V_E &= \alpha_1^{\gamma} T_E - K_6 V_E, \\
{}^{C}D^{\gamma} V_L &= \alpha_2^{\gamma} T_L - K_7 V_L, \\
{}^{C}D^{\gamma} R_D &= \sigma_1^{\gamma} \rho_1 T_L + \sigma_2^{\gamma} \rho_2 V_L - K_8 R_D, \\
{}^{C}D^{\gamma} R_W &= r_1^{\gamma} T_E + r_2^{\gamma} V_E + \sigma_1^{\gamma} \Pi_3 T_L + \sigma_2^{\gamma} \Pi_4 V_L - K_9 R_W, \\
{}^{C}D^{\gamma} N_S &= \Lambda_S^{\gamma} N_S \left(1 - \frac{N_S}{K_S}\right) - \mu_S^{\gamma} N_S, \\
{}^{C}D^{\gamma} D &= \delta_1^{\gamma} I + (T_L + V_L)\delta_2^{\gamma},
\end{aligned} \tag{3}$$

$$\lambda(t) = \frac{\beta^{\gamma} N_S}{N_H + N_S},$$

where,

$K_1 = \epsilon^{\gamma} + \mu_H^{\gamma},\ K_2 = \mu_H^{\gamma},\ K_3 = \tau^{\gamma} + \delta_1^{\gamma} + \mu_H^{\gamma},\ K_4 = \alpha_1^{\gamma} + r_1^{\gamma} + \mu_H^{\gamma},\ K_5 = \alpha_2^{\gamma} + \sigma_1^{\gamma} + \delta_2^{\gamma} + \mu_H^{\gamma},\ K_6 = r_2^{\gamma} + \mu_H^{\gamma},\ K_7 = \sigma_2^{\gamma} + \delta_2^{\gamma} + \mu_H^{\gamma},\ K_8 = \phi_1^{\gamma} + \mu_H^{\gamma},\ K_9 = \phi_2^{\gamma} + \mu_H^{\gamma},\ \Pi_1 = 1 - \theta,\ \Pi_2 = 1 - k, \Pi_3 = 1 - \rho_1,\ \Pi_4 = 1 - \rho_2,$ with the initial conditions

$$S_U(0) > 0,\ S_E(0) \geq 0,\ I(0) \geq 0,\ T_E(0) \geq 0,\ T_L(0) \geq 0,\ V_E(0) \geq 0,\ V_L(0) \geq 0,\ R_D(0) \geq 0,\ R_W(0) \geq 0,\ N_S(0) \geq 0,\ D(0) \geq 0. \tag{4}$$

The definitions of the model parameters alongwith their numerical values are given in Table 1.

Table 1: Description of model parameters [30]

Parameter	Description	Values
Λ_H	Birth rate of unaware susceptible humans	1324
Λ_S	Growth rate of snake population	0.1925
μ_H	Natural death rate of humans	5.04×10^{-6}
μ_S	Natural death rate of snakes	2.283×10^{-4}
β	Snakebite envenomation rate	0.0742
ϵ	Public health awareness campaign rate	0.0051
θ	Public health awareness campaign's efficacy	1.7729×10^{-4}
τ	Rate of treatment with antivenom received by SBE individuals	0.9997
k	SBE individuals proportion getting early remedy with antivenom	0.8073
δ_1	SBE induced death rate in I compartment	0.0025
δ_2	SBE induced death rate in T_L and V_L compartments	4.2564×10^{-4}
α_1	Rate at which humans getting remedy with antivenom suffering from EAR in T_E compartment	0.1215
α_2	Rate at which humans getting remedy with antivenom suffering from EAR in T_L compartment	0.1708
r_1	Recovery rate of T_E class without disability	0.9310
r_2	Recovery rate of V_E class without disability	0.9310
σ_1	Recovery rate of T_L class with disability	0.9924
σ_2	Recovery rate of V_L class with disability	0.9924
ρ_1	Proportion of humans recovered with disabilities in T_L class	0.1500
ρ_2	Proportion of humans recovered with disabilities in V_L class	0.9985
ϕ_1	Transition rates of individuals in R_D class to S_E	0.5233
ϕ_2	Transition rates of individuals in R_W class to S_E	0.9416
K_S	Carrying capacity of snake	6.6604×10^4
$S_U(0)$	Initial size of unaware susceptible individuals	2.1459×10^7
$S_E(0)$	Initial size of aware susceptible individuals	6.5132×10^3
$I(0)$	Initial size of SBE individuals	99
$T_E(0)$	Initial size of humans getting early treatment with antivenom	76
$T_L(0)$	Initial size of humans getting late treatment with antivenom	7
$V_E(0)$	Initial size of humans facing EAR during early treatment	8
$V_L(0)$	initial size of humans facing EAR during late treatment	0
$R_D(0)$	Initial size of individuals recovered with disabilities	4
$R_W(0)$	Initial size of individuals recovered without disabilities	93
$N_S(0)$	Initial population of snakes	1.2250×10^4
$D(0)$	Initial cumulative number of deaths due to snakebite	2

Theorem 2. *There exists a unique solution for the model (3) and (4) which belongs to* $(R_0^+)^{11} := \{(S_U^*, S_E^*, I^*, T_E^*, T_L^*, V_E^*, V_L^*, R_D^*, R_W^*, N_S^*, D^*) \in \mathbb{R}^{+11}\}$.

Proof. By using the Theorem 3.1 and Remark 3.2 of [39], the global existence of the unique solution can easily be proved. Now to prove the non-negativity of the solution, let us write the following auxiliary system of fractional differential equations:

$$\begin{aligned}
{}^{C}D^{\gamma} S_U &= \Lambda_H^{\gamma} - (\lambda + K_1) S_U + \frac{1}{k},\\
{}^{C}D^{\gamma} S_E &= \epsilon^{\gamma} S_U + \phi_1^{\gamma} R_D + \phi_2^{\gamma} R_W - (\Pi_1 \lambda + K_2) S_E + \frac{1}{k},\\
{}^{C}D^{\gamma} I &= (\Pi_1 S_E + S_U) \lambda - K_3 I + \frac{1}{k},\\
{}^{C}D^{\gamma} T_E &= \tau^{\gamma} k I - K_4 T_E + \frac{1}{k},\\
{}^{C}D^{\gamma} T_L &= \tau^{\gamma} \Pi_2 I - K_5 T_L + \frac{1}{k},\\
{}^{C}D^{\gamma} V_E &= \alpha_1^{\gamma} T_E - K_6 V_E + \frac{1}{k},\\
{}^{C}D^{\gamma} V_L &= \alpha_2^{\gamma} T_L - K_7 V_L + \frac{1}{k},\\
{}^{C}D^{\gamma} R_D &= \sigma_1^{\gamma} \rho_1 T_L + \sigma_2^{\gamma} \rho_2 V_L - K_8 R_D + \frac{1}{k},\\
{}^{C}D^{\gamma} R_W &= r_1^{\gamma} T_E + r_2^{\gamma} V_E + \sigma_1^{\gamma} \Pi_3 T_L + \sigma_2^{\gamma} \Pi_4 V_L - K_9 R_W + \frac{1}{k},\\
{}^{C}D^{\gamma} N_S &= \Lambda_S^{\gamma} N_S \left(1 - \frac{N_S}{K_S}\right) - \mu_S^{\gamma} N_S + \frac{1}{k},\\
{}^{C}D^{\gamma} D &= \delta_1^{\gamma} I + (T_L + V_L) \delta_2^{\gamma} + \frac{1}{k},
\end{aligned} \tag{5}$$

with $k \in \mathbb{N}$. We will show that solution of (5) $(S_{U_k}^*(t), S_{E_k}^*(t), I_k^*(t), T_{E_k}^*(t), T_{L_k}^*(t), V_{E_k}^*(t), V_{L_k}^*(t), R_{D_k}^*(t), R_{W_k}^*(t), N_{S_k}^*(t), D_k^*(t))$ belongs to $(R_0^+)^{11}$, $\forall\, t \geq 0$. For obtaining a contradiction, we opine that there exists a point of time where the condition fails. Let $t_0 := \inf\{t > 0 | (S_{U_k}^*(t), S_{E_k}^*(t), I_k^*(t), T_{E_k}^*(t), T_{L_k}^*(t), V_{E_k}^*(t), V_{L_k}^*(t), R_{D_k}^*(t), R_{W_k}^*(t), N_{S_k}^*(t), D_k^*(t)) \in (R_0^+)^{11}\}$. Thus, $(S_{U_k}^*(t_0), S_{E_k}^*(t_0), I_k^*(t_0), T_{E_k}^*(t_0), T_{L_k}^*(t_0), V_{E_k}^*(t_0), V_{L_k}^*(t_0), R_{D_k}^*(t_0), R_{W_k}^*(t_0), N_{S_k}^*(t_0), D_k^*(t_0)) \in (R_0^+)^{11}$ and one of the quantities $(S_{U_k}^*(0), S_{E_k}^*(0), I_k^*(0), T_{E_k}^*(0), T_{L_k}^*(0), V_{E_k}^*(0), V_{L_k}^*(0), R_{D_k}^*(0), R_{W_k}^*(0), N_{S_k}^*(0), D_k^*(0))$ is zero. Suppose that $I_k^*(t_0) = 0$. Since

$${}^{C}D^{\gamma} I_k^*(t_0) = \left(\Pi_1 S_{U_k}^*(t_0) + S_{E_k}^*(t_0)\right) \lambda + \frac{1}{k} > 0$$

by continuity of ${}^{C}D_{0+}^{\gamma} I_k^*$, we conclude that ${}^{C}D_{0+}^{\gamma} I_k^*([t_0, t_0 + \xi) \subset \mathbb{R}^+$, for some $\xi > 0$. By Theorem 1, $I_k^*([t_0, t_0 + \xi) \subset \mathbb{R}_0^+$ and so I_k^* is non negative. In an analogous way we can justify that the remaining functions $S_{U_k}^*, S_{E_k}^*, T_{E_k}^*, T_{L_k}^*, V_{E_k}^*, V_{L_k}^*, R_{D_k}^*, R_{W_k}^*, N_{S_k}^*$, and D_k^* are non-negative, establishing a contradiction. Using Lemma 1 as $k \to \infty$, we get that the solution $(S_{U_k}^*(t), S_{E_k}^*(t), I_k^*(t), T_{E_k}^*(t), T_{L_k}^*(t), V_{E_k}^*(t), V_{L_k}^*(t), R_{D_k}^*(t), R_{W_k}^*(t), N_{S_k}^*(t), D_k^*(t))$ of (5) belongs to $(R_0^+)^{11}$, $\forall\, t \geq 0$, giving the required result.

Now before performing the further numerical simulations on the proposed fractional-order model (3), we rewrite the model into a compact form by representing it in terms of an initial value problem, defined as follows:

Let us consider

$$\begin{cases} f_1(t, S_U, \ldots, D) = \Lambda_H^\gamma - (\lambda + K_1) S_U, \\ f_2(t, S_U, \ldots, D) = \epsilon^\gamma S_U + \phi_1^\gamma R_D + \phi_2^\gamma R_W - (\Pi_1 \lambda + K_2) S_E, \\ f_3(t, S_U, \ldots, D) = (\Pi_1 S_E + S_U)\lambda - K_3 I, \\ f_4(t, S_U, \ldots, D) = \tau^\gamma k I - K_4 T_E, \\ f_5(t, S_U, \ldots, D) = \tau^\gamma \Pi_2 I - K_5 T_L, \\ f_6(t, S_U, \ldots, D) = \alpha_1^\gamma T_E - K_6 V_E, \\ f_7(t, S_U, \ldots, D) = \alpha_2^\gamma T_L - K_7 V_L, \\ f_8(t, S_U, \ldots, D) = \sigma_1^\gamma \rho_1 T_L + \sigma_2^\gamma \rho_2 V_L - K_8 R_D, \\ f_9(t, S_U, \ldots, D) = r_1^\gamma T_E + r_2^\gamma V_E + \sigma_1^\gamma \Pi_3 T_L + \sigma_2^\gamma \Pi_4 V_L - K_9 R_W, \\ f_{10}(t, S_U, \ldots, D) = \Lambda_S^\gamma N_S \left(1 - \frac{N_S}{K_S}\right) - \mu_S^\gamma N_S, \\ f_{11}(t, S_U, \ldots, D) = \delta_1^\gamma I + (T_L + V_L)\delta_2^\gamma. \end{cases} \tag{6}$$

By using (6), we have

$$^{C}D^\gamma \mathcal{A}(t) = \Phi(t, \mathcal{A}(t)),\ t \in [0, T], 0 < \gamma \le 1,$$
$$\mathcal{A}(0) = \mathcal{A}_0, \tag{7}$$

where

$$\mathcal{A}(t) = \begin{cases} S_U(t) \\ S_E(t) \\ I(t) \\ T_E(t) \\ T_L(t) \\ V_E(t) \\ V_L(t) \\ R_D(t) \\ R_W(t) \\ N_S(t) \\ D(t) \end{cases}, \quad \mathcal{A}_0(t) = \begin{cases} S_{U_0}(t) \\ S_{E_0}(t) \\ I_0(t) \\ T_{E_0}(t) \\ T_{L_0}(t) \\ V_{E_0}(t) \\ V_{L_0}(t) \\ R_{D_0}(t) \\ R_{W_0}(t) \\ N_{S_0}(t) \\ D_0(t) \end{cases}, \quad \Phi(t, \mathcal{A}(t)) = \begin{cases} f_1(t, S_U, \ldots, D) \\ f_2(t, S_U, \ldots, D) \\ f_3(t, S_U, \ldots, D) \\ f_4(t, S_U, \ldots, D) \\ f_5(t, S_U, \ldots, D) \\ f_6(t, S_U, \ldots, D) \\ f_7(t, S_U, \ldots, D) \\ f_8(t, S_U, \ldots, D) \\ f_9(t, S_U, \ldots, D) \\ f_{10}(t, S_U, \ldots, D) \\ f_{11}(t, S_U, \ldots, D) \end{cases}. \tag{8}$$

4 Numerical Analysis on the Model

In this section, we perform the necessary numerical simulations (solution derivation, error estimation and stability) to derive the solution of the proposed fractional-order model (3) by using the L1-predictor-corrector scheme [29].

Consider the above given initial value problem (IVP) for $0 < \gamma < 1$,

$$
\begin{aligned}
&{}^{C}D^{\gamma}\mathcal{A}(t) = \Phi(t, \mathcal{A}(t)),\ t \in [0, T],\\
&\quad \mathcal{A}(0) = \mathcal{A}_0.
\end{aligned} \tag{9}
$$

where ${}^{C}D^{\gamma}$ represents the Caputo derivatives and $\Phi : [0, T] \times D \to \mathbb{R}, D \subset \mathbb{R}$. Split the time span $[0,\ T]$ into N subintervals. Take an uniform grid with step size of $h = \frac{T}{N}$ with $t_k = kh, k = 0, 1, \ldots, N$.

4.1 Derivation of the Solution

According to the L1-PC method, the Caputo fractional derivative is numerically defined by

$$
\begin{aligned}
\left[{}^{C}D^{\gamma}\mathcal{A}\left(t\right)\right]_{t=t_n} &= \frac{1}{\Gamma(1-\gamma)}\int_0^{t_k}(t_n - s)^{-\gamma}\mathcal{A}'\left(s\right)\mathrm{d}s\\
&= \frac{1}{\Gamma(1-\gamma)}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(t_n - s)^{-\gamma}\mathcal{A}'\left(s\right)\mathrm{d}s\\
&\approx \frac{1}{\Gamma(1-\gamma)}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(t_n - s)^{-\gamma}\frac{\mathcal{A}(t_{k+1}) - \mathcal{A}(t_k)}{h}\mathrm{d}s\\
&= \sum_{k=0}^{n-1} b_{n-k-1}\left(\mathcal{A}\left(t_{k+1}\right) - \mathcal{A}\left(t_k\right)\right),
\end{aligned} \tag{10}
$$

where,

$$
b_k = \frac{h^{-\gamma}}{\Gamma(2-\gamma)}\left[(k+1)^{1-\gamma} - k^{1-\gamma}\right].
$$

We approximate ${}^{C}D^{\gamma}\mathcal{A}(t)$ by the Eq. (10) and put it into (9) to get

$$
[{}^{C}D^{\gamma}\mathcal{A}\left(t\right)]_{t=t_n} = \sum_{k=0}^{n-1} b_{n-k-1}\left(\mathcal{A}\left(t_{k+1}\right) - \mathcal{A}\left(t_k\right)\right) = \Phi\left(t_n, \mathcal{A}_n\right), \tag{11}
$$

where $\mathcal{A}_k$ defines the approximate value of the solution of (9) at $t = t_k$ and

$$
b_{n-k-1} = \frac{h^{-\gamma}}{\Gamma(2-\gamma)}\left[(n-k)^{1-\gamma} - (n-k-)^{1-\gamma}\right].
$$

(11) can be rewritten as

$$
b_{n-1}(\mathcal{A}_1 - \mathcal{A}_0) + b_{n-2}(\mathcal{A}_2 - \mathcal{A}_1) + \cdots + b_0(\mathcal{A}_n - \mathcal{A}_{n-1}) = \Phi(t_n, \mathcal{A}_n). \tag{12}
$$

After rewriting the terms (12), we get the following from:

$$
b_0\mathcal{A}_n = b_0\mathcal{A}_{n-1} - \sum_{k=0}^{n-2} b_{k+1}\mathcal{A}_{n-1-k} + \sum_{k=1}^{n-1} b_k\mathcal{A}_{n-1-k} + \Phi\left(t_n, \mathcal{A}_n\right). \tag{13}
$$

Substituting

$$
b_0 = \frac{h^{-\gamma}}{\Gamma(2-\gamma)} \text{ and } b_k = \frac{h^{-\gamma}}{\Gamma(2-\gamma)}\left[(n-k)^{1-\gamma} - (n-k-)^{1-\gamma}\right].
$$

in (12), we get

$$\mathcal{A}_n = \mathcal{A}_{n-1} - \left(2^{1-\gamma} - 1^{1-\gamma}\right)\mathcal{A}_{n-1} - \sum_{k=1}^{n-2}\left((2+k)^{1-\gamma} - (1+k)^{(1-\gamma)}\right)\mathcal{A}_{n-1-k}$$
$$+ \sum_{k=1}^{n-2}\left((1+k)^{1-\gamma} - (k)^{(1-\gamma)}\right)\mathcal{A}_{n-1-k} + \left(n^{1-\gamma} - (n-1)^{1-\gamma}\right)\mathcal{A}_0$$
$$+ \Gamma(2-\gamma)h^{\gamma}\Phi(t_n, \mathcal{A}_n) \tag{14}$$
$$= \left(n^{1-\gamma} - (n-1)^{1-\gamma}\right)\mathcal{A}_0 + \sum_{k=1}^{n-1}\left[2\,(n-k)^{1-\gamma} - (n+1-k)^{(1-\gamma)} - (n-1-k)^{(1-\gamma)}\right]\mathcal{A}_k$$
$$+ \Gamma(2-\gamma)h^{\gamma}\Phi(t_n, \mathcal{A}_n).$$

Define

$$a_k := k + 1^{(1-\gamma)} - k^{(1-\gamma)}. \tag{15}$$

Remark that $a_k's$ have the following characteristics:

- $a_k > 0, k = 0, 1, \ldots, n-1.$
- $a_0 = 1 > a_1 > \cdots > a_k$, and $a_k \to 0$ as $k \to \infty$.
- $\sum_{k=0}^{n-1}(a_k - a_{k+1}) + a_n = (1 - a_1) + \sum_{k=1}^{n-2}(a_k - a_{k+1}) + a_{n-1} = 1.$

Given Eqs. (14) and (15), the following form can be obtained:

$$\mathcal{A}_n = a_{n-1}\mathcal{A}_0 + \sum_{k=1}^{n-1}(a_{n-1-k} - a_{n-k})\,\mathcal{A}_k + \Gamma\,(2-\gamma)\,h^{\gamma}\,\Phi\,(t_n, \mathcal{A}_n)\,. \tag{16}$$

We can see that Eq. (16) is of the form $\mathcal{A}_n = g + N(\mathcal{A}_n)$, if we identify

$$g = a_{n-1}\mathcal{A}_0 + \sum_{k=1}^{n-1}(a_{n-1-k} - a_{n-k})\,\mathcal{A}_k$$

and

$$N(\mathcal{A}_n) = \Gamma(2-\gamma)h^{\gamma}\Phi(t_n, \mathcal{A}_n).$$

Hence using the scheme of the DGJ method gives approximate value of $\mathcal{A}_n$, given by

$$\mathcal{A}_{n,0} = g = a_{n-1}\mathcal{A}_0 + \sum_{k=1}^{n-1}(a_{n-1-k} - a_{n-k})\,\mathcal{A}_k,$$
$$\mathcal{A}_{n,0} = N(\mathcal{A}_{n,0}) = \Gamma(2-\gamma)h^{\gamma}\Phi(t_n, \mathcal{A}_n),$$
$$\mathcal{A}_{n,2} = N(\mathcal{A}_{n,0} + \mathcal{A}_{n,1} - N(\mathcal{A}_{n,0}).$$

The three term approximation of $\mathcal{A}_n \approx \mathcal{A}_{n,0} + \mathcal{A}_{n,0} + \mathcal{A}_{n,2}$. Therefore, this approximated solution of DGJ scheme gives the following predictor-corrector algorithm called as L1-PCM.

$$\begin{aligned}
\mathcal{A}_n^p &= a_{n-1}\mathcal{A}_0 + \sum_{k=1}^{n-1} (a_{n-1-k} - a_{n-k})\,\mathcal{A}_k, \\
z_n^p &= N(\mathcal{A}_n) = \Gamma(2-\gamma)h^\gamma \Phi(t_n, \mathcal{A}_n^p), \\
\mathcal{A}_n^c &= \mathcal{A}_n^p + \Gamma(2-\gamma)h^\gamma \Phi(t_n, \mathcal{A}_n^p + z_n^p),
\end{aligned} \tag{17}$$

where $\mathcal{A}_n^p$ and z_n^p are predictors and $\mathcal{A}_n^c$ is the corrector.

Using the above given methodology, the approximation equations of the proposed model (3) in terms of L1-PC method are derived as follows:

$$\begin{cases}
S_{U_n}^{\ c} = S_{U_n}^{\ p} + \Gamma(2-\gamma)h^\gamma f_1(t_n, S_{U_n}^{\ p} + z_{1n}^p, \ldots, D_n^p + z_{11n}^p), \\
S_{E_n}^{\ c} = S_{E_n}^{\ p} + \Gamma(2-\gamma)h^\gamma f_2(t_n, S_{U_n}^{\ p} + z_{1n}^p, \ldots, D_n^p + z_{11n}^p), \\
I_n^c = I_n^p + \Gamma(2-\gamma)h^\gamma f_3(t_n, S_{U_n}^{\ p} + z_{1n}^p, \ldots, D_n^p + z_{11n}^p), \\
T_{E_n}^{\ c} = T_{E_n}^{\ p} + \Gamma(2-\gamma)h^\gamma f_4(t_n, S_{U_n}^{\ p} + z_{1n}^p, \ldots, D_n^p + z_{11n}^p), \\
T_{L_n}^{\ c} = T_{L_n}^{\ p} + \Gamma(2-\gamma)h^\gamma f_5(t_n, S_{U_n}^{\ p} + z_{1n}^p, \ldots, D_n^p + z_{11n}^p), \\
V_{E_n}^{\ c} = V_{E_n}^{\ p} + \Gamma(2-\gamma)h^\gamma f_6(t_n, S_{U_n}^{\ p} + z_{1n}^p, \ldots, D_n^p + z_{11n}^p), \\
V_{L_n}^{\ c} = V_{L_n}^{\ p} + \Gamma(2-\gamma)h^\gamma f_7(t_n, S_{U_n}^{\ p} + z_{1n}^p, \ldots, D_n^p + z_{11n}^p), \\
R_{D_n}^{\ c} = R_{D_n}^{\ p} + \Gamma(2-\gamma)h^\gamma f_8(t_n, S_{U_n}^{\ p} + z_{1n}^p, \ldots, D_n^p + z_{11n}^p), \\
R_{W_n}^{\ c} = R_{W_n}^{\ p} + \Gamma(2-\gamma)h^\gamma f_9(t_n, S_{U_n}^{\ p} + z_{1n}^p, \ldots, D_n^p + z_{11n}^p), \\
N_{S_n}^{\ c} = N_{S_n}^{\ p} + \Gamma(2-\gamma)h^\gamma f_{10}(t_n, S_{U_n}^{\ p} + z_{1n}^p, \ldots, D_n^p + z_{11n}^p), \\
D_n^c = D_n^p + \Gamma(2-\gamma)h^\gamma f_{11}(t_n, S_{U_n}^{\ p} + z_{1n}^p, \ldots, D_n^p + z_{11n}^p),
\end{cases} \tag{18}$$

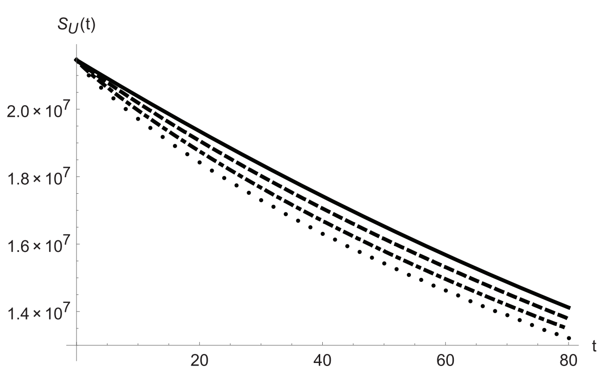
where,

$$\begin{cases}
S_{U_n}^{\ p} = a_{n-1}S_{U_0} + \sum_{k=1}^{n-1} (a_{n-1-k} - a_{n-k})\, S_{U_k}, \\
S_{E_n}^{\ p} = a_{n-1}S_{E_0} + \sum_{k=1}^{n-1} (a_{n-1-k} - a_{n-k})\, S_{E_k}, \\
I_n^p = a_{n-1}I_0 + \sum_{k=1}^{n-1} (a_{n-1-k} - a_{n-k})\, I_k, \\
T_{E_n}^{\ p} = a_{n-1}T_{E_0} + \sum_{k=1}^{n-1} (a_{n-1-k} - a_{n-k})\, T_{E_k}, \\
T_{L_n}^{\ p} = a_{n-1}T_{L_0} + \sum_{k=1}^{n-1} (a_{n-1-k} - a_{n-k})\, T_{L_k}, \\
V_{E_n}^{\ p} = a_{n-1}V_{E_0} + \sum_{k=1}^{n-1} (a_{n-1-k} - a_{n-k})\, V_{E_k}, \\
V_{L_n}^{\ p} = a_{n-1}V_{L_0} + \sum_{k=1}^{n-1} (a_{n-1-k} - a_{n-k})\, V_{L_k}, \\
R_{D_n}^{\ p} = a_{n-1}R_{D_0} + \sum_{k=1}^{n-1} (a_{n-1-k} - a_{n-k})\, R_{D_k}, \\
R_{W_n}^{\ p} = a_{n-1}R_{W_0} + \sum_{k=1}^{n-1} (a_{n-1-k} - a_{n-k})\, R_{W_k}, \\
N_{S_n}^{\ p} = a_{n-1}N_{S_0} + \sum_{k=1}^{n-1} (a_{n-1-k} - a_{n-k})\, N_{S_k}, \\
D_n^p = a_{n-1}D_0 + \sum_{k=1}^{n-1} (a_{n-1-k} - a_{n-k})\, D_k,
\end{cases} \tag{19}$$

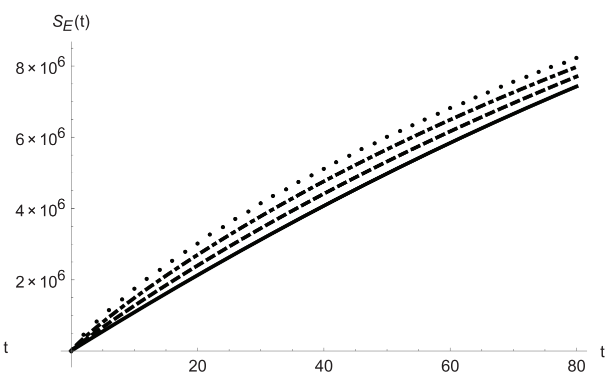
and

$$\begin{cases} z_{1n}^p = N(S_{U_n}) = \Gamma(2-\gamma)h^\gamma \mathrm{f}_1(t_n, S_{U_n}^p, \ldots, D_n^p), \\ z_{2n}^p = N(S_{E_n}) = \Gamma(2-\gamma)h^\gamma \mathrm{f}_2(t_n, S_{U_n}^p, \ldots, D_n^p), \\ z_{3n}^p = N(I_n) = \Gamma(2-\gamma)h^\gamma \mathrm{f}_3(t_n, S_{U_n}^p, \ldots, D_n^p), \\ z_{4n}^p = N(T_{E_n}) = \Gamma(2-\gamma)h^\gamma \mathrm{f}_4(t_n, S_{U_n}^p, \ldots, D_n^p), \\ z_{5n}^p = N(T_{L_n}) = \Gamma(2-\gamma)h^\gamma \mathrm{f}_5(t_n, S_{U_n}^p, \ldots, D_n^p), \\ z_{6n}^p = N(V_{E_n}) = \Gamma(2-\gamma)h^\gamma \mathrm{f}_6(t_n, S_{U_n}^p, \ldots, D_n^p), \\ z_{7n}^p = N(V_{L_n}) = \Gamma(2-\gamma)h^\gamma \mathrm{f}_7(t_n, S_{U_n}^p, \ldots, D_n^p), \\ z_{8n}^p = N(R_{D_n}) = \Gamma(2-\gamma)h^\gamma \mathrm{f}_8(t_n, S_{U_n}^p, \ldots, D_n^p), \\ z_{9n}^p = N(R_{W_n}) = \Gamma(2-\gamma)h^\gamma \mathrm{f}_9(t_n, S_{U_n}^p, \ldots, D_n^p), \\ z_{10n}^p = N(N_{S_n}) = \Gamma(2-\gamma)h^\gamma \mathrm{f}_{10}(t_n, S_{U_n}^p, \ldots, D_n^p), \\ z_{11n}^p = N(D_n) = \Gamma(2-\gamma)h^\gamma \mathrm{f}_{11}(t_n, S_{U_n}^p, \ldots, D_n^p). \end{cases} \tag{20}$$

The above given algorithm is coded in *Mathematica* and the bunch of Figs. 1 and 2 plotted to explore the outputs of the scheme at various fractional-order values. From the Figs. 1c–1f, we identify when the fractional order decreases, the population of the SBE humans; $I(t)$, peoples getting early and late treatment with antivenom; $T_E(t)$ and $T_L(t)$, respectively, and the humans facing EAR at the time of early treatment; $V_E(t)$ increases, respectively. The time-respective variations in the rest of the model classes can be seen from the Figs. 2a–2e. Overall, we can say that the proposed L1-PC scheme performed well to explore the proposed model dynamics in the sense of the Caputo fractional derivative. The accuracy and stability of the given scheme are now discussed in our further analysis.

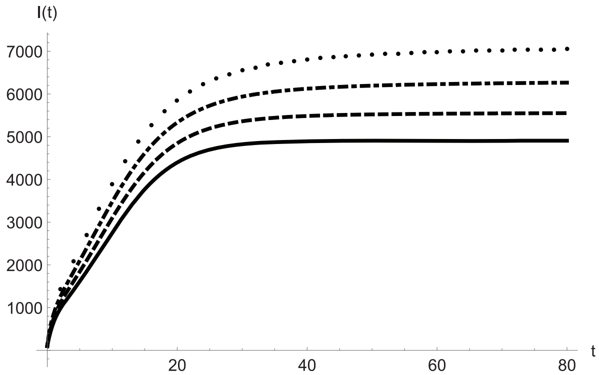


(a) S_U versus time t

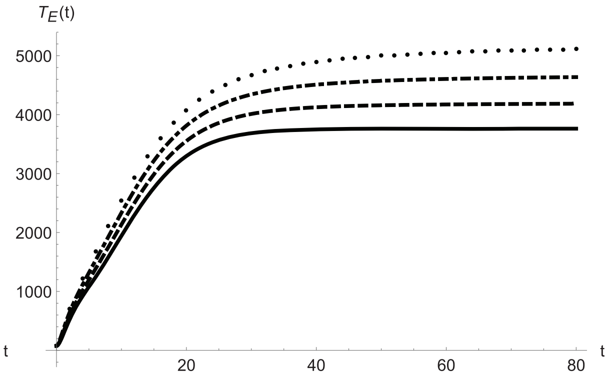


(b) S_E versus time t

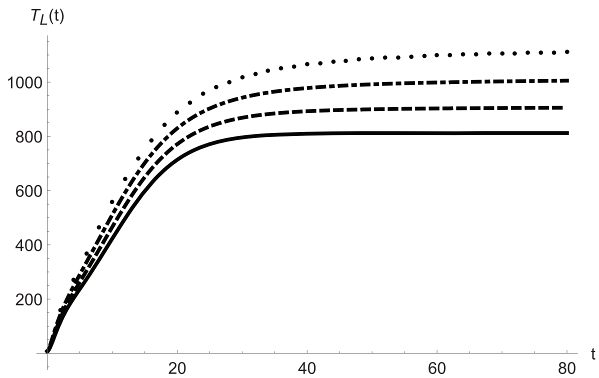
Figure 1: (Continued)



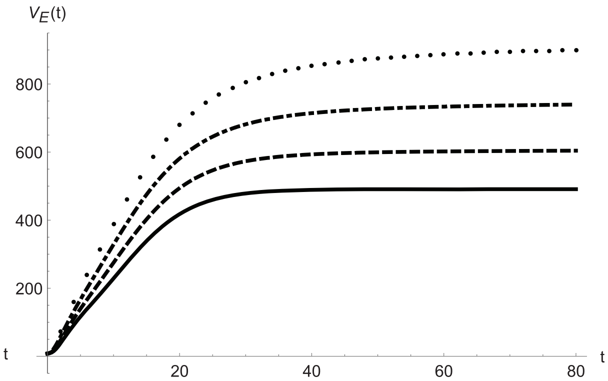
(c) I versus time t



(d) T_E versus time t

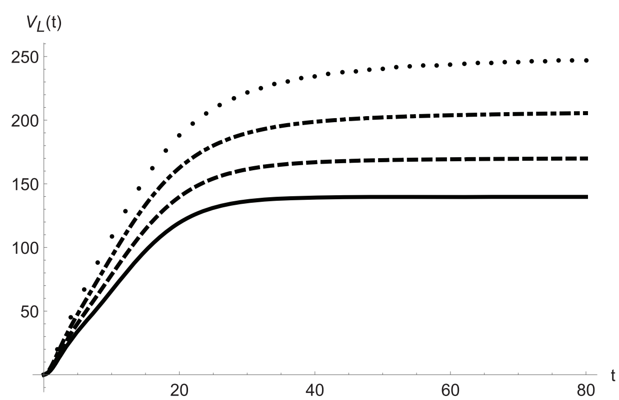


(e) T_L versus time t

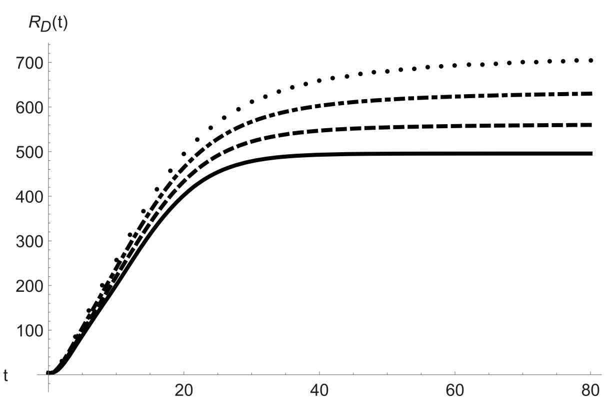


(f) V_E versus time t

Figure 1: Dynamics of the model classes $S_U, S_E, I, T_E, T_L, V_E$ at fractional orders $\gamma = 1$ (solid), $\gamma = 0.95$ (dashed), $\gamma = 0.90$ (dot-dashed), and $\gamma = 0.85$ (dotted)

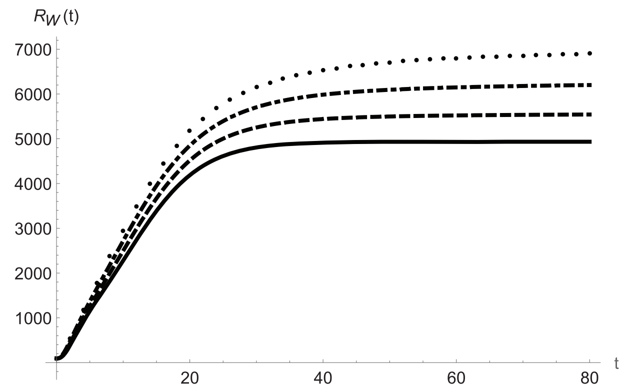


(a) V_L versus time t

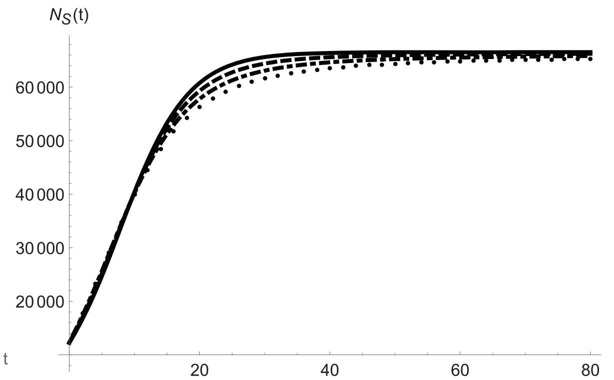


(b) R_D versus time t

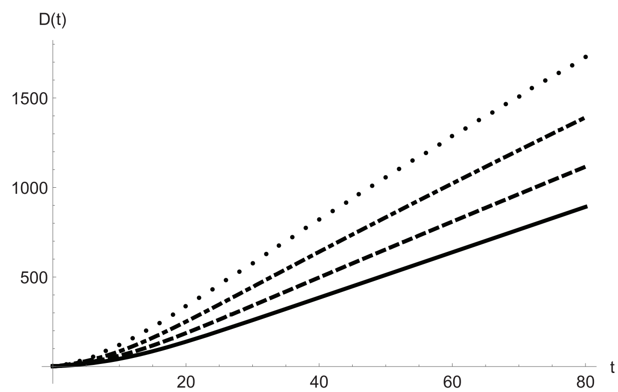
Figure 2: (Continued)



(c) R_W versus time t



(d) N_S versus time t



(e) D versus time t

Figure 2: Dynamics of the model classes V_L, R_D, R_W, N_S, D at fractional orders $\gamma = 1$ (solid), $\gamma = 0.95$ (dashed), $\gamma = 0.90$ (dot-dashed), and $\gamma = 0.85$ (dotted)

4.2 Error Analysis

The brief analysis on the error estimation of L1-PC scheme has been given in the studies [29,40,41] and now investigated below. The error estimate is given by

$$\left| \left[{}^{C}D^{\gamma}\mathcal{A}(t) \right]_{t=t_n} - \sum_{k=0}^{n-1} b_{n-k-1}\left(\mathcal{A}_{k+1} - \mathcal{A}_k\right) \right| \leq Ch^{2-\gamma}, \tag{21}$$

here C is a positive constant depends on γ and $\mathcal{A}$.

Derive r_n by

$$r_n := \Gamma(2-\gamma)\, h^{\gamma} \left[\left[{}^{C}D^{\gamma}\mathcal{A}(t) \right]_{t=t_n} - \sum_{k=0}^{n-1} b_{n-k-1}\left(\mathcal{A}_{k+1} - \mathcal{A}_k\right) \| \right]. \tag{22}$$

In view of (21)

$$|r_n| = \Gamma(2-\gamma)\, h^{\gamma} \left| \left[{}^{C}D^{\gamma}\mathcal{A}(t) \right]_{t=t_n} - \sum_{k=0}^{n-1} b_{n-k-1}\left(\mathcal{A}_{k+1} - \mathcal{A}_k\right) \| \right| \leq \Gamma(2-\gamma)\, Ch^{2}. \tag{23}$$

To derive the error estimation, we will use the lemmas given below:

Lemma 2. [42] *For* $0 < \gamma < 1$ *and* a_k'*s (as given in* Eq. (15)*), we have*

$$k^{-\gamma}a_{k-1}^{-1} \leq \frac{1}{1-\gamma}, k = 1, 2, \ldots, N.$$

Lemma 3. [29] *Consider* $\mathcal{A}(t_k)$ *as exact solution of the proposed IVP and* $\mathcal{A}_k^p$ *be the approximate solution calculated from the algorithm* (17)*. Then for* $0 < \gamma < 1$*, we have*

$$|\mathcal{A}(t_k) - \mathcal{A}_k^p| \leq Ca_{k-1}^{-1}, k = 1, 2, \ldots N,$$

where a_k's *are given in* Eq. (15).

Lemma 4. [29] *Consider* $\mathcal{A}(t_k)$ *as exact solution of the proposed IVP and* $\mathcal{A}_k^p$ *be the approximate value evaluated from* Eq. (17)*. Then for* $0 < \gamma < 1$*, we have*

$$|\mathcal{A}(t_k) - \mathcal{A}_k^p| \leq C_\gamma T^\gamma h^{2-\gamma}, k = 1, 2, \cdots N,$$

where $C_\gamma = C/(1-\gamma)$.

Theorem 3. *Consider* $\mathcal{A}(t)$ *as exact solution of the proposed IVP* (9), $\Phi(t, \mathcal{A}(t))$ *satisfies the Lipschitz property respect to the variable* $\mathcal{A}$ *with a constant L, and* $\Phi(t, \mathcal{A}(t)), \mathcal{A}(t) \in C^1[0, T]$*. Also,* $\mathcal{A}_k^c$ *defines the approximate solutions at* $t = t_k$ *calculated by using L1-PC method. Then for* $0 < \gamma < 1$*, we have*

$$|\mathcal{A}(t_k) - \mathcal{A}_k^c| \leq C_1 T^\gamma h^{2-\gamma}, k = 1, 2, \ldots N, \tag{24}$$

where $C_1 = d/(1-\gamma)$ *and d is a constant.*

Proof. Let $e_k = \mathcal{A}(t_k) - \mathcal{A}_k^c$ and $e_k^p = \mathcal{A}(t_k) - \mathcal{A}_k^p$. Using Eqs. (9), (17), and (22), we get

$$e_n = e_n^p + \Gamma(2-\gamma)h^\gamma(\Phi(t_n, \mathcal{A}(t_n)) + N(\mathcal{A}(t_n))) - \Phi(t_n, \mathcal{A}_n^p + N((\mathcal{A}_n^p))).$$

Further observe that

$$\begin{aligned}
|e_n| &\leq |e_n^p| + \Gamma(2-\gamma)h^\gamma|\Phi(t_n, \mathcal{A}(t_n)) + N(\mathcal{A}(t_n))) - \Phi(t_n, \mathcal{A}_n^p + N((\mathcal{A}_n^p)))| \\
&\leq |e_n^p| + L\Gamma(2-\gamma)h^\gamma|\mathcal{A}(t_n) - \mathcal{A}_n^p + N(\mathcal{A}(t_n))) - N((\mathcal{A}_n^p))| \\
&\leq |e_n^p| + L\Gamma(2-\gamma)h^\gamma|e_n^p| + L^2(\Gamma(2-\gamma))^2h^{2\gamma}|\mathcal{A}(t_n) - \mathcal{A}_n^p| \\
&\leq |e_n^p| + L\Gamma(2-\gamma)h^\gamma|e_n^p| + L^2(\Gamma(2-\gamma))^2h^{2\gamma}|e_n^p| \\
&\leq [1 + L\Gamma(2-\gamma)h^\gamma + L^2(\Gamma(2-\gamma))^2h^{2\gamma}]|e_n^p|.
\end{aligned} \tag{25}$$

Using Lemma 4 in Eq. (25), we get

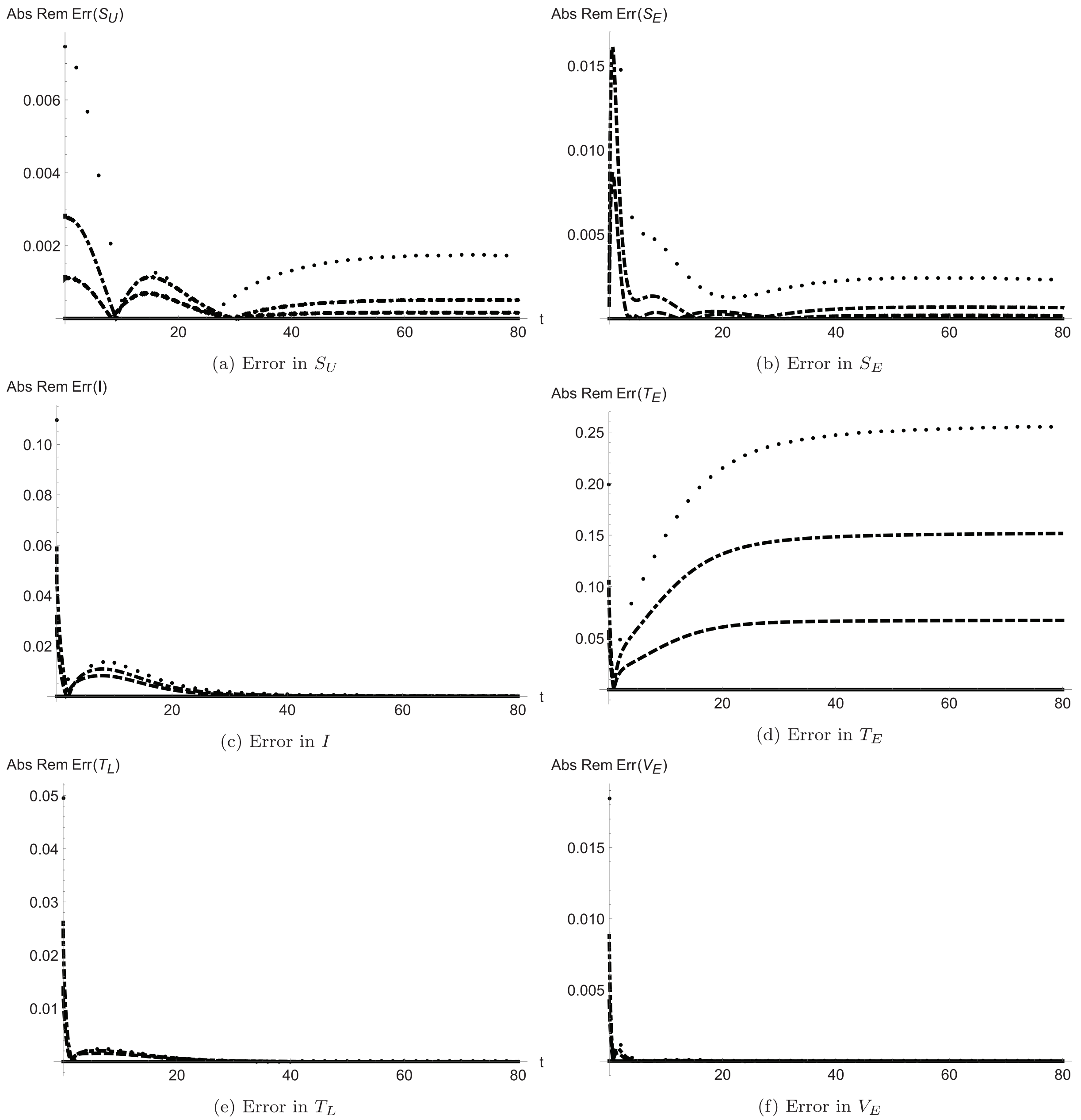
$$\begin{aligned}
|e_n| &\leq [1 + L\Gamma(2-\gamma)h^\gamma + L^2(\Gamma(2-\gamma))^2h^{2\gamma}]C_\gamma T^\gamma h^{2-\gamma} \\
&\leq [1 + L\Gamma(2-\gamma) + L^2(\Gamma(2-\gamma))^2]C_\gamma T^\gamma h^{2-\gamma}.
\end{aligned}$$

Therefore

$$|e_n| \quad \leq \quad C_\gamma T^\gamma h^{2-\gamma},$$

where C_1 is a constant.

The behaviour of the absolute remainder error is plotted in the bundle of Figs. 3 and 4 at the previously used fractional-order values $\gamma = 1, 0.95, 0.90, 0.85$. For all classes, when the fractional order γ increases the error decreases as we move away from the left endpoint, namely 0. For S_U, S_E, V_L, R_D, and D classes, very small errors are obtained at the other points of the considered interval while the errors deteriorate at points very close to the origin for T_E, R_W and N_S classes. Moreover, it is obvious that quite small errors are obtained for the integer-order case.



(a) Error in S_U

(b) Error in S_E

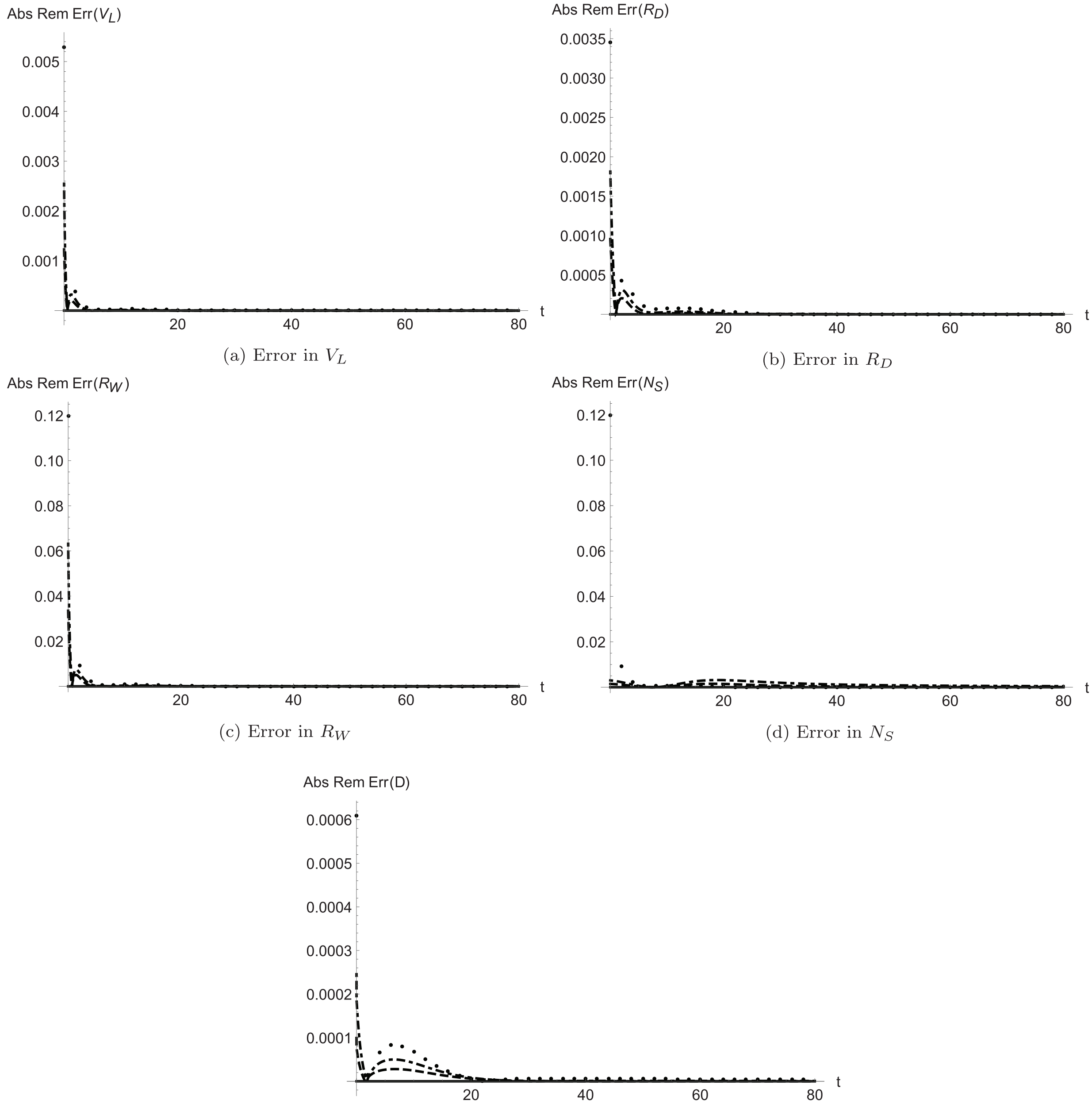
(c) Error in I

(d) Error in T_E

(e) Error in T_L

(f) Error in V_E

Figure 3: Absolute remainder error in the solution of the model classes S_U, S_E, I, T_E, T_L, V_E at fractional orders $\gamma = 1$ (solid), $\gamma = 0.95$ (dashed), $\gamma = 0.90$ (dot-dashed), and $\gamma = 0.85$ (dotted)



(a) Error in V_L
(b) Error in R_D
(c) Error in R_W
(d) Error in N_S
(e) Error in D

Figure 4: Absolute remainder error in the solution of the model classes V_L, R_D, R_W, N_S, D at fractional orders $\gamma = 1$ (solid), $\gamma = 0.95$ (dashed), $\gamma = 0.90$ (dot-dashed), and $\gamma = 0.85$ (dotted)

4.3 Stability Analysis

The term stability means that small deviations in the initial values do not make the big changes in the numerical solutions. Consider that $\mathcal{A}_n^c$ and v_n^c $(n = 1, 2 \ldots, N)$ are two solutions calculated by the numerical scheme (17). For $\delta_0 = |\mathcal{A}_0 - v_0|$, there exists two positive quantities k and h', such that

$|\mathcal{A}_n^c - v_n^c| \le k\delta_0$ *for* $h \in (0, h')$, $1 \le n \le N$,

here h is the step size given in Eq. (9).

Theorem 4. *Suppose* $\Phi(t, \mathcal{A})$ *follows the Lipschitz property respect to the variable* $\mathcal{A}$ *with a constant* L *and* $\mathcal{A}_n^c$ $(n = 1, 2 \ldots, N)$ *are the solutions established from the scheme (17), then the scheme (17) is stable.*

Proof. We have to prove that

$$|\mathcal{A}_n^c - v_n^c| \le C|\mathcal{A}_0 - v_0|.$$

Denote by $\eta_0 := (1 + (L\Gamma(2-\gamma)) + L^2(\Gamma(2-\gamma))^2 h^\gamma)$. Note that

$$|\mathcal{A}_n^c - v_n^c| \le |\mathcal{A}_n^p - v_n^p| + L\Gamma(2-\gamma)h^\gamma(|\mathcal{A}_n^p - v_n^p| + |N(\mathcal{A}_n^p) - N(v_n^p)|). \tag{26}$$

Further observe that

$$\begin{aligned}
|\mathcal{A}_n^c - v_n^c| &= \left| a_{n-1}(\mathcal{A}_0 - v_0) + \sum_{k=1}^{n-1}(a_{n-1-k} - a_{n-k})(\mathcal{A}_k - v_k) \right| \\
&\le a_{n-1}|\mathcal{A}_0 - v_0| + \sum_{k=1}^{n-1}(a_{n-1-k} - a_{n-k})|\mathcal{A}_k - v_k| \\
&\le |\mathcal{A}_0 - v_0| + \sum_{k=1}^{n-1}(a_{n-1-k} - a_{n-k})|\mathcal{A}_k - v_k|.
\end{aligned}$$

Using discrete form of Gronwalls inequality and Eq. (15), we obtain

$$|\mathcal{A}_n^p - v_n^p| \le c|\mathcal{A}_0 - v_0|, \tag{27}$$

where c is a constant and

$$\begin{aligned}
|N(\mathcal{A}_n^p) - N(v_n^p)| &= |\Gamma(2-\gamma)h^\gamma(\Phi(t_n, \mathcal{A}_n^p)) - \Phi(t_n, v_n^p))| \\
&\le L\Gamma(2-\gamma)h^\gamma|\mathcal{A}_n^p - v_n^p|.
\end{aligned} \tag{28}$$

Using (27) and (28) in (26), we get

$$\begin{aligned}
|\mathcal{A}_n^c - v_n^c| &\le |\mathcal{A}_n^p - v_n^p| + L\Gamma(2-\gamma)h^\gamma|\mathcal{A}_n^p - v_n^p| + L^2(\Gamma(2-\gamma))^2 h^{2\gamma}|\mathcal{A}_n^p - v_n^p| \\
&\le |\mathcal{A}_n^p - v_n^p| + L\Gamma(2-\gamma)h^\gamma|\mathcal{A}_n^p - v_n^p| + L^2(\Gamma(2-\gamma))^2 h^{\gamma}|\mathcal{A}_n^p - v_n^p| \\
&\le (1 + L\Gamma(2-\gamma) + L^2(\Gamma(2-\gamma))^2)h^\gamma|\mathcal{A}_n^p - v_n^p| \\
&\le \eta_0 c|\mathcal{A}_0 - v_0| \\
&\le C|\mathcal{A}_0 - v_0|,
\end{aligned}$$

where C is a constant.

5 Conclusion

In this research work, we have performed a novel implementation of a finite-difference predictor-corrector scheme on a fractional-order nonlinear snakebite envenoming model in terms of the Caputo fractional derivative. The numerical solution of the model has been plotted by using several graphs to justify the behavior of the model at various fractional-order values. The analysis related to the error estimation and stability of the scheme has also been derived from exploring method's accuracy. In the error estimation, the fractional order γ increases and the error decreases which justifies that our fractional-order analysis should not be performed at small values of order γ. From the given observations, it is clear that the proposed method can easily be implemented on various epidemic models, and the algorithm is time-efficient for more accurate solutions. In the future, the given scheme can be widely used to solve various nonlinear mathematical models related to real-life phenomena.

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