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ARTICLE

Construction of a Computational Scheme for the Fuzzy HIV/AIDS Epidemic Model with a Nonlinear Saturated Incidence Rate

Muhammad Shoaib Arif[1,2,*], Kamaleldin Abodayeh[1] and Yasir Nawaz[2]

[1]Department of Mathematics and Sciences, College of Humanities and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia

[2]Department of Mathematics, Air University, Islamabad, 44000, Pakistan

*Corresponding Author: Muhammad Shoaib Arif. Email: marif@psu.edu.sa

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ABSTRACT

This work aimed to construct an epidemic model with fuzzy parameters. Since the classical epidemic model does not elaborate on the successful interaction of susceptible and infective people, the constructed fuzzy epidemic model discusses the more detailed versions of the interactions between infective and susceptible people. The next-generation matrix approach is employed to find the reproduction number of a deterministic model. The sensitivity analysis and local stability analysis of the system are also provided. For solving the fuzzy epidemic model, a numerical scheme is constructed which consists of three time levels. The numerical scheme has an advantage over the existing forward Euler scheme for determining the conditions of getting the positive solution. The established scheme also has an advantage over existing non-standard finite difference methods in terms of order of accuracy. The stability of the scheme for the considered fuzzy model is also provided. From the plotted results, it can be observed that susceptible people decay by rising interaction parameters.

KEYWORDS

Epidemic model; fuzzy rate parameters; next generation matrix; local stability; proposed numerical scheme

1 Introduction

Epidemiology is the branch that offers predictions against infectious diseases, their presence, and control among a given population. It explains the factors governing infection spread and provides a suitable precautionary measure adapted for its control. Therefore, one can say that epidemiology is the branch that improves the healthcare system by regulating disease spread [1]. In 1981, the USA was the first country to report early cases of HIV (a retrovirus) among transgender people, plotting it as a life-threatening ailment. To date, 25 million deaths have been reported, with 14,000 new cases/day forming a nightmare with no cure as no vaccine has yet been prepared. It requires 6 months to 15 years for an individual to become completely susceptible to HIV and reveal the symptoms.

On the other hand, the virus severely disrupts the CD4+ T-cells, affecting the autoimmunity system of cells. Ultimately, individuals lacking immunity became vulnerable to even minute diseases like influenza, leading them straight to death. The routes for HIV movement are unprotected sexual

intercourse along with contaminated syringes, needles, and blood, which may infect the person when in use. Blood containing HIV, when transferred through vertical transmissions [2], could be fatal for a receiver.

Mathematics has given researchers enormous tools to build the epidemiology of HIV/AIDS and understand the involvement of variable factors contributing to the epidemic. Three approaches to the population-based study of the HIV/AIDS epidemic were made by Zafar et al. [3]. The Adams-Bashforth Moulton method, Grunwald Letnikov approach, and Grunwald Letnikov approach with binomial coefficients. In this study, the stability if $R_o < 1$ and if $R_o > 1$ and disability are discussed along with endemic equilibrium. A delayed fractional-order SIR model with saturated incidence and treatment functions was investigated by Wang et al. [4].

A fuzzy differential equation is commonly used to omit uncertainty modeling a dynamic system. Such a system may involve cell growth and the dynamic of population, tumor growth, and nuclear disintegration under uncertainty. The transition of HIV to AIDS can be modeled using a fuzzy transference rate correlated with the viral load and CD4+ T-cells level by rule bases [5]. Several drugs have been designed that are optimal for drug therapy and prescribed to HIV patients. Considering the drug therapy, several questions have been asked about the best medicinal therapy for the HIV patient, although the therapeutic period is confined. The answer lies in the traditional differential equation, which efficiently describes the interaction between HIV and an infective individual's immune system. This equation provides an appropriate optimal control on the system of equations by a suitable medicine program [6].

Individuals carrying HIV depicted variable levels of their immune system regarding the number of immunity cells *vs*. concentration of virus inside the body during a virus attack. This could be modeled using mathematical modeling that interprets the ratio between HIV and immune cells. Perelson et al. in [7] elaborated on three state variables, i.e., healthy CD4+ T-cells, infected CD4+ T-cells, and viruses, and created the basis of a simple mathematical model for HIV. Complexity arises when considering other variables like cytotoxic T lymphocytes and macrophages, as reported by [8]. Ambiguity in the given publication arises due to an unclarified uncertainty description without interpreting fuzzy parameters. The major concern of this paper is to focus on uncertain parameters and halt them using the fuzzy equation in preparation for the HIV model.

Hukuhara derivative [9] was the pioneer who suggested the concept of differentiability for fuzzy mappings about the fuzzy differential equation. Time revealed that the fuzzy differential equation used by the Hukuhara derivative could be termed fuzzier and possesses different characters compared to a crisp solution [10]. To remove this obstacle, another strong variable, fuzzy-number-valued functions, was derived, which exclusively generalized differentiability as mentioned in [11–14].

The simplest are first-order fuzzy differential equations that as many applications. Whereas such simplest equation creates hurdles under different fuzzy differential equation concepts, i.e., the behavior of solutions is different (based on interpretation). Several researchers have utilized first-order fuzzy linear equations, as mentioned in [11]. Under the umbrella of the generalized differentiability notion, the author proposed a general form of the solutions to the first-order fuzzy differential equations with crisp coefficients. Recently operator approach has been proposed for solving first- and second-order linear fuzzy differential equations under strongly generalized differentiability [12]. Boundary value conditions have been introduced to the impulses [13]. A first-order linear fuzzy differential equation's explicit solutions are found by calculating them on each level set and proving their existence and uniqueness. They are using the generalized Euler approximation approach. The numerical solution of the fuzzy differential equation under generalized differentiability has been studied in [14]. A

generalization of Hukuhara's differentiability for interval-valued functions has been introduced in [15]. Under this sort of differentiability, interval differential equations have local existence and uniqueness.

Linear fuzzy differential dynamical systems with complex number representation of level sets and solutions derived under this representation were the focus of [16], which focused on linear fuzzy differential dynamical systems with initial conditions defined by a fuzzy number vector. When solving linear differential dynamical systems, including fuzzy matrices, it is helpful to keep this in mind, but [17] goes one step further by providing solutions to these systems using fuzzy matrices in addition to those provided by [18].

This publication provided sufficient information for the existence of equilibrium and global stability concerning the endemic and infection-free medium by formulating suitable Lyapunov functions. Fractional SEIR epidemic models and fractional differential equations were examined by Almeida [19] in an attempt to understand the dynamics of specific epidemics. Carvalho et al. developed a fractional-order model of HIV/HCV coinfection to understand better how the HIV viral load affects the coinfection [20]. Their study was confined to providing a numeric of individuals suffering from various infections like HIV, HCV, dengue fever, and many others. However, results revealed that the HIV viral load impacts remarkably on the harshness of the HCV infection, and the treatment efficacy is dominated by the natural progression of HCV against HIV/HCV coinfection. Using a multi-patch HIV/AIDS epidemic model with fractional-order derivatives, Kheiri et al. [21] studied the effect of human drive on HIV/AIDS pandemic propagation among patches. The major objective of the study was to bring the basic reproduction number R_o of the model to observe local and global stability of the equilibrium based on R_o values. Fractional optimal control brings the system towards stability by making $R_o < 1$ and disability when $R_o > 1$, and hence proved that the endemic equilibrium is unique and globally asymptotically stable beside the presentation of state and co-state equation as left fractional derivatives.

The model formulated by [21] comprised time-dependent control to minimize the spread of HIV/AIDS and the derived fractional optimal control that varies the fractional order of the disease spread. Nowadays, plenty of literature on fractional-order models of HIV disease dynamics reveals the engagement of mathematical techniques. Such techniques solve the models and the disease dynamics (HIV) spread [22,23]. Studies focused on fractional-order modeling can be seen in the literature [24–26] and references therein. The most significant application of fractional order derivative is modeling infectious disease, vibration equations [27,28], and much more.

Moreover, massive optimal control theory applications have made it applicable in several fields of pure and applied sciences [29]. The theory of linear systems has been extensively worked on [30]. Contrary, the theory of nonlinear optimal control (OCP) needed a thorough description and further analysis [31,32]. The modal series method and eigenvalue decomposition technique were used by Jajarmi et al. [33] to illustrate a class of nonlinear optimal control problems and convergence analysis. Jajarmi et al. [34] investigated the optimal control of time-varying delay systems with persistent matched external disturbances. The uniqueness of their study, which made the whole working thing interesting, is the conversion of the disturbed original time-delay model into an augmented system without any disturbance. A quadratic performance index was chosen for the enhanced system to establish an undisturbed time-delay optimum control issue. Two-point boundary value problems drive to advance and delay optimality criteria. An iterative algorithm for solving the latter advance-delay boundary value problem with a new iterative technique was also offered. To approximate the solution of the HIV infection of CD4+ T-cells model, the authors of [35] employed the continuous Galerkin-Petrov method (cGP(2)-method). Different clinical parameters' effects on the model are

discussed and analyzed. In the study [36], we applied a Legendre wavelet collocation scheme to an HIV infection model and compared our results to those found elsewhere. The results show how accurate and useful the proposed method is for estimating the HIV model's solutions. The purpose of the research presented in [37] was to create a new fuzzy fractional model for the human liver. This model would incorporate the ABC fractional differentiability, which is also referred to as the ABC gH-differentiability. It would be based on the generalized Hukuhara derivative. Using the gH-differentiability of Caputo fractional order, this study [38] developed and analyzed a robust double-parametric analytical approach for the nonlinear fuzzy fractional KdV equation (FFKdVE). With the time-dependent variable of carbon flux in the atmosphere, carbon flux in soil, and the carbon flux of animals and plants, the dynamical behaviors and mathematics of the fractional order atmosphere-soil-land plant carbon cycle system were studied in [39].

This study concerns a fuzzy epidemic model comprising five compartments: Susceptible, exposed, unaware, infective, aware, and recovered. The fuzzy-based interaction parameters are defined and further used in the model. The fuzzy parameters are split into three parts that discuss the unsuccessful, successful, and fully successful meetings of infective with susceptible. If the value of the fuzzy parameter falls between 0 and 1, then a higher value corresponds to higher chances of successful interaction of infective with susceptible. So, two fuzzy parameters in a model provide nine different cases because each fuzzy parameter is divided into three parts. The discussions related to three cases exist in this study, but the remaining cases can be discussed similarly. In addition, a numerical scheme is established for three of the discussed cases. The numerical scheme has the advantage over some existing classical schemes in determining conditions for a positive solution. The numerical scheme is established on three time levels, so either a different scheme is adopted, or some initial guess is chosen for the second time level. In this work, the forward Euler scheme is chosen for finding a solution at the second time level. Since the given scheme and existing forward Euler schemes are conditionally stable, both require suitable step sizes for obtaining a convergent solution. Among existing finite difference schemes for solving differential equations, a non-standard finite difference method has been chosen by various researchers for solving epidemic models. The non-standard finite difference scheme has been constructed on two-time levels. It is unconditionally stable and provides surety to get positive solutions for some of the existing epidemic models. But, the scheme has its drawback: it does not achieve first-order accuracy in the obtained solution. So, the solution obtained by a non-standard scheme is doubtful unless the proper step size is chosen. The scheme constructed in this work has the advantage over the non-standard scheme due to the order of accuracy. In addition, the constructed scheme has an advantage over the forward Euler method in terms of enlarged stability region, but the constructed scheme is conditionally stable.

2 Fuzzy Epidemic Model

Before starting the mathematical modeling and analysis of the fuzzy epidemic model, some basic definitions are given.

2.1 Fuzzy Subset [40]

A fuzzy subset of a Universe X is denoted by the function $P_f(x)\colon X \to [0,1]$ where $P_f(x)$ is a membership function representing the degree of membership of element x in the fuzzy set f.

2.2 Fuzzy Number

A fuzzy number is a generalization of a regular real number. It refers to a connected set of potential values instead of just one specific value, where each potential value weights 0 and 1. The membership function is the name of this weight. Consequently, a convex, normalized fuzzy set of the real line is a particular instance of a fuzzy number.

2.3 Triangular Fuzzy Number [40]

The number $f = (a, b, c)$ is a triangular fuzzy number if its membership function is denoted by:

$$P_f(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a < x \le b \\ \dfrac{x-c}{b-c}, & b < x \le c \\ 0, & c \le x \end{cases} \tag{1}$$

where $a \le b \le c$.

2.4 Trapezoidal Fuzzy Number [40]

The number $f = (a, b, c, d)$ is called a trapezoidal fuzzy number if its membership function is defined as:

$$\mu_f(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & b \le x \le c \\ \dfrac{d-x}{d-c}, & c < x \le d \\ 0, & otherwise \end{cases} \tag{2}$$

where $a \le b \le c \le d$.

2.5 Expected Value [41]

The expected value of a triangular fuzzy number (p, q, r) is given as:

$$E(f) = \frac{p + 2q + r}{4}. \tag{3}$$

where p, q, and r are constants.

2.6 Fuzzy Basic Reproductive Number [41]

The fuzzy basic reproductive number of a triangular fuzzy number $R_0(u_1, u_2)$ is given by:

$$R_0^f = E[R_0(u_1, u_2)]. \tag{4}$$

The deterministic model given in [42] is expressed as:

$$\frac{dS}{dt} = \Lambda - S(\beta_1 I_1 + \beta_2 I_2) - \lambda S, \tag{5}$$

$$\frac{dE}{dt} = S(\beta_1 I_1 + \beta_2 I_2) - (\lambda + \sigma)E, \tag{6}$$

$$\frac{dI_1}{dt} = \sigma E - (\rho + \lambda + \theta) I_1, \tag{7}$$

$$\frac{dI_2}{dt} = \theta I_1 - (\lambda + \rho) I_2. \tag{8}$$

$$\frac{dR}{dt} = \rho(I_1 + I_2) - (d + \lambda) R. \tag{9}$$

where S denotes the susceptible, E denotes the exposed, I_1 denotes the infected but unaware, I_2 represents the infected but aware, R denotes the recovered individuals, Λ denotes the recruited rate of susceptible, β_1 is the rate of transmission from susceptible to infected but unaware, β_2 denotes the transmission rate from susceptible to becoming infected but aware, λ is the natural death rate which is not related to acquired immune deficiency syndrome (AIDS), θ is the rate of unaware infective to becoming aware infectives, σ is the rate of exposed to become unaware infectives, ρ denotes the rate of transferring infective to those having AIDS, and d denotes the AIDS-related to the death rate.

Now, a fuzzy mathematical model with the incidence rate of AIDS is represented as:

$$\frac{dS}{dt} = -S\left(\frac{\beta_1(u_1) I_1}{1 + \alpha_1 I_1} + \frac{\beta_2(u_2) I_2}{1 + \alpha_2 I_2}\right), \tag{10}$$

$$\frac{dE}{dt} = S\left(\frac{\beta_1(u_1) I_1}{1 + \alpha_1 I_1} + \frac{\beta_2(u_2) I_2}{1 + \alpha_2 I_2}\right) - \sigma E, \tag{11}$$

$$\frac{dI_1}{dt} = \sigma E - (\rho + \theta) I_1, \tag{12}$$

$$\frac{dI_2}{dt} = \theta I_1 - \rho I_2, \tag{13}$$

$$\frac{dR}{dt} = \rho(I_1 + I_2) - dR, \tag{14}$$

where $\beta_1(u_1), \beta_2(u_2), \sigma, \rho, \theta, d$ are positive constants. Let $\beta_1(u_1)$ and $\beta_2(u_2)$ denote the chances of transmission disease rate from susceptible to unaware infective and aware infective, respectively, $\beta_1(u_1)$ and $\beta_2(u_2)$ are defined as:

$$\beta_1(u_1) = \begin{cases} 0, & u_1 < u_{m,1} \\ \dfrac{u_1 - u_{m,1}}{u_{0,1} - u_{m,1}}, & u_{m,1} < u_1 < u_{0,1} \\ 1, & u_{0,1} \le u_1 < u_{M,1} \end{cases} \tag{15}$$

$$\beta_2(u_2) = \begin{cases} 0, & u_2 < u_{m,2} \\ \dfrac{u_2 - u_{m,2}}{u_{0,2} - u_{m,1}}, & u_{m,2} < u_2 < u_{0,2} \\ 1, & u_{0,2} \le u_2 < u_{M,2} \end{cases} \tag{16}$$

The $\beta_1(u)$ shows that a higher value of u_1 produces higher $\beta_1(u_1)$ which is greater than some minimum $u_{m,1}$. The transferring of disease from susceptible to infectives is ignorable if u_1 is smaller than $u_{m,1}$. Also, the rate of transmission of disease from susceptible to infectives ones will be higher if u_1 is greater than $u_{0,1}$.

3 Mathematical Analysis

The equilibrium point of the model (10)–(13) will be found in this section.

3.1 Fuzzy Equilibrium Analysis

The system (10)–(13) has disease-free equilibrium points

$$E^0\left(S^0, E^0, I_1^0, I_2^0\right) = E^0\left(S^0, 0, 0, 0\right) \tag{17}$$

3.2 Fuzzy Basic Reproductive Number

The first basic reproductive number of the deterministic model will be found. For doing so, the next-generation matrix approach is considered. According to this approach, the following matrices can be found:

$$\mathcal{F}(\psi) = \begin{bmatrix} S\left(\dfrac{\beta_1 I_1}{1+\alpha_1 I_1} + \dfrac{\beta_2 I_2}{1+\alpha_2 I_2}\right) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{18}$$

$$\upsilon(\psi) = \begin{bmatrix} \sigma E \\ -\sigma E + (\rho+\theta) I_1 \\ -\theta I_1 + \rho I_2 \\ S\left(\dfrac{\beta_1 I_1}{1+\alpha_1 I_1} + \dfrac{\beta_2 I_2}{1+\alpha_2 I_2}\right) \end{bmatrix}, \tag{19}$$

where

$$\psi' = \mathcal{F}(\psi) - \upsilon(\psi), \tag{20}$$

and $\psi = (E, I_1, I_2, S)^T$.

Since infected compartments are E, I_1 and I_2, so

$$F = \left[\frac{\partial \mathcal{F}_P(E^\circ)}{\partial y_m}\right] \text{ and } V = \left[\frac{\partial \upsilon_P(E^\circ)}{\partial y_m}\right] \text{ for } 1 \le m, P \le 3, \tag{21}$$

F and V are given by

$$F = \begin{bmatrix} \sigma & 0 & 0 \\ -\sigma & \rho+\theta & 0 \\ 0 & -\rho-\theta & \rho \end{bmatrix} \text{ and } V = \begin{bmatrix} \sigma & 0 & 0 \\ -\sigma & \rho+\theta & 0 \\ 0 & -\rho-\theta & \rho \end{bmatrix}. \tag{22}$$

The spectral radius of FV^{-1} gives the basic reproductive number, which is given as:

$$R_0 = \frac{\rho\beta_1 + \beta_2(\rho+\theta)}{\rho(\rho+\theta)}. \tag{23}$$

The fuzzy basic reproductive number is given by:

$$\overline{R}_0 = \frac{\rho\beta(u_1) + \beta_2(u_2)(\rho+\theta)}{\rho(\rho+\theta)}. \tag{24}$$

Since there are three parts for each $\beta_1(u)$ and $\beta_2(u)$, so nine possible cases can be discussed. But in this work, only three will be discussed for convenience.

Case I: In this case $u_1 < u_{m,1}$ and $u_2 < u_{m,2}$ then $\beta_1(u_1) = 0 = \beta_2(u_2)$, so $\overline{R}_0$ is given by:

$$\overline{R}_0 = 0. \tag{25}$$

Case II: For this case, let $u_{m,1} \leq u_1 \leq u_{0,1}$ and $u_{m,2} \leq u_2 \leq u_{0,2}$, then

$$\beta_1(u_1) = \frac{u_1 - u_{m,1}}{u_{0,1} - u_{m,1}} \text{ and } \beta_2(u_2) = \frac{u_2 - u_{m,2}}{u_{0,2} - u_{m,2}}.$$

The fuzzy basic reproductive number is given by:

$$\overline{R}_0 = \frac{\rho\beta_1(u_1) + \beta_2(u_2)(\rho + \theta)}{\rho(\rho + \theta)}. \tag{26}$$

Case III: When $u_{0,1} < u_1 < u_{M,1}$ and $u_{0,2} < u_2 < u_{M,2}$ then $\beta_1(u_1) = 1 = \beta_2(u_2)$ and so

$$\overline{R}_0 = \frac{2\rho + \theta}{\rho(\rho + \theta)}. \tag{27}$$

$R_0(u)$ can be expressed as:

$$R_0(u_1, u_2) = \left(0, \frac{\rho\beta_1(u_1) + \beta_2(u_2)(\rho + \theta)}{\rho(\rho + \theta)}, \frac{2\rho + \theta}{\rho(\rho + \theta)}\right),$$

where $\beta_1(u_1) = \dfrac{u_1 - u_{m,1}}{u_{0,1} - u_{m,1}}$ and $\beta_2(u_2) = \dfrac{u_2 - u_{m,2}}{u_{0,2} - u_{m,2}}$.

Given the definition of the expected value, R_0^f is expressed as:

$$R_0^f = E[R_0(u_1, u_2)],$$

$$R_0^f = \frac{\rho(2\beta_1(u_1) + 2\beta_2(u_2) + 2) + \theta(1 + 2\beta_2(u_2))}{4\rho(\rho + \theta)}. \tag{28}$$

3.3 Sensitivity Analysis

For a parameter S, the normalized forward parameter sensitivity index is given by:

$$\phi(S) = \frac{S}{R_0^f}\frac{\partial R_0^f}{\partial S}. \tag{29}$$

The sensitivity of parameters is given by:

$$\begin{aligned}\phi(\beta_1(u_1)) &= \frac{\beta_1(u_1)}{R_0^f}\frac{\partial}{\partial\beta_1(u_1)}\left(\frac{\rho(2\beta_1(u_1) + 2\beta_2(u_2) + 2) + \theta(1 + 2\beta_2(u_2))}{4\rho(\rho + \theta)}\right) \\ &= \frac{2\rho\beta_1(u_1)}{\rho(2\beta_1(u_1) + 2\beta_2(u_2) + 2) + \theta(1 + 2\beta_2(u_2))} > 0, \end{aligned} \tag{30}$$

$$\begin{aligned}\phi(\beta_2(u_2)) &= \frac{\beta_2(u_2)}{R_0^f}\frac{\partial}{\partial\beta_2(u_2)}\left(\frac{\rho(2\beta_1(u_1) + 2\beta_2(u_2) + 2) + \theta(1 + 2\beta_2(u_2))}{4\rho(\rho + \theta)}\right) \\ &= \frac{\beta_2(u_2)(2\rho + 2\theta)}{\rho(2\beta_1(u_1) + 2\beta_2(u_2) + 2) + \theta(1 + 2\beta_2(u_2))} > 0, \end{aligned} \tag{31}$$

$$\phi(\rho) = \frac{\rho}{R_0^f}\frac{\partial}{\partial\rho}\left(R_0^f\right) = -\frac{2(1 + \beta_1 + \beta_2)\rho^2 + (1 + 2\beta_2)\theta^2 + 2\rho(\theta + 2\beta_2\theta)}{(\rho + \theta)(2(1 + \beta_1 + \beta_2)\rho + \theta + 2\beta_2\theta)} < 0, \tag{32}$$

$$\phi(\theta) = \frac{\theta}{R_0^f}\frac{\partial}{\partial\theta}\left(R_0^f\right) = \frac{(1+2\beta_1)\,\rho\theta}{(\rho+\theta)(2\,(1+\beta_1+\beta_2)\,\rho+\theta+2\beta_2\theta)} < 0. \tag{33}$$

According to sensitivity analysis, $\beta_1(u_1)$ and $\beta_2(u_2)$ are sensitive parameters. The sensitive parameters and R_0^f have a direct relationship. By increasing sensitive parameters, R_0^f increases and vice versa. The insensitive parameters have an indirect relationship with R_0^f. It means that a decrease in insensitive parameters will increase R_0^f and vice versa.

3.4 Theorem

If $R_0 \leq 1$ and $\rho^2 + 2\rho\sigma + \rho\theta + \sigma\theta > \beta_1(u_1)\,\sigma$ holds, then disease-free equilibrium point $E^0(S^0, 0, 0, 0)$ of the system (10)–(13) will be locally asymptotically stable.

Proof: For proving this theorem, eigenvalues for the Jacobian system (10)–(13) must have non-positive eigenvalues. For doing so, consider the Jacobian of the system (10)–(13) in the form of

$$\begin{bmatrix} 0 & 0 & -\beta_1(u_1) & -\beta_2(u_2) \\ 0 & -\sigma & \beta_1(u_1) & \beta_2(u_2) \\ 0 & \sigma & -\rho-\theta & 0 \\ 0 & 0 & \rho+\theta & -\rho \end{bmatrix}. \tag{34}$$

Case I: When $u_1 < u_{m,1}$ and $u_2 < u_{m,2}$ then $\beta_1(u_1) = 0 = \beta_2(u_2)$, so the Jacobian (34) can be written as:

$$J_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sigma & 0 & 0 \\ 0 & \sigma & -\rho-\theta & 0 \\ 0 & 0 & \rho+\theta & -\rho \end{bmatrix}. \tag{35}$$

The eigenvalues of (35) are:

$$\overline{\lambda}_1 = 0, -\rho-\theta, -\rho, -\sigma. \tag{36}$$

Since all eigenvalues are negative, system (10)–(13) is stable for this case.

Case II: In this case, $u_{m,1} \leq u_1 < u_{0,1}$ and $u_{m,2} \leq u_2 < u_{0,2}$, then $\beta_1(u_1) = \dfrac{u_1 - u_{m,1}}{u_{0,1} - u_{m,1}}$ and $\beta_2(u_2) = \dfrac{u_2 - u_{m,2}}{u_{0,2} - u_{m,2}}$, and characteristic polynomial for (34) can be written as:

$$R\left(\overline{\lambda}\right) = \overline{\lambda}\left(\overline{\lambda}^3 + a_2\overline{\lambda}^2 + a_1\overline{\lambda} + a_0\right), \tag{37}$$

where $a_0 = -\beta_1(u_1)\,\rho\sigma - \beta_2(u_2)\,\rho\sigma + \rho^2\sigma - \beta_2(u_2)\,\sigma\theta + \rho\sigma\theta$.

$a_1 = \rho^2 - \beta_1\sigma + 2\rho\sigma + \rho\theta + \sigma\theta$

$a_2 = 2\rho + \sigma + \theta$

According to Routh Hurwitz's criterion for cubic polynomials, the eigenvalues will lie in the left half plane, if and only if $a_0, a_1, a_2 > 0$ and $a_1 a_2 > a_0$.

The condition a_2 is satisfied because all terms are positive. The condition $a_0 > 0$ follows from $R_0 \leq 1$. For proving $a_1 a_2 > a_0$, first the expression for $a_1 a_2 - a_0$ will be found, which is given as:

$$a_1 a_2 - a_0 = 2\rho^3 - \beta_1(u_1)\rho\sigma + \beta_2(u_2)\rho\sigma + 4\rho^2\sigma - \beta_1(u_1)\sigma^2 + 2\rho\sigma^2 + 3\rho^2\theta - \beta_1(u_1)\sigma\theta + \beta_2(u_2)\sigma\theta + 4\rho\sigma\theta + \sigma^2\theta + \rho\theta^2 + \sigma\theta^2. \tag{38}$$

All terms in (38) are positive except for three terms. Now, the second inequality in the theorem statement will be utilized.

Since $\rho^2 + 2\rho\sigma + \rho\theta + \sigma\theta > \beta_1(u_1)\sigma$ holds. From here, three more inequalities can be constructed as:

$$\rho^3 + 2\rho^2\sigma + \rho^2\theta + \rho\sigma\theta > \beta_1(u_1)\sigma\rho, \tag{39}$$

$$\rho^2\sigma + 2\rho\sigma^2 + \rho\theta\sigma + \sigma^2\theta > \beta_1(u_1)\sigma^2, \tag{40}$$

$$\rho^2\theta + 2\rho\sigma\theta + \rho\theta^2 + \sigma\theta^2 > \beta_1(u_1)\sigma\theta. \tag{41}$$

By using inequalities (39)–(41) in (38), it can be shown that $a_1 a_2 - a_0 > 0$. Therefore, all the conditions of the Routh Hurwitz criterion are satisfied, so eigenvalues lie in the left half plane.

Case III: When $u_1 \geq u_{0,1}$ and $u_2 > u_{0,2}$ then $\beta_1(u_1) = 1 = \beta_2(u_2)$, then it can be shown that eigenvalues are non-positive.

4 Numerical Scheme

A numerical scheme is presented for solving systems (10)–(13). The scheme will be constructed for the model's first Eq. (10). For doing so, consider the discretized equation for Eq. (10) as:

$$S^{n+1} = \frac{(S^{n-1} + S^n)}{2} + \bar{a}_1 h\left[-S^{n+1}\left(\frac{\beta_1(u_1)I_1^n}{1+\alpha_1 I_1^n} + \frac{\beta_2(u_2)I_2^n}{1+\alpha_2 I_2^n}\right)\right], \tag{42}$$

where h is step size and $\bar{a}_1$ will be determined later.

For finding $\bar{a}_1$ Taylor series expansions for S^{n+1} and S^{n-1} will be considered, which are given as follows:

$$S^{n+1} = S^n + h\left(\frac{dS}{dt}\right)^n + O(h^2), \tag{43}$$

$$S^{n-1} = S^n - h\left(\frac{dS}{dt}\right)^n + O(h^2). \tag{44}$$

Substituting Taylor series expansions (43) and (44) into Eq. (42) yields

$$S^n + h\left(\frac{dS}{dt}\right)^n = S^n - h\left(\frac{dS}{dt}\right)^n + \bar{a}_1 h\left[\begin{array}{l} -S^n\left(\frac{\beta_1(u_1)I_1^n}{1+\alpha_1 I_1^n} + \frac{\beta_2(u_2)I_2^n}{1+\alpha_2 I_2^n}\right) - \\ h\left(\frac{dS}{dt}\right)^n\left(\frac{\beta_1(u_1)I_1^n}{1+\alpha_1 I_1^n} + \frac{\beta_2(u_2)I_2^n}{1+\alpha_2 I_2^n}\right) \end{array}\right]. \tag{45}$$

Eq. (45) can be expressed as:

$$S^n + h\left(\frac{dS}{dt}\right)^n = S^n - \frac{h}{2}\left(\frac{dS}{dt}\right)^n + \bar{a}_1 h\left[\left(\frac{dS}{dt}\right)^n - h\left(\frac{dS}{dt}\right)^n\left(\frac{\beta_1(u_1)I_1^n}{1+\alpha_1 I_1^n} + \frac{\beta_2(u_2)I_2^n}{1+\alpha_2 I_2^n}\right)\right]. \tag{46}$$

Equating coefficients of S^n and $h\left(\frac{dS}{dt}\right)^n$ on both sides of Eq. (46), it is obtained:

$$1 = -\frac{1}{2} + \overline{a}_1\left(1 - h\left(\frac{\beta_1(u_1)I_1^n}{1+\alpha_1 I_1^n} + \frac{\beta_2(u_2)I_2^n}{1+\alpha_2 I_2^n}\right)\right). \tag{47}$$

The unknown $\overline{a}_1$ can be found as:

$$\overline{a}_1 = \frac{3/2}{1 - h\left(\frac{\beta_1(u_1)I_1^n}{1+\alpha_1 I_1^n} + \frac{\beta_2(u_2)I_2^n}{1+\alpha_2 I_2^n}\right)}. \tag{48}$$

Given Eqs. (48), (42) can be written as:

$$S^{n+1} = \frac{\frac{1}{2}(S^{n-1}+S^n)}{1 + \overline{a}_1 h\left(\frac{\beta_1(u_1)I_1^n}{1+\alpha_1 I_1^n} + \frac{\beta_2(u_2)I_2^n}{1+\alpha_2 I_2^n}\right)}. \tag{49}$$

Similarly, the following equations can be obtained:

$$E^{n+1} = \frac{\frac{1}{2}\left(E^{n-1}+E^n\right) + S^n\left(\frac{\beta_1(u_1)I_1^n}{1+\alpha_1 I_1^n} + \frac{\beta_2(u_2)I_2^n}{1+\alpha_2 I_2^n}\right)\overline{a}_2 h}{1+\overline{a}_2 h\sigma}, \tag{50}$$

$$I_1^{n+1} = \frac{\frac{1}{2}\left(I_1^{n-1}+I_1^n\right) + \overline{a}_3 h\sigma}{1+\overline{a}_3 h(\rho+\theta)}, \tag{51}$$

$$I_2^{n+1} = \frac{\frac{1}{2}\left(I_2^{n-1}+I_2^n\right) + \overline{a}_4 h\theta}{1+\overline{a}_4 h\rho}, \tag{52}$$

where

$$\overline{a}_2 = \frac{3/2}{1-h\sigma},$$

$$\overline{a}_3 = \frac{3/2}{1-h(\rho+\theta)},$$

$$\overline{a}_4 = \frac{3/2}{1-h\rho}.$$

Case I: When $u_1 < u_{m,1}$ and $u_2 < u_{m,2}$ then $\beta_1(u_1) = 0 = \beta_2(u_2)$ and Eqs. (49) and (50) can be written as:

$$S^{n+1} = \frac{1}{2}(S^{n-1}+S^n), \tag{53}$$

$$E^{n+1} = \frac{\frac{1}{2}(E^{n-1}+E^n)}{1+\overline{a}_2 h\sigma}. \tag{54}$$

where $\overline{a}_2$ is the same as given before. The rest of the difference equations for this case will be the same.

Case II: When $u_{m,1} \le u_1 < u_{0,1}$ and $u_{m,2} \le u_2 < u_{0,2}$ then $\beta_1(u_1) = \frac{u_1 - u_{m,1}}{u_{0,1} - u_{m,1}}$, and

$\beta_2(u_2) = \dfrac{u_2 - u_{m,2}}{u_{0,2} - u_{m,2}}$. The difference equation for this case is already given in (49)–(52).

Case III: When $u_{0,1} \leq u_1 < u_{M,1}$ and $u_{0,2} \leq u_2 < u_{M,2}$ then $\beta_1(u_1) = 1 = \beta_2(u_2)$. Therefore, Eqs. (49) and (50) are expressed as:

$$S^{n+1} = \frac{\frac{1}{2}\left(S^{n-1} + S^n\right)}{1 + \overline{a}_1 h\left(\frac{I_1^n}{1+\alpha_1 I_1^n} + \frac{I_2^n}{1+\alpha_2 I_2^n}\right)}, \tag{55}$$

$$E^{n+1} = \frac{\frac{1}{2}\left(E^{n-1} + E^n\right) + S^n\left(\frac{I_1^n}{1+\alpha_1 I_1^n} + \frac{I_2^n}{1+\alpha_2 I_2^n}\right)}{1 + \overline{a}_2 h\sigma}. \tag{56}$$

The rest of the equations will be the same as in this work.

4.1 Stability Analysis

For finding the stability conditions system of Eqs. (10)–(13) can be expressed as:

$$\frac{d\boldsymbol{H}}{dt} = (J_1 - J_2)\boldsymbol{H}, \tag{57}$$

where $J = J_1 - J_2$ and $\boldsymbol{H} = (S, E, I_1, I_2)^T$, J is the Jacobian of the system (10)–(13) evaluated at equilibrium points $E^0(1, 0, 0, 0)$ and J_1 & J_2 are given as:

$$J_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1(u_1) & \beta_2(u_2) \\ 0 & \sigma & 0 & 0 \\ 0 & 0 & \rho+\theta & 0 \end{bmatrix} \text{ and } J_2 = \begin{bmatrix} 0 & 0 & \beta_1(u_1) & \beta_2(u_2) \\ 0 & \sigma & 0 & 0 \\ 0 & 0 & \rho+\theta & 0 \\ 0 & 0 & 0 & \rho \end{bmatrix}.$$

Discretizing Eq. (57) using the proposed scheme gives:

$$\boldsymbol{H}^{n+1} = \frac{1}{2}\left(\boldsymbol{H}^{n-1} + \boldsymbol{H}^n\right) + bh\left(J_1\boldsymbol{H}^n - J_2\boldsymbol{H}^{n+1}\right), \tag{58}$$

where $b = \dfrac{3}{2}(I - hJ_2)^{-1}$ and I is an identity matrix. Eq. (58) can be written as:

$$\boldsymbol{H}^{n+1} = (I + bhJ_2)^{-1}\left(\frac{\boldsymbol{H}^n + \boldsymbol{H}^{n-1}}{2} + bhJ_1\boldsymbol{H}^n\right). \tag{59}$$

The characteristic polynomial for Eq. (59) can be expressed as:

$$\omega^2 - (I + bhJ_2)^{-1}\left(\frac{1}{2}I + bhJ_1\right)\omega + (I + bhJ_2)^{-1}\frac{I}{2} = 0. \tag{60}$$

The scheme will be stable if the following condition holds:

$$\left|\frac{\left(1 + bh\overline{\lambda}_2\right)^{-1}\left(\frac{1}{2} + bh\overline{\lambda}_1\right) \pm \sqrt{\left(1 + bh\overline{\lambda}_2\right)^{-2}\left(\frac{1}{2} + bh\overline{\lambda}_1\right)^2 - 2\left(1 + bh\overline{\lambda}_2\right)^{-1}}}{2}\right| < 1, \tag{61}$$

where $\overline{\lambda}_2$ denotes the maximum eigenvalue of J_2. For three cases, some entries of J_1 and J_2 will be changed, and the remaining proceeds will be the same.

5 Results and Discussions

The fuzzy epidemic model has been constructed in this work. The model comprised fuzzy parameters that can be more generalized than classical ones. These given fuzzy parameters depend on the quantities of the interaction of people. The interaction parameters are divided into three categories. Suppose the quantities for the interaction of infective with susceptible are less than some minimum number. In that case, none of the susceptible will shift to the category of exposed people. If the interaction of people falls in a specific range, then the classical model can be used in which different values of parameters can be chosen. If the interaction of people is greater than some number, then there will be a 100% chance that the susceptible will become an exposed individual. So, nine possible cases can be discussed based on these three cases for one parameter. But, in this work, only a few cases have been discussed, and the rest can be discussed similarly. The presented numerical scheme has been constructed for three cases with specific values of $\beta_1\,(u_1)$ and $\beta_2\,(u_2)$. The consistency of the numerical scheme has not been discussed. It can be proved by applying Taylor series expansions. But, since the order of accuracy of the proposed scheme is one, it is consistent. The convergence of the scheme mainly depends on the choice of parameters and the considered value of step size. The convergence of numerical solutions can be controlled by choosing a suitable step size. Since the presented scheme utilizes three grid points, a solution at the second grid point is required for employing the scheme. For this work, the forward Euler scheme has been employed on the second grid points, and the presented scheme has been employed for the rest of the grid points. Since the solution remains positive for some specific values of the parameters, any numerical scheme can be considered for those parameters. So, forward Euler can also be considered for finding numerical solutions to epidemic disease models. Also, forward Euler does not produce the conditions for getting the positive solution, but the presented scheme determines the conditions for checking the positivity of the solution.

Fig. 1 shows the effect of $\beta_1\,(u_1)$ on susceptible and exposed people. The clear decay in the susceptible people can be seen, and little growth in the exposed people can be seen in Fig. 1. Since the increase in the $\beta_1\,(u_1)$ leads to shifting the susceptible people to exposed people, so susceptible people will decrease, and exposed people will increase due to the entrance of susceptible into the category of exposed people. The effect of parameter α_1 on susceptible and exposed people is deliberated in Fig. 2. The small increase in the susceptible people and dual behavior in exposed individuals can be seen in Fig. 2. The growth in α_1 will decrease the interaction of susceptible and infective resulting in changes in the susceptible and exposed individuals. The influence of σ on infective people can be seen in Fig. 3. Both categories of infective have dual behavior by incrementing the parameter σ. The effect of σ on both types of infectives can be seen in particular intervals. Initially, both types of infectives rise, but with time, the rise in infectives people converts to decay. Fig. 4 shows the infective people with a variation of the parameter ρ. By incrementing ρ, decay in both types of infective people can be seen in Fig. 4. Since higher values of ρ show that more people have transferred to people who have AIDS. So, due to this fact, the number of infective decays. The effect of parameter σ on infective people can be seen in Fig. 5 for case 3 discussed in this work. Again, the dual behavior of aware and unaware infective can be seen in Fig. 5. The decay in the infective people by incrementing the parameter ρ for case 3 is displayed in Fig. 6. The variation of $\beta_2\,(u_2)$ when $\beta_1\,(u_1) = 0$ on susceptible and exposed people is displayed in Fig. 7. The decay in the susceptible people and escalation in the exposed people can be observed by looking at Fig. 7. The escalating values of the parameter $\beta_2\,(u_2)$ will tend to shift the susceptible people to the category of exposed people, leading to decay in the susceptible people and growth in the exposed people. Since the model, in this case, does not consider the birth rate of susceptible people, and so the equilibrium point is fixed at $E^0\left(S^0, 0, 0, 0\right)$. The relationship between susceptible, exposed, and unaware infective can be seen in Fig. 8 without including the birth rate of susceptible people. Figs. 9–11

show the relationships of susceptible, exposed, and unaware infective people with different birth rates of susceptible people. Since choosing different birth rates of susceptible people, the equilibrium points of the system change, resulting in a different plot for the relationship between susceptible, exposed, and unaware infective.

Table 1 shows the comparison of schemes for finding errors. Since the exact solution of the considered system is not available, a numerical solution is adopted. The Matlab solver ode45 obtains the numerical solution. Looking at Table 1, it can be concluded that the proposed scheme performs better than the existing non-standard finite difference method in most cases.

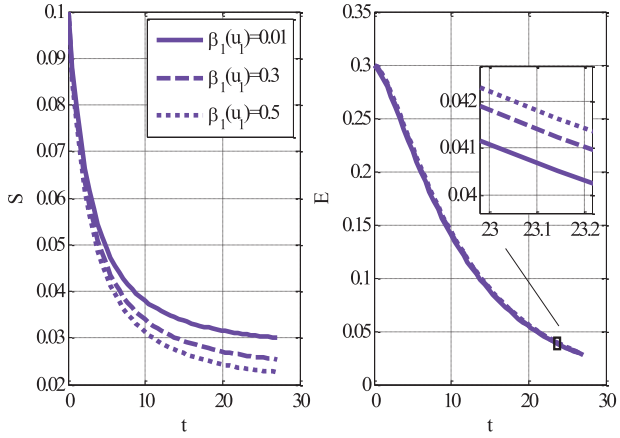

Figure 1: Effect of parameter β_1 (u_1) on susceptible and exposed people for case 2 using $\alpha_1 = 0.1, \alpha_2 = 0.3, \beta_2 = 0.5, \sigma = 0.1, \rho = 0.3, \theta = 0.5, N = 50$

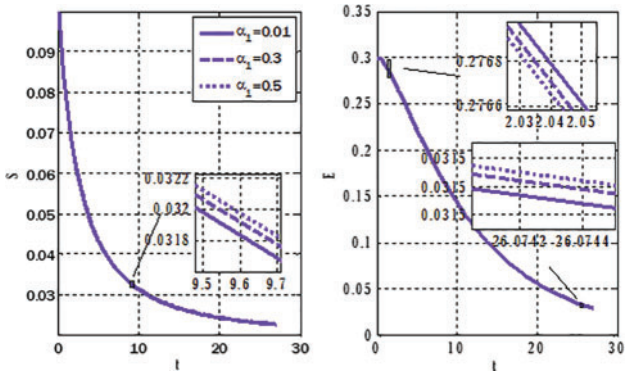

Figure 2: Effect of parameter α_1 on susceptible and exposed people for case 2 using $\beta_1 = 0.5, \alpha_2 = 0.3, \beta_2 = 0.5, \sigma = 0.1, \rho = 0.3, \theta = 0.5, N = 50$

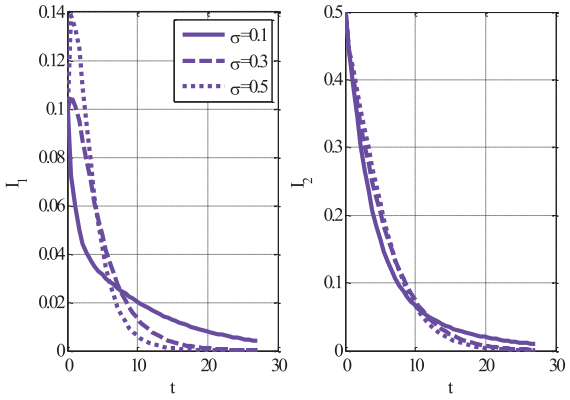

Figure 3: Effect of parameter σ on unaware and aware infective people for case 2 using $\alpha_1 = 0.5, \alpha_2 = 0.3, \beta_2 = 0.5, \beta_1 = 0.5, \rho = 0.3, \theta = 0.5, N = 50$

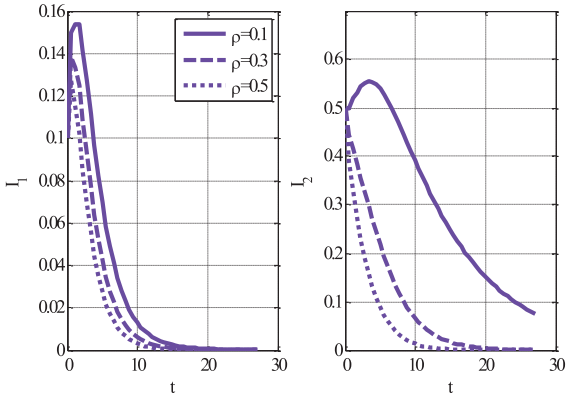

Figure 4: Effect of parameter ρ on unaware and aware infective people for case 2 using $\alpha_1 = 0.5, \alpha_2 = 0.3, \beta_2 = 0.5, \beta_1 = 0.5, \sigma = 0.5, \theta = 0.5, N = 50$

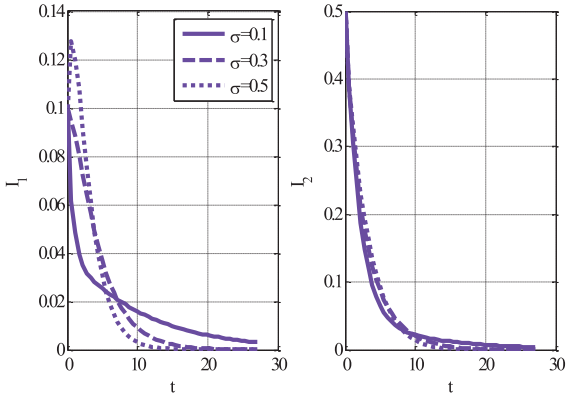

Figure 5: Effect of parameter σ on unaware and aware infective people for case 3 using $\alpha_1 = 0.5, \alpha_2 = 0.3, \beta_2 = 1, \beta_1 = 1, \rho = 0.5, \theta = 0.5, N = 50$

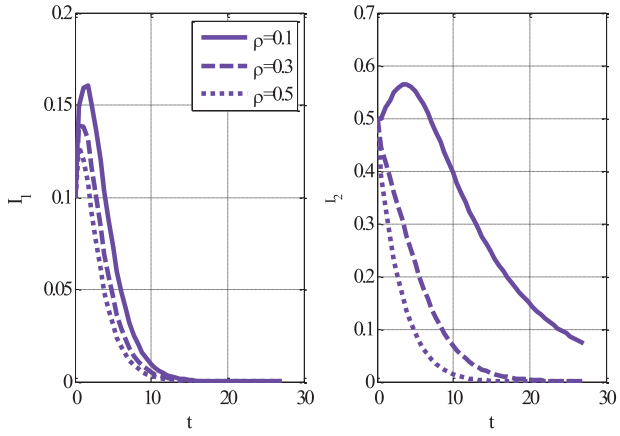

Figure 6: Effect of parameter ρ on unaware and aware infective people for case 3 using $\alpha_1 = 0.5, \alpha_2 = 0.3, \beta_2 = 1, \beta_1 = 1, \sigma = 0.5, \theta = 0.5, N = 50$

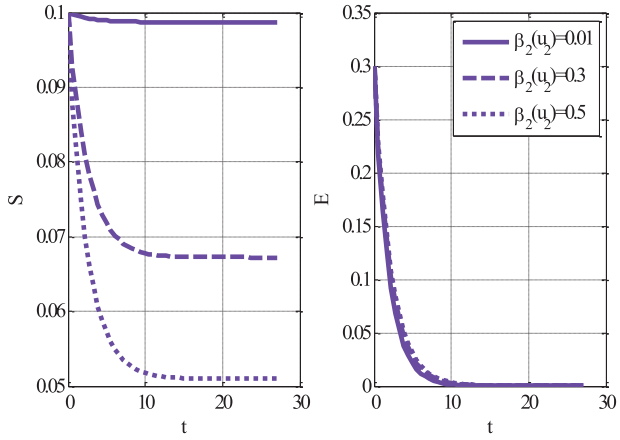

Figure 7: Effect of parameter $\beta_2\,(u_2)$ on susceptible and exposed people using $\alpha_1 = 0.5, \alpha_2 = 0.3, \sigma = 0.5, \beta_1 = 0, \rho = 0.5, \theta = 0.5, N = 50$

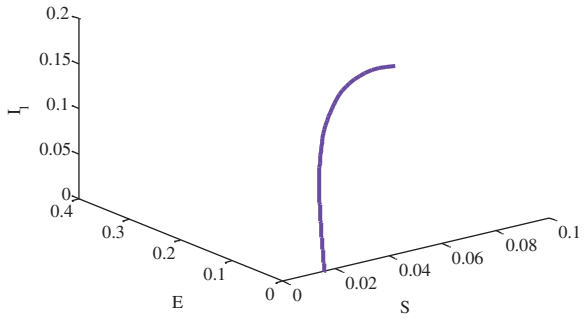

Figure 8: Relationship between susceptible, exposed and unaware infective people without birth rate of susceptible people using $\alpha_1 = 0.5, \alpha_2 = 0.3, \beta_2 = 1, \beta_1 = 1, \rho = 0.5, \sigma = 0.5, \theta = 0.5, N = 50$

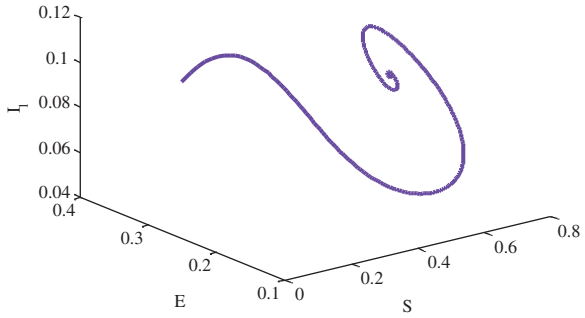

Figure 9: Relationship between susceptible, exposed and unaware infective people with a birth rate of 0.1 for susceptible people for case 3 using $\alpha_1 = 0.5, \alpha_2 = 0.3, \beta_2 = 1, \beta_1 = 1, \rho = 0.5, \sigma = 0.5, \theta = 0.5, N = 50$

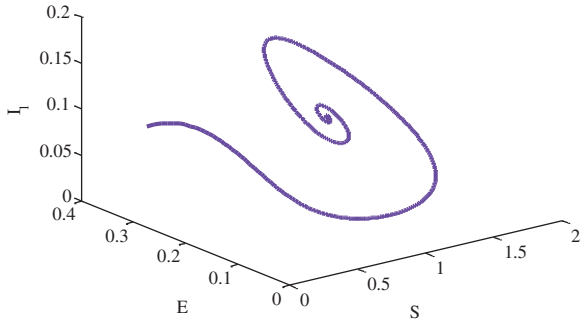

Figure 10: Relationship between susceptible, exposed and unaware infective people with a birth rate of 0.1 for susceptible people for case 2 using $\alpha_1 = 0.5, \alpha_2 = 0.3, \beta_2 = 1, \beta_1 = 1, \rho = 0.5, \sigma = 0.5, \theta = 0.5, N = 50$

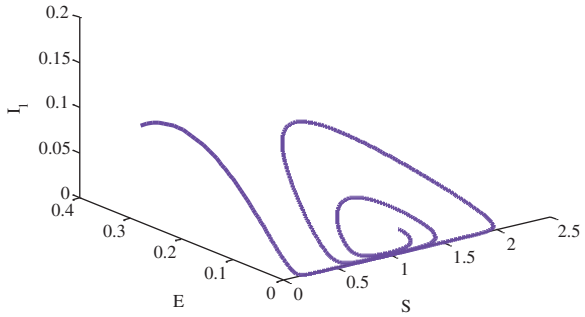

Figure 11: Relationship between susceptible, exposed and unaware infective people with a birth rate of 0.01 for susceptible people for case 2 using $\alpha_1 = 0.5, \alpha_2 = 0.3, \beta_2 = 1, \beta_1 = 1, \rho = 0.5, \sigma = 0.5, \theta = 0.5, N = 50$

Table 1: Comparison of schemes for finding L_2 error

Stepsize	L_2 error	
	NSFD	Proposed
2.0408	0.2894	0.2973
1.0101	0.4997	0.4337
0.6711	0.6521	0.5290
0.5025	0.7768	0.6061
0.4016	0.8846	0.6728

6 Conclusion

A fuzzy epidemic model has been proposed in this work. The model consists of five compartments. The fuzzy transmission rates have been defined in the model. The fuzzy transmission rate parameters have been divided into three parts based on three possibilities for the interaction of unaware and aware infective with susceptible people. The numerical scheme has been established for some fuzzy parameter cases. The concluding points can be expressed as:

- The established scheme had an advantage over the existing forward Euler method in finding conditions for positive solutions.
- Also, the constructed scheme produced less error than the existing NSFD method in small step sizes.
- Susceptible people de-escalated, and exposed people escalated by incrementing fuzzy transmission parameters for unaware infective.
- Both types of infectives were de-escalated by growth in the AIDS parameter.
- Both types of infectives had dual behavior by raising transferring parameters from exposed to unaware infective.
- The plots for relationships between susceptible, exposed, and infective converged to the model's equilibrium points with the susceptible people's birth rate.

The established numerical scheme in this work can be employed in different branches of science and engineering to solve similar types of fuzzy differential equations. The findings of this study may lead to novel uses for the present method. Different types of theoretical and applied fuzzy differential equations might be amenable to the proposed technique. It can be put into action with little effort. After this study is over, the current method could be used in a variety of contexts [43, 44].

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