Computer Modeling in
Engineering & Sciences

Tech Science Press

DOI: 10.32604/cmes.2023.029180

ARTICLE

# Results Involving Partial Differential Equations and Their Solution by Certain Integral Transform

**Rania Saadah[1], Mohammed Amleh[1], Ahmad Qazza[1], Shrideh Al-Omari[2,*] and Ahmet Ocak Akdemir[3]**

[1]Department of Mathematics, Faculty of Science, Zarqa University, Zarqa, 13110, Jordan

[2]Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Amman, 11134, Jordan

[3]Department of Mathematics, Faculty of Science and Letters, Agri Ibrahim Çeçen University, Agri, 04100, Turkey

*Corresponding Author: Shrideh Al-Omari. Email: shridehalomari@bau.edu.jo

Received: 06 February 2023 Accepted: 25 April 2023 Published: 17 November 2023

**ABSTRACT**

In this study, we aim to investigate certain triple integral transform and its application to a class of partial differential equations. We discuss various properties of the new transform including inversion, linearity, existence, scaling and shifting, etc. Then, we derive several results enfolding partial derivatives and establish a multi-convolution theorem. Further, we apply the aforementioned transform to some classical functions and many types of partial differential equations involving heat equations, wave equations, Laplace equations, and Poisson equations as well. Moreover, we draw some figures to illustrate 3-D contour plots for exact solutions of some selected examples involving different values in their variables.

**KEYWORDS**

ARA transform; double ARA transform; triple ARA transform; partial differential equations; integral transform

## 1 Introduction

Mathematical modeling is one of the main aspects of mathematics that deals with the real-life problems involved in various fields of science. Mathematical modeling transforms life problems into mathematical models and analyzes models mathematically. It is generally used in the computational theory of the field of natural sciences, computer sciences, engineering and social sciences as well. Mathematical models are the cornerstone of the modern scientific understanding of sound, heat, diffusion, static electricity, thermodynamics, fluid dynamics, elasticity, quantum mechanics and general relativity. They can be generated from various mathematical considerations comprising differential geometry and calculus of variation. Partial differential equations (PDEs) occupy a large sector of oriented scientific fields in physics and engineering. Various methods have been developed over the years to find solutions of partial differential equations, some of which reduce partial differential equations to one or more than one ordinary differential equation (ODE). However, a number of approaches have been proposed to deal with partial differential equations, such as the method of group static solutions [1,2], the non-classical method of group static [3], the method of partially invariant solutions [1] and the general method of variable separation for both linear systems. In the non-linear equations we recall the Hamilton-Jacobi equation, residual power series [4], reproducing kernel reproduction [5,6], Laplace residual power series and others [7–9].

Partial differential equations can also be handled using integral transformations to determine the exact solution to the target equations without the need for linearization or discretization. The transformation techniques are powerful and straightforward. Therefore, scientists have put a lot of efforts into understanding and improving techniques. Many integral transforms have been developed such as the Laplace transform [10], Novel transform [11], Sumudu transform [12], Elzaki transform [13], Natural transform [14], ARA transform [15], formable transform [16] and others [17–19].

In the sequel, extensions of double transformations are extensively developed for solving PDEs and obtaining results by performing solutions based on numerical approaches [20]. Double transforms such as the double Laplace transform [21–23], double Shehu transform [24], double Kamal transform [25], double Sumudu transform [26–31], double Elzaki transform [32], double Laplace-Sumudu transform [33–35] and ARA–Sumudu transform [36,37] have been considered in the literature. The ARA transform [15] was used for solving ordinary fractional differential equations [38] and some nonlinear PDEs [39]. The double ARA transform was employed for some kinds of partial and integral equations in [40–47].

In this work, we extend the idea of the single and double ARA transforms to a triple ARA transform which can be used in solving larger classes of PDEs. In this context, we define the new transform and present its main properties. More results regarding the existence, partial derivatives, and convolution theorems are also presented. However, the novelty of this study fulfills the generalization of the double ARA transform and its application to wider classes of PDEs.

Motivation in this work emerges from introducing a new approach that solves partial differential equations of multi-variables. We present characteristics of the transform and compute its values for some basic functions. Further, we show its applicability to various types of PDEs of great importance in physics. However, we organize this paper as follows. In Sections 2 and 3, we discuss the single and double ARA transforms and obtain their main properties. In Section 4, we introduce the triple ARA transform and prove some basic results on existence, elementary functions, convolutions, and partial derivatives. In Section 5, we apply the TARAT in solving PDEs and consider some examples and their 3D graphs. In Section 6, we give a conclusion section.

## 2 Basic Definitions and Preliminary Results

In this section, we recall some definitions and notations from [15] and establish several basic properties of the ARA transform.

**Definition 2.1.** The $n$th order ARA transform of a continuous function $\psi$ is defined over an infinite interval $(0,\infty)$ as

$$\mathcal{G}_n[\psi(x)](s) = \Psi(n,s) = s\int_0^\infty x^{n-1}e^{-sx}\psi(x)\,dx, \quad s>0, \tag{1}$$

provided the integral exists. In particular, for $n = 1$, the ARA integral transform of a continuous function $\psi$ is given by

$$\mathcal{G}_1[\psi(x)](s) = \Psi(s) = s\int_0^\infty e^{-sx}\psi(x)\,dx, \quad s>0, \tag{2}$$

when the integral exists. For simplicity, we make use of the notation $\mathcal{G}[\psi]$ of the ARA transform to replacce the notation $\mathcal{G}_1[\psi]$.

The inverse ARA transform is defined by [15]

$$\mathcal{G}_x^{-1}\left[\Psi\left(s\right)\right] = \psi\left(x\right) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{e^{sx}}{s}\Psi\left(s\right)ds. \tag{3}$$

Following is a result which is very needful in the sequel.

**Theorem 2.1.** Let the function $\psi$ be piecewise continuous in every finite closed interval $0 \le x \le \alpha$ [15], and satisfies the condition

$$\left|x^{n-1}\psi\left(x\right)\right| \le Re^{\alpha x},$$

where $R$ is a positive constant. Then, the ARA transform exists for all $s > \alpha, \alpha$ being real constant.

**Proof.** By employing the definition of the ARA transform we obtain

$$\left|\Psi\left(n,s\right)\right| = \left|s\int_0^{\infty}x^{n-1}e^{-sx}\psi\left(x\right)dx\right|,$$

which reveals

$$\left|\Psi\left(n,s\right)\right| \le s\left|\int_0^{\infty}x^{n-1}e^{-sx}\psi\left(x\right)dx\right| \le s\int_0^{\infty}e^{-sx}\left|x^{n-1}\psi\left(x\right)\right|dx \le s\int_0^{\infty}e^{-sx}Re^{\alpha x}dx$$

$$= sR\int_0^{\infty}e^{-(s-\alpha)x}dx = \frac{sR}{s-\alpha}e^{-(s-\alpha)}, \text{ for all } s > \alpha.$$

It is clear from the above equation that the improper integral converges for all $s > \alpha$. Therefore, the $\mathcal{G}_{n+1}\left[\psi\right]$ exists for all $s > \alpha, \alpha$ being a real constant.

This completes the proof of the theorem.

Note that, the existence condition of the ARA transform fulfils when the function $\psi$ is continuous and possessing the property.

$$\left|\psi\left(x\right)\right| \le Re^{\alpha x}, \quad \text{for some real constant } R.$$

Let $\Psi\left(s\right) = \mathcal{G}\left[\psi\right]$, $Q(s) = \mathcal{G}\left[q\right]$ and $a, b \in \mathbb{R}$. Then, the following are some basic properties of the ARA transform:

- $\mathcal{G}\left[aq\left(x\right) + b\psi\left(x\right)\right] = a\mathcal{G}\left[q\left(x\right)\right] + b\mathcal{G}\left[\psi\left(x\right)\right].$
- $\mathcal{G}^{-1}\left[aQ\left(s\right) + b\Psi\left(s\right)\right] = a\mathcal{G}^{-1}\left[Q\left(s\right)\right] + b\mathcal{G}^{-1}\left[\Psi\left(s\right)\right].$
- $\mathcal{G}\left[x^{\alpha}\right] = \dfrac{\Gamma(\alpha+1)}{s^{\alpha}}, \alpha > 0.$
- $\mathcal{G}\left[e^{ax}\right] = \dfrac{s}{s-a}, a \in \mathbb{R}.$
- $\mathcal{G}\left[\psi^{(n)}\left(x\right)\right] = s^n\mathcal{G}\left[\psi\left(x\right)\right] - \sum_{k=1}^{n}s^{n-k+1}\psi^{(k-1)}\left(0\right).$
- $\mathcal{G}\left[x^n\psi\left(x\right)\right] = \Psi(1+n,s), n \in \mathbb{N}.$

## 3 Double ARA Transform

This section discusses the new double ARA transform DARAT. It provides fundamental properties and characteristics of the transform including existence conditions, linearity and inversion formula for the proposed new double transform. Moreover, it provides some important properties and results and computes the double ARA transform of some elementary functions.

**Definition 3.1.** Let $\psi$ be a continuous function of two positive variables $x$ and $y$. Then, the DARAT of $\psi$ is defined by

$$\mathcal{G}_{x,y}\left[\psi\left(x,y\right)\right]=\Psi\left(s,p\right)=sp\int_{0}^{\infty}\int_{0}^{\infty}e^{-(sx+py)}\psi\left(x,y\right)dxdy,\quad s,p>0, \tag{4}$$

provided the integral exists. Clearly, the double ARA transform is linear, which can be shown from the equation

$$\begin{aligned}\mathcal{G}_{x,y}\left[Aq\left(x,y\right)+B\psi\left(x,y\right)\right]&=sp\int_{0}^{\infty}\int_{0}^{\infty}e^{-(sx+py)}\left(Aq\left(x,y\right)+B\psi\left(x,y\right)\right)dxdy\\&=Asp\int_{0}^{\infty}\int_{0}^{\infty}e^{-(sx+py)}q\left(x,y\right)dxdy\\&\quad+Bsp\int_{0}^{\infty}\int_{0}^{\infty}e^{-(sx+py)}\psi\left(x,y\right)dxdy=A\mathcal{G}_{x,y}\left[q\left(x,y\right)\right]+B\mathcal{G}_{x,y}\left[\psi\left(x,y\right)\right],\end{aligned}$$

provided $A$ and $B$ are constants and, $\mathcal{G}_{x,y}[q]$ and $\mathcal{G}_{x,y}[\psi]$ exist. The inverse DARAT is defined by

$$\mathcal{G}_{x,y}^{-1}\left[\Psi\left(s,p\right)\right]=\mathcal{G}_{x}^{-1}\left[\mathcal{G}_{y}^{-1}\left[\Psi\left(s,p\right)\right]\right]=\left(\frac{1}{2\pi i}\right)\int_{c-i\infty}^{c+i\infty}\frac{e^{sx}}{s}ds\left(\frac{1}{2\pi i}\right)\int_{r-i\infty}^{r+i\infty}\frac{e^{py}}{p}\Psi\left(s,p\right)dp=\psi\left(x,y\right).$$

Now, we state without proof the following theorems [40,41].

**Theorem 3.1 [40]** Let $\psi$ be a continuous function of exponential orders $\alpha$ and $\beta$, defined on the region $[0, X)\times[0, Y)$. Then $\Psi(s,p)$ exists for $s$ and $p$, provided $Re(s)>\alpha$ and $Re(p)>\beta$.

**Theorem 3.2** (Periodic Function) **[40]** Let $\mathcal{G}_{x,y}[\psi]$ exist, where $\psi$ is a periodic function of periods $\alpha>0$ and $\beta>0$, and $\psi\left(x+\alpha,y+\beta\right)=\psi\left(x,y\right),\forall x,y$. Then,

$$\mathcal{G}_{x,y}\left[\psi\left(x,y\right)\right]=\frac{1}{\left(1-e^{-(s\alpha+p\beta)}\right)}\left(sp\int_{0}^{\alpha}\int_{0}^{\beta}e^{-(sx+py)}\left(\psi\left(x,y\right)\right)dxdy\right).$$

**Theorem 3.3** (Heaviside Function) **[40]** Let $\mathcal{G}_{x,y}\left[\psi(x,y)\right]=\Psi\left(s,p\right)$ be exist. Then, we have

$$\mathcal{G}_{x,y}\left[\psi\left(x-\delta,y-\varepsilon\right)H\left(x-\delta,y-\varepsilon\right)\right]=e^{-s\delta-p\varepsilon}\Psi\left(s,p\right),$$

where $H(x-\delta,y-\varepsilon)$ is the Heaviside function defined by

$$H\left(x-\delta,y-\varepsilon\right)=\begin{cases}1,& x>\delta,y>\varepsilon,\\0,&\text{otherwise.}\end{cases}$$

**Theorem 3.4** (Convolution Theorem). **[40]** Let $\mathcal{G}_{x,y}[q]$ and $\mathcal{G}_{x,y}[\psi]$ be given such that $\mathcal{G}_{x,y}[q(x,y)]=Q(s,p)$ and $\mathcal{G}_{x,y}\left[\psi\left(x,y\right)\right]=\Psi(s,p)$. Then, we have

$$\mathcal{G}_{x,y}\left[q\left(x,y\right)**\psi\left(x,y\right)\right]=\frac{1}{sp}Q\left(s,p\right)\Psi\left(s,p\right), \tag{5}$$

where $q(x,y)**\psi(x,y)=\int_0^x\int_0^y q(x-\delta,y-\varepsilon)\psi(\delta,\varepsilon)\,d\delta d\varepsilon$ and the symbol $**$ denotes the double convolution with respect to $x$ and $y$.

In the following, we present Table 1 to illustrate the values of the double ARA transform of some elementary functions and give main properties of the transform.

**Table 1:** DARAT of some basic functions and its main properties

| $\psi(x,y)$ | $\mathcal{G}_{x,y}[\psi(x,y)] = \Psi(s,p)$ |
|---|---|
| $w(x)\,u(y)\,,x,y>0$ | $\mathcal{G}_x[w(x)]\,\mathcal{G}_y[u(y)]$ |
| $1$ | $1$ |
| $x^{\alpha}y^{\beta}$ | $\frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{s^{\alpha}p^{\beta}}, Re(\alpha)>-1, Re(\beta)>-1$ |
| $e^{\alpha x+\beta y}$ | $\frac{sp}{(s-\alpha)(p-\beta)}$ |
| $\sin(\alpha x+\beta y)$ | $\frac{sp(s\beta+p\alpha)}{(s^2+\alpha^2)(p^2+\beta^2)}$ |
| $\cos(\alpha x+\beta y)$ | $\frac{sp(sp-\alpha\beta)}{(s^2+\alpha^2)(p^2+\beta^2)}$ |
| $\sinh(\alpha x+\beta y)$ | $\frac{sp(s\beta+p\alpha)}{(s^2-\alpha^2)(p^2-\beta^2)}$ |
| $\cosh(\alpha x+\beta y)$ | $\frac{sp(sp+\alpha\beta)}{(s^2-\alpha^2)(p^2-\beta^2)}$ |
| $e^{\alpha x+\beta y}\psi(x,y)$ | $\frac{sp}{(s-\alpha)(p-\beta)}\Psi(s-\alpha,p-\beta)$ |
| $\frac{\partial\psi(x,y)}{\partial y}$ | $p\Psi(s,p)-p\mathcal{G}[\psi(x,0)]$ |
| $\frac{\partial\psi(x,y)}{\partial x}$ | $s\Psi(s,p)-s\mathcal{G}[\psi(0,y)]$ |
| $\frac{\partial^2\psi(x,y)}{\partial y^2}$ | $p^2\Psi(s,p)-p^2\mathcal{G}[\psi(x,0)]-p\mathcal{G}\left[\frac{\partial\psi(x,0)}{\partial y}\right]$ |
| $\frac{\partial^2\psi(x,\mathrm{y})}{\partial x^2}$ | $s^2\Psi(s,p)-s^2\mathcal{G}[\psi(0,y)]-s\mathcal{G}\left[\frac{\partial\psi(0,y)}{\partial x}\right]$ |
| $\frac{\partial^2\psi(x,y)}{\partial x\partial y}$ | $sp\Psi(s,p)-sp\mathcal{G}[\psi(x,0)]-sp\mathcal{G}[\psi(0,y)]+sp\psi(0,0)$ |

## 4 The Triple ARA Transform

In this section, we introduce a new triple transform, called the triple ARA transform (TARAT), and derive some basic properties and establish certain results involving partial derivatives.

**Definition 4.1.** Let $\psi$ be a continuous function. Then, the triple ARA transform of $\psi$ is defined by

$$\begin{aligned}\mathcal{G}_{x,y,t}[\psi(x,y,t)] = \Psi(s,p,k) &= \mathcal{G}_x\{\mathcal{G}_y\{\mathcal{G}_t\{\psi(x,y,t); t\to k\}; y\to p\}; x\to s\}\\ &= spk\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-sx-py-kt}\psi(x,y,t)dxdydt,\end{aligned}$$

where $x,y,t>0$ and, $s,p$ and $k$ are complex variables of the transform. The inverse triple ARA transform is given by

$$\psi\,(x,y,t)=\mathcal{G}^{-1}_{x,y,t}\left[\Psi\,(s,p,k)\right]=\frac{1}{2\pi}\int_{a-i\infty}^{a+i\infty}\frac{e^{sx}}{s}\times\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{e^{py}}{p}\times\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{kt}}{k}\Psi\,(s,p,k)\,dk\,dp\,ds.$$

Note that, the triple ARA transform is a linear transform, due to the fact that if $\psi$ and $q$ are two continuous functions, then

$$\mathcal{G}_{x,y,t}\left[\alpha\psi\,(x,y,t)+\beta q\,(x,y,t)\right]=spk\int_0^\infty\int_0^\infty\int_0^\infty\left[\alpha\psi\,(x,y,t)+\beta q\,(x,y,t)\right]dxdydt$$
$$=\alpha spk\int_0^\infty\int_0^\infty\int_0^\infty\psi\,(x,y,t)\,dxdydt+\beta spk\int_0^\infty\int_0^\infty\int_0^\infty q\,(x,y,t)\,dxdydt$$
$$=\alpha\Psi\,(s,p,k)+\beta Q\,(s,p,k)\,.$$

To simplify the application of the triple ARA transform, we need to recall the following notations:

$\mathcal{G}_x\left[\psi\,(x,y,t)\right]=\overline{\Psi}(s,y,t)$ is the single ARA transform of the function $\psi$ with respect to $x$, and

$\mathcal{G}_{xy}\left[\psi\,(x,y,t)\right]=\overline{\overline{\Psi}}(s,p,t)$ is the double ARA transform of the function $\psi$ with respect to $x$ and $y$.

By a similarity in the variables $x, y$ and $t$, we may consider the following notations:

$$\mathcal{G}_y\left[\psi\,(x,y,t)\right]=\overline{\Psi}(x,p,t),\mathcal{G}_t\left[\psi\,(x,y,t)\right]=\overline{\Psi}\,(x,y,k)\,,\mathcal{G}_{x,t}\left[\psi\,(x,y,t)\right]=\overline{\overline{\Psi}}(s,y,k),$$
$$\mathcal{G}_{yt}\left[\psi\,(x,y,t)\right]=\overline{\overline{\Psi}}\,(x,p,k)\,.$$

The existence conditions of the triple ARA transform are presented in the following theorem.

**Theorem 4.1.** Let $\psi$ be a continuous function on $I^3$, where $I=[0,\infty)$, and satisfy the following condition

$|\psi\,(x,y,t)|\le Me^{\alpha x+\beta y+\gamma t}$,

where $\alpha,\beta,\gamma$ and $M>0$. Then, the triple ARA transform exists for all $Re\,(s)>\alpha$, $Re\,(p)>\beta$ and $Re\,(k)>\gamma$. In addition, the triple ARA transform is unique, i.e., if $\Psi=Q$, then $\psi=q$.

**Proof.** Assume that $\alpha,\beta,\gamma>0$ such that $Re(s)>\alpha$, $Re(p)>\beta$ and $Re(k)>\gamma$. Then, we have

$$\|\Psi\,(s,p,k)\|=\left\|\mathcal{G}_{x,y,t}\left[\psi\,(x,y,t)\,,(s,p,k)\right]\right\|\le|spk|\int_0^\infty\int_0^\infty\int_0^\infty|\psi\,(x,y,t)|\,e^{-Re(s)x-Re(p)y-Re(k)t}dxdydt$$
$$\le M\frac{|s|}{\alpha-Re\,(s)}\times\frac{|p|}{\beta-Re\,(p)}\times\frac{|k|}{\gamma-Re\,(k)}.$$

Also, the inverse triple ARA transform implies that

$$\psi\,(x,y,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{sx}}{s}\times\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{e^{py}}{p}\times\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{kt}}{k}\Psi\,(s,p,k)\,dk\,dp\,ds$$
$$=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{sx}}{s}\times\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{e^{py}}{p}\times\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{kt}}{k}\mathrm{Q}\,(s,p,k)\,dk\,dp\,ds=q\,(x,y,t)\,.$$

This finishes the proof of the theorem.

In Table 2, we provide some values of the triple ARA transform for some basic functions.

**Table 2:** TARAT of some basic functions

| $\psi\ (x,y,t)$ | $\mathcal{G}_{xyt}\,[\psi\ (x,y,t)] = \Psi\,(s,p,k)$ |
|---|---|
| $C \in \mathbb{R}$ | $C$ |
| $x^{\alpha} y^{\beta} t^{\gamma}$ | $\dfrac{\Gamma(\alpha+1)}{s^{\alpha}} \dfrac{\Gamma(\beta+1)}{p^{\beta}} \dfrac{\Gamma(\gamma+1)}{k^{\gamma}}$, for all $\alpha,\beta,\gamma > -1$ |
| $\cos\,(ax+by+ct)$ | $spk\dfrac{(sp-ab)\,k-(ap+bs)\,c}{(s^2+a^2)\,(p^2+b^2)\,(k^2+c^2)}$ |
| $\sin\,(ax+by+ct)$ | $spk\dfrac{(sp-ab)\,c+(ap+bs)\,k}{(s^2+a^2)\,(p^2+b^2)\,(k^2+c^2)}$ |
| $\cosh\,(ax+by+ct)$ | $\dfrac{spk}{2}\left(\dfrac{1}{(s-a)\,(p-b)\,(k-c)}+\dfrac{1}{(s+a)\,(p+b)\,(k+c)}\right)$ |
| $\sinh\,(ax+by+ct)$ | $\dfrac{spk}{2}\left(\dfrac{1}{(s-a)\,(p-b)\,(k-c)}-\dfrac{1}{(s+a)\,(p+b)\,(k+c)}\right)$ |

Now, we introduce some properties of the new approach.

**Property 4.1.** (Change of scale property) Let $\psi$ be a continuous function on which the triple ARA transform exists. Then, we have

$$\mathcal{G}_{x,y,t}\,[\psi\ (ax,by,ct)] = \Psi\left(\frac{s}{a},\frac{p}{b},\frac{k}{c}\right), \tag{6}$$

where $a, b$ and $c$ are real constants.

**Proof.** Based on the definition of the triple ARA transform, we have

$$\mathcal{G}_{x,y,t}\,[\psi\ (ax,by,ct)] = spk\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-sx-py-kt}.\psi\,(ax,by,ct)\ dxdydt. \tag{7}$$

Putting $ax = u, by = v$ and $ct = w$ ensures that Eq. (7) becomes

$$\mathcal{G}_{x,y,t}\,[\psi\ (ax,by,ct)] = spk\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-\frac{s}{a}u-\frac{p}{b}v-\frac{k}{c}w}\psi\ (u,v,w)\ \frac{du}{a}\frac{dv}{b}\frac{dw}{c}$$

$$= \frac{s}{a}\frac{p}{b}\frac{k}{c}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-\frac{s}{a}u-\frac{p}{b}v-\frac{k}{c}w}\psi\ (u,v,w)\ du\ dv\ dw = \Psi\left(\frac{s}{a},\frac{p}{b},\frac{k}{c}\right).$$

This finishes the proof of our result.

**Property 4.2.** (First shifting property) Let $\psi$ be a continuous function on which the triple ARA transform exists. Then, we have

$$\mathcal{G}_{x,y,t}\left[e^{ax+by+ct}\psi\ (x,y,t)\right] = \frac{spk}{(s-a)\,(p-b)\,(k-c)}\Psi\,(s-a,p-b,k-c)\,,$$

where $a, b$ and $c$ are real constants.

**Proof.** Based on the definition of the triple ARA transform, we obtain

$$\mathcal{G}_{x,y,t}\left[e^{ax+by+ct}\psi\ (x,y,t)\right] = spk\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{ax+by+ct}.e^{-sx-py-kt}.\psi\ (x,y,t)\ dxdydt$$

$$= spk\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s-a)x}.e^{-(p-b)y}e^{-(k-c)t}.\psi\left(x,y,t\right)dxdydt$$

$$= \frac{spk.\left(s-a\right)\left(p-b\right)\left(k-c\right)}{\left(s-a\right)\left(p-b\right)\left(k-c\right)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s-a)x}.e^{-(p-b)y}e^{-(k-c)t}.\psi\left(x,y,t\right)dx\,dy\,dt$$

$$= \frac{spk}{\left(s-a\right)\left(p-b\right)\left(k-c\right)}\Psi\left(s-a,p-b,k-c\right).$$

This finishes the proof of our result.

The following result is very useful in solving some kinds of partial differential equations.

**Theorem 4.2.** If $\mathcal{G}_{x,y,t}\left[\psi\left(x,y,t\right)\right]=\Psi\left(s,p,k\right)$ be given. Then, we have

$$\mathcal{G}_{x,y,t}\left[x^n y^m t^r\psi\left(x,y,t\right)\right]=\left(-1\right)^{n+m+r}\times spk\frac{\partial^{n+m+r}}{\partial s^n\partial p^m\partial k^r}\left(\frac{\Psi\left(s,p,k\right)}{spk}\right), \tag{8}$$

where $n,m$ and $r\in\mathrm{N}$.

**Proof:** The definition of the triple ARA transform implies

$$\mathcal{G}_{x,y,t}\left[x^n y^m t^r\psi\left(x,y,t\right)\right]=spk\int_0^\infty\int_0^\infty\int_0^\infty e^{-sx-py-kt}x^n y^m t^r\psi\left(x,y,t\right)dx\,dy\,dt$$

$$=pk\int_0^\infty\int_0^\infty e^{-py-kt}y^m t^r\left(s\int_0^\infty e^{-sx}x^n\psi\left(x,y,t\right)dx\right)dy\,dt.$$

From the properties of the single ARA, we get

$$\mathcal{G}_{x,y,t}\left[x^n y^m t^r\psi\left(x,y,t\right)\right]=pk\int_0^\infty\int_0^\infty y^m t^r e^{-py-kt}s\left(-1\right)^n\frac{\partial^n}{\partial s^n}\left[\frac{\mathcal{G}_x\left[\psi\left(x,y,t\right)\right]}{s}\right]dy\,dt.$$

Again, using some properties of the ARA transform to both variables $y$ and $t$ leads to write

$$\mathcal{G}_{x,y,t}\left[x^n y^m t^r\psi\left(x,y,t\right)\right]=\left(-1\right)^n\left(-1\right)^m\left(-1\right)^r spk\frac{\partial^{n+m+r}}{\partial s^n\partial p^m\partial k^r}\left(\frac{\mathcal{G}_{x,y,t}\left[\psi\left(x,y,t\right)\right]}{spk}\right).$$

This finishes the proof of the theorem.

The application of triple ARA on partial derivatives of different types is illustrated on the following theorem.

**Theorem 4.3.** Let $\Psi$ be the triple ARA transform of the function $\psi$. Then, the following hold true:

i) $\mathcal{G}_{x,y,t}\left[\frac{\partial\psi\left(x,y,t\right)}{\partial x}\right]=s\Psi\left(s,p,k\right)-s\overline{\overline{\Psi}}\left(0,p,k\right).$

ii) $\mathcal{G}_{x,y,t}\left[\frac{\partial^2\psi\left(x,y,t\right)}{\partial x^2}\right]=s^2\Psi\left(s,p,k\right)-s^2\overline{\overline{\Psi}}\left(0,p,k\right)-s\frac{\partial\overline{\overline{\Psi}}\left(0,p,k\right)}{\partial x}.$

iii) $\mathcal{G}_{x,y,t}\left[\frac{\partial^3\psi\left(x,y,t\right)}{\partial x^3}\right]=s^3\Psi\left(s,p,k\right)-s^3\overline{\overline{\Psi}}\left(0,p,k\right)-s^2\frac{\partial\overline{\overline{\Psi}}\left(0,p,k\right)}{\partial x}-s\frac{\partial^2\overline{\overline{\Psi}}\left(0,p,k\right)}{\partial x^2}.$

iv) $\mathcal{G}_{x,y,t}\left[\frac{\partial^3\psi\left(x,y,t\right)}{\partial x\partial y\partial t}\right]=spk\Psi\left(s,p,k\right)-spk\overline{\overline{\Psi}}\left(0,p,k\right)-spk\overline{\overline{\Psi}}\left(s,0,k\right)-spk\overline{\overline{\Psi}}\left(s,p,0\right)+$
$spk\overline{\Psi}\left(s,0,0\right)+spk\overline{\Psi}\left(0,p,0\right)+spk\overline{\Psi}\left(0,0,k\right)-spk\psi\left(0,0,0\right).$

**Proof (i)**. From the definition of the triple ARA transform, we write

$$\mathcal{G}_{x,y,t}\left[\frac{\partial\psi\,(x,y,t)}{\partial x}\right] = spk\int_0^\infty\int_0^\infty\int_0^\infty e^{-sx-py-kt}\frac{\partial\psi\,(x,y,t)}{\partial x}dx\,dy\,dt$$
$$= pk\int_0^\infty e^{-kt}dt\int_0^\infty e^{-py}dy\left(s\int_0^\infty e^{-sx}\frac{\partial\psi\,(x,y,t)}{\partial x}dx\right).$$

To solve the integration inside the brackets, we use the idea of integration by parts with the assumptions that

$u = e^{-sx}$ and $dv = \dfrac{\partial\psi\,(x,y,t)}{\partial x}$ to get

$$\mathcal{G}_{x,y,t}\left[\frac{\partial\psi\,(x,y,t)}{\partial x}\right] = pk\int_0^\infty e^{-kt}dt\int_0^\infty e^{-py}dt\left(-s\psi\,(0,y,t) + s^2\int_0^\infty \psi\,(x,y,t)\,dx\right)$$
$$= s\Psi\,(s,p,k) - s\overline{\overline{\Psi}}\,(0,p,k)\,.$$

**(ii).** By using part (i) of the theorem, it follows that:

$$\mathcal{G}_{x,y,t}\left[\frac{\partial^2\psi\,(x,y,t)}{\partial x^2}\right] = \mathcal{G}_{x,y,t}\left[\frac{\partial}{\partial x}\left(\frac{\partial\psi\,(x,y,t)}{\partial x}\right)\right] = s\mathcal{G}_{x,y,t}\left[\frac{\partial\psi\,(x,y,t)}{\partial x}\right] - s\frac{\partial\overline{\overline{\Psi}}\,(0,p,k)}{\partial x}$$
$$= s^2\Psi\,(s,p,k) - s^2\overline{\overline{\Psi}}\,(0,p,k) - s\frac{\partial\overline{\overline{\Psi}}\,(0,p,k)}{\partial x},$$

where,

$$\frac{\partial\overline{\overline{\Psi}}\,(0,p,k)}{\partial x} = \mathcal{G}_{x,y,t}\left[\frac{\partial\psi\,(0,y,t)}{\partial x}\right].$$

Similarly, one can follow the proof of (iii) and (iv).

This proves the theorem.

In view of the similarity between $x, y$ and $t$, one can conclude the rest of derivative rules which can enlisted as follows.

The following property is a basic result from Theorem 4.3, and the proof can be obtained by similar arguments to the above theorem.

**Corollary 4.1.** Let $\Psi\,(s,p,k) = \mathcal{G}_{x,y,t}\,[\psi\,(x,y,t)]$. Then, we have the following results hold:

- $\mathcal{G}_{x,y,t}\left[\dfrac{\partial\psi\,(x,y,t)}{\partial y}\right] = p\Psi\,(s,p,k) - p\overline{\Psi}\,(s,0,k)\,.$
- $\mathcal{G}_{x,y,t}\left[\dfrac{\partial\psi\,(x,y,t)}{\partial t}\right] = k\Psi\,(s,p,k) - k\overline{\Psi}\,(s,p,0)\,.$
- $\mathcal{G}_{x,y,t}\left[\dfrac{\partial^2\psi\,(x,y,t)}{\partial x\partial y}\right] = sp\Psi\,(s,p,k) - sp\overline{\overline{\Psi}}\,(s,0,k) - sp\overline{\overline{\Psi}}\,(0,p,k) + sp\overline{\Psi}\,(0,0,k)\,.$
- $\mathcal{G}_{x,y,t}\left[\dfrac{\partial^2\psi\,(x,y,t)}{\partial x\partial t}\right] = sk\Psi\,(s,p,k) - sk\overline{\overline{\Psi}}\,(s,p,0) - sk\overline{\overline{\Psi}}\,(0,p,k) + sk\overline{\Psi}\,(0,p,0)\,.$
- $\mathcal{G}_{x,y,t}\left[\dfrac{\partial^2\psi(x,y,t)}{\partial y\partial t}\right] = pk\Psi\,(s,p,k) - pk\overline{\overline{\Psi}}\,(s,0,k) - pk\overline{\overline{\Psi}}\,(s,p,0) + pk\overline{\Psi}\,(s,0,0).$

The following corollary presents a new application of the triple ARA transform, that is useful in solving some kinds of integral equations.

**Corollary 4.2.** If $\mathcal{G}_{x,y,t}[\psi\,(x,y,t)] = \Psi(s,p,k)$, then we have

$$\mathcal{G}_{x,y,t}\left[\int_0^t\int_0^y\int_0^x \psi\,(u,v,w)\,du\,dv\,dw\right] = \frac{\Psi\,(s,p,k)}{spk}, s,p,k > 0.$$

**Proof.** Suppose that

$$g\,(x,y,t) = \int_0^t\int_0^y\int_0^x \psi\,(u,v,w)\,du\,dv\,dw,$$

and $g\,(0,0,0) = 0$. Now, consider

$$\frac{\partial^3 g(x,y,t)}{\partial x\partial y\partial t} = \psi\,(x,y,t)\,. \tag{9}$$

Applying the triple ARA transform to both sides of Eq. (4), and using Theorem 4.3 suggest to write

$$\mathcal{G}_{x,y,t}\left[\frac{\partial^3 g\,(x,y,t)}{\partial x\partial y\partial t}\right] = \mathcal{G}_{x,y,t}[\psi\,(x,y,t)] = \Psi\,(s,p,k)\,.$$

Thus, we write

$$\begin{aligned}\Psi\,(s,p,k) &= \mathcal{G}_{x,y,t}\left[\frac{\partial^3 g\,(x,y,t)}{\partial x\partial y\partial t}\right]\\ &= spk\;G\,(s,p,k) - spk\;\overline{\overline{\overline{\mathrm{G}}}}\,(s,p,0) - spk\;\overline{\overline{\overline{\mathrm{G}}}}\,(S,0,K) - spk\;\overline{\overline{\overline{\mathrm{G}}}}\,(0,p,k) + psk\;\overline{G}\,(0,p,0)\\ &\quad + ksp\;\overline{G}\,(0,0,k) + spk\;\overline{G}\,(s,0,0) - spk\;g\,(0,0,0)\,, = spk\;G\,(s,p,k)\,,\end{aligned}$$

which implies that

$$G\,(s,p,k) = \frac{1}{spk}\Psi\,(s,p,k)\,.$$

This ends the proof of our corollary.

Now we study the effect of the triple ARA on the Heaviside function.

**Theorem 4.4.** Let $\mathcal{G}_{x,y,t}[\psi\,(x,y,t)] = \Psi(s,p,k)$. Then, we have

$$\mathcal{G}_{x,y,t}[\psi\,(x-\delta,y-\varepsilon,t-\sigma)\,H\,(x-\delta,y-\varepsilon,t-\sigma)] = e^{-\delta s-\epsilon p-\sigma k}\Psi\,(s,p,k)\,,$$

where $H(x,y,t)$ denotes the Heaviside function defined by

$$H\,(x-\delta,y-\varepsilon,t-\sigma) = \begin{cases}1, & x>\delta, y>\epsilon, t>\sigma\\ 0, & otherwise.\end{cases}$$

**Proof.** From the definition of the triple ARA transform, we have

$$\begin{aligned}&\mathcal{G}_{x,y,t}[\psi\,(x-\delta,y-\varepsilon,t-\sigma)\,H\,(x-\delta,y-\varepsilon,t-\sigma)]\\ &\quad = spk\int_0^\infty\int_0^\infty\int_0^\infty e^{-sx-py-kt}[\psi\,(x-\delta,y-\varepsilon,t-\sigma)\,H\,(x-\delta,y-\varepsilon,t-\sigma)]\,dxdydt\\ &\quad = spk\int_\sigma^\infty\int_\varepsilon^\infty\int_\delta^\infty e^{-sx-py-kt}[\psi\,(x-\delta,y-\varepsilon,t-\sigma)]\,dxdydt.\end{aligned}$$

Putting $x-\delta=\rho$, $y-\epsilon=\beta$ and $t-\sigma=\lambda$ in Eq. (32), we obtain

$$\mathcal{G}_{x,y,t}\left[\psi\left(x-\delta,y-\varepsilon,t-\sigma\right)H\left(x-\delta,y-\varepsilon,t-\sigma\right)\right]$$
$$=spk\int_0^\infty\int_0^\infty\int_0^\infty e^{-s(\rho+\delta)-p(\beta+\epsilon)-k(\lambda+\sigma)}\left[\psi\left(\rho,\beta,\lambda\right)\right]d\rho d\beta d\lambda.$$

Thus, we infer

$$\mathcal{G}_{x,y,t}\left[\psi\left(x-\delta,y-\varepsilon,t-\sigma\right)H\left(x-\delta,y-\varepsilon,t-\sigma\right)\right]$$
$$=e^{-\delta s-\epsilon p-\sigma k}\left(spk\int_0^\infty\int_0^\infty\int_0^\infty e^{-s\rho-p\beta-\sigma\lambda}\left[\psi\left(\rho,\beta,\lambda\right)\right]d\rho d\beta d\lambda\right)$$
$$=e^{-\delta s-\epsilon p-\sigma k}\Psi\left(s,p,k\right).$$

This ends the proof of our theorem.

The following theorem presents the application of the triple ARA on the multi convolution property.

**Theorem 4.5.** (Convolution Theorem) If $\mathcal{G}_{x,y,t}[\psi]=\Psi$ and $\mathcal{G}_{x,y,t}[q]=\mathrm{Q}$, then we have

$$\mathcal{G}_{x,y,t}\left[\left(\psi***q\right)\left(x,y,t\right)\right]=\frac{1}{spk}\Psi\left(s,p,k\right)\mathrm{Q}\left(s,p,k\right),$$

where

$$\left(\psi***q\right)\left(x,y,t\right)=\int_0^x\int_0^y\int_0^t\psi\left(x-\delta,y-\varepsilon,t-\sigma\right)q\left(\delta,\varepsilon,\sigma\right)d\delta d\varepsilon d\sigma.$$

**Proof.** Following the definition of the triple ARA transform, we write

$$\mathcal{G}_{x,y,t}\left[\left(\psi***q\right)\left(x,y,t\right)\right]$$
$$=spk\int_0^\infty\int_0^\infty\int_0^\infty e^{-sx-py-kt}\left(\int_0^x\int_0^y\int_0^t\psi\left(x-\delta,y-\varepsilon,t-\sigma\right)q\left(\delta,\varepsilon,\sigma\right)d\delta d\varepsilon d\sigma\right)dxdydt.$$

The definition of the Heaviside function implies

$$\mathcal{G}_{x,y,t}\left[\left(\psi***q\right)\left(x,y,t\right)\right]$$
$$=spk\int_0^\infty\int_0^\infty\int_0^\infty e^{-sx-py-kt}\left(\int_0^\infty\int_0^\infty\int_0^\infty\psi\left(x-\delta,y-\varepsilon,t-\sigma\right)H\left(x-\delta,y-\varepsilon,t-\sigma\right)\right.$$
$$\left.\times q\left(\delta,\varepsilon,\sigma\right)d\delta d\varepsilon d\sigma\right)dxdydt. \tag{10}$$

Thus, we get

$$\mathcal{G}_{x,y,t}\left[\left(\psi***q\right)\left(x,y,t\right)\right]$$
$$=\int_0^\infty\int_0^\infty\int_0^\infty q\left(\delta,\varepsilon,\sigma\right)d\delta d\varepsilon d\sigma\left(spk\int_0^\infty\int_0^\infty\int_0^\infty e^{-sx-py-kt}\psi\left(x-\delta,y-\varepsilon,t-\sigma\right)\right.$$
$$\left.\times\ H\left(x-\delta,y-\varepsilon,t-\sigma\right)\right)dxdydt.$$

Using Theorem 4.4 reveals

$$\mathcal{G}_{x,y,t}\left[(\psi *** q)(x,y,t)\right] = \int_0^\infty \int_0^\infty \int_0^\infty q(\delta,\varepsilon,\sigma)\, d\delta d\varepsilon d\sigma\, e^{-\delta s-\epsilon p-\sigma k}\Psi(s,p,k)$$
$$= \Psi(s,p,k) \int_0^\infty \int_0^\infty \int_0^\infty e^{-\delta s-\epsilon p-\sigma k} q(\delta,\varepsilon,\sigma)\, d\delta d\varepsilon d\sigma = \frac{1}{spk}\Psi(s,p,k)\, \mathrm{Q}(s,p,k).$$

This completes the proof of the theorem.

## 5 Algorithm and Applications

In this section, we present an algorithm for using the triple ARA transform in solving partial differential equations and provide solutions of some examples including partial differential equations.

### *5.1 Algorithm of the TARAT Method*

To illustrate the method of using the triple ARA transform in solving partial differential equations, let us consider the following partial differential equation:

$$L(\psi(x,y,t)) + A(\psi(x,y,t)) = g(x,y,t), \tag{11}$$

where $L$ is any linear differential operator, with respect to $x, y$ or $t$, $A$ is any constant, $\psi$ is the unknown function and $g$ is the source term. To apply the triple ARA transform method, we operate the TARAT on both sides of Eq. (7) to get

$$\mathcal{G}_{x,y,t}\left[L(\psi(x,y,t)) + \mathcal{G}_{x,y,t}\left[A\psi(x,y,t)\right]\right] = \mathcal{G}_{x,y,t}\left[g(x,y,t)\right],$$

which yields

$$\mathcal{G}_{x,y,t}\left[L(\psi(x,y,t))\right] + A\Psi(s,p,k) = G(s,p,k), \tag{12}$$

where $\Psi = \mathcal{G}_{x,y,t}[\psi]$ and $G = \mathcal{G}_{x,y,t}[g]$. Now, since $L$ and $\mathcal{G}_{x,y,t}$ are linear operators, then by running the triple ARA transform on the operator $L$ and doing some algebraic simplifications lead to the following form

$$\Psi(s,p,k) H_1(s,p,k) + H_2(s,p,k) + A\Psi(s,p,k) = G(s,p,k), \tag{13}$$

where $H_1$ and $H_2$ are functions depending on the discussed equation to be determined. Simplifying Eq. (13) suggests to have

$$\Psi(s,p,k) = \frac{G(s,p,k) - H_2(s,p,k)}{H_1(s,p,k) + A}. \tag{14}$$

To obtain the solution in the original space, we apply the inverse triple ARA transform to Eq. (14) to have

$$\psi(x,y,t) = \mathcal{G}^{-1}_{x,y,t}\left[\frac{G(s,p,k) - H_2(s,p,k)}{H_1(s,p,k) + A}\right].$$

Hence, the desired result is therefore established.

### *5.2 Applications*

In this section, we apply the triple ARA transform operator to solve the wave equation, heat equation, Poisson equation and some few others.

**Example 1:** Consider the following wave equation

$$2\frac{\partial^2\psi(x,y,t)}{\partial t^2}-\frac{\partial^2\psi(x,y,t)}{\partial x^2}-\frac{\partial^2\psi(x,y,t)}{\partial y^2}=24t^2+4y, \tag{15}$$

where $x, y \epsilon \mathbb{R}$ *and* $t > 0$ subject to the conditions:

$$\begin{aligned}
&\psi(0,y,t)=t^4+yt^2, \psi(x,0,t)=t^4, \psi(x,y,0)=0,\\
&\frac{\partial\psi(0,y,t)}{\partial x}=\sin y\sin t, \frac{\partial\psi(x,0,t)}{\partial y}=t^2+\sin x\sin t,\\
&\frac{\partial\psi(x,y,0)}{\partial t}=\sin x\sin y.
\end{aligned} \tag{16}$$

**Solution**: Operating the triple ARA transform to both sides of Eq. (5) yields

$$\begin{aligned}
2\mathcal{G}_{x,y,t}\left[\frac{\partial^2\psi(x,y,t)}{\partial t^2}\right]-\mathcal{G}_{x,y,t}\left[\frac{\partial^2\psi(x,y,t)}{\partial x^2}\right]-\mathcal{G}_{x,y,t}\left[\frac{\partial^2\psi(x,y,t)}{\partial y^2}\right]\\
=\mathcal{G}_{x,y,t}\left[24t^2+4y\right].
\end{aligned}$$

Hence, using the derivative properties implies

$$\begin{aligned}
2\left(k^2\Psi(s,p,k)-k^2\overline{\overline{\Psi}}(s,p,0)-k\frac{\partial\overline{\overline{\Psi}}(s,p,0)}{\partial t}\right)\\
-\left(s^2\Psi(s,p,k)-s^2\overline{\overline{\Psi}}(0,p,k)-s\frac{\partial\overline{\overline{\Psi}}(0,p,k)}{\partial x}\right)\\
-\left(p^2\Psi(s,p,k)-p^2\overline{\overline{\Psi}}(s,0,k)-p\frac{\partial\overline{\overline{\Psi}}(s,0,k)}{\partial y}\right)=\frac{48}{k^2}+\frac{4}{p}.
\end{aligned} \tag{17}$$

To simplify Eq. (17), we find the transformed values of the conditions (16) as follows:

$$\begin{aligned}
&\overline{\overline{\Psi}}(0,p,k)=\mathcal{G}_{y,t}[\psi(0,y,t)]=\mathcal{G}_{y,t}\left[t^4+yt^2\right]=\frac{24}{k^4}+\frac{2}{pk^2},\\
&\overline{\overline{\Psi}}(s,0,k)=\mathcal{G}_{x,t}[\psi(x,0,t)]=\mathcal{G}_{x,t}\left[t^4\right]=\frac{24}{k^4},\\
&\overline{\overline{\Psi}}(s,p,0)=\mathcal{G}_{x,y}[\psi(x,y,0)]=\mathcal{G}_{x,y}[0]=0,\\
&\frac{\partial\overline{\overline{\Psi}}(0,p,k)}{\partial x}=\mathcal{G}_{y,t}\left[\frac{\partial\psi(0,y,t)}{\partial x}\right]=\mathcal{G}_{y,t}[\sin y\sin t]=\frac{pk}{(p^2+1)(k^2+1)},\\
&\frac{\partial\overline{\overline{\Psi}}(s,0,k)}{\partial y}=\mathcal{G}_{x,t}\left[\frac{\partial\psi(x,0,t)}{\partial y}\right]=\mathcal{G}_{x,t}\left[t^2+\sin x\sin t\right]=\frac{2}{k^2}+\frac{k}{(s^2+1)(k^2+1)},\\
&\frac{\partial\overline{\overline{\Psi}}(s,p,0)}{\partial t}=\mathcal{G}_{x,y}\left[\frac{\partial\psi(x,y,0)}{\partial t}\right]=\mathcal{G}_{x,y}[\sin x\sin y]=\frac{sp}{(s^2+1)(p^2+1)}.
\end{aligned}$$

By substituting the above values and using simple calculations, we can derive

$$\Psi(s,p,k)=\frac{24}{k^4}+\frac{2}{pk^2}+\frac{spk}{(s^2+1)(p^2+1)(k^2+1)}. \tag{18}$$

Applying the inverse triple ARA transform $\mathcal{G}^{-1}_{x,y,t}$ to Eq. (18), we obtain a solution to Example 1 in the form

$\psi\,(x, y, t) = t^4 + yt^2 + \sin x \sin y \sin t.$

The following Fig. 1 illustrates the contours 3D plot of the exact solution to Example 1, with different values of the variables as follows.

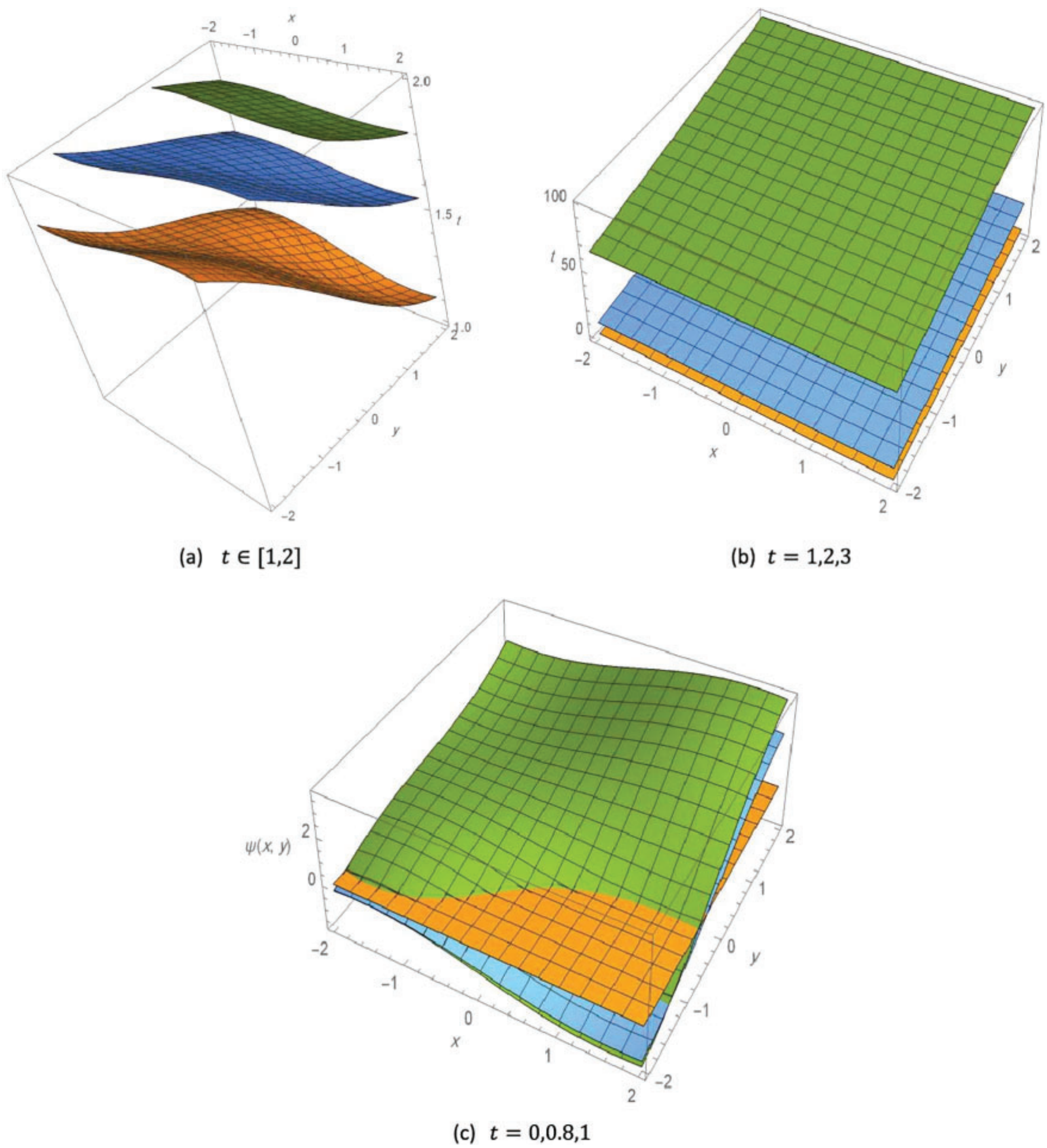



**Figure 1:** Contour 3D plot of the solution to Example 1, $\psi\,(x, y, t)$ with the values $x \in [-2, 2]$, $y \in [-2, 2]$ and three values of $t$ in (a) $t \in [1, 2]$, (b) $t = 1, 2, 3$, (c) $t = 1, 2, 3$

**Example 2:** Consider the heat partial differential equation

$$\frac{\partial \psi (x,y,t)}{\partial t} = \frac{\partial^2 \psi (x,y,t)}{\partial x^2} + \frac{\partial^2 \psi (x,y,t)}{\partial y^2} + 2\cos(x+y), \tag{19}$$

where $x, y \epsilon \mathbb{R}$ and $t > 0$ subject to the conditions:

$$\psi(0,y,t) = e^{-2t}\sin y + \cos y, \psi(x,0,t) = e^{-2t}\sin x + \cos x,$$

$$\psi(x,y,0) = \sin(x+y) + \cos(x+y), \frac{\partial \psi(0,y,t)}{\partial x} = e^{-2t}\cos y - \sin y,$$

$$\frac{\partial \psi(x,0,t)}{\partial y} = e^{-2t}\cos x - \sin x. \tag{20}$$

**Solution:** Operating the triple ARA transform to both sides of Eq. (19) gives rise to

$$\mathcal{G}_{x,y,t}\left[\frac{\partial \psi(x,y,t)}{\partial t}\right] = \mathcal{G}_{x,y,t}\left[\frac{\partial^2 \psi(x,y,t)}{\partial x^2}\right] + \mathcal{G}_{x,y,t}\left[\frac{\partial^2 \psi(x,y,t)}{\partial y^2}\right]. + \mathcal{G}_{x,y,t}[2\cos(x+y)]. \tag{21}$$

Employing properties of the derivative infer

$$\begin{aligned} &k\Psi(s,p,k) - k\overline{\overline{\Psi}}(s,p,0) \\ &\quad = s^2\overline{\overline{\Psi}}(\mathrm{s},\mathrm{p},\mathrm{k}) - s^2\overline{\overline{\Psi}}(0,p,k) - s\frac{\partial\overline{\overline{\Psi}}(0,p,k)}{\partial x} + p^2\Psi\left(\mathrm{s},p,\mathrm{k}\right) - p^2\overline{\overline{\Psi}}(s,0,k)\Big) \\ &\quad\quad - p\overline{\overline{\Psi}}(s,0,k) + \frac{sp(sp-1)}{s^2+1}. \end{aligned} \tag{22}$$

To simplify Eq. (22), we compute the transformed values of the conditions as follows:

$$\overline{\overline{\Psi}}(0,p,k) = \mathcal{G}_{y,t}\left[e^{-2t}\sin y + \cos y\right] = \frac{pk}{(p^2+1)(k+2)} + \frac{p^2}{p^2+1},$$

$$\overline{\overline{\Psi}}(s,0,k) = \mathcal{G}_{x,t}\left[e^{-2t}\sin x + \cos x\right] = \frac{sk}{(s^2+1)(k+2)} + \frac{s^2}{s^2+1},$$

$$\overline{\overline{\Psi}}(s,p,0) = \mathcal{G}_{x,y}[\sin(x+y) + \cos(x+y)] = \frac{sp(s+p)}{p^2+1} + \frac{sp(sp-1)}{s^2+1},$$

$$\frac{\partial\overline{\overline{\Psi}}(0,p,k)}{\partial x} = \mathcal{G}_{y,t}\left[e^{-2t}\cos y - \sin y\right] = \frac{p^2k}{(p^2+1)(k+2)} - \frac{p}{p^2+1},$$

$$\frac{\partial\overline{\overline{\Psi}}(s,0,k)}{\partial y} = \mathcal{G}_{x,t}\left[e^{-2t}\cos x - \sin x\right] = \frac{s^2k}{(s^2+1)(k+2)} - \frac{s}{s^2+1}.$$

Substituting the above values included in Eq. (22) and applying some calculations we reach the conclusion

$$\Psi(s,p,k) = \frac{ksp(s+p)}{(k+2)(p^2+1)} + \frac{sp(sp-1)}{s^2+1}. \tag{23}$$

Applying the inverse $\mathcal{G}^{-1}_{x,y,t}$ to Eq. (23), we get a solution in the original space as

$$\psi(x,y,t) = e^{-2t}\sin(x+y) + \cos(x+y).$$

Our example has, therefore, been solved.

The following Fig. 2 illustrates the contours 3D plot of the exact solution to Example 2, with different values of the variables as follows.

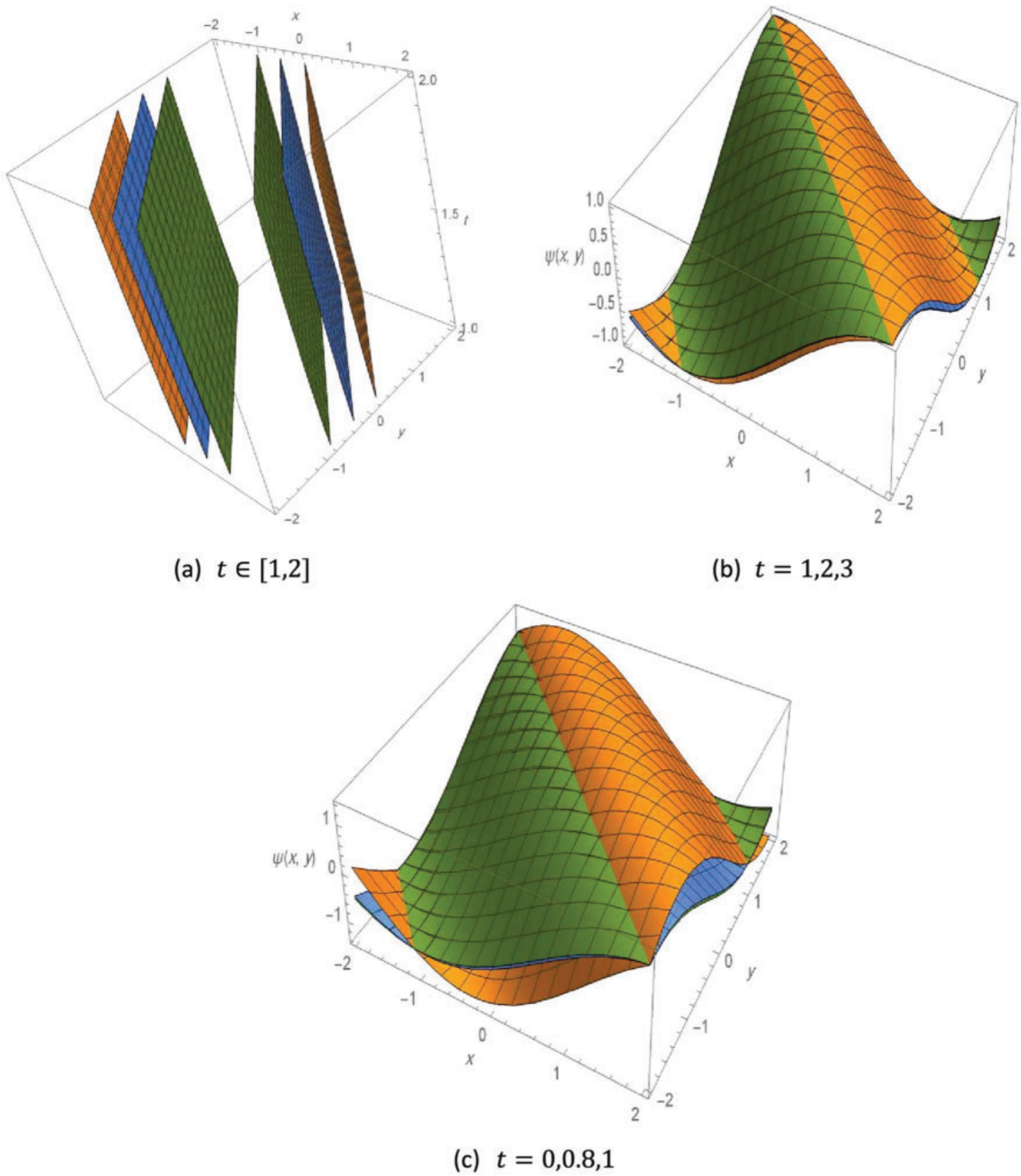



**Figure 2:** Contour the 3D plot of the solution to Example 2, $\psi\ (x, y, t)$ with the values $x \in [-2, 2]$ , $y \in [-2, 2]$ and three values of $t$ in (a) $t \in [1, 2]$, (b) $t = 1, 2, 3$, (c) $t = 1, 2, 3$

**Example 3:** Consider the following Poisson partial differential equation:

$$\frac{\partial^2\psi\,(x,y,t)}{\partial x^2}+\frac{\partial^2\psi\,(x,y,t)}{\partial y^2}+\frac{\partial^2\psi\,(x,y,t)}{\partial t^2}=2\sin x\cos y\sinh 2t, \tag{24}$$

where $x, y$ and $t$ are positive real numbers subject to the initial conditions:

$$\psi\,(0,y,t)=0,\psi\,(x,0,t)=\sin x\sinh 2t,\psi\,(x,y,0)=0,$$

$$\frac{\partial\psi\,(0,y,t)}{\partial x}=\cos y\sinh 2t,\frac{\partial\psi\,(x,0,t)}{\partial y}=0,\frac{\partial\psi\,(x,y,0)}{\partial t}=2\sin x\cos y. \tag{25}$$

**Solution:** By arguments alike to those employed for Examples 1 and 2, we apply the TARAT to Eq. (24) and compute the transformed values of the initial conditions to infer

$$\overline{\overline{\Psi}}\,(0,p,k)=\mathcal{G}_{y,t}\,[0]=0,$$

$$\overline{\overline{\Psi}}\,(s,0,k)=\mathcal{G}_{x,t}\,[\sin x\sinh 2t]=\frac{sk}{(s^2+1)\,(k^2-1)},$$

$$\overline{\overline{\Psi}}\,(s,p,0)=\mathcal{G}_{x,y}\,[0]=0,$$

$$\frac{\partial\overline{\overline{\Psi}}\,(0,p,k)}{\partial x}=\mathcal{G}_{y,t}\,[\cos y-\sinh 2t]=\frac{2p^2k}{(p^2+1)\,(k^2-1)},$$

$$\frac{\partial\overline{\overline{\Psi}}\,(s,0,k)}{\partial y}=\mathcal{G}_{x,t}\,[0]=0,$$

$$\frac{\partial\overline{\overline{\Psi}}\,(s,p,0)}{\partial t}=\mathcal{G}_{x,y}\,[2sin\,x\,cosy]=\frac{2sp^2}{(s^2+1)\,(p^2+1)}.$$

Thus, we have the solution, in the triple ARA space, as

$$\Psi\,(s,p,k)=\frac{2sp^2k}{(s^2+1)(p^2+1)(k^2-4)}. \tag{26}$$

Now, we reach the solution to Example 3 in the original space, by applying the inverse TARAT $\mathcal{G}^{-1}_{x,y,t}$ to Eq. (26) to have

$$\psi\,(x,y,t)=\mathcal{G}^{-1}_{x,y,t}\left[\frac{2sp^2k}{(p^2+1)\,(s^2+1)\,(k^2-4)}\right]=\sin x\cos y\sinh 2t.$$

Our example has been solved.

The following Fig. 3 illustrates the contours 3D plot of the exact solution to Example 3, with different values of the variables as follows.

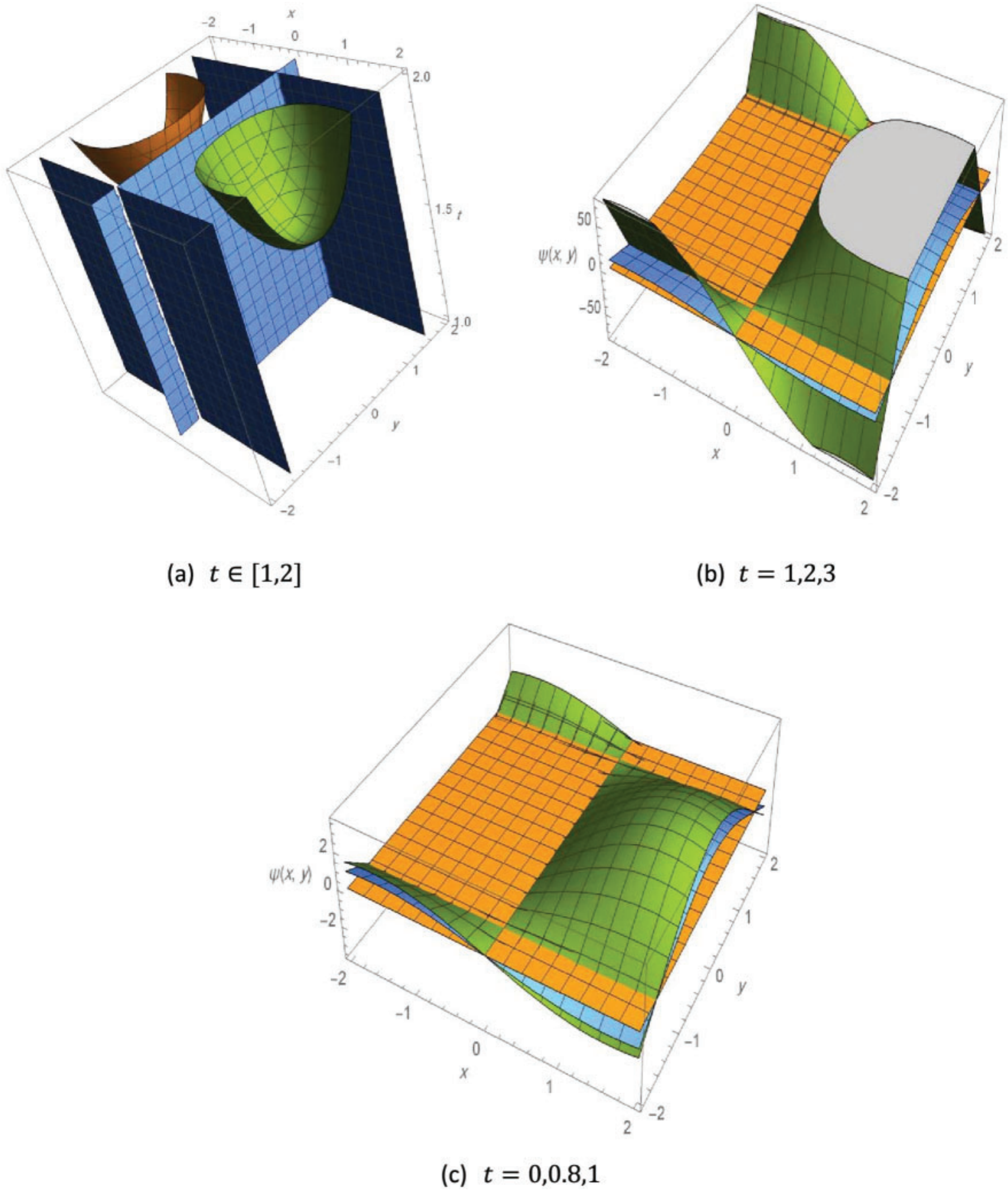



**Figure 3:** Contour 3D plot of the solution to Example 3, $\psi\,(x, y, t)$ with the values $x \in [-2, 2]\,, y \in [-2, 2]$ and three values of $t$ in (a) $t \in [1, 2]$, (b) $t = 1, 2, 3$, (c) $t = 1, 2, 3$

**Example 4:** Consider the following Laplace partial differential equation:

$$\frac{\partial^2\psi\,(x, y, t)}{\partial x^2} + \frac{\partial^2\psi\,(x, y, t)}{\partial y^2} + \frac{\partial^2\psi\,(x, y, t)}{\partial t^2} = 0, \tag{27}$$

where $x, y$ and $t$ are positive real numbers subject to the conditions:

$$\psi(0,y,t)=0,\ \psi(x,0,t)=0,\ \psi(x,y,0)=0,$$
$$\frac{\partial\psi(0,y,t)}{\partial x}=\text{siny}\sinh\sqrt{2}t,\ \frac{\partial\psi(x,0,t)}{\partial y}=sinx\sinh\sqrt{2}t,$$
$$\frac{\partial\psi(x,y,0)}{\partial t}=\sqrt{2}\sin x\sin y. \tag{28}$$

**Solution:** Following steps alike to those used in the previous examples, then by applying the TARAT to the conditions in (28) and operating the triple ARA transform to Eq. (27) we establish that

$$\overline{\overline{\Psi}}(0,p,k)=\overline{\overline{\Psi}}(s,0,k)=\overline{\overline{\Psi}}(s,p,0)=0,$$

$$\frac{\partial\overline{\overline{\Psi}}(0,p,k)}{\partial x}=\frac{\sqrt{2}pk}{(p^2+1)(k^2-2)},$$
$$\frac{\partial\overline{\overline{\Psi}}(s,0,k)}{\partial y}=\frac{\sqrt{2}sk}{(s^2+1)(k^2-2)},$$
$$\frac{\partial\overline{\overline{\Psi}}(s,p,0)}{\partial t}=\frac{\sqrt{2}sp}{(s^2+1)(p^2+1)}.$$

Thus, we get the solution in the triple ARA space in the form

$$\Psi(s,p,k)=\frac{\sqrt{2}spk}{(s^2+1)(p^2+1)(k^2-2)}. \tag{29}$$

Following that, we get a solution to Example 4 in the original space after applying the inverse triple ARA transform $\mathcal{G}^{-1}_{x,y,t}$ to Eq. (29) as

$$\psi(x,y,t)=\mathcal{G}^{-1}_{x,y,t}\left[\frac{\sqrt{2}spk}{(s^2+1)(p^2+1)(k^2-2)}\right]=\sin x\sin y\sinh\left(\sqrt{2}t\right).$$

Our example has been solved.

The following Fig. 4 illustrates the contours 3D plot of the exact solution to Example 4, with different values of the variables as follows.

**Example 5:** Consider the following partial differential equation

$$\frac{\partial^3\psi(x,y,t)}{\partial x\partial y\partial t}+\psi(x,y,t)=\cos x\cos y\cos t-\sin x\sin y\sin t, \tag{30}$$

where $x, y$ and $t$ are positive real numbers subject to the initial conditions

$$\psi(x,y,0)=\cos x\cos y,\ \psi(x,0,t)=\cos x\cos t,\ \psi(0,y,t)=\cos y\cos t,$$

$$\psi(x,0,0)=\cos x,\ \psi(0,y,0)=\cos y,$$

$$\psi(0,0,t)=\cos t,\ \psi(0,0,0)=1. \tag{31}$$

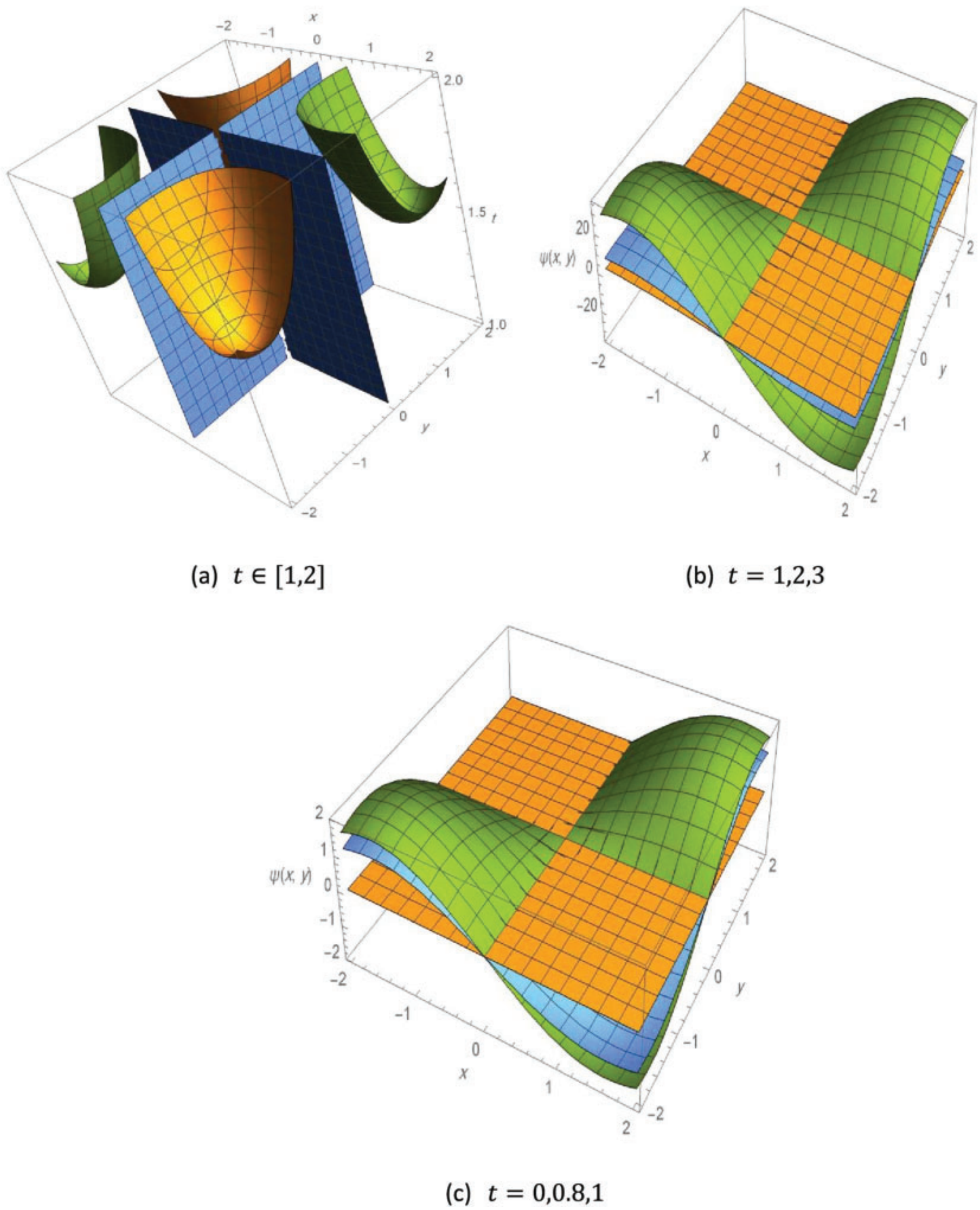



**Figure 4:** Contour 3D plot of the solution to Example 4, $\psi(x, y, t)$ with the values $x \in [-2, 2]$, $y \in [-2, 2]$ and three values of $t$ in (a) $t \in [1, 2]$, (b) $t = 1, 2, 3$, (c) $t = 1, 2, 3$

**Solution:** Applying the triple ARA transform to the condition (31) infers

$$\overline{\overline{\Psi}}(s, p, 0) = \mathcal{G}_{x,y}[\cos x \cos y] = \frac{s^2 p^2}{(s^2 + 1)(p^2 + 1)}.$$

$$\overline{\overline{\Psi}}(s, 0, k) = \mathcal{G}_{x,t}[\cos x \cos t] = \frac{s^2 k^2}{(s^2 + 1)(k^2 + 1)}.$$

$$\overline{\overline{\Psi}}(0,p,k) = \mathcal{G}_{y,t}[\cos y \cos t] = \frac{p^2k^2}{(p^2+1)(k^2+1)}.$$

$$\overline{\Psi}(s,0,0) = \mathcal{G}_{x}[\cos x] = \frac{s^2}{s^2+1}.$$

$$\Psi(0,p,0) = \mathcal{G}_{y}[\cos y] = \frac{p^2}{p^2+1}.$$

$$\overline{\Psi}(0,0,k) = \mathcal{G}_{t}[\cos t] = \frac{k^2}{k^2+1}.$$

Now, allowing the triple ARA transform to act on both sides of Eq. (31) and using the transformed values of the conditions with simple calculations reveal

$$\mathcal{G}_{x,y,t}\left[\frac{\partial\psi(x,y,t)}{\partial x\,\partial y\,\partial t}\right] + \mathcal{G}_{x,y,t}[\psi(x,y,t)] = \mathcal{G}_{x,y,t}[\cos x\cos y\cos t - \sin x\sin y\sin t].$$

This indeed implies

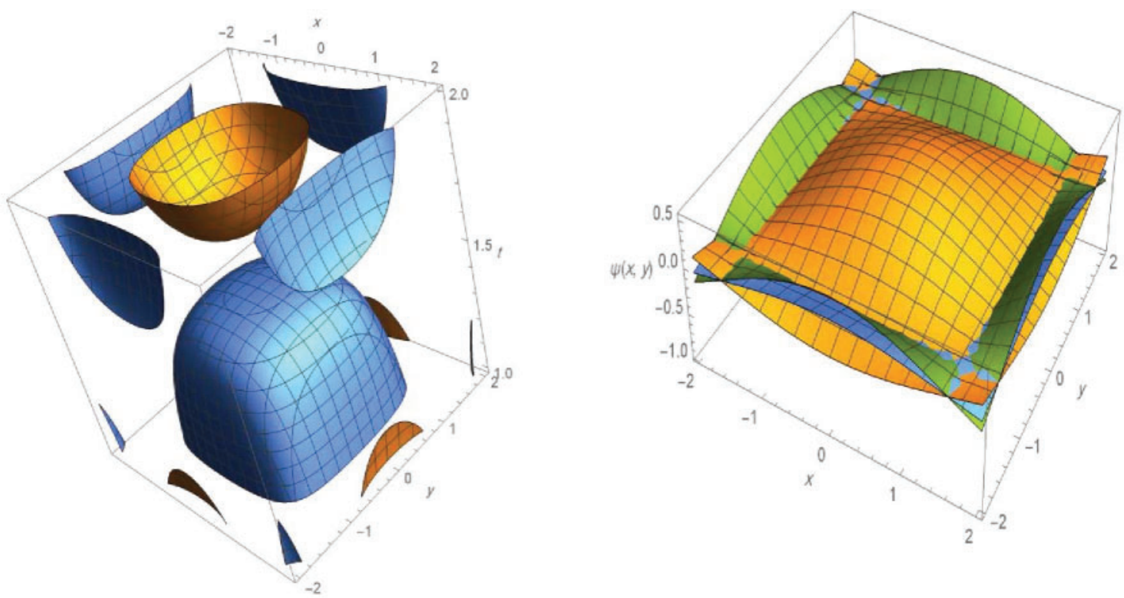
$$\Psi(s,p,k) = \frac{s^2p^2k^2}{(s^2+1)(p^2+1)(k^2+1)}. \tag{32}$$

Operating the inverse TARAT $\mathcal{G}^{-1}_{x,y,t}$ to Eq. (32) gives

$$\psi(x,y,t) = \mathcal{G}^{-1}_{x,y,t}\left[\frac{s^2p^2k^2}{(s^2+1)(p^2+1)(k^2+1)}\right] = \cos x\cos y\cos t.$$

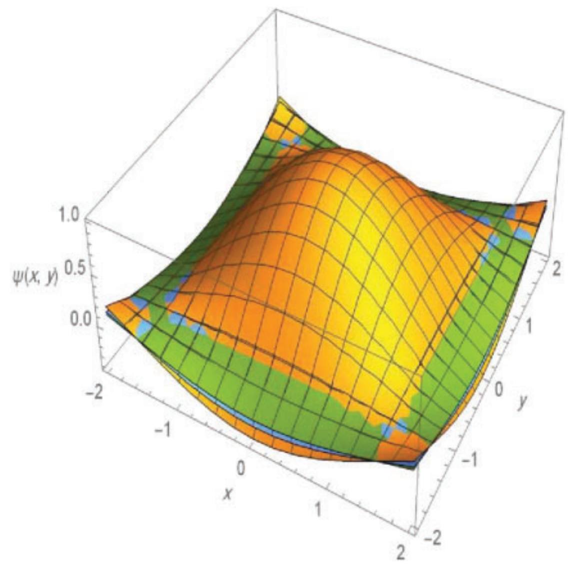
Our example has then been solved.

The following Fig. 5 illustrates the contours 3D plot of the exact solution to Example 5, with different values of the variables as follows.




(a) $t \in [1,2]$ (b) $t = 1,2,3$

**Figure 5:** (Continued)




(c) $t = 0{,}0.8{,}1$

**Figure 5:** Contour 3D plot of the solution to Example 4, $\psi\,(x, y, t)$ with the values $x \in [-2, 2]\,, y \in [-2, 2]$ and three values of $t$ in (a) $t \in [1, 2]$, (b) $t = 1, 2, 3$, (c) $t = 1, 2, 3$

## 6 Conclusion

In this paper, a new triple transform called the triple ARA transform is presented and several properties and theorems including linearity, existence, partial derivatives, and the multiple convolution theorem are introduced. Tables that summarize the new results are established to illustrate the new approach. The methodology of using the triple ARA transform in solving partial differential equations is given and further examples of several types of partial differential equations involving heat, wave, and Poisson equations are discussed. The outcomes have shown the simplicity of using the proposed transform in different aspects. In future work, we aim to solve integral equations and nonlinear partial differential equations based on the triple ARA transform.

**Acknowledgement:** Not applicable.

**Funding Statement:** This research received no external funding.

**Author Contributions:** All authors have read and agreed to the published version of the manuscript.

**Availability of Data and Materials:** Not applicable.

**Conflicts of Interest:** The authors declare that they have no conflicts of interest to report regarding the present study.

## References

1. Ovsiannikov, L. V. E. (1982). *Group analysis of differential equations*. Academic Press.
2. Ovsiannikov, L. V. (1958). Groups and group-invariant solutions of partial differential equations. *Doklady Akademii Nauk SSSR, 118(3),* 439–442.

3. Bluman, G. W., Cole, J. D. (1969). The general similarity solution to the heat equation. *Journal of Mathematics and Mechanics, 18(11),* 1025–1042.
4. Saadeh, R., Alaroud, M., Al-Smadi, M., Ahmad, R. R., Salma Din, U. K. (2019). Application of fractional residual power series algorithm to solve Newell-Whitehead–Segel equation of fractional order. *Symmetry, 11(12),* 1431.
5. Saadeh, R. (2021). Numerical algorithm to solve a coupled system of fractional order using a novel reproducing kernel method. *Alexandria Engineering Journal, 60(5),* 4583–4591.
6. Saadeh, R., Al-Smadi, M., Gumah, G., Khalil, H., Khan, R. A. (2016). Numerical investigation for solving two-point fuzzy boundary value problems by reproducing kernel approach. *Applied Mathematics and Information Sciences, 10(6),* 2117–2129.
7. Saadeh, R., Burqan, A., El-Ajou, A. (2022). Reliable solutions to fractional Lane-Emden equations via Laplace transform and residual error function. *Alexandria Engineering Journal, 61(12),* 10551–10562.
8. Burqan, A., Saadeh, R., Qazza, A., Momani, S. (2023). ARA-residual power series method for solving partial fractional differential equations. *Alexandria Engineering Journal, 62,* 47–62.
9. Qazza, A., Burqan, A., Saadeh, R. (2022). Application of ARA-residual power series method in solving systems of fractional differential equations. *Mathematical Problems in Engineering, 2022,* 1–17.
10. Widder, V. (1941). *The Laplace transform*. Princeton, NJ, USA: Princeton University Press.
11. Atangana, A., Kiliçman, A. (2013). A novel integral operator transform and its application to some FODE and FPDE with some kind of singularities. *Mathematical Problems in Engineering, 2013,* 531984.
12. Al-Omari, S., Kilicman, A. (2013). An estimate of Sumudu transform for Boehmians. *Advances in Difference Equations, 2013,* 77.
13. Elzaki, T. M. (2011). The new integral transform "Elzaki transform". *Global Journal of Pure and Applied Mathematics, 7,* 57–64.
14. Al-Omari, S. (2015). Natural transform in Boehmian spaces. *Nonlinear Studies, 22(2),* 291–297.
15. Saadeh, R., Qazza, A., Burqan, A. (2020). A new integral transform: ARA transform and its properties and applications. *Symmetry, 12,* 925.
16. Saadeh, R. Z., Ghazal, B. F. A. (2021). A new approach on transforms: Formable integral transform and its applications. *Axioms, 10(4),* 332.
17. Al-Omari, S. K., Kilicman, A. (2014). On the exponential Radon transform and its extension to certain functions spaces. *Abstract and Applied Analysis, 2014,* 612391.
18. Sedeeg, A. H. (2016). The new integral transform "Kamal Transform". *Advances in Theoretical and Applied Mathematics, 11,* 451–458.
19. Al-Omari, S. K. (2016). On $q$-Analogues of the Mangontarum transform for certain $q$-Bessel functions and some application. *Journal of King Saud University-Science, 28(4),* 375–379.
20. Aghili, A., Parsa Moghaddam, B. (2011). Certain theorems on two dimensional Laplace transform and non-homogeneous parabolic partial differential equations. *Surveys in Mathematics and its Applications, 6,* 165–174.
21. Dhunde, R. R., Bhondge, N. M., Dhongle, P. R. (2013). Some remarks on the properties of double Laplace transforms. *International Journal of Applied Physics and Mathematics, 3,* 293–295.
22. Dhunde, R. R., Waghmare, G. L. (2017). Double Laplace transform method in mathematical physics. *International Journal of Theoritical and Mathematical Physics, 7,* 14–20.
23. Eltayeb, H., Kiliçman, A. A. (2013). A note on double Laplace transform and telegraphic equations. *Abstract and Applied Analysis, 2013,* 932578.
24. Alfaqeih, S., Misirli, E. (2020). On double Shehu transform and its properties with applications. *International Journal of Analysis and Applications, 18,* 381–395.
25. Sonawane, M., Kiwne, S. B. (2019). Double Kamal transforms: Properties and applications. *Journal of Applied Science and Computation, 4,* 1727–1739.

26. Ganie, J. A., Ahmad, A., Jain, R. (2018). Basic analogue of double Sumudu transform and its applicability in population dynamics. *Asian Journal of Mathematics and Statistics, 11,* 12–17.
27. Eltayeb, H., Kiliçman, A. (2010). On double Sumudu transform and double Laplace transform. *Malaysian Journal of Mathematical Sciences, 4,* 17–30.
28. Tchuenche, J. M., Mbare, N. S. (2007). An application of the double Sumudu transform. *Applied Mathematical Sciences, 1,* 31–39.
29. Al-Omari, S. K. Q. (2012). Generalized functions for double Sumudu transformation. *International Journal of Algebra, 6,* 139–146.
30. Eshag, M. O. (2017). On double Laplace transform and double Sumudu transform. *American Journal of Engineering Research, 6,* 312–317.
31. Ahmed, Z., Idrees, M. I., Belgacem, F. B. M., Perveenb, Z. (2020). On the convergence of double Sumudu transform. *Journal of Nonlinear Science and Applications, 13,* 154–162.
32. Idrees, M. I., Ahmed, Z., Awais, M., Perveen, Z. (2018). On the convergence of double Elzaki transform. *International Journal of Advanced and Applied Sciences, 5,* 19–24.
33. Ahmed, S., Elzaki, T., Elbadri, M., Mohamed, M. Z. (2021). Solution to partial differential equations by new double integral transform (Laplace-Sumudu transform). *Ain Shams Engineering Journal, 12,* 4045–4049.
34. Ahmed, S. A., Qazza, A., Saadeh, R. (2022). Exact solutions of nonlinear partial differential equations via the new double integral transform combined with iterative method. *Axioms, 11(6),* 247.
35. Ahmed, S. A., Qazza, A., Saadeh, R., Elzaki, T. (2022). Conformable double Laplace-Sumudu iterative method. *Symmetry, 15,* 78.
36. Saadeh, R., Qazza, A., Burqan, A. (2022). On the double ARA-Sumudu transform and its applications. *Mathematics, 10,* 2581.
37. Qazza, A., Burqan, A., Saadeh, R., Khalil, R. (2022). Applications on double ARA–Sumudu transform in solving fractional partial differential equations. *Symmetry, 14,* 1817.
38. Qazza, A., Burqan, A., Saadeh, R. (2021). A new attractive method in solving families of fractional differential equations by a new transform. *Mathematics, 9(23),* 3039.
39. Burqan, A., Saadeh, R., Qazza, A. (2021). A novel numerical approach in solving fractional neutral pantograph equations via the ARA integral transform. *Symmetry, 14(1),* 50.
40. Saadeh, R. (2022). Applications of double ARA integral transform. *Computation, 10(12),* 216.
41. Saadeh, R. (2022). Application of the ARA method in solving integro-differential equations in two dimensions. *Computation, 11(1),* 4.
42. Eltayeb, H., Bachar, I., Abdalla, Y. T. (2020). A note on time-fractional Navier-Stokes equation and multi-Laplace transform decomposition method. *Advances in Difference Equations, 2020(1),* 1–19.
43. Eltayeb, H., Bachar, I. (2020). A note on singular two-dimensional fractional coupled Burgers' equation and triple Laplace Adomian decomposition method. *Boundary Value Problems, 2020,* 1–17.
44. Al-Omari, S. (2014). On the generalized Sumudu transform. *Journal of Applied Functional Analysis, 9(1–2),* 42–53.
45. Issa, A., Al-Omari, S., Naser, M., Al-Shawaqfeh, S. (2016). On Sumudu transforms and their applications to some initial value problems. *Advances in Differential Equations and Control Processes, 17(4),* 285–300.
46. Agarwal, P., Al-Omari, S. (2016). Some general properties of a fractional Sumudu transform in the class of Boehmians. *Kuwait Journal of Science, 43(2),* 206–220.
47. Al-Omari, S. K. (2018). On the quantum theory of the natural transform and some applications. *Journal of Difference Equations and Applications, 25(1),* 21–37.