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ARTICLE

A Stochastic Model to Assess the Epidemiological Impact of Vaccine Booster Doses on COVID-19 and Viral Hepatitis B Co-Dynamics with Real Data

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ABSTRACT

A patient co-infected with COVID-19 and viral hepatitis B can be at more risk of severe complications than the one infected with a single infection. This study develops a comprehensive stochastic model to assess the epidemiological impact of vaccine booster doses on the co-dynamics of viral hepatitis B and COVID-19. The model is fitted to real COVID-19 data from Pakistan. The proposed model incorporates logistic growth and saturated incidence functions. Rigorous analyses using the tools of stochastic calculus, are performed to study appropriate conditions for the existence of unique global solutions, stationary distribution in the sense of ergodicity and disease extinction. The stochastic threshold estimated from the data fitting is given by: $\mathscr{R}_0^S = 3.0651$. Numerical assessments are implemented to illustrate the impact of double-dose vaccination and saturated incidence functions on the dynamics of both diseases. The effects of stochastic white noise intensities are also highlighted.

KEYWORDS

Viral hepatitis B; COVID-19; stochastic model; extinction; ergodicity; real data

1 Introduction

According to the report issued on April 24, 2023, by Johns Hopkins University Coronavirus Center: The "Coronavirus Disease 2019" (COVID-19) caused by the "Severe Acute Respiratory Syndrome Coronavirus-2" (SARS-CoV-2) has infected 676,609,955 individuals which resulted in 6,881,955 deaths globally, and 13,338,833,198 COVID-19 vaccine doses have been administered [1]. Viral hepatitis B virus (HBV) poses a great threat to public health globally [2]. The "Centers for Disease Control and Prevention" has estimated that over 290 million individuals are infected with HBV with nearly 0.9 million deaths globally [3]. HBV is among the high-risk factors for chronic liver infections: cirrhosis, liver fibrosis and hepatocellular carcinoma. It is worth mentioning that HBV is responsible for more than 40% of hepatocellular carcinoma cases and more than 25% of liver cirrhosis cases [4]. The "Global Burden of Disease" has reported that HBV infection is one of the leading causes of adult mortality worldwide with more than 750,000 deaths annually associated with it [5]. The prevalence of HBV cases differs across different countries and regions [6]. Most HBV cases have been reported in the Middle East countries, Asia, and Africa [7]. In many of these countries, most of the reported cases are transmitted via mother-child [7]. Particularly, in Pakistan, the prevalence of viral hepatitis B is around 5 million [8]. Recent studies have indicated that between 2% and 11% of individuals infected with COVID-19 were already suffering from the liver infection and about 25% of liver problems are linked with COVID-19 [9]. SARS-CoV-2 infection can be a big risk factor for severe illness among under-diagnosed patients with viral hepatitis B [10].

According to the report issued by the "McGill COVID-19 vaccine tracker" system on December 02, 2022; 50 out of 242 developed vaccines were approved and are now available in over 200 countries [11]. Some of the approved against COVID-19 include: "BNT162b2 (Pfizer/BioNTech), AZD-1222/ChAdOx1-nCoV (Oxford/AstraZeneca), NVX-CoV2373 (Novavax), CoronaVac (Sinovac), mRNA-1273 (Moderna), rAd26-S+rAd5-S (Sputnik V), and Ad26.COV2.S (Janssen)" [12]. The effectiveness of many recommended vaccines ranges between 60% and above 90% [12]. Although antiviral agents can treat HBV, medicines alone cannot eradicate the virus from a host. Thus, vaccination has become an important resource to eliminate HBV infections [13]. That is why, different countries like China and the USA have made infant and adult vaccination programme a government priority [13,14]. This measure has significantly reduced the prevalence of HBV, as reported by Zheng et al. [14] and Zhao et al. [13]. However, developing nations in Asia and sub-Saharan Africa are still far away from achieving the target of mass vaccination, especially among the adult population.

On the other hand, mathematical modeling play a significant role not only in understanding the dynamics of the disease but also in suggesting the cost effective measures to eliminate the disease. Different epidemiological models have been proposed and analyzed to understand the dynamics of COVID-19 [15–21], Hepatitis B virus [22,23] as well as the co-infection both diseases [24–26]. Specifically, the authors [24] considered an HBV-COVID-19 model in resource limitation settings. Omame et al. [25] investigated the impact of incident co-infection in a model for HBV and COVID-19, and showed that this phenomenon could trigger backward bifurcation. Din et al. [26] investigated a co-dynamical model for HBV and COVID-19 with a bilinear incidence rate. Mathematical modelling can mainly be classified into deterministic and stochastic modelling. Though, deterministic models have their own merits and advantages but stochastic models have a potential to incorporate uncertainties and randomness and to explain the dynamics of epidemics more effectively. This is why, stochastic models have been applied successfully in understanding the patterns of diseases in recent years [27–32].

As a matter of fact, the nature of epidemic growth and spread is random due to the unpredictability in human-to-human contacts [33]. Therefore, handling the variability and randomness in the different states of the disease dynamical system is a big challenge [34] (see also, [35]). In many such cases, stochastic models could best describe the randomness of infectious contacts which takes place at different infectious stages [36]. It has been shown that stochastic models provide a higher level of realistic outputs when compared with their deterministic counterparts [37]. It was shown in [37] that an endemic equilibrium appearing in a deterministic model could disappear in its corresponding stochastic model because of stochastic fluctuations. Also, the authors in [38] explained that stochastic models give better interpretation to the question of disease extinction than their deterministic equivalents. Nasell [39] pointed out that stochastic models present better approaches in describing epidemics given large range of realistic parameter values when compared with the corresponding deterministic models.

Vaccination has become an important measure in managing the co-spread of both COVID-19 and viral hepatitis B; a feature that has not been considered by previous studies. This is the main motivation behind this work which aims to fill the gap in the existing studies by proposing a robust stochastic model for the two viral diseases co-dynamics incorporating vaccine booster doses, logistic growth and saturated incidence rates. The logistic growth deals with the population growth over time taking the carrying capacity into consideration when there is limited resources whereas, in exponential growth assumption, the population of interest grows over time and the carrying capacity is not considered which is not realistic and hence the logistic growth is most suitable [40]. Also, effective contacts between infected and uninfected humans may saturate at high infection peaks as a result of the crowding effect of infected individuals or as a result of control measures by the uninfected individuals [41]. The saturated incidence function best explains this, and has been adopted successfully in many epidemic models [42–44]. In biological models which deal with high co-endemicity of two diseases, this serves as the best incidence rate.

This study develops and analyzes the model using stochastic calculus tools. It seeks to suggest comprehensive intervention measures against the two viral diseases with the inclusion of vaccination programs for both infections. The study's findings contribute significantly to aid in the battle against COVID-19 and viral hepatitis.

This paper contributes in the following:

(i) A novel comprehensive stochastic model incorporating the logistic growth and saturated incidence rates is designed to assess the epidemiological effect of vaccine booster doses on the co-dynamics of viral hepatitis B and COVID-19.

(ii) The perturbed system is fitted to the real data from Pakistan and stochastic thresholds for both diseases are estimated.

(iii) Rigorous analysis using the tools of stochastic calculus are performed to establish appropriate conditions needed for existence of unique global solution, stationary distribution in the sense of ergodicity and disease extinction.

(iv) The optimal levels for COVID-19 and viral hepatitis B primary and booster vaccination rates that eliminate both infections are determined.

(v) Numerical assessments are carried out which highlight the impact of saturated incidence functions and stochastic white noise intensities on the dynamics of both diseases.

1.1 Preliminaries

We shall now recall some basic tools of stochastic calculus required in the sequel.

Theorem 1.1. [45] Let $\varepsilon(t)$ be a "one-dimensional Ito process" on $t \geq 0$ with "stochastic differential" given by:

$$d\varepsilon(t) = p(t, \varepsilon(t))dt + g(t, \varepsilon(t))dB_t, \quad \varepsilon(0) = \varepsilon_0, \tag{1}$$

where, $p \in \mathscr{L}^1(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$, $g \in \mathscr{L}^2(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$, and B_t is a "one-dimensional Brownian motion". If $\mathscr{H} \in \mathscr{C}^2(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R}_+)$. Then $\mathscr{H}(t, \varepsilon(t))$ is another Ito process whose "stochastic differential" is given by:

$$\begin{aligned} d\mathscr{H}(t, \varepsilon(t)) = {} & \left[\frac{\partial \mathscr{H}(t, \varepsilon(t))}{\partial t} + \frac{\partial \mathscr{H}(t, \varepsilon(t))}{\partial \varepsilon} p(t, \varepsilon(t)) + \frac{1}{2}\frac{\partial^2 \mathscr{H}(t, \varepsilon(t))}{\partial \varepsilon^2} g^2(t, \varepsilon(t))\right]dt \\ & + \frac{\partial \mathscr{H}(t, \varepsilon(t))}{\partial \varepsilon} g(t, \varepsilon(t))dB_t. \end{aligned} \tag{2}$$

The "infinitesimal generator" $\mathscr{L}$ associated with system (1) [45] is given by:

$$\mathscr{L} = \frac{\partial}{\partial t} + \frac{\partial}{\partial \varepsilon} p(t, \varepsilon(t)) + \frac{1}{2} \frac{\partial^2}{\partial \varepsilon^2} g^2(t, \varepsilon(t)). \tag{3}$$

Lemma 1.1. [45] If $\mathscr{L}$ applies to a function $\mathscr{H}(t, \varepsilon(t)) \in C^2(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R}_+)$, then

$$\mathscr{L}\mathscr{H}(t, \varepsilon(t)) = \frac{\partial \mathscr{H}(t, \varepsilon(t))}{\partial t} + \frac{\partial \mathscr{H}(t, \varepsilon(t))}{\partial \varepsilon} p(t, \varepsilon(t)) + \frac{1}{2} \frac{\partial^2 \mathscr{H}(t, \varepsilon(t))}{\partial \varepsilon^2} g^2(t, \varepsilon(t)). \tag{4}$$

Also, the one dimensional Ito's lemma [44] can be re-written as:

$$d\mathscr{H}(t, \varepsilon(t)) = \mathscr{L}\mathscr{H}(t, \varepsilon(t))dt + \frac{\partial \mathscr{H}(t, \varepsilon(t))}{\partial \varepsilon} g(t, \varepsilon(t))dB_t. \tag{5}$$

2 Model Formulation

The compartments for the formulation of the proposed model are defined as follows: $S(t)$: susceptibles, $V_a(t)$: those vaccinated against COVID-19, $V_b(t)$: those vaccinated against viral hepatitis B, $I_a(t)$: infected with COVID-19, $I_b(t)$: infected with viral hepatitis B, $I_{cb}(t)$: infected with the dual infections, $R_a(t)$: those who have recored from COVID-19, $R_b(t)$: those who have recovered from viral hepatitis B, $R_{ab}(t)$: those who have recovered from the dual infections, and at any time t, the total population is given by $N(t) = S(t) + V_a(t) + V_b(t) + I_c(t) + I_a(t) + I_{ab}(t) + R_a(t) + R_b(t) + R_{ab}(t)$. As the constant recruitment rate which has been used in many epidemic models is unrealistic, this study assumes the logistic growth in the proposed model, where r denotes per capita birth rate, K stands for the carrying capacity of the environment. The COVID-19 and viral hepatitis B transmission rates are defined as: β_a and β_b, respectively. Unlike several existing models dealing with COVID-19 and hepatitis B which assume the bilinear or standard incidence rates, the saturated incidence functions are used in this model, where the saturation effects associated with COVID-19 and viral hepatitis B are denoted by α_1 and α_2, respectively. This form of incidence has been considered most favorable when dealing with disease models involving large number of infectives. COVID-19 and viral hepatitis B primary vaccination rates are defined by ψ and ρ. The parameters: θ_c and θ_h denote the COVID-19 and viral hepatitis B booster dose vaccination rates. Immunity due to COVID-19 and viral hepatitis B vaccines are not lifelong, and wanes at the rates δ_a and δ_b, respectively. It is assumed that Individuals who recovered from COVID-19 are immune to re-infection. Similar assumption is made for individuals who have recovered from viral hepatitis B. However, infection with the other disease is possible. Due to the imperfect nature of the COVID-19 and viral hepatitis B vaccines, vulnerable or unvaccinated individuals have reduced rates of getting infected with either of the viral diseases with $\sigma(0 < \sigma < 1)$ and $\gamma(0 < \gamma < 1)$ denoting the COVID-19 and viral hepatitis B vaccine inefficacies. The model flow chart is given in Fig. 1, and parameters description is given in Table 1. The proposed model (both unperturbed and equivalent perturbed form) is described by the nonlinear systems defined by Eqs. (6) and (7), respectively.

$$\begin{aligned}
\frac{dS(t)}{dt} &= rN\left(1 - \frac{N}{K}\right) - \frac{\beta_a I_a}{1 + \alpha_1 I_a} S - \frac{\beta_b I_b}{1 + \alpha_2 I_b} S + \delta_a V_a + \delta_b V_b - (\mu + \psi + \rho) S \\
\frac{dV_a(t)}{dt} &= \psi S - \frac{\sigma \beta_a I_a}{1 + \alpha_1 I_a} V_a - \frac{\beta_b I_b}{1 + \alpha_2 I_b} V_a - (\delta_a + \theta_a + \mu) V_a \\
\frac{dV_b(t)}{dt} &= \rho S - \frac{\beta_a I_a}{1 + \alpha_1 I_a} V_b - \frac{\gamma \beta_b I_b}{1 + \alpha_2 I_b} V_b - (\delta_b + \theta_b + \mu) V_b
\end{aligned}$$

$$
\begin{aligned}
\frac{dI_a(t)}{dt} &= \frac{\beta_a I_a}{1+\alpha_1 I_a}(S + R_b + \sigma V_a + V_b) - (\xi_a + \eta_a + \mu)I_a - \varphi_1 \frac{\beta_b I_b}{1+\alpha_2 I_b} I_a \\
\frac{dI_b(t)}{dt} &= \frac{\beta_b I_b}{1+\alpha_2 I_b}(S + R_a + V_a + \gamma V_b) - (\xi_b + \eta_b + \mu)I_b - \varphi_2 \frac{\beta_a I_a}{1+\alpha_1 I_a} I_b \\
\frac{dI_{ab}(t)}{dt} &= \varphi_1 \frac{\beta_b I_b}{1+\alpha_2 I_b} I_a + \varphi_2 \frac{\beta_a I_a}{1+\alpha_1 I_a} I_b - (\xi_{ab} + \eta_{ab} + \mu) I_{ab} \\
\frac{dR_a(t)}{dt} &= \theta_a V_a + \xi_a I_a - \mu R_a - \frac{\beta_b I_b}{1+\alpha_2 I_b} R_a \\
\frac{dR_b(t)}{dt} &= \theta_b V_b + \xi_b I_b - \mu R_b - \frac{\beta_a I_a}{1+\alpha_1 I_a} R_b \\
\frac{dR_{ab}(t)}{dt} &= \xi_{ab} I_{ab} - \mu R_{ab}
\end{aligned}
\tag{6}
$$

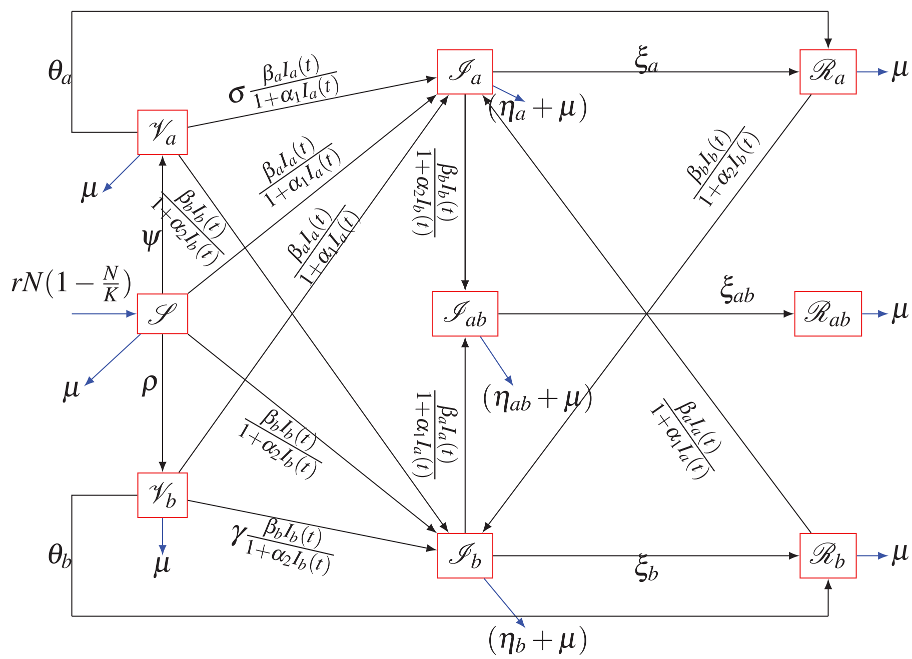


Figure 1: Model's schematic diagram

Table 1: Model parameters and variables

Parameter	Description	Value	Reference
δ_a	Waning immunity due to COVID-19 vaccination	0.010–0.015	[31]
β_a	COVID-19 transmission rate	$5.1024 \times 10^{-9}\ day^{-1}$	Fitted
β_b	Viral hepatitis B transmission rate	$1.0 \times 10^{-9}\ day^{-1}$	Estimated
ξ_a	COVID-19 recovery rate	$\left[\frac{1}{30}, \frac{1}{3}\right]\ day^{-1}$	[46]

(Continued)

Table 1 (continued)

Parameter	Description	Value	Reference
ξ_{ab}	Recovery rate for co-infected persons	$\frac{1}{21}$	[46]
η_{ab}	Co-infection death rate	0.05	[46]
η_b	Viral hepatitis B induced death rate	0.05	[47]
ξ_b	Viral hepatitis B recovery rate	$\frac{1}{21}\ day^{-1}$	[47]
γ	Viral hepatitis B vaccine inefficacy rate	(1–0.85)	[47]
η_a	COVID-19 induced death rate	0.158	Fitted
α_1	Saturation effect	2.6891×10^{-6}	Assumed
α_2	Saturation effect	7.3×10^{-5}	Assumed
ψ	COVID-19 primary vaccination rate	0.0015	Fitted
ρ	Viral hepatitis B primary vaccination rate	0.0010	Estimated
θ_a	COVID-19 booster vaccination rate	0.0010	Estimated
θ_b	Viral hepatitis B booster vaccination rate	0.0010	Assumed
δ_b	Waning immunity due to viral hepatitis B vaccination	0.010–0.015	Assumed
σ	COVID-19 vaccine inefficacy rate	(1–0.85)	[48,49]
r	*Per capita* birth rate	0.0199	[50]
K	Carrying capacity of the environment	$238, 181, 034$	[50]
μ	Natural death rate	$\frac{1}{69.37 \times 365}\ day^{-1}$	[50]
φ_1, φ_2	Modification parameter for vulnerability to a second infection	0.15	Assumed

The stochastic model analogue of the above deterministic model is given by:

$$dS(t) = \left(rN\left(1 - \frac{N}{K}\right) - \frac{\beta_a I_a}{1+\alpha_1 I_a}S - \frac{\beta_b I_b}{1+\alpha_2 I_b}S + \delta_a V_a + \delta_b V_b - (\mu + \psi + \rho)S\right)dt + \zeta_1 S dB_1(t)$$

$$dV_a(t) = \left(\psi S - \frac{\sigma\beta_a I_a}{1+\alpha_1 I_a}V_a - \frac{\beta_b I_b}{1+\alpha_2 I_b}V_a - (\delta_a + \theta_a + \mu)V_a\right)dt + \zeta_2 V_a dB_2(t)$$

$$dV_b(t) = \left(\rho S - \frac{\beta_a I_a}{1+\alpha_1 I_a}V_b - \frac{\gamma\beta_b I_b}{1+\alpha_2 I_b}V_b - (\delta_b + \theta_b + \mu)V_b\right)dt + \zeta_3 V_b dB_3(t)$$

$$dI_a(t) = \left(\frac{\beta_a I_a}{1+\alpha_1 I_a}(S + R_b + \sigma V_a + V_b) - (\xi_a + \eta_a + \mu)I_a - \varphi_1\frac{\beta_b I_b}{1+\alpha_2 I_b}I_a\right)dt + \zeta_4 I_a dB_4(t)$$

$$dI_b(t) = \left(\frac{\beta_b I_b}{1+\alpha_2 I_b}(S + R_a + V_a + \gamma V_b) - (\xi_b + \eta_b + \mu)I_b - \varphi_2\frac{\beta_a I_a}{1+\alpha_1 I_a}I_b\right)dt + \zeta_5 I_b dB_5(t)$$

$$dI_{ab}(t) = \left(\varphi_1\frac{\beta_b I_b}{1+\alpha_2 I_b}I_a + \varphi_2\frac{\beta_a I_a}{1+\alpha_1 I_a}I_b - (\xi_{ab} + \eta_{ab} + \mu)I_{ab}\right)dt + \zeta_6 I_{ab} dB_6(t)$$

$$dR_a(t) = \left(\theta_a V_a + \xi_a I_a - \mu R_a - \frac{\beta_b I_b}{1+\alpha_2 I_b} R_a\right) dt + \zeta_7 R_a dB_7(t)$$
$$dR_b(t) = \left(\theta_b V_b + \xi_b I_b - \mu R_b - \frac{\beta_a I_a}{1+\alpha_1 I_a} R_b\right) dt + \zeta_8 R_a dB_8(t)$$
$$dR_{ab}(t) = \left(\xi_{ab} I_{ab} - \mu R_{ab}\right) dt + \zeta_9 R_a dB_9(t) \tag{7}$$

where, $B_i(t)$, $i = 1, \dots, 9$ denote "independent Brownian motions" with $B_i(0) = 0$, and ζ_i, $i = 1, \dots, 9$ are the intensities of white noise which reflects all random components that could impact the disease dynamics in each compartment.

3 Analysis of the Unperturbed System (6)

In this section, we now present the analysis of the unperturbed system (6).

3.1 Unperturbed System's Reproduction Number

The disease free equilibrium (DFE) for the deterministic system is:

$$Q_0 = \left(S^0, V_a^0, V_b^0, I_a^0, I_b^0, I_{ab}^0, R_a^0, R_b^0, R_{ab}^0\right),$$

with,

$$S^0 = \frac{\mu K(r-\mu)(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)}{r[\psi(\theta_a+\mu)(\delta_b+\theta_b+\mu)+\rho(\theta_b+\mu)(\delta_a+\theta_a+\mu)+\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)]},$$
$$V_a^0 = \frac{\psi\mu K(r-\mu)(\delta_b+\theta_b+\mu)}{r[\psi(\theta_a+\mu)(\delta_b+\theta_b+\mu)+\rho(\theta_b+\mu)(\delta_a+\theta_a+\mu)+\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)]},$$
$$V_b^0 = \frac{\rho\mu K(r-\mu)(\delta_a+\theta_a+\mu)}{r[\psi(\theta_a+\mu)(\delta_b+\theta_b+\mu)+\rho(\theta_b+\mu)(\delta_a+\theta_a+\mu)+\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)]},$$
$$R_a^0 = \frac{\theta_a\psi K(r-\mu)(\delta_b+\theta_b+\mu)}{r[\psi(\theta_a+\mu)(\delta_b+\theta_b+\mu)+\rho(\theta_b+\mu)(\delta_a+\theta_a+\mu)+\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)]},$$
$$R_b^0 = \frac{\theta_b\rho K(r-\mu)(\delta_a+\theta_a+\mu)}{r[\psi(\theta_a+\mu)(\delta_b+\theta_b+\mu)+\rho(\theta_b+\mu)(\delta_a+\theta_a+\mu)+\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)]}. \tag{8}$$

Associated transfer matrices are:

$$F = \begin{pmatrix} \beta_a\left(S^0+\sigma V_a^0+V_b^0+R_b^0\right) & 0 & 0 \\ 0 & \beta_a\left(S^0+V_a^0+\gamma V_b^0+R_a^0\right) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and}$$
$$V = \begin{pmatrix} \xi_a+\eta_a+\mu & 0 & 0 \\ 0 & \xi_b+\eta_b+\mu & 0 \\ 0 & 0 & \xi_{ab}+\eta_{ab}+\mu \end{pmatrix}. \tag{9}$$

The basic reproduction employing the approach in [51] is: $\mathscr{R}_0 = \rho(FV^{-1}) = \max\{\mathscr{R}_{0a}, \mathscr{R}_{0b}\}$, where $\mathscr{R}_{0b}$ and $\mathscr{R}_{0a}$ denote the deterministic reproduction numbers for viral hepatitis B and COVID-19, and are given by:

$$\mathscr{R}_{0a} = \frac{\beta_a[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\sigma\psi(\delta_a+\theta_a+\mu)+\rho(\mu+\theta_b)(\delta_a+\theta_a+\mu)]S^0}{\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)(\xi_a+\eta_a+\mu)},$$

$$\mathscr{R}_{0b} = \frac{\beta_b[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\gamma\rho(\delta_b+\theta_b+\mu)+\psi(\mu+\theta_a)(\delta_b+\theta_b+\mu)]S^0}{\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)(\xi_b+\eta_b+\mu)}.$$

3.2 Local Asymptotic Stability the Unperturbed System's DFE (6)

Theorem 3.1. The unperturbed system's DFE, Q_0, "is locally asymptotically stable" whenever $\mathscr{R}_0 < 1$, and unstable for $\mathscr{R}_0 > 1$.

Proof:

Note that, the stability within neighbourhood of DFE for unperturbed system (6) is analyzed with the help of its Jacobian matrix evaluated at DFE given by:

$$J = \begin{pmatrix} -(r-\mu+\psi+\rho) & \Delta+\delta_a & \Delta+\delta_b & \Delta-\beta_a S^0 & \Delta-\beta_b S^0 & \Delta-(\beta_a+\beta_b)S^0 & \Delta & \Delta & \Delta-\beta_a S^0 \\ \psi & -\Upsilon_1 & 0 & -\sigma\beta_a V_a^0 & -\beta_b V_a^0 & -(\sigma\beta_a+\beta_b)V_a^0 & 0 & 0 & 0 \\ \rho & 0 & -\Upsilon_2 & 0 & -\beta_a V_b^0 & -\gamma\beta_b V_a^0 & -(\beta_a+\gamma\beta_b)V_b^0 & 0 & 0 \\ 0 & 0 & 0 & \beta_a A^0-K_1 & 0 & \beta_a A^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_b B^0-K_2 & \beta_b B^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -K_3 & 0 & 0 & 0 \\ 0 & \theta_a & 0 & \xi_a & 0 & 0 & -\mu & 0 & 0 \\ 0 & 0 & \theta_b & 0 & \xi_b & 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & -\xi_{ab} & 0 & 0 & -\mu \end{pmatrix}, \tag{10}$$

where,

$$\Delta = 2\mu - r,\ A^0 = \left(S^0+\sigma V_a^0+V_b^0+R_b^0\right),\ B^0 = \left(S^0+V_a^0+\gamma V_b^0+R_a^0\right),\ \Upsilon_1 = (\delta_a+\theta_a+\mu),$$
$$\Upsilon_2 = (\delta_b+\theta_b+\mu),\ K_1 = \xi_a+\eta_a+\mu,\ K_2 = \xi_b+\eta_b+\mu,\ K_3 = \xi_{ab}+\eta_{ab}+\mu(\delta_a+\theta_a+\mu).$$

The first seven eigenvalue are given by:

$\Phi_1 = -\mu$(with multiplicity of three), $\Phi_2 = -(r-\mu+\psi+\rho)$, $\Phi_3 = -(\delta_a+\theta_a+\mu)\Phi_4 = -(\delta_b+\theta_b+\mu)$, $\Phi_5 = -(\xi_{ab}+\eta_{ab}+\mu)$,
and the solutions of these equations:

$$(\Phi+(\xi_a+\eta_a+\mu)(1-\mathscr{R}_{0a}))=0, \quad (\Phi+(\xi_b+\eta_b+\mu)(1-\mathscr{R}_{0b}))=0. \tag{11}$$

As all parameters of the model are positive, it is concluded that the equations in (11) will both possess roots with negative real parts whenever $\mathscr{R}_0 = \max\{\mathscr{R}_{0a}, \mathscr{R}_{0b}\} < 1$. Thus, the unperturbed system's DFE, Q_0 is locally asymptotically stable if $\mathscr{R}_0 = \max\{\mathscr{R}_{0a}, \mathscr{R}_{0b}\} < 1$.

4 Analysis of the Perturbed System (6)

4.1 Existence of Solution

In this section, we study appropriate conditions for the existence of a unique global solution to the stochastic system with the help of well defined stochastic Lyapunov functions. We now present the following result:

Theorem 4.1. Given the initial conditions

$$\mathscr{G}_0 = (S(0), V_a(0), V_b(0), I_a(0), I_b(0), I_{ab}(0), R_a(0), R_b(0), R_{ab}(0)) \in \mathscr{M},$$

the perturbed system (7) has a unique global solution

$\mathscr{G}_t = \mathscr{G}(t) = (S(t), V_a(t), V_b(t), I_a(t), I_b(t), I_{ab}(t), R_a(t), R_b(t), R_{ab}(t))$

for $t \geq 0$, which is invariant in $\mathscr{M}$ with unit probability, where,

$$\mathscr{M} = \Big\{(S(t), V_a(t), V_b(t), I_a(t), I_b(t), I_{ab}(t), R_a(t), R_b(t), R_{ab}(t)) \in \mathbb{R}^9 :$$
$$S(t) > 0, V_a(t) > 0, V_b(t) > 0, I_a(t) > 0, I_b(t) > 0, I_{ab}(t) > 0, R_a(t), R_b(t) > 0, R_{ab}(t) > 0,$$
$$S(t) + V_a(t) + V_b(t) + I_a(t) + I_b(t) + I_{ab}(t) + R_a(t) + R_b(t) + R_{ab}(t) \leq \frac{\mu K(r-\mu)}{r}\Big\}.$$

Proof:

For given initial states $\mathscr{G}_0 = (S(0), V_a(0), V_b(0), I_a(0), I_b(0), I_{ab}(0), R_a(0), R_b(0), R_{ab}(0)) \in \mathscr{M}$, unique local solution $\mathscr{G}_t = \mathscr{G}(t) = (S(t), V_a(t), V_b(t), I_a(t), I_b(t), I_{ab}(t), R_a(t), R_b(t), R_{ab}(t))$ exists for $t \in [0, \tau_e)$ with τ_e denoting explosion time [45]. Suppose $\phi_0 > 0$ is such that

$S(0), V_a(0), V_b(0), I_a(0), I_b(0), I_{ab}(0), R_a(0), R_b(0), R_{ab}(0)$ stay within $\left[\frac{1}{\phi_0}, \phi_0\right]$. Then, for every $\phi \geq \phi_0$, we define the stopping time [29]:

$$\tau_\phi = \inf\Big\{t \in [0, \tau_e) : \min\{S(t), V_a(t), V_b(t), I_a(t), I_b(t), I_{ab}(t), R_a(t), R_b(t), R_{ab}(t)\} \leq \frac{1}{\phi}$$
$$or \max\{S(t), V_a(t), V_b(t), I_a(t), I_b(t), I_{ab}(t), R_a(t), R_b(t), R_{ab}(t)\} \geq \phi\Big\} \tag{12}$$

Note that, τ_ϕ increases as $\phi \to \infty$. Therefore, $\tau_\infty =^{\lim}_{\phi\to\infty} \tau_\phi$. It is to be shown that $\tau_\infty = \infty$ a.s., so that $\tau_e = \infty$ and

$(S(t), V_a(t), V_b(t), I_a(t), I_b(t), I_{ab}(t), R_a(t), R_b(t), R_{ab}(t)) \in \mathscr{M}$ a.s for all $t \geq 0$.

If $\tau_\infty < \infty$, then $\exists\ T > 0$ and $\vartheta \in (0,1)$ such that $\mathbb{P}\{\tau_{\infty \leq T}\} > \vartheta$. Thus, there is an integer $\phi_1 \geq \phi_0$ such that

$$\mathbb{P}\{\tau_{\phi \leq T}\} \geq \vartheta \forall \phi \geq \phi_1. \tag{13}$$

Define a $\mathscr{C}^2$ stochastic Lyapunov function $\mathscr{H}_1 : \mathbb{R}^9_+ \to \mathbb{R}_+$ as:

$$\begin{aligned}\mathscr{H}_1(S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) = {} & (S - 1 - \ln S) + (V_a - 1 - \ln V_a) + (V_b - 1 - \ln V_b) + (I_a - 1 - \ln I_a) \\ & + (I_b - 1 - \ln I_b) + (I_{ab} - 1 - \ln I_{ab}) + (R_a - 1 - \ln R_a) + (R_b - 1 - \ln R_b) \\ & + (R_{ab} - 1 - \ln R_{ab}),\end{aligned} \tag{14}$$

Applying Ito's lemma [45] to (14), we have

$$\begin{aligned}d\mathscr{H}_1 = {} & \left(1 - \frac{1}{S}\right) dS + \frac{1}{2}\frac{1}{S^2}(dS)^2 + \left(1 - \frac{1}{V_a}\right) dV_a + \frac{1}{2}\frac{1}{V_a^2}(dV_a)^2 + \left(1 - \frac{1}{V_b}\right) dV_b + \frac{1}{2}\frac{1}{V_b^2}(dV_b)^2 \\ & + \left(1 - \frac{1}{I_a}\right) dI_a + \frac{1}{2}\frac{1}{I_a^2}(dI_a)^2 + \left(1 - \frac{1}{I_b}\right) dI_b + \frac{1}{2}\frac{1}{I_b^2}(dI_b)^2 + \left(1 - \frac{1}{I_{ab}}\right) dI_{ab} + \frac{1}{2}\frac{1}{I_{ab}^2}(dI_{ab})^2\end{aligned}$$

$$
\begin{aligned}
&+\left(1-\frac{1}{R_a}\right)dR_a+\frac{1}{2}\frac{1}{R_a^2}(dR_a)^2+\left(1-\frac{1}{R_b}\right)dR_b+\frac{1}{2}\frac{1}{R_b^2}(dR_b)^2+\left(1-\frac{1}{R_b}\right)dR_b+\frac{1}{2}\frac{1}{R_b^2}(dR_b)^2\\
=&\left(1-\frac{1}{S}\right)\left(rN\left(1-\frac{N}{K}\right)-\frac{\beta_aI_a}{1+\alpha_1I_a}S-\frac{\beta_bI_b}{1+\alpha_2I_b}S+\delta_aV_a+\delta_bV_b-(\mu+\psi+\rho)S\right)dt\\
&+\left(1-\frac{1}{S}\right)\zeta_1SdB_1+\frac{1}{2}\zeta_1^2dt\\
&+\left(1-\frac{1}{V_a}\right)\left(\psi S-\frac{\sigma\beta_aI_a}{1+\alpha_1I_a}V_a-\frac{\beta_bI_b}{1+\alpha_2I_b}V_a-(\delta_a+\theta_a+\mu)V_a\right)dt+\left(1-\frac{1}{V_a}\right)\zeta_2V_adB_2+\frac{1}{2}\zeta_2^2dt\\
&+\left(1-\frac{1}{V_b}\right)\left(\rho S-\frac{\beta_aI_a}{1+\alpha_1I_a}V_b-\frac{\gamma\beta_bI_b}{1+\alpha_2I_b}V_b-(\delta_b+\theta_b+\mu)V_b\right)dt+\left(1-\frac{1}{V_b}\right)\zeta_3V_bdB_3+\frac{1}{2}\zeta_3^2dt\\
&+\left(1-\frac{1}{I_a}\right)\left(\frac{\beta_aI_a}{1+\alpha_1I_a}(S+R_b+\sigma V_a+V_b)-(\xi_a+\eta_a+\mu)I_a-\varphi_1\frac{\beta_bI_b}{1+\alpha_2I_b}I_a\right)dt\\
&+\left(1-\frac{1}{I_a}\right)\zeta_4I_adB_4+\frac{1}{2}\zeta_4^2dt\\
&+\left(1-\frac{1}{I_b}\right)\left(\frac{\beta_bI_b}{1+\alpha_2I_b}(S+R_a+V_a+\gamma V_b)-(\xi_b+\eta_b+\mu)I_b-\varphi_2\frac{\beta_aI_a}{1+\alpha_1I_a}I_b\right)dt\\
&+\left(1-\frac{1}{I_b}\right)\zeta_5I_bdB_5+\frac{1}{2}\zeta_5^2dt\\
&+\left(1-\frac{1}{I_{ab}}\right)\left(\varphi_1\frac{\beta_bI_b}{1+\alpha_2I_b}I_a+\varphi_2\frac{\beta_aI_a}{1+\alpha_1I_a}I_b-(\xi_{ab}+\eta_{ab}+\mu)I_{ab}\right)dt+\left(1-\frac{1}{I_b}\right)\zeta_6I_{ab}dB_6+\frac{1}{2}\zeta_6^2dt\\
&+\left(1-\frac{1}{R_a}\right)\left(\theta_aV_a+\xi_aI_a-\mu R_a-\frac{\beta_bI_b}{1+\alpha_2I_b}R_a\right)dt+\left(1-\frac{1}{R_a}\right)\zeta_7R_adB_7+\frac{1}{2}\zeta_7^2dt\\
&+\left(1-\frac{1}{R_b}\right)\left(\theta_bV_b+\xi_bI_b-\mu R_b-\frac{\beta_aI_a}{1+\alpha_1I_a}R_b\right)dt+\left(1-\frac{1}{R_b}\right)\zeta_8R_bdB_8+\frac{1}{2}\zeta_8^2dt\\
&+\left(1-\frac{1}{R_{ab}}\right)\left(\xi_{ab}I_{ab}-\mu R_{ab}\right)dt+\left(1-\frac{1}{R_{ab}}\right)\zeta_9R_{ab}dB_9+\frac{1}{2}\zeta_9^2dt.
\end{aligned}
$$

On simplification, we obtain that

$$
\begin{aligned}
d\mathscr{H}_1=&\Bigg[\left(1-\frac{1}{S}\right)\left(rN\left(1-\frac{N}{K}\right)-\frac{\beta_aI_a}{1+\alpha_1I_a}S-\frac{\beta_bI_b}{1+\alpha_2I_b}S+\delta_aV_a+\delta_bV_b-(\mu+\psi+\rho)S\right)\\
&+\frac{1}{2}\zeta_1^2+\left(1-\frac{1}{V_a}\right)\left(\psi S-\frac{\sigma\beta_aI_a}{1+\alpha_1I_a}V_a-\frac{\beta_bI_b}{1+\alpha_2I_b}V_a-(\delta_a+\theta_a+\mu)V_a\right)+\frac{1}{2}\zeta_2^2\\
&+\left(1-\frac{1}{V_b}\right)\left(\rho S-\frac{\beta_aI_a}{1+\alpha_1I_a}V_b-\frac{\gamma\beta_bI_b}{1+\alpha_2I_b}V_b-(\delta_b+\theta_b+\mu)V_b\right)+\frac{1}{2}\zeta_3^2\\
&+\left(1-\frac{1}{I_a}\right)\left(\frac{\beta_aI_a}{1+\alpha_1I_a}(S+R_b+\sigma V_a+V_b)-(\xi_a+\eta_a+\mu)I_a-\varphi_1\frac{\beta_bI_b}{1+\alpha_2I_b}I_a\right)\\
&+\frac{1}{2}\zeta_4^2+\left(1-\frac{1}{I_b}\right)\left(\frac{\beta_bI_b}{1+\alpha_2I_b}(S+R_a+V_a+\gamma V_b)-(\xi_b+\eta_b+\mu)I_b\right.
\end{aligned}
$$

$$-\varphi_2\frac{\beta_a I_a}{1+\alpha_1 I_a}I_b\bigg)+\frac{1}{2}\zeta_5^2+\left(1-\frac{1}{I_{ab}}\right)\bigg(\varphi_1\frac{\beta_b I_b}{1+\alpha_2 I_b}I_a+\varphi_2\frac{\beta_a I_a}{1+\alpha_1 I_a}I_b$$

$$-(\xi_{ab}+\eta_{ab}+\mu)I_{ab}\bigg)+\frac{1}{2}\zeta_6^2+\left(1-\frac{1}{R_a}\right)\left(\theta_a V_a+\xi_a I_a-\mu R_a-\frac{\beta_b I_b}{1+\alpha_2 I_b}R_a\right)+\frac{1}{2}\zeta_7^2$$

$$+\left(1-\frac{1}{R_b}\right)\left(\theta_b V_b+\xi_b I_b-\mu R_b-\frac{\beta_a I_a}{1+\alpha_1 I_a}R_b\right)+\frac{1}{2}\zeta_8^2$$

$$+\left(1-\frac{1}{R_{ab}}\right)\left(\xi_{ab}I_{ab}-\mu R_{ab}\right)+\frac{1}{2}\zeta_9^2\bigg]dt$$

$$+\zeta_1(S-a_1)dB_1+\zeta_2(V_a-a_2)dB_2+\zeta_3(V_b-a_3)dB_3+\zeta_4(I_a-1)dB_4+\zeta_5(I_b-1)dB_5+\zeta_6(I_{ab}-1)dB_6$$
$$+\zeta_7(R_a-a_4)dB_7+\zeta_8(R_b-a_5)dB_8+\zeta_9(R_{ab}-a_6)dB_9.$$

Hence,

$$d\mathscr{H}_1=\mathscr{L}\mathscr{H}_1dt+\zeta_1(S-a_1)dB_1+\zeta_2(V_a-a_2)dB_2+\zeta_3(V_b-a_3)dB_3+\zeta_4(I_a-1)dB_4+\zeta_5(I_b-1)dB_5$$
$$+\zeta_6(I_{ab}-1)dB_6+\zeta_7(R_a-a_4)dB_7+\zeta_8(R_b-a_5)dB_8+\zeta_9(R_{ab}-a_6)dB_9$$

where,

$$\mathscr{L}\mathscr{H}_1=\left(1-\frac{1}{S}\right)\left(rN\left(1-\frac{N}{K}\right)-\frac{\beta_a I_a}{1+\alpha_1 I_a}S-\frac{\beta_b I_b}{1+\alpha_2 I_b}S+\delta_a V_a+\delta_b V_b-(\mu+\psi+\rho)S\right)$$
$$+\frac{1}{2}\zeta_1^2+\left(1-\frac{1}{V_a}\right)\left(\psi S-\frac{\sigma\beta_a I_a}{1+\alpha_1 I_a}V_a-\frac{\beta_b I_b}{1+\alpha_2 I_b}V_a-(\delta_a+\theta_a+\mu)V_a\right)+\frac{1}{2}\zeta_2^2$$
$$+\left(1-\frac{1}{V_b}\right)\left(\rho S-\frac{\beta_a I_a}{1+\alpha_1 I_a}V_b-\frac{\gamma\beta_b I_b}{1+\alpha_2 I_b}V_b-(\delta_b+\theta_b+\mu)V_b\right)+\frac{1}{2}\zeta_3^2$$
$$+\left(1-\frac{1}{I_a}\right)\left(\frac{\beta_a I_a}{1+\alpha_1 I_a}(S+R_b+\sigma V_a+V_b)-(\xi_a+\eta_a+\mu)I_a-\varphi_1\frac{\beta_b I_b}{1+\alpha_2 I_b}I_a\right)$$
$$+\frac{1}{2}\zeta_4^2+\left(1-\frac{1}{I_b}\right)\left(\frac{\beta_b I_b}{1+\alpha_2 I_b}(S+R_a+V_a+\gamma V_b)-(\xi_b+\eta_b+\mu)I_b-\varphi_2\frac{\beta_a I_a}{1+\alpha_1 I_a}I_b\right)+\frac{1}{2}\zeta_5^2$$
$$+\left(1-\frac{1}{I_{ab}}\right)\bigg(\varphi_1\frac{\beta_b I_b}{1+\alpha_2 I_b}I_a+\varphi_2\frac{\beta_a I_a}{1+\alpha_1 I_a}I_b$$
$$-(\xi_{ab}+\eta_{ab}+\mu)I_{ab}\bigg)+\frac{1}{2}\zeta_6^2+\left(1-\frac{1}{R_a}\right)\left(\theta_a V_a+\xi_a I_a-\mu R_a-\frac{\beta_b I_b}{1+\alpha_2 I_b}R_a\right)+\frac{1}{2}\zeta_7^2$$
$$+\left(1-\frac{1}{R_b}\right)\left(\theta_b V_b+\xi_b I_b-\mu R_b-\frac{\beta_a I_a}{1+\alpha_1 I_a}R_b\right)+\frac{1}{2}\zeta_8^2$$
$$+\left(1-\frac{1}{R_{ab}}\right)\left(\xi_{ab}I_{ab}-\mu R_{ab}\right)+\frac{1}{2}\zeta_9^2.$$

$$\mathscr{L}\mathscr{H}_1\le\mu K(r-\mu)-\frac{\mu K(r-\mu)}{S}+\beta_a I_a+\beta_b I_b-\frac{\delta_a V_b}{S}+(\mu+\psi+\rho)-\frac{\psi S}{V_a}+\sigma\beta_a I_a+\beta_b I_b$$
$$+(\delta_a+\theta_a+\mu)-\frac{\varphi S}{V_b}+\beta_a I_a+\gamma\beta_b I_b+(\delta_b+\theta_b+\mu)-\frac{\theta_a V_a}{R_a}-\frac{\xi_a I_a}{R_a}+\mu+\beta_b I_b$$

$$- \frac{\theta_b V_b}{R_b} - \frac{\xi_b I_b}{R_b} + \mu + \beta_a I_a - \eta_a - \eta_b - \eta_{ab} - \frac{\beta_a I_a}{I_a} + (\xi_a + \eta_a + \mu) + \varphi_1 \beta_b I_b + \varphi_2 \beta_a I_a$$
$$- \frac{\beta_b I_b}{I_b} + (\xi_b + \eta_b + \mu) - \frac{\varphi_1 \beta_b I_b I_a}{I_{ab}} - \frac{\varphi_2 \beta_a I_a I_b}{I_{ab}} + (\xi_{ab} + \eta_{ab} + \mu) - \frac{\xi_{ab} I_{ab}}{R_{ab}} + \mu$$
$$+ \frac{\zeta_1^2}{2} + \frac{\zeta_2^2}{2} + \frac{\zeta_3^2}{2} + \frac{\zeta_4^2}{2} + \frac{\zeta_5^2}{2} + \frac{\zeta_6^2}{2} + \frac{\zeta_7^2}{2} + \frac{\zeta_8^2}{2} + \frac{\zeta_9^2}{2}.$$

$$\begin{aligned}\mathscr{L}\mathscr{H}_1 \leq{}& \mu K(r-\mu) + \beta_a I_a + \beta_b I_b + (\mu + \psi + \rho) + \sigma \beta_a I_a + \beta_b I_b + (\delta_a + \theta_a + \mu) + \beta_a I_a \\ &+ \gamma \beta_b I_b + (\delta_b + \theta_b + \mu) + \mu + \beta_b I_b + \mu + \beta_a I_a + (\xi_a + \eta_a + \mu) + \varphi_1 \beta_b I_b + \varphi_2 \beta_a I_a \\ &+ (\xi_b + \eta_b + \mu) + (\xi_{ab} + \eta_{ab} + \mu) + \mu + \frac{\zeta_1^2}{2} + \frac{\zeta_2^2}{2} + \frac{\zeta_3^2}{2} + \frac{\zeta_4^2}{2} + \frac{\zeta_5^2}{2} + \frac{\zeta_6^2}{2} + \frac{\zeta_7^2}{2} + \frac{\zeta_8^2}{2} + \frac{\zeta_9^2}{2}.\end{aligned} \tag{15}$$

$$\begin{aligned}\mathscr{L}\mathscr{H}_1 \leq{}& \mu K(r-\mu) + \frac{(3+\sigma)\beta_a K(r-\mu)}{r} + \frac{(3+\gamma)\beta_b K(r-\mu)}{r} + (\mu + \psi + \rho) + (\delta_a + \theta_a + \mu) \\ &+ (\delta_b + \theta_b + \mu) + \mu + \mu + (\xi_a + \eta_a + \mu) + (\xi_b + \eta_b + \mu) + (\xi_{ab} + \eta_{ab} + \mu) + \mu + \frac{\zeta_1^2}{2} + \frac{\zeta_2^2}{2} \\ &+ \frac{\zeta_3^2}{2} + \frac{\zeta_4^2}{2} + \frac{\zeta_5^2}{2} + \frac{\zeta_6^2}{2} + \frac{\zeta_7^2}{2} + \frac{\zeta_8^2}{2} + \frac{\zeta_9^2}{2} =: \Psi\end{aligned}$$

Thus, we have

$$\begin{aligned}d\mathscr{H}_1 ={}& \Psi dt + \Big(\zeta_1(S-1)dB_1 + \zeta_2(V_a-1)dB_2 + \zeta_3(V_b-1)dB_3 + \zeta_4(I_a-1)dB_4 + \zeta_5(I_b-1)dB_5 \\ &+ \zeta_6(I_{ab}-1)dB_6 + \zeta_7(R_a-1)dB_7 + \zeta_8(R_b-1)dB_8 + \zeta_9(R_{ab}-1)dB_9\Big).\end{aligned} \tag{16}$$

If both sides of Eq. (16) are integrated from 0 to $\tau_\lambda \wedge T$, then we have

$$\begin{aligned}&\int_0^{\tau_\phi \wedge T} d\mathscr{H}_1(S(\nu), V_a(\nu), V_b(\nu), I_a(\nu), I_b(\nu), I_{ab}(\nu), R_a(\nu), R_b(\nu), R_{ab}(\nu)) \\ &\leq \int_0^{\tau_\phi \wedge T} \Psi d\nu + \int_0^{\tau_\phi \wedge T} \Big(\zeta_1(S(\nu)-1)dB_1 + \zeta_2(V_a(\nu)-1)dB_2 + \zeta_3(V_b(\nu)-1)dB_3 + \zeta_4(I_a(\nu)-1)dB_4 \\ &+ \zeta_5(I_b(\nu)-1)dB_5 + \zeta_6(I_{ab}(\nu)-1)dB_6 + \zeta_7(R_a(\nu)-1)dB_7 + \zeta_8(R_a(\nu)-1)dB_8 + \zeta_9(R_b(\nu)-1)dB_9\Big).\end{aligned} \tag{17}$$

On taking expectation on both sides of the above inequality, we obtain that

$$\begin{aligned}&\mathbb{E}\mathscr{H}_1(S(\tau_\phi \wedge T), V_a(\tau_\phi \wedge T), V_a(\tau_\phi \wedge T), I_a(\tau_\phi \wedge T), I_b(\tau_\phi \wedge T), I_{ab}(\tau_\phi \wedge T), R_a(\tau_\phi \wedge T), R_b(\tau_\phi \wedge T), R_{ab}(\tau_\phi \wedge T)) \\ &\leq \mathscr{H}_1(S(0), V_a(0), V_b(0), I_a(0), I_b(0), I_{ab}(0), R_a(0), R_b(0), R_{ab}(0)) + \mathbb{E}\int_0^{\tau_\phi \wedge T} \Psi d\nu, \\ &\leq \mathscr{H}_1(S(0), V_a(0), V_b(0), I_a(0), I_b(0), I_{ab}(0), R_a(0), R_b(0), R_{ab}(0)) + \Psi T.\end{aligned}$$

Let $\Omega_\phi = \{\tau_\phi \le T\} \forall \phi \ge \phi_1$. Then, by (13), we have $\mathbb{P}(\Omega_\phi) \ge \vartheta$. Note that, for every $\omega \in \Omega$, at the least one of $S(\tau_\phi, \omega)$ or $V_a(\tau_\phi, \omega)$ or $V_b(\tau_\phi, \omega)$ or $I_a(\tau_\phi, \omega)$ or $I_b(\tau_\phi, \omega)$ or $I_{ab}(\tau_\phi, \omega)$ or $R_a(\tau_\phi, \omega)$ or $R_b(\tau_\phi, \omega)$ or $R_{ab}(\tau_\phi, \omega)$ is equivalent to ϕ or $\frac{1}{\phi}$. Thus,

$$\mathscr{H}_1\left(S(\tau_\phi, \omega), V_a(\tau_\phi, \omega), V_b(\tau_\phi, \omega), I_a(\tau_\phi, \omega), I_b(\tau_\phi, \omega), I_{ab}(\tau_\phi, \omega), R_a(\tau_\phi, \omega), R_b(\tau_\phi, \omega), R_{ab}(\tau_\phi, \omega),\right)$$

is not less than

$$\phi - 1 - \ln\phi \text{or} \frac{1}{\phi} - 1 - \ln\phi \left(\text{which equals:} \frac{1}{\phi} - 1 + \ln\phi\right).$$

Therefore,

$$\mathscr{H}_1\left(S(\tau_\phi, \omega), V_a(\tau_\phi, \omega), V_b(\tau_\phi, \omega), I_a(\tau_\phi, \omega), I_b(\tau_\phi, \omega), I_{ab}(\tau_\phi, \omega), R_a(\tau_\phi, \omega), R_b(\tau_\phi, \omega), R_{ab}(\tau_\phi, \omega),\right)$$
$$\ge \min\left\{(\phi - 1 - \ln\phi), \left(\frac{1}{\phi} - 1 + \ln\phi\right)\right\}.$$

Finally, we have

$$\begin{aligned}
&\mathscr{H}_1(S(0), V_a(0), V_b(0), I_a(0), I_b(0), I_{ab}(0), R_a(0), R_b(0), R_{ab}(0)) + \Psi T \\
&\ge \mathbb{E}(1_{\Omega\phi}\mathscr{H}_1(S(\tau_\phi \wedge T), V_a(\tau_\phi \wedge T), V_b(\tau_\phi \wedge T), I_a(\tau_\phi \wedge T), I_b(\tau_\phi \wedge T), I_{ab}(\tau_\phi \wedge T), R_a(\tau_\phi \wedge T), \\
&\quad R_b(\tau_\phi \wedge T), R_{ab}(\tau_\phi \wedge T))) \\
&= \mathbb{E}\left(1_{\Omega\phi}(\omega)\mathscr{H}_1(S(\tau_\phi, \omega), V_a(\tau_\phi, \omega), V_b(\tau_\phi, \omega), I_a(\tau_\phi, \omega), I_b(\tau_\phi, \omega), I_{ab}(\tau_\phi, \omega), R_a(\tau_\phi, \omega), R_b(\tau_\phi, \omega), R_{ab}(\tau_\phi, \omega))\right), \\
&\ge \mathbb{E}\left(\min\left\{1_{\Omega\phi}(\omega)(\phi - 1 - \ln\phi), \left(\frac{1}{\phi} - 1 + \ln\phi\right)\right\}\right) \\
&= \min\left\{(\phi - 1 - \ln\phi), \left(\frac{1}{\phi} - 1 + \ln\phi\right)\right\}\mathbb{E}\left(1_{\Omega\phi}(\omega)\right) \\
&\ge \vartheta \min\left\{(\phi - 1 - \ln\phi), \left(\frac{1}{\phi} - 1 + \ln\phi\right)\right\},
\end{aligned} \tag{18}$$

where, $1_{\Omega\phi}(\omega)$ is "indicator function" of $\Omega_\phi(\omega)$. If $\phi \to \infty$, then

$\infty > \mathscr{H}_1(S(0), V_a(0), V_b(0), I_a(0), I_b(0), I_{ab}(0), R_a(0), R_b(0), R_{ab}(0)) + \Psi T = \infty$, which is a contradiction and hence $\tau_\infty = \infty$.

4.2 Extinction of the Disease

The following notation and results are stated first:

$$\langle \mathscr{K}(t) \rangle = \frac{1}{t}\int_0^t \mathscr{K}(s)ds.$$

Theorem 4.2 ("Strong Law of Large Numbers"). [45] Let $G = \{G_t\}_{t\ge0}$ be continuous and real valued local martingale vanishing at $t = 0$ and $\langle G, G\rangle_t$ be its quadratic variation. Then

$$\lim_{t\to\infty}\langle G, G\rangle_t = \infty, \text{ a.s. } \Rightarrow \lim_{t\to\infty}\frac{G_t}{\langle G, G\rangle_t} = 0, \text{ a.s. and,} \quad \limsup_{t\to\infty}\frac{\langle G, G\rangle_t}{t} < 0, \text{ a.s. } \Rightarrow \lim_{t\to\infty}\frac{G_t}{t} = 0, \text{ a.s.} \tag{19}$$

Lemma 4.1. Let $(S(t), V_a(t), V_b(t), I_a(t), I_b(t), I_{ab}(t), R_a(t), R_b(t), R_{ab}(t))$ be a solution of perturbed system (7) subject to

$(S(0), V_a(0), V_b(0), I_a(0), I_b(0), I_{ab}(0), R_a(0), R_b(0), R_{ab}(0)) \in (R)^9_+$, then

$$\lim_{t\to\infty} \frac{S(t) + V_a(t) + V_b(t) + I_a(t) + I_b(t) + I_{ab}(t) + R_a(t) + R_b(t) + R_{ab}(t)}{t} = 0, \text{ a.s.} \tag{20}$$

Moreover, if $\mu > \dfrac{\left(\zeta_1^2 \vee \zeta_2^2 \vee \zeta_3^2 \vee \zeta_4^2 \vee \zeta_5^2 \vee \zeta_6^2 \vee \zeta_7^2 \vee \zeta_8^2 \vee \zeta_9^2\right)}{2}$, then

$$\lim_{t\to\infty} \frac{\int_0^t S(\nu)dB_1(\nu)}{t} = 0,\ \lim_{t\to\infty} \frac{\int_0^t V_a(\nu)dB_2(\nu)}{t} = 0,\ \lim_{t\to\infty} \frac{\int_0^t V_b(\nu)dB_3(\nu)}{t} = 0,\ \lim_{t\to\infty} \frac{\int_0^t I_a(\nu)dB_4(\nu)}{t} = 0,$$
$$\lim_{t\to\infty} \frac{\int_0^t I_b(\nu)dB_5(\nu)}{t} = 0,\ \lim_{t\to\infty} \frac{\int_0^t I_{ab}(\nu)dB_6(\nu)}{t} = 0,\ \lim_{t\to\infty} \frac{\int_0^t R_a(\nu)dB_7(\nu)}{t} = 0,\ \lim_{t\to\infty} \frac{\int_0^t R_b(\nu)dB_8(\nu)}{t} = 0,$$
$$\lim_{t\to\infty} \frac{\int_0^t R_{ab}(\nu)dB_9(\nu)}{t} = 0. \tag{21}$$

This Proof follows using the arguments similar to those given in Lemmas (2.1) and (2.2) in [32] and is omitted.

The threshold parameter $\mathscr{R}_0^S$ for the perturbed system (7) is given as:

$$\mathscr{R}_0^S = \max\{\mathscr{R}_{0a}^S, \mathscr{R}_{0b}^S\}, \tag{22}$$

where,

$$\mathscr{R}_{0a}^S = \frac{\beta_a[\mu(\delta_a + \theta_a + \mu)(\delta_b + \theta_b + \mu) + \mu\sigma\psi(\delta_a + \theta_a + \mu) + \rho(\mu + \theta_b)(\delta_a + \theta_a + \mu)]S^0}{\mu(\delta_a + \theta_a + \mu)(\delta_b + \theta_b + \mu)(\xi_a + \eta_a + \mu)}$$
$$- \frac{\zeta_4^2}{2(\xi_a + \eta_a + \mu)} = \mathscr{R}_{0a} - \frac{\zeta_4^2}{2(\xi_a + \eta_a + \mu)},$$

$$\mathscr{R}_{0b}^S = \frac{\beta_b[\mu(\delta_a + \theta_a + \mu)(\delta_b + \theta_b + \mu) + \mu\gamma\rho(\delta_b + \theta_b + \mu) + \psi(\mu + \theta_a)(\delta_b + \theta_b + \mu)]S^0}{\mu(\delta_a + \theta_a + \mu)(\delta_b + \theta_b + \mu)(\xi_b + \eta_b + \mu)}$$
$$- \frac{\zeta_5^2}{2(\xi_b + \eta_b + \mu)} = \mathscr{R}_{0b} - \frac{\zeta_5^2}{2(\xi_b + \eta_b + \mu)}.$$

The theorem below provides necessary requirement for disease extinction from a population.

Theorem 4.3. Given the initial states $(S(0), V_a(0), V_b(0), I_a(0), I_b(0), I_{ab}(0), R_a(0), R_b(0), R_{ab}(0)) \in \mathbb{R}^9_+$, the solution

$(S(t), V_a(t), V_b(t), I_a(t), I_b(t), I_{ab}(t), R_a(t), R_b(t), R_{ab}(t))$ of the stochastic system (7) has the properties defined below: If

(a) (i) $\zeta_4^2 > \dfrac{\beta_a^2[\mu(\delta_a + \theta_a + \mu)(\delta_b + \theta_b + \mu) + \mu\sigma\psi(\delta_a + \theta_a + \mu) + \rho(\mu + \theta_b)(\delta_a + \theta_a + \mu)]^2(S^0)^2}{2(\delta_a + \theta_a + \mu)\mu^2(\delta_b + \theta_b + \mu)^2}$,
then COVID-19 goes extinct "almost surely" (a.s).

(ii) $\zeta_5^2 > \dfrac{\beta_b^2[\mu(\delta_b + \theta_b + \mu)(\delta_a + \theta_a + \mu) + \mu\rho\gamma(\delta_a + \theta_a + \mu) + \psi(\mu + \theta_a)(\delta_b + \theta_b + \mu)]^2(S^0)^2}{2\mu^2(\delta_b + \theta_b + \mu)(\delta_a + \theta_a + \mu)^2}$,
viral hepatitis B goes extinct a.s.

(b) (i) $\mathscr{R}_{0a}^S < 1$, then COVID-19 is eliminated from the population with unit probability.

(ii) $\mathscr{R}_{0b}^{S} < 1$, then viral hepatitis B is eliminated from the population with unit probability.

Thus, if the requirements (a) and (b) above are satisfied, then

$$\lim_{t\to\infty}\frac{\langle \log I_a(t)\rangle}{t} < 0, \quad \text{and} \quad \lim_{t\to\infty}\frac{\langle \log I_b(t)\rangle}{t} < 0, \quad \text{and} \quad \lim_{t\to\infty}\frac{\langle \log I_{ab}(t)\rangle}{t} < 0, \quad \text{a.s.}$$

That is, both viral diseases will be eliminated with probability one.

More-over,

$$\begin{aligned}
&\lim_{t\to\infty}\langle S(t)\rangle = \frac{\mu K(r-\mu)(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)}{r[\psi(\theta_a+\mu)(\delta_b+\theta_b+\mu)+\rho(\theta_b+\mu)(\delta_a+\theta_a+\mu)+\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)]},\\
&\lim_{t\to\infty}\langle V_a(t)\rangle = \frac{\psi\mu K(r-\mu)(\delta_b+\theta_b+\mu)}{r[\psi(\theta_a+\mu)(\delta_b+\theta_b+\mu)+\rho(\theta_b+\mu)(\delta_a+\theta_a+\mu)+\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)]},\\
&\lim_{t\to\infty}\langle V_a(t)\rangle = \frac{\rho\mu K(r-\mu)(\delta_a+\theta_a+\mu)}{r[\psi(\theta_a+\mu)(\delta_b+\theta_b+\mu)+\rho(\theta_b+\mu)(\delta_a+\theta_a+\mu)+\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)]},\\
&\lim_{t\to\infty}\langle I_a(t)\rangle = 0, \quad \lim_{t\to\infty}\langle I_b(t)\rangle = 0, \quad \lim_{t\to\infty}\langle I_{ab}(t)\rangle = 0,\\
&\lim_{t\to\infty}\langle R_a(t)\rangle = \frac{\theta_a\psi K(r-\mu)(\delta_b+\theta_b+\mu)}{r[\psi(\theta_a+\mu)(\delta_b+\theta_b+\mu)+\rho(\theta_b+\mu)(\delta_a+\theta_a+\mu)+\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)]},\\
&\lim_{t\to\infty}\langle R_b(t)\rangle = \frac{\theta_b\rho K(r-\mu)(\delta_a+\theta_a+\mu)}{r[\psi(\theta_a+\mu)(\delta_b+\theta_b+\mu)+\rho(\theta_b+\mu)(\delta_a+\theta_a+\mu)+\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)]},\\
&\lim_{t\to\infty}\langle R_{ab}(t)\rangle = 0.
\end{aligned} \tag{23}$$

Proof:

(a) If Ito's lemma is applied to the 2nd equation of (7), then we have

$$\begin{aligned}
d\log I_a(t) &= \left[\frac{\beta_a I_a}{1+\alpha_1 I_a}(S+R_b+\sigma V_a+V_b)-(\xi_a+\eta_a+\mu)I_a-\varphi_1\frac{\beta_b I_b}{1+\alpha_2 I_b}I_a\right]\frac{1}{I_a}dt-\frac{\zeta_4^2}{2}dt+\zeta_4 dB_4(t)\\
&\le \left[\frac{\beta_a I_a}{1+\alpha_1 I_a}(S+R_b+\sigma V_a+V_b)-(\xi_a+\eta_a+\mu)I_a\right]\frac{1}{I_a}dt-\frac{\zeta_4^2}{2}dt+\zeta_4 dB_4(t)\\
&\le \left[\beta_a I_a(S+R_b+\sigma V_a+V_b)-(\xi_a+\eta_a+\mu)I_a\right]\frac{1}{I_a}dt-\frac{\zeta_4^2}{2}dt+\zeta_4 dB_4(t)\\
&\le \left[\beta_a I_a\left(S^0+R_b^0+\sigma V_a^0+V_b^0\right)-(\xi_a+\eta_a+\mu)I_a\right]\frac{1}{I_a}dt-\frac{\zeta_4^2}{2}dt+\zeta_4 dB_4(t)\\
&= \left[\beta_a(S^0+R_b^0+\sigma V_a^0+V_b^0)-(\xi_a+\eta_a+\mu)\right]dt-\frac{\zeta_4^2}{2}dt+\zeta_4 dB_4(t)\\
&= \left[\frac{\beta_a[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\sigma\psi(\delta_a+\theta_a+\mu)+\rho(\mu+\theta_b)(\delta_a+\theta_a+\mu)]S^0}{\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)}\right.\\
&\quad \left.-(\xi_a+\eta_a+\mu)\right]dt-\frac{\zeta_4^2}{2}dt+\zeta_4 dB_4(t)
\end{aligned} \tag{24}$$

Re-writing (24) in the form of integral gives that

$$\log I_a(t) \le \int_0^t \left(\frac{\beta_a[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\sigma\psi(\delta_a+\theta_a+\mu)+\rho(\mu+\theta_b)(\delta_a+\theta_a+\mu)]S^0}{\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)} - \frac{\zeta_4^2}{2}\right) d\nu (\xi_a+\eta_a+\mu)t + \int_0^t \zeta_4 dB_4(\nu) + \log I_a(0)$$

This can also be written in the form given by:

$$\begin{aligned}\log I_a(t) \le& -\frac{\zeta_4^2}{2}\int_0^t \left(1 - \frac{\beta_a[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\sigma\psi(\delta_a+\theta_a+\mu)+\rho(\mu+\theta_b)(\delta_a+\theta_a+\mu)]S^0}{\zeta_4^2\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)}\right)^2 dr \\ &+ \int_0^t \left(\frac{\beta_a^2[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\sigma\psi(\delta_a+\theta_a+\mu)+\rho(\mu+\theta_b)(\delta_a+\theta_a+\mu)]^2(S^0)^2}{2\zeta_4^2\mu^2(\delta_a+\theta_a+\mu)^2(\delta_b+\theta_b+\mu)^2}\right) dr \\ &- (\delta_a+\theta_a+\mu)t + \int_0^t \zeta_4 dB_4(\nu) + \log I_a(0) \\ \le& -\left((\xi_a+\eta_a+\mu)t - \frac{\beta_a^2[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\sigma\psi(\delta_a+\theta_a+\mu)+\rho(\mu+\theta_b)(\delta_a+\theta_a+\mu)]^2(S^0)^2}{2\zeta_4^2\mu^2(\delta_a+\theta_a+\mu)^2(\delta_b+\theta_b+\mu)^2}\right) t \\ &+ \int_0^t \zeta_4 dB_4(\nu) + \log I_a(0)\end{aligned}$$

Dividing by t, we get that

$$\begin{aligned}\frac{\log I_a(t)}{t} \le& -\left((\xi_a+\eta_a+\mu)t - \frac{\beta_a^2[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\sigma\psi(\delta_a+\theta_a+\mu)+\rho(\mu+\theta_b)(\delta_a+\theta_a+\mu)]^2(S^0)^2}{2\zeta_4^2\mu^2(\delta_a+\theta_a+\mu)^2(\delta_b+\theta_b+\mu)^2}\right) \\ &+ \frac{1}{t}\int_0^t \zeta_4 dB_4(\nu) + \frac{\log I_a(0)}{t},\end{aligned} \tag{25}$$

Using the "Strong Law of Large numbers", $\lim_{t\to\infty} \frac{1}{t}\int_0^t \zeta_4 dB_4(\nu) = 0. a.s.$,

$\zeta_4^2 > \frac{\beta_a^2[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\sigma\psi(\delta_a+\theta_a+\mu)+\rho(\mu+\theta_b)(\delta_a+\theta_a+\mu)]^2(S^0)^2}{2\mu^2(\delta_a+\theta_a+\mu)^2(\delta_b+\theta_b+\mu)^2}$, and on taking "limit superior on both sides of" (25), we have

$$\begin{aligned}&\lim_{t\to\infty}\sup \frac{\log I_a(t)}{t} \\ &\le -\left((\xi_a+\eta_a+\mu)t - \frac{\beta_a^2[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\sigma\psi(\delta_a+\theta_a+\mu)+\rho(\mu+\theta_b)(\delta_a+\theta_a+\mu)]^2(S^0)^2}{2\zeta_4^2\mu^2(\delta_a+\theta_a+\mu)^2(\delta_b+\theta_b+\mu)^2}\right) < 0.\end{aligned}$$

That is, $\lim_{t\to\infty} I_a(0) = 0$.

Similarly, it can be shown that

$$\begin{aligned}&\lim_{t\to\infty}\sup \frac{\log I_b(t)}{t} \\ &\le -\left((\xi_b+\eta_b+\mu) - \frac{\beta_b^2[\mu(\delta_b+\theta_b+\mu)(\delta_a+\theta_a+\mu)+\mu\rho\gamma(\delta_a+\theta_a+\mu)+\psi(\mu+\theta_a)(\delta_b+\theta_b+\mu)]^2(S^0)^2}{2\zeta_5^2\mu^2(\delta_a+\theta_a+\mu)^2(\delta_b+\theta_b+\mu)^2}\right) < 0,\end{aligned}$$

which implies that $\lim_{t\to\infty} I_b(0) = 0$.

(b) Also, if Eq. (24) is integrated over the interval [0, t] and divided by t, we obtain

$$\frac{\log I_a(t) - \log I_a(0)}{t} = \frac{\beta_a I_a}{1+\alpha_1 I_a}\frac{(S+R_b+\sigma V_a+V_b)}{I_a} - (\xi_a+\eta_a+\mu) - \varphi_1\frac{\beta_b I_b}{1+\alpha_2 I_b} - \frac{\zeta_4^2}{2} + \frac{\zeta_4}{t}\int_0^t dB_4(\nu),$$

$$\leq \frac{\beta_a[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\sigma\psi(\delta_a+\theta_a+\mu)+\rho(\mu+\theta_b)(\delta_a+\theta_a+\mu)]S^0}{\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)}$$

$$-(\xi_a+\eta_a+\mu) - \frac{\zeta_4^2}{2} + \frac{\zeta_4}{t}\int_0^t dB_4(\nu)$$

$$= (\xi_a+\eta_a+\mu)\left(\frac{\beta_a[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\sigma\psi(\delta_a+\theta_a+\mu)+\rho(\mu+\theta_b)(\delta_a+\theta_a+\mu)]S^0}{\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)(\xi_a+\eta_a+\mu)}\right.$$

$$\left.-\frac{\zeta_4^2}{2(\xi_a+\eta_a+\mu)} - 1\right) + \frac{\zeta_4}{t}\int_0^t dB_4(\nu)$$

$$= (\xi_a+\eta_a+\mu)\left(\mathscr{R}_{0a}^S - 1\right) + \frac{\zeta_4}{t}\int_0^t dB_4(\nu). \tag{26}$$

Moreover, $G(t) = \frac{\zeta_4}{t}\int_0^t dB_4(\nu)$ is “locally continuous and $G(0) = 0$”. Employing the Lemma (4.1) on taking $t \to \infty$, we get that

$$\limsup_{t\to\infty}\frac{G(t)}{t} = 0 \tag{27}$$

If $\mathscr{R}_{0a}^S < 1$, then Eq. (26) becomes

$$\limsup_{t\to\infty}\frac{\log I_a(t)}{t} \leq \left(\xi_a+\eta_a+\mu\right)\left(\mathscr{R}_{0a}^S - 1\right) < 0 \quad \text{a.s.} \tag{28}$$

Eq. (28) implies

$$\lim_{t\to\infty} I_a(t) = 0 \quad \text{a.s.} \tag{29}$$

Applying *Itô* Lemma to the 5th equation of system (7), we have

$$\frac{\log I_b(t) - \log I_b(0)}{t} \leq (\xi_b+\eta_b+\mu)\left(\frac{\beta_b[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\gamma\rho(\delta_b+\theta_b+\mu)+\psi(\mu+\theta_a)(\delta_b+\theta_b+\mu)]S^0}{\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)(\xi_b+\eta_b+\mu)}\right.$$

$$\left.-\frac{\zeta_5^2}{2(\xi_b+\eta_b+\mu)} - 1\right) + \frac{\zeta_4}{t}\int_0^t dB_4(\nu)$$

$$= (\xi_b+\eta_b+\mu)\left(\mathscr{R}_{0b}^S - 1\right) + \frac{\zeta_4}{t}\int_0^t dB_4(\nu). \tag{30}$$

Note that $G(t) = \frac{\zeta_4}{t}\int_0^t dB_4(\nu)$, is “locally continuous martingale” and $G(0) = 0$. By applying Lemma (4.1) with $t \to \infty$, we obtain

$$\limsup_{t\to\infty}\frac{G(t)}{t} = 0. \tag{31}$$

If $\mathscr{R}_{0b}^{S} < 1$, then Eq. (30) results in

$$\lim_{t\to\infty} \sup \frac{\log I_b(t)}{t} \le \left(\xi_b + \eta_b + \mu\right)\left(\mathscr{R}_{0b}^{S} - 1\right) < 0 \quad \text{a.s.} \tag{32}$$

Eq. (32) gives

$$\lim_{t\to\infty} I_b(t) = 0 \quad \text{a.s.} \tag{33}$$

Using the last equation of the stochastic system, we have

$$\lim_{t\to\infty} \langle R_{ab}(t) \rangle = 0, \quad \text{a.s.}$$

From the first equations of system (7), we get that

$$\begin{aligned} S(t) - S(0) = &\int_0^t \left(rN\left(1 - \frac{N}{K}\right) - \frac{\beta_a I_a}{1+\alpha_1 I_a} S - \frac{\beta_b I_b}{1+\alpha_2 I_b} S + \delta_a V_a + \delta_b V_b \right) d\nu \\ &- (\mu + \psi + \rho) \int_0^t S(\nu) d\nu + \zeta_1 \int_0^t S(\nu) B_1(\nu), \end{aligned} \tag{34}$$

Considering the bound for N, and recalling (29) and (33), gives

$$\begin{aligned} S(t) - S(0) = &\frac{\mu K(r-\mu)}{r} t + \delta_a \int_0^t V_a(\nu) d\nu + \delta_b \int_0^t V_b(\nu) d\nu - (\mu + \psi + \rho) \int_0^t S(\nu) d\nu \\ &+ \zeta_1 \int_0^t S(\nu) B_1(\nu), \end{aligned} \tag{35}$$

Dividing by t and taking $\lim\limits_{t\to\infty}$, we obtain that

$$\begin{aligned} \lim_{t\to\infty} \frac{S(t) - S(0)}{t} &= \frac{\mu K(r-\mu)}{r} + \delta_a \lim_{t\to\infty} \frac{1}{t} \int_0^t V_a(\nu) d\nu + \delta_b \lim_{t\to\infty} \frac{1}{t} \int_0^t V_b(\nu) d\nu \\ &\quad - (\mu + \psi + \rho) \lim_{t\to\infty} \frac{1}{t} \int_0^t S(\nu) d\nu + \zeta_1 \lim_{t\to\infty} \frac{1}{t} \int_0^t S(\nu) B_1(\nu), \\ &= \frac{\mu K(r-\mu)}{r} + \delta_a \lim_{t\to\infty} \langle V_a(t) \rangle + \delta_b \lim_{t\to\infty} \langle V_b(t) \rangle - (\mu + \psi + \rho) \lim_{t\to\infty} \langle S(t) \rangle \\ &\quad + \zeta_1 \lim_{t\to\infty} \frac{1}{t} \int_0^t S(\nu) B_1(\nu), \end{aligned} \tag{36}$$

which can be re-written as:

$$(\mu + \psi + \rho) \lim_{t\to\infty} \langle S(t) \rangle - \delta_a \lim_{t\to\infty} \langle V_a(t) \rangle - \delta_b \lim_{t\to\infty} \langle V_b(t) \rangle = \frac{\mu K(r-\mu)}{r} - \lim_{t\to\infty} \frac{S(t) - S(0)}{t} + \zeta_1 \lim_{t\to\infty} \frac{1}{t} \int_0^t S(\nu) B_1(\nu) \tag{37}$$

On taking $t \to \infty$, the above becomes

$$(\mu + \psi + \rho) \lim_{t\to\infty} \langle S(t) \rangle - \delta_a \lim_{t\to\infty} \langle V_a(t) \rangle - \delta_b \lim_{t\to\infty} \langle V_b(t) \rangle = \frac{K(r-\mu)\mu}{r} \tag{38}$$

Following the similar arguments to those given above, the following expressions can also be obtained from the stochastic system (7):

$$\begin{aligned}
&\psi \lim_{t\to\infty}\left\langle S(t)\right\rangle = (\delta_a+\theta_a+\mu)\lim_{t\to\infty}\left\langle V_a(t)\right\rangle,\\
&\rho \lim_{t\to\infty}\left\langle S(t)\right\rangle = (\delta_b+\theta_b+\mu)\lim_{t\to\infty}\left\langle V_b(t)\right\rangle,\\
&\theta_a \lim_{t\to\infty}\left\langle V_a(t) = \mu \lim_{t\to\infty}\left\langle R_a(t)\right\rangle,\right.\\
&\theta_b \lim_{t\to\infty}\left\langle V_b(t) = \lim_{t\to\infty}\left\langle R_b(t)\right\rangle\right.
\end{aligned} \tag{39}$$

Solving the Eqs. (38) and (39) simultaneously, the following bounds are obtained

$$\begin{aligned}
\lim_{t\to\infty}\langle S(t)\rangle &= \frac{\mu K(r-\mu)(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)}{r[\psi(\theta_a+\mu)(\delta_b+\theta_b+\mu)+\rho(\theta_b+\mu)(\delta_a+\theta_a+\mu)+\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)]},\\
\lim_{t\to\infty}\langle V_a(t)\rangle &= \frac{\psi\mu K(r-\mu)(\delta_b+\theta_b+\mu)}{r[\psi(\theta_a+\mu)(\delta_b+\theta_b+\mu)+\rho(\theta_b+\mu)(\delta_a+\theta_a+\mu)+\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)]},\\
\lim_{t\to\infty}\langle V_a(t)\rangle &= \frac{\rho\mu K(r-\mu)(\delta_a+\theta_a+\mu)}{r[\psi(\theta_a+\mu)(\delta_b+\theta_b+\mu)+\rho(\theta_b+\mu)(\delta_a+\theta_a+\mu)+\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)]},\\
\lim_{t\to\infty}\langle R_a(t)\rangle &= \frac{\theta_a\psi K(r-\mu)(\delta_b+\theta_b+\mu)}{r[\psi(\theta_a+\mu)(\delta_b+\theta_b+\mu)+\rho(\theta_b+\mu)(\delta_a+\theta_a+\mu)+\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)]},\\
\lim_{t\to\infty}\langle R_b(t)\rangle &= \frac{\theta_b\rho K(r-\mu)(\delta_a+\theta_a+\mu)}{r[\psi(\theta_a+\mu)(\delta_b+\theta_b+\mu)+\rho(\theta_b+\mu)(\delta_a+\theta_a+\mu)+\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)]}.
\end{aligned} \tag{40}$$

4.3 Existence of Ergodic Stationary Distribution

Definition 4.1. [52] "The transition probability function $P(s,\varepsilon,t,A)$ is said to be time-homogeneous (and the corresponding Markov process is called time-homogeneous) if the function $P(s,\varepsilon,t+s,A)$ is independent of s, where $0\leq s\leq t$, $t\in\mathbb{R}^q$, $A\in\mathscr{B}$ and $\mathscr{B}$ denotes the σ-algebra of Borel sets in $\mathbb{R}^q$".

Let $Y(t)$ represent a "time-homogeneous regular Markov" process in $\mathbb{R}^9_+$ defined by the SDE below:

$$dY(t) = p(Y)dt + \sum_{r=1}^{k} h_r(Y)dB_r(t). \tag{41}$$

The corresponding diffusion matrix is defined as:

$$A(\varepsilon) = (a_{ij}(\varepsilon)), \quad a_{ij(\varepsilon)} = \sum_{r=1}^{k} h_r^i(\varepsilon)h_r^j(\varepsilon).$$

Lemma 4.2. [52] "Suppose there exists a bounded open domain $\mathscr{M}\subset\mathbb{R}^9_+$ having regular boundary Γ, with the following properties":

A1: "there exists a positive number G such that $\sum_{i,j=1}^{q} a_{ij}(\varepsilon)\rho_i\rho_j \geq G|\rho|^2, \varepsilon\in\mathscr{M}, \rho\in\mathbb{R}^q$".

A2: "there exists a non-negative $\mathscr{C}^2$-function $\mathscr{H}$ such that $\mathscr{L}\mathscr{H}$ is negative for any $\mathbb{R}^q\backslash\mathscr{M}$".

"Then the Markov process $Y(t)$ have a unique ergodic stationary distribution $\mu()$". That is

$$\mathbb{P}_\varepsilon\left\{\lim_{T\longrightarrow\infty}\frac{1}{T}\int_0^T p(Y)dt=\int_{\mathbb{R}^9_+}p(\varepsilon)\mu(d\varepsilon)\right\}=1,\quad \forall\varepsilon\in\mathbb{R}^9_+,$$

with $p(\varepsilon)$ denotes an integrable function relative to measure μ.

Theorem 4.4. Define the threshold

$$\bar{\mathscr{R}}_0^S=\frac{\mu\Delta_1\Delta_2\beta_a\beta_b}{\left\{\left(\mu+\psi+\rho+\frac{\zeta_1^2}{2}\right)+\left(\delta_a+\theta_a+\mu+\frac{\zeta_2^2}{2}\right)+\left(\delta_b+\theta_b+\mu+\frac{\zeta_3^2}{2}\right)\right\}\left(\xi_a+\eta_a+\mu+\frac{\zeta_4^2}{2}\right)\left(\xi_b+\eta_b+\mu+\frac{\zeta_5^2}{2}\right)},\tag{42}$$

where,

$$\Delta_1=\frac{\beta_a[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\sigma\psi(\delta_a+\theta_a+\mu)+\rho(\mu+\theta_b)(\delta_a+\theta_a+\mu)]S^0}{\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)},$$

$$\Delta_2=\frac{\beta_b[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\gamma\rho(\delta_b+\theta_b+\mu)+\psi(\mu+\theta_a)(\delta_b+\theta_b+\mu)]S^0}{\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)},$$

$$S^0=\frac{\mu K(r-\mu)(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)}{r[\psi(\theta_a+\mu)(\delta_b+\theta_b+\mu)+\rho(\theta_b+\mu)(\delta_a+\theta_a+\mu)+\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)]}.$$

Then the perturbed system (7) has a "unique ergodic stationary distribution" $\mu(\cdot)$ whenever $\bar{\mathscr{R}}_0^S>1$.

Proof:

For $(S(0),V_a(0),V_b(0),I_a(0),I_b(0),I_{ab}(0),R_a(0),R_b(0),R_{ab}(0))\in\mathbb{R}^9_+$, we have unique solution $(S(t),V_a(t),V_b(t),I_a(t),I_b(t),I_{ab}(t),R_a(t),R_b(t),R_{ab}(t))\in\mathbb{R}^9_+$. Moreover, the diffusion matrix of (7) is given as:

$$A=\begin{bmatrix}\zeta_1^2S^2&0&0&0&0&0&0&0&0\\0&\zeta_2^2V_a^2&0&0&0&0&0&0&0\\0&0&\zeta_3^2V_b^2&0&0&0&0&0&0\\0&0&0&\zeta_4^2I_a^2&0&0&0&0&0\\0&0&0&0&\zeta_5^2I_b^2&0&0&0&0\\0&0&0&0&0&\zeta_6^2I_{ab}^2&0&0&0\\0&0&0&0&0&0&\zeta_7^2R_a^2&0&0\\0&0&0&0&0&0&0&\zeta_8^2R_b^2&0\\0&0&0&0&0&0&0&0&\zeta_9^2R_{ab}^2\end{bmatrix}_{(S(t),V_a(t),V_b(t),I_a(t),I_b(t),I_{ab}(t),R_a(t),R_b(t),R_{ab}(t))\in\mathbb{R}^9}$$

Suppose that $G=\min\{\zeta_1^2S^2,\zeta_2^2V_a^2,\zeta_3^2V_b^2,\zeta_4^2I_a^2,\zeta_5^3I_b^2,\zeta_6^2I_{ab}^2,\zeta_7^2R_a^2,\zeta_8^2R_b^2,\zeta_9^2R_{ab}^2\}$.
where, $(S(t),V_a(t),V_b(t),I_a(t),I_b(t),I_{ab}(t),R_a(t),R_b(t),R_{ab}(t))\in\overline{D}\in\mathbb{R}^9_+$

Then we have

$$\sum_{i,j=1}^{9}a_{ij}(S(t),V_a(t),V_b(t),I_a(t),I_b(t),I_{ab}(t),R_a(t),R_b(t),R_{ab}(t))\bar{\rho}_i\bar{\rho}_j$$

$$=\zeta_1^2S^2\bar{\rho}_1^2+\zeta_2^2V_a^2\bar{\rho}_2^2+\zeta_3^2V_b^2\bar{\rho}_3^2+\zeta_4^2I_a^2\bar{\rho}_4^2+\zeta_5^2I_b^2\bar{\rho}_5^2+\zeta_6^2I_{ab}^2\bar{\rho}_6^2+\zeta_7^2\bar{\rho}_7^2R_a^2+\zeta_8^2\bar{\rho}_8^2R_b^2+\zeta_9^2R_{ab}^2\bar{\rho}_9^2$$

$\geq G|\bar{\rho}|^2, where, (S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in \overline{D},\ and\ \bar{\rho}$
$= (\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_4, \bar{\rho}_5, \bar{\rho}_6, \bar{\rho}_7, \bar{\rho}_8, \bar{\rho}_9) \in \mathbb{R}^9_+$.

Therefore, the requirement *A1* of Lemma (4.2) is fulfilled.

Now, consider a C^2- function V: $\mathbb{R}^9_+ \to \mathbb{R}_+$:

Let

$$\mathscr{H}_3 = -lnS - \ lnV_a - lnV_b - \omega_1 lnI_a - \omega_2 lnI_b + (S + V_a + V_b + I_a + I_b + I_{ab} + R_a + R_b + R_{ab}),$$

where $\omega_1 > 0$ and $\omega_2 > 0$ are to be determined. Applying *Ito's* Lemma, we have

$$\mathscr{L}(S + V_a + V_b + I_a + I_b + I_{ab} + R_a + R_b + R_{ab})$$
$$= rN\left(1 - \frac{N}{K}\right) - \mu(S + V_a + V_b + I_a + I_b + I_{ab} + R_a + R_b + R_{ab}) - \eta_a I_a - \eta_b I_b - \eta_{ab} I_{ab},$$

$$\mathscr{L}(-lnS) = -\frac{rN}{S}\left(1 - \frac{N}{K}\right) + \frac{\beta_a I_a}{1 + \alpha_1 I_a} + \frac{\beta_b I_b}{1 + \alpha_2 I_b} - \frac{\delta_a V_a}{S} - \frac{\delta_b V_b}{S} + (\mu + \psi + \rho) + \frac{\zeta_1^2}{2}$$

$$\mathscr{L}(-lnV_a) = -\frac{\psi S}{V_a} + \frac{\sigma \beta_a I_a}{1 + \alpha_1 I_a} + \frac{\beta_b I_b}{1 + \alpha_2 I_b} + (\delta_a + \theta_a + \mu) + \frac{\zeta_2^2}{2}$$

$$\mathscr{L}(-lnV_b) = -\frac{\rho S}{V_b} + \frac{\beta_a I_a}{1 + \alpha_1 I_a} + \frac{\gamma \beta_b I_b}{1 + \alpha_2 I_b} + (\delta_b + \theta_b + \mu) + \frac{\zeta_3^2}{2}$$

$$\mathscr{L}(-lnI_a) = -\frac{\beta_a I_a}{1 + \alpha_1 I_a}\frac{(S + R_b + \sigma V_a + V_b)}{I_a} + (\xi_a + \eta_a + \mu) + \varphi_1 \frac{\beta_b I_b}{1 + \alpha_2 I_b} + \frac{\zeta_4^2}{2}$$

$$\mathscr{L}(-lnI_b) = -\frac{\beta_b I_b}{1 + \alpha_2 I_b}\frac{(S + R_a + V_a + \gamma V_b)}{I_b} + (\xi_b + \eta_b + \mu) + \varphi_2 \frac{\beta_a I_a}{1 + \alpha_1 I_a} + \frac{\zeta_5^2}{2}$$

$$\mathscr{L}(-lnI_{ab}) = -\varphi_1 \frac{\beta_b I_b}{1 + \alpha_2 I_b}\frac{I_a}{I_{ab}} - \varphi_2 \frac{\beta_a I_a}{1 + \alpha_1 I_a}\frac{I_b}{I_{ab}} + (\xi_{ab} + \eta_{ab} + \mu) + \frac{\zeta_6^2}{2}$$

$$\mathscr{L}(-lnR_a) = -\frac{\theta_a V_a}{R_a} - \frac{\xi_a I_a}{R_a} + \mu + \frac{\beta_b I_b}{1 + \alpha_2 I_b} + \frac{\zeta_7^2}{2},$$

$$\mathscr{L}(-lnR_b) = -\frac{\theta_b V_b}{R_b} - \frac{\xi_b I_b}{R_b} + \mu + \frac{\beta_a I_a}{1 + \alpha_1 I_a} + \frac{\zeta_8^2}{2}$$

$$\mathscr{L}(-lnR_{ab}) = -\frac{\xi_{ab} I_{ab}}{R_{ab}} + \mu + \frac{\zeta_9^2}{2}. \tag{43}$$

Thus, we get that

$$\mathscr{L}\mathscr{H}_3 = -\frac{rN}{S}\left(1 - \frac{N}{K}\right) + \frac{\beta_a I_a}{1 + \alpha_1 I_a} + \frac{\beta_b I_b}{1 + \alpha_2 I_b} - \frac{\delta_a V_a}{S} - \frac{\delta_b V_b}{S} + \left(\mu + \psi + \rho + \frac{\zeta_1^2}{2}\right)$$
$$- \frac{\psi S}{V_a} + \frac{\sigma \beta_a I_a}{1 + \alpha_1 I_a} + \frac{\beta_b I_b}{1 + \alpha_2 I_b} + \left(\delta_a + \theta_a + \mu + \frac{\zeta_2^2}{2}\right)$$
$$- \frac{\rho S}{V_b} + \frac{\beta_a I_a}{1 + \alpha_1 I_a} + \frac{\gamma \beta_b I_b}{1 + \alpha_2 I_b} + \left(\delta_b + \theta_b + \mu + \frac{\zeta_3^2}{2}\right)$$

$$
\begin{aligned}
&+\omega_1\left[-\frac{\beta_a I_a}{1+\alpha_1 I_a}\frac{(S+R_b+\sigma V_a+V_b)}{I_a}+\left(\xi_a+\eta_a+\mu+\frac{\zeta_4^2}{2}\right)+\varphi_1\frac{\beta_b I_b}{1+\alpha_2 I_b}\right]\\
&+\omega_2\left[-\frac{\beta_b I_b}{1+\alpha_2 I_b}\frac{(S+R_a+V_a+\gamma V_b)}{I_b}+\left(\xi_b+\eta_b+\mu+\frac{\zeta_5^2}{2}\right)+\varphi_2\frac{\beta_a I_a}{1+\alpha_1 I_a}\right]\\
&+\left[rN\left(1-\frac{N}{K}\right)-\mu(S+V_a+V_b+I_a+I_b+I_{ab}+R_a+R_b+R_{ab})-\eta_a I_a-\eta_b I_b-\eta_{ab}I_{ab}\right]\\
&\leq -\frac{K(r-\mu)\mu}{rS}-\frac{\omega_1\beta_a(S+R_b+\sigma V_a+V_b)}{1+\alpha_1 I_a}-\frac{\omega_2\beta_b(S+R_a+V_a+\gamma V_b)}{1+\alpha_2 I_b}\\
&+\omega_1\left(\xi_a+\eta_a+\mu+\frac{\zeta_4^2}{2}\right)+\omega_2\left(\xi_b+\eta_b+\mu+\frac{\zeta_5^2}{2}\right)\\
&+\left(\mu+\psi+\rho+\frac{\zeta_1^2}{2}\right)+\left(\delta_a+\theta_a+\mu+\frac{\zeta_2^2}{2}\right)+\left(\delta_b+\theta_b+\mu+\frac{\zeta_3^2}{2}\right)\\
&+\frac{\beta_a(2+\sigma)I_a}{1+\alpha_1 I_a}+\frac{\beta_b(2+\gamma)I_b}{1+\alpha_2 I_b}+\omega_1\varphi_1\frac{\beta_b I_b}{1+\alpha_2 I_b}+\omega_2\varphi_2\frac{\beta_a I_a}{1+\alpha_1 I_a}\\
&+\frac{K(r-\mu)\mu}{r}-\frac{K(r-\mu)\mu}{r}-\frac{\delta_a V_a}{S}-\frac{\delta_b V_b}{S}-\frac{\psi S}{V_a}-\frac{\rho S}{V_b}\\
&\leq -\mu-\omega_1\frac{\beta_a[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\sigma\psi(\delta_a+\theta_a+\mu)+\rho(\mu+\theta_b)(\delta_a+\theta_a+\mu)]S^0}{\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)}\\
&-\omega_2\frac{\beta_b[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\gamma\rho(\delta_b+\theta_b+\mu)+\psi(\mu+\theta_a)(\delta_b+\theta_b+\mu)]S^0}{\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)}\\
&+\omega_1\left(\xi_a+\eta_a+\mu+\frac{\zeta_4^2}{2}\right)+\omega_2\left(\xi_b+\eta_b+\mu+\frac{\zeta_5^2}{2}\right)\\
&+\left(\mu+\psi+\rho+\frac{\zeta_1^2}{2}\right)+\left(\delta_a+\theta_a+\mu+\frac{\zeta_2^2}{2}\right)+\left(\delta_b+\theta_b+\mu+\frac{\zeta_3^2}{2}\right)\\
&+\frac{\beta_a(2+\sigma)I_a}{1+\alpha_1 I_a}+\frac{\beta_b(2+\gamma)I_b}{1+\alpha_2 I_b}+\omega_1\varphi_1\frac{\beta_b I_b}{1+\alpha_2 I_b}+\omega_2\varphi_2\frac{\beta_a I_a}{1+\alpha_1 I_a}\\
&+\frac{K(r-\mu)\mu}{r}-\frac{K(r-\mu)\mu}{r}-\frac{\delta_a V_a}{S}-\frac{\delta_b V_b}{S}-\frac{\psi S}{V_a}-\frac{\rho S}{V_b}-\eta_a I_a-\eta_b I_b-\eta_{ab}I_{ab},
\end{aligned}
$$

Since the arithmetic mean is always greater than or equal to the geometric mean, it implies that

$$
\begin{aligned}
\mathscr{L}\mathscr{H}_3 \leq{}& -3\left[\mu\Delta_1\Delta_2\omega_1\omega_2\beta_a\beta_b\right]^{\frac{1}{3}}+\omega_1\left(\xi_a+\eta_a+\mu+\frac{\zeta_4^2}{2}\right)+\omega_2\left(\xi_b+\eta_b+\mu+\frac{\zeta_5^2}{2}\right)\\
&+\frac{K(r-\mu)\mu}{r}+\left(\mu+\psi+\rho+\frac{\zeta_1^2}{2}\right)+\left(\delta_a+\theta_a+\mu+\frac{\zeta_2^2}{2}\right)+\left(\delta_b+\theta_b+\mu+\frac{\zeta_3^2}{2}\right)+\frac{\beta_a(2+\sigma)I_a}{1+\alpha_1 I_a}\\
&+\frac{\beta_b(2+\gamma)I_b}{1+\alpha_2 I_b}+\omega_1\varphi_1\frac{\beta_b I_b}{1+\alpha_2 I_b}+\omega_2\varphi_2\frac{\beta_a I_a}{1+\alpha_1 I_a}-\frac{\delta_a V_a}{S}-\frac{\delta_b V_b}{S}-\frac{\psi S}{V_a}-\frac{\rho S}{V_b}-\eta_a I_a-\eta_b I_b-\eta_{ab}I_{ab}.
\end{aligned}
$$

where,

$$\Delta_1 = \frac{\beta_a[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\sigma\psi(\delta_a+\theta_a+\mu)+\rho(\mu+\theta_b)(\delta_a+\theta_a+\mu)]S^0}{\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)}$$

$$\Delta_2 = \frac{\beta_b[\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)+\mu\gamma\rho(\delta_b+\theta_b+\mu)+\psi(\mu+\theta_a)(\delta_b+\theta_b+\mu)]S^0}{\mu(\delta_a+\theta_a+\mu)(\delta_b+\theta_b+\mu)}.$$

Let

$$\omega_1\left(\xi_a+\eta_a+\mu+\frac{\zeta_4^2}{2}\right)=\omega_2\left(\xi_b+\eta_b+\mu+\frac{\zeta_5^2}{2}\right)=\frac{\mu\Delta_1\Delta_2\beta_a\beta_b}{\left(\xi_a+\eta_a+\mu+\frac{\zeta_4^2}{2}\right)\left(\xi_b+\eta_b+\mu+\frac{\zeta_5^2}{2}\right)},$$

with

$$\omega_1=\frac{\mu\Delta_1\Delta_2\beta_a\beta_b}{\left(\xi_a+\eta_a+\mu+\frac{\zeta_4^2}{2}\right)^2\left(\xi_b+\eta_b+\mu+\frac{\zeta_5^2}{2}\right)},\quad \omega_2=\frac{\mu\Delta_1\Delta_2\beta_a\beta_b}{\left(\xi_a+\eta_a+\mu+\frac{\zeta_4^2}{2}\right)\left(\xi_b+\eta_b+\mu+\frac{\zeta_5^2}{2}\right)^2}. \tag{44}$$

Consequently, we have

$$\begin{aligned}\mathscr{L}\mathscr{H}_3 \le& -\left[\left(\mu+\psi+\rho+\frac{\zeta_1^2}{2}\right)+\left(\delta_a+\theta_a+\mu+\frac{\zeta_2^2}{2}\right)+\left(\delta_b+\theta_b+\mu+\frac{\zeta_3^2}{2}\right)\right]-\eta_aI_a-\eta_bI_b-\eta_{ab}I_{ab}\\ &\times\left[\frac{\mu\Delta_1\Delta_2\beta_a\beta_b}{\left\{\left(\mu+\psi+\rho+\frac{\zeta_1^2}{2}\right)+\left(\delta_a+\theta_a+\mu+\frac{\zeta_2^2}{2}\right)+\left(\delta_b+\theta_b+\mu+\frac{\zeta_3^2}{2}\right)\right\}\left(\xi_a+\eta_a+\mu+\frac{\zeta_4^2}{2}\right)\left(\xi_b+\eta_b+\mu+\frac{\zeta_5^2}{2}\right)}\right.\\ &\left.-1\right]+\frac{\beta_a(2+\sigma)I_a}{1+\alpha_1I_a}+\frac{\beta_b(2+\gamma)I_b}{1+\alpha_2I_b}+\omega_1\varphi_1\frac{\beta_bI_b}{1+\alpha_2I_b}+\omega_2\varphi_2\frac{\beta_aI_a}{1+\alpha_1I_a}-\frac{\delta_aV_a}{S}-\frac{\delta_bV_b}{S}-\frac{\psi S}{V_a}-\frac{\rho S}{V_b}.\end{aligned} \tag{45}$$

It implies that

$$\begin{aligned}\mathscr{L}\mathscr{H}_3 \le& -\left[\left(\mu+\psi+\rho+\frac{\zeta_1^2}{2}\right)+\left(\delta_a+\theta_a+\mu+\frac{\zeta_2^2}{2}\right)+\left(\delta_b+\theta_b+\mu+\frac{\zeta_3^2}{2}\right)\right]\left[(\bar{\mathscr{R}}_0^S)-1\right]-\eta_aI_a-\eta_bI_b-\eta_{ab}I_{ab}\\ &+\frac{\beta_a(2+\sigma)I_a}{1+\alpha_1I_a}+\frac{\beta_b(2+\gamma)I_b}{1+\alpha_2I_b}+\omega_1\varphi_1\frac{\beta_bI_b}{1+\alpha_2I_b}+\omega_2\varphi_2\frac{\beta_aI_a}{1+\alpha_1I_a}-\frac{\delta_aV_a}{S}-\frac{\delta_bV_b}{S}-\frac{\psi S}{V_a}-\frac{\rho S}{V_b}.\end{aligned}$$

Moreover, define

$$\begin{aligned}\mathscr{H}_4 =& \omega_3\Big[-lnS-lnV_a-lnV_b-\omega_1lnI_a-\omega_2lnI_b+(S+V_a+V_b+I_a+I_b+I_{ab}+R_a+R_b+R_{ab})\Big]\\ &-\ln S-\ln V_a-\ln V_b-\ln R_a-\ln R_b-\ln R_{ab}+S+V_a+V_b+I_a+I_b+I_{ab}+R_a+R_b+R_{ab}.\end{aligned}$$

where, $\omega_3>0$, shall be determined later.

Now, consider a C^2-function $\mathscr{H}:\mathbb{R}_+^9\to R_+$:

$$\begin{aligned}\mathscr{H}(S,V_a,V_b,I_a,I_b,I_{ab},R_a,R_b,R_{ab}) =& \mathscr{H}_4(S,V_a,V_b,I_a,I_b,I_{ab},R_a,R_b,R_{ab})\\ &-\mathscr{H}_4(S(0),I_a(0),I_b(0),I_{ab}(0),R(0),V(0)).\end{aligned}$$

Applying *Ito's* Lemma, we have

$$\begin{aligned}
\mathscr{L}\mathscr{H} \leq \omega_3\Bigg\{ &-\left[\left(\mu+\psi+\rho+\frac{\zeta_1^2}{2}\right)+\left(\delta_a+\theta_a+\mu+\frac{\zeta_2^2}{2}\right)+\left(\delta_b+\theta_b+\mu+\frac{\zeta_3^2}{2}\right)\right]\left[(\bar{\mathscr{R}}_0^S)-1\right]\\
&-\eta_a I_a-\eta_b I_b-\eta_{ab}I_{ab}+\frac{\beta_a(2+\sigma)I_a}{1+\alpha_1 I_a}+\frac{\beta_b(2+\gamma)I_b}{1+\alpha_2 I_b}+\omega_1\varphi_1\frac{\beta_b I_b}{1+\alpha_2 I_b}+\omega_2\varphi_2\frac{\beta_a I_a}{1+\alpha_1 I_a}-\frac{\delta_a V_a}{S}\\
&-\frac{\delta_b V_b}{S}-\frac{\psi S}{V_a}-\frac{\rho S}{V_b}\Bigg\}+rN\left(1-\frac{N}{K}\right)-\mu(S+V_a+V_b+I_a+I_b+I_{ab}+R_a+R_b+R_{ab})\\
&-\eta_a I_a-\eta_b I_b-\eta_{ab}I_{ab}-\frac{rN}{S}\left(1-\frac{N}{K}\right)+\frac{\beta_a I_a}{1+\alpha_1 I_a}+\frac{\beta_b I_b}{1+\alpha_2 I_b}-\frac{\delta_a V_a}{S}-\frac{\delta_b V_b}{S}+(\mu+\psi+\rho)+\frac{\zeta_1^2}{2}\\
&-\frac{\psi S}{V_a}+\frac{\sigma\beta_a I_a}{1+\alpha_1 I_a}+\frac{\beta_b I_b}{1+\alpha_2 I_b}+(\delta_a+\theta_a+\mu)+\frac{\zeta_2^2}{2}-\frac{\rho S}{V_b}+\frac{\beta_a I_a}{1+\alpha_1 I_a}+\frac{\gamma\beta_b I_b}{1+\alpha_2 I_b}+(\delta_b+\theta_b+\mu)\\
&+\frac{\zeta_3^2}{2}-\frac{\theta_a V_a}{R_a}-\frac{\xi_a I_a}{R_a}+\mu+\frac{\beta_b I_b}{1+\alpha_2 I_b}+\frac{\zeta_7^2}{2}-\frac{\theta_b V_b}{R_b}-\frac{\xi_b I_b}{R_b}+\mu+\frac{\beta_a I_a}{1+\alpha_1 I_a}+\frac{\zeta_8^2}{2}-\frac{\xi_{ab}I_{ab}}{R_{ab}}+\mu+\frac{\zeta_9^2}{2},
\end{aligned} \tag{46}$$

which results in

$$\begin{aligned}
\mathscr{L}\mathscr{H} \leq &-\omega_3\omega_4+rN\left(1-\frac{N}{K}\right)-\mu(S+V_a+V_b+I_a+I_b+I_{ab}+R_a+R_b+R_{ab})-\eta_a I_a-\eta_b I_b-\eta_{ab}I_{ab}\\
&-\frac{rN}{S}\left(1-\frac{N}{K}\right)+\frac{\beta_a I_a}{1+\alpha_1 I_a}+\frac{\beta_b I_b}{1+\alpha_2 I_b}-\frac{\delta_a V_a}{S}-\frac{\delta_b V_b}{S}+(\mu+\psi+\rho)+\frac{\zeta_1^2}{2}\\
&-\frac{\psi S}{V_a}+\frac{\sigma\beta_a I_a}{1+\alpha_1 I_a}+\frac{\beta_b I_b}{1+\alpha_2 I_b}+(\delta_a+\theta_a+\mu)+\frac{\zeta_2^2}{2}-\frac{\rho S}{V_b}+\frac{\beta_a I_a}{1+\alpha_1 I_a}+\frac{\gamma\beta_b I_b}{1+\alpha_2 I_b}+(\delta_b+\theta_b+\mu)\\
&+\frac{\zeta_3^2}{2}-\frac{\theta_a V_a}{R_a}-\frac{\xi_a I_a}{R_a}+\mu+\frac{\beta_b I_b}{1+\alpha_2 I_b}+\frac{\zeta_7^2}{2}-\frac{\theta_b V_b}{R_b}-\frac{\xi_b I_b}{R_b}+\mu+\frac{\beta_a I_a}{1+\alpha_1 I_a}+\frac{\zeta_8^2}{2}-\frac{\xi_{ab}I_{ab}}{R_{ab}}+\mu+\frac{\zeta_9^2}{2},
\end{aligned} \tag{47}$$

where,

$$\omega_4=\left[\left(\mu+\psi+\rho+\frac{\zeta_1^2}{2}\right)+\left(\delta_a+\theta_a+\mu+\frac{\zeta_2^2}{2}\right)+\left(\delta_b+\theta_b+\mu+\frac{\zeta_3^2}{2}\right)\right]\left[(\bar{\mathscr{R}}_0^S)-1\right]>0.$$

Define the domain

$$\begin{aligned}
D=\Bigg\{&\varepsilon_1<S<\frac{1}{\varepsilon_2},\ \varepsilon_1<V_a<\frac{1}{\varepsilon_2},\ \varepsilon_1<V_b<\frac{1}{\varepsilon_2},\ \varepsilon_1<I_a<\frac{1}{\varepsilon_2},\ \varepsilon_1<I_b<\frac{1}{\varepsilon_2},\ \varepsilon_1<I_{ab}<\frac{1}{\varepsilon_2},\ \varepsilon_1\\
&<R_a<\frac{1}{\varepsilon_2},\ \varepsilon_1<R_b<\frac{1}{\varepsilon_2},\ \varepsilon_1<R_{ab}<\frac{1}{\varepsilon_2}\Bigg\}.
\end{aligned}$$

where $\varepsilon_i>0$ for $i=1,2$ will be estimated later. First, we subdivide the domain $\mathbb{R}_+^9\backslash D$ as follows:

$$D_1=\left\{(S,V_a,V_b,I_a,I_b,I_{ab},R_a,R_b,R_{ab})\in\mathbb{R}_+^9,0<S\leq\varepsilon_1\right\},$$

$$D_2 = \left\{ (S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in \mathbb{R}_+^9, 0 < V_a \le \varepsilon_1, S > \varepsilon_2 \right\},$$

$$D_3 = \left\{ (S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in \mathbb{R}_+^9, 0 < V_b \le \varepsilon_1, V_a > \varepsilon_2 \right\},$$

$$D_4 = \left\{ (S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in \mathbb{R}_+^9, 0 < I_a \le \varepsilon_1, V_b > \varepsilon_2 \right\},$$

$$D_5 = \left\{ (S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in \mathbb{R}_+^9, 0 < I_b \le \varepsilon_1, I_a > \varepsilon_2 \right\},$$

$$D_6 = \left\{ (S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in \mathbb{R}_+^9, 0 < I_{ab} \le \varepsilon_2, I_b > \varepsilon_1 \right\},$$

$$D_7 = \left\{ (S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in \mathbb{R}_+^9, 0 < R_a < \frac{1}{\varepsilon_2}, I_{ab} \ge \varepsilon_1 \right\},$$

$$D_8 = \left\{ (S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in \mathbb{R}_+^9, 0 < R_b < \frac{1}{\varepsilon_2}, R_a \ge \varepsilon_1 \right\},$$

$$D_9 = \left\{ (S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in \mathbb{R}_+^9, 0 < R_{ab} < \frac{1}{\varepsilon_2}, R_b \ge \varepsilon_1 \right\},$$

$$D_{10} = \left\{ (S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in \mathbb{R}_+^9, S \ge \frac{1}{\varepsilon_2} \right\},$$

$$D_{11} = \left\{ (S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in \mathbb{R}_+^9, V_a \ge \frac{1}{\varepsilon_2} \right\},$$

$$D_{12} = \left\{ (S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in \mathbb{R}_+^9, V_b \ge \frac{1}{\varepsilon_2} \right\},$$

$$D_{13} = \left\{ (S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in \mathbb{R}_+^9, I_a \ge \frac{1}{\varepsilon_2} \right\},$$

$$D_{14} = \left\{ (S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in \mathbb{R}_+^9, I_b \ge \frac{1}{\varepsilon_2} \right\},$$

$$D_{15} = \left\{ (S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in \mathbb{R}_+^9, I_{ab} \ge \frac{1}{\varepsilon_2} \right\},$$

$$D_{16} = \left\{ (S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in \mathbb{R}_+^9, R_a \ge \frac{1}{\varepsilon_2} \right\},$$

$$D_{17} = \left\{ (S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in \mathbb{R}_+^9, R_b \ge \frac{1}{\varepsilon_2} \right\},$$

$$D_{18} = \left\{ (S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in \mathbb{R}_+^9, R_{ab} \ge \frac{1}{\varepsilon_2} \right\}.$$

Next, we show that $\mathscr{L}\mathscr{H} < 0$ in all the above eighteen domains which then implies that $\mathscr{L}\mathscr{H}(S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) < 0$ on $\mathbb{R}_+^9 \backslash D$.

Case 1. Suppose $(S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in D_1$, then using Eq. (47), we have

$$\begin{aligned}
\mathscr{L}\mathscr{H} \leq & -\omega_3\omega_4 + rN\left(1 - \frac{N}{K}\right) - \mu(S + V_a + V_b + I_a + I_b + I_{ab} + R_a + R_b + R_{ab}) - \eta_a I_a - \eta_b I_b - \eta_{ab} I_{ab} \\
& - \frac{rN}{S}\left(1 - \frac{N}{K}\right) + \frac{\beta_a I_a}{1 + \alpha_1 I_a} + \frac{\beta_b I_b}{1 + \alpha_2 I_b} + (\mu + \psi + \rho) \\
& + \frac{\zeta_1^2}{2} - \frac{\psi S}{V_a} + \frac{\sigma\beta_a I_a}{1 + \alpha_1 I_a} + \frac{\beta_b I_b}{1 + \alpha_2 I_b} + (\delta_a + \theta_a + \mu) + \frac{\zeta_2^2}{2} - \frac{\rho S}{V_b} + \frac{\beta_a I_a}{1 + \alpha_1 I_a} + \frac{\gamma\beta_b I_b}{1 + \alpha_2 I_b} \\
& + (\delta_b + \theta_b + \mu) + \frac{\zeta_3^2}{2} - \frac{\theta_a V_a}{R_a} - \frac{\xi_a I_a}{R_a} + \mu + \frac{\beta_b I_b}{1 + \alpha_2 I_b} + \frac{\zeta_7^2}{2} - \frac{\theta_b V_b}{R_b} - \frac{\xi_b I_b}{R_b} + \mu + \frac{\beta_a I_a}{1 + \alpha_1 I_a} \\
& + \frac{\zeta_8^2}{2} - \frac{\xi_{ab} I_{ab}}{R_{ab}} + \mu + \frac{\zeta_9^2}{2} - \frac{\delta_a V_a}{S} - \frac{\delta_b V_b}{S} \\
\leq & -\omega_3\omega_4 + rN\left(1 - \frac{N}{K}\right) - \mu(S + V_a + V_b + I_a + I_b + I_{ab} + R_a + R_b + R_{ab}) - \eta_a I_a - \eta_b I_b - \eta_{ab} I_{ab} \\
& - \frac{rN}{S}\left(1 - \frac{N}{K}\right) + \frac{\beta_a I_a}{1 + \alpha_1 I_a} + \frac{\beta_b I_b}{1 + \alpha_2 I_b} + (\mu + \psi + \rho) + \frac{\zeta_1^2}{2} \\
& - \frac{\psi S}{V_a} + \frac{\sigma\beta_a I_a}{1 + \alpha_1 I_a} + \frac{\beta_b I_b}{1 + \alpha_2 I_b} + (\delta_a + \theta_a + \mu) + \frac{\zeta_2^2}{2} - \frac{\rho S}{V_b} + \frac{\beta_a I_a}{1 + \alpha_1 I_a} + \frac{\gamma\beta_b I_b}{1 + \alpha_2 I_b} + (\delta_b + \theta_b + \mu) \\
& + \frac{\zeta_3^2}{2} - \frac{\theta_a V_a}{R_a} - \frac{\xi_a I_a}{R_a} + \mu + \frac{\beta_b I_b}{1 + \alpha_2 I_b} + \frac{\zeta_7^2}{2} - \frac{\theta_b V_b}{R_b} - \frac{\xi_b I_b}{R_b} + \mu + \frac{\beta_a I_a}{1 + \alpha_1 I_a} + \frac{\zeta_8^2}{2} - \frac{\xi_{ab} I_{ab}}{R_{ab}} \\
& + \mu + \frac{\zeta_9^2}{2} - \frac{\delta_a V_a}{\epsilon_1} - \frac{\delta_b V_b}{\epsilon_1}.
\end{aligned} \tag{48}$$

If $\epsilon_1 > 0$ is chosen to be sufficiently small such that the right hand sides (48) is not greater than zero, then $\mathscr{L}\mathscr{H} < 0$ for $(S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in D_1$.

Similarly, it can also be shown that $\mathscr{L}\mathscr{H} < 0$ for $(S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in D_2$,
$(S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in D_3$, $(S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in D_4$,
$(S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in D_5$,
$(S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in D_6$, $(S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in D_7$,
$(S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in D_8$,
$(S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in D_9$.

Case 2. If $(S, V_a, V_b, I_a, I_b, I_{ab}, R_a, R_b, R_{ab}) \in D_{10}$, then by Eq. (47), we get

$$\begin{aligned}
\mathscr{L}\mathscr{H} \leq & -\omega_3\omega_4 + rN\left(1 - \frac{N}{K}\right) - \mu(S + V_a + V_b + I_a + I_b + I_{ab} + R_a + R_b + R_{ab}) - \eta_a I_a - \eta_b I_b - \eta_{ab} I_{ab} \\
& - \frac{rN}{S}\left(1 - \frac{N}{K}\right) + \frac{\beta_a I_a}{1 + \alpha_1 I_a} + \frac{\beta_b I_b}{1 + \alpha_2 I_b} + (\mu + \psi + \rho) + \frac{\zeta_1^2}{2} \\
& - \frac{\psi S}{V_a} + \frac{\sigma\beta_a I_a}{1 + \alpha_1 I_a} + \frac{\beta_b I_b}{1 + \alpha_2 I_b} + (\delta_a + \theta_a + \mu) + \frac{\zeta_2^2}{2} - \frac{\rho S}{V_b} + \frac{\beta_a I_a}{1 + \alpha_1 I_a} + \frac{\gamma\beta_b I_b}{1 + \alpha_2 I_b} + (\delta_b + \theta_b + \mu)
\end{aligned}$$

$$+\frac{\zeta_3^2}{2}-\frac{\theta_a V_a}{R_a}-\frac{\xi_a I_a}{R_a}+\mu+\frac{\beta_b I_b}{1+\alpha_2 I_b}+\frac{\zeta_7^2}{2}-\frac{\theta_b V_b}{R_b}-\frac{\xi_b I_b}{R_b}+\mu+\frac{\beta_a I_a}{1+\alpha_1 I_a}+\frac{\zeta_8^2}{2}-\frac{\xi_{ab} I_{ab}}{R_{ab}}$$
$$+\mu+\frac{\zeta_9^2}{2}-\frac{\delta_a V_a}{S}-\frac{\delta_b V_b}{S}$$
$$\leq -\omega_3\omega_4+rN\left(1-\frac{N}{K}\right)-\mu(S+V_a+V_b+I_a+I_b+I_{ab}+R_a+R_b+R_{ab})-\eta_a I_a-\eta_b I_b-\eta_{ab} I_{ab}$$
$$-\frac{rN}{S}\left(1-\frac{N}{K}\right)+\frac{\beta_a I_a}{1+\alpha_1 I_a}+\frac{\beta_b I_b}{1+\alpha_2 I_b}+(\mu+\psi+\rho)+\frac{\zeta_1^2}{2}$$
$$-\frac{\psi S}{V_a}+\frac{\sigma\beta_a I_a}{1+\alpha_1 I_a}+\frac{\beta_b I_b}{1+\alpha_2 I_b}+(\delta_a+\theta_a+\mu)+\frac{\zeta_2^2}{2}-\frac{\rho S}{V_b}+\frac{\beta_a I_a}{1+\alpha_1 I_a}+\frac{\gamma\beta_b I_b}{1+\alpha_2 I_b}+(\delta_b+\theta_b+\mu)$$
$$+\frac{\zeta_3^2}{2}-\frac{\theta_a V_a}{R_a}-\frac{\xi_a I_a}{R_a}+\mu+\frac{\beta_b I_b}{1+\alpha_2 I_b}+\frac{\zeta_7^2}{2}-\frac{\theta_b V_b}{R_b}-\frac{\xi_b I_b}{R_b}+\mu+\frac{\beta_a I_a}{1+\alpha_1 I_a}+\frac{\zeta_8^2}{2}-\frac{\xi_{ab} I_{ab}}{R_{ab}}$$
$$+\mu+\frac{\zeta_9^2}{2}-\frac{\delta_a\varepsilon_2}{\varepsilon_1}-\frac{\delta_b\varepsilon_3}{\varepsilon_1} \tag{49}$$

If we take, $\varepsilon_1=\varepsilon_2^2$, for large value of $\omega_3>0$ and smallest value of $\varepsilon_2>0$ such that right hand sides of (49) is not greater than zero, then $\mathscr{L}\mathscr{H}<0$ for every $(S,V_a,V_b,I_a,I_b,I_{ab},R_a,R_b,R_{ab})\in D_7$.

Similarly, we can obtain that $(S,V_a,V_b,I_a,I_b,I_{ab},R_a,R_b,R_{ab})\in D_{11}$,$(S,V_a,V_b,I_a,I_b,I_{ab},R_a,R_b,R_{ab})\in D_{12}$, $(S,V_a,V_b,I_a,I_b,I_{ab},R_a,R_b,R_{ab})\in D_{13}$, $(S,V_a,V_b,I_a,I_b,I_{ab},R_a,R_b,R_{ab})\in D_{14}$, $(S,V_a,V_b,I_a,I_b,I_{ab},R_a,R_b,R_{ab})\in D_{15}$, $(S,V_a,V_b,I_a,I_b,I_{ab},R_a,R_b,R_{ab})\in D_{16}$.

Hence, there exist $W>0$ such that

$$\mathscr{L}\mathscr{H}(S,V_a,V_b,I_a,I_b,I_{ab},R_a,R_b,R_{ab})<-W<0 \quad \text{for every} \quad (S,V_a,V_b,I_a,I_b,I_{ab},R_a,R_b,R_{ab})\in\mathbb{R}_+^9\backslash D.$$

Therefore, we have

$$\begin{aligned}d\mathscr{H}(S,I_a,I_b,I_{ab},R)<{}&-Wdt+[(\omega_3+1)\,S-(\omega_1+1)\zeta_1]dB_1(t)\\&+[(\omega_3+1)I_a-\omega_2\omega_3\zeta_2]dB_2(t)+[(\omega_3+1)I_b-\omega_3\zeta_3]dB_3(t)\\&+[(\omega_3+1)I_{ab}-\omega_3\zeta_4]dB_4(t)+[(\omega_3+1)R-\zeta_5]dB_5(t)+[(\omega_3+1)R-\zeta_5]dB_6(t).\end{aligned} \tag{50}$$

Let $(S(0),V_a(0),V_b(0),I_a(0),I_b(0),I_{ab}(0),R_a(0),R_b(0),R_{ab}(0))=\varepsilon=(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6,\varepsilon_7,\varepsilon_8,\varepsilon_9)\in\mathbb{R}_+^9\backslash D$, and τ^ε denotes the time in moving from ε to D

$$and\tau_n=\inf\{t: z=|Y(t)|\},\text{and}\tau^{(z)}(t)=\min\{\tau^\varepsilon,t,\tau_z\}.$$

Taking expectation, applying Dynkin's formula [53] and integrating both sides of Eq. (50) over $[0,\tau^{(z)}(t)]$, gives

$$\mathbb{E}\mathscr{H}(S(\tau^{(z)}(t)),V_a(\tau^{(z)}(t)),V_b(\tau^{(z)}(t)),I_a(\tau^{(z)}(t)),I_b(\tau^{(z)}(t)),I_{ab}(\tau^{(z)}(t)),R_a(\tau^{(z)}(t)),R_b(\tau^{(z)}(t)),R_{ab}(\tau^{(z)}(t)))$$
$$-\mathscr{H}(0)=\mathbb{E}\int_0^{\tau(z)(t)}\mathscr{L}\mathscr{H}(S(\nu),V_a(\nu),V_b(\nu),I_a(\nu),I_b(\nu),I_{ab}(\nu),R_a(\nu),R_b(\nu),R_{ab}(\nu))du$$
$$\leq\mathbb{E}\int_0^{\tau(z)(t)}(-W)du$$

$$= -W\mathbb{E}\tau^{(z)}(t). \tag{51}$$

Since $\mathscr{H}$ is non-negative, we have

$$\mathbb{E}\tau^{(z)}(t) \leq \frac{1}{W}\mathscr{H}(\varepsilon).$$

Following arguments similar to those in the proof of the Theorem (4.4), we can show that $P\{\tau_e = \infty\} = 1$. Hence, the system (7) is regular. But, if $z \to \infty$ and $t \to \infty$, then $\tau(z)(t) \to \tau^{\varepsilon}$ a.s. Hence, using Fatou's lemma [54], it is obtained that

$$E\tau^{(z)}(t) \leq \frac{1}{W}\mathscr{H}(\varepsilon) < \infty.$$

Now, $\sup_{\varepsilon \in K} E\tau^{\varepsilon} < \infty$, where K denotes the compact subset of $\mathbb{R}^9_+$; condition *A2* in Lemma (4.2) is satisfied. Therefore, it is concluded that system (7) has "unique stationary distribution".

5 Numerical Scheme/Simulations

The perturbed system (7) shall now be experimented numerically in this section. The higher order scheme by Milstein [55] shall be used and is defined below:

$$\begin{aligned}
S_{i+1} &= S_i + \left[rN_i\left(1 - \frac{N_i}{K}\right) - \frac{\beta_a S_i I_{c,i}}{1+\alpha_1 I_{c,i}} + \frac{\beta_b S_i I_{h,i}}{1+\alpha_2 I_{h,i}} + \delta_a V_{c,i} + +\delta_b V_{h,i} - (\mu+\psi+\rho)S_i\right]\triangle t \\
&\quad + \zeta_1 S_i\sqrt{\triangle t}\varsigma_{1,i} + \frac{\zeta_1^2}{2}S_i(\varpi_{1,i}^2 - 1)\triangle t, \\
V_{c,i+1} &= V_{c,i} + \left[\delta_a S_i - \frac{\sigma\beta_a V_{c,i} I_{c,i}}{1+\alpha_1 I_{c,i}} - \frac{\beta_b V_{c,i} I_{h,i}}{1+\alpha_2 I_{h,i}} - (\psi+\theta_a+\mu)V_{c,i}\right]\triangle t + \zeta_2 V_{c,i}\sqrt{\triangle t}\varpi_{2,i} \\
&\quad + \frac{\zeta_2^2}{2}V_{c,i}(\varpi_{2,i}^2 - 1)\triangle t, \\
V_{hi+1} &= V_{h,i} + \left[\delta_b S_i - \frac{\beta_a V_{h,i} I_{c,i}}{1+\alpha_1 I_{c,i}} - \frac{\gamma\beta_b V_{h,i} I_{h,i}}{1+\alpha_2 I_{h,i}} - (\rho+\theta_b+\mu)V_{h,i}\right]\triangle t + \zeta_3 V_{h,i}\sqrt{\triangle t}\varpi_{3,i} \\
&\quad + \frac{\zeta_3^2}{2}V_{h,i}(\varpi_{3,i}^2 - 1)\triangle t, \\
I_{c,i+1} &= I_{c,i} + \left[\frac{\beta_a(S_i+\sigma V_{c,i}+V_{h,i}+R_{h,i})I_{c,i}}{1+\alpha_1 I_{c,i}} - (\xi_a+\eta_a+\mu)I_{c,i} - \frac{\varphi_1 I_{h,i} I_{c,i}}{1+\alpha_2 I_{h,i}}\right]\triangle t + \zeta_4 I_{c,i}\sqrt{\triangle t}\varpi_{4,i} \\
&\quad + \frac{\zeta_4^2}{2}I_{c,i}(\varpi_{4,i}^2 - 1)\triangle t, \\
I_{h,i+1} &= I_{h,i} + \left[\frac{\beta_b(S_i+V_{c,i}+\gamma V_{h,i}+R_{c,i})I_{h,i}}{1+\alpha_2 I_{h,i}} + (\xi_b+\eta_b+\mu)I_{h,i} - \frac{\varphi_2 I_{c,i} I_{h,i}}{1+\alpha_1 I_{c,i}}\right]\triangle t + \zeta_5 I_{h,i}\sqrt{\triangle t}\varpi_{5,i} \\
&\quad + \frac{\zeta_5^2}{2}I_{h,i}(\varpi_{5,i}^2 - 1)\triangle t, \\
I_{ch,i+1} &= I_{ch,i} + \left[\frac{\varphi_1 I_{h,i} I_{c,i}}{1+\alpha_2 I_{h,i}} + \frac{\varphi_2 I_{c,i} I_{h,i}}{1+\alpha_1 I_{c,i}} - (\xi_a+\eta_a+\mu)I_{ch,i}\right]\triangle t + \zeta_6 I_{ch,i}\sqrt{\triangle t}\varpi_{6,i} + \frac{\zeta_6^2}{2}I_{ch,i}(\varpi_{6,i}^2 - 1)\triangle t,
\end{aligned}$$

$$R_{c,i+1} = R_{c,i} + \left[\theta_a V_{c,i} + \xi_a I_{c,i} - \mu R_{c,i} - \frac{I_{h,i} R_{c,i}}{1 + \alpha_2 I_{h,i}}\right] \triangle t + \zeta_7 R_{c,i} \sqrt{\triangle t} \varpi_{7,i} + \frac{\zeta_7^2}{2} R_{c,i} (\varpi_{7,i}^2 - 1) \triangle t,$$

$$R_{hi+1} = R_{h,i} + \left[\theta_b V_{h,i} + \xi_b I_{h,i} - \mu R_{h,i} - \frac{I_{c,i} R_{h,i}}{1 + \alpha_1 I_{c,i}}\right] \triangle t + \zeta_8 R_{h,i} \sqrt{\triangle t} \varpi_{8,i} + \frac{\zeta_8^2}{2} R_{h,i} (\varpi_{8,i}^2 - 1) \triangle t,$$

$$R_{chi+1} = R_{ch,i} + \left[\xi_{ab} I_{ch,i} - \mu R_{ch,i}\right] \triangle t + \zeta_9 R_{ch,i} \sqrt{\triangle t} \varpi_{9,i} + \frac{\zeta_9^2}{2} R_{ch,i} (\varpi_{9,i}^2 - 1) \triangle t, \tag{52}$$

with, $\varpi_{j,i} (j = 1, 2, 3, 4, 5, 6, 7, 8, 9)$ the "independent Gaussian random variables", $N(0, 1)$, "normal distribution" and Δt represents step size. $\zeta_j > 0, (i = 1, 2, 3, 4, 5, 6, 7, 8, 9)$ represent white noise intensities. In the sequel, numerical experiments are implemented to substantiate qualitative results established in preceding sections. Demographic data related to Pakistan have been used for the simulations. The initial conditions for the state variables are assumed thus: $S(0) = 175{,}000{,}000$, $V_a(0) = 15{,}000{,}000$, $V_b(0) = 15{,}000{,}000$, $I_a(0) = 1{,}296{,}527$, $I_b(0) = 250{,}895$, $I_{ab}(0) = 5{,}000$, $R_a(0) = 0$, $R_b(0) = 0$, $R_{ab}(0) = 0$. For the fitting of the model to data, available records for daily reported COVID-19 cases in Pakistan [56] between January 01, 2022 and April 10, 2022 are used. The fitting shown in Fig. 2 reveals that our perturbed model (7) behaves very well with the data. Important parameters estimated from the fitting exercise are presented in Table 1. Other parameters are estimated or derived from relevant literature. As depicted by Figs. 3–5, assessments of the perturbed system (7) are carried out, when the white noise intensities are: $\zeta_i^2 = 0.015$, for $i = 1, 2, \ldots, 9$ and when $\alpha_1 = 2.6891 \times 10^{-6}$, $\alpha_2 = 7.3 \times 10^{-7}$, $\beta_a = 5.1024 \times 10^{-9}$, $\beta_b = 1.0 \times 10^{-9}$, so that $\mathscr{R}_0^S = \max\{\mathscr{R}_{0a}^S, \mathscr{R}_{0b}^S\} = \max\{3.0323, 1.5789\} = 3.0323 > 1$. These assessments show that both viruses will persist in the population with unit probability. This experiment is also compatible with the conclusion of Theorem 4.4. The associated frequency distributions showing the intensity of random fluctuations, for the different compartments under this scenario are also presented alongside their solution profiles.

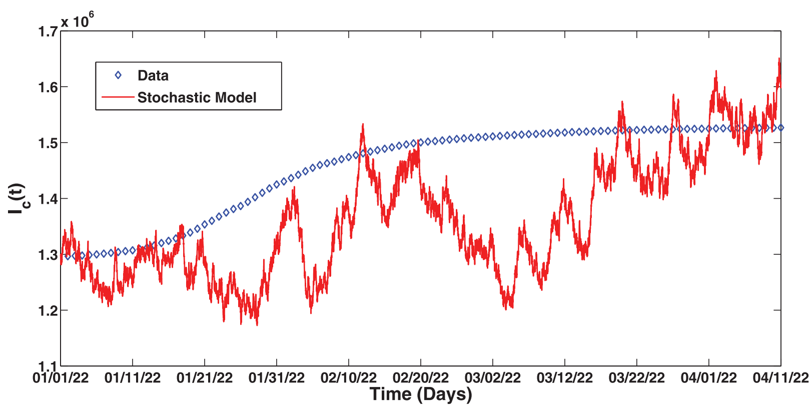
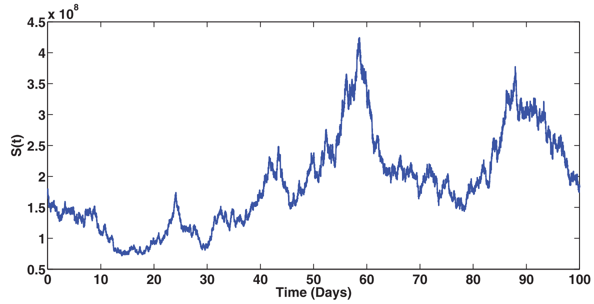
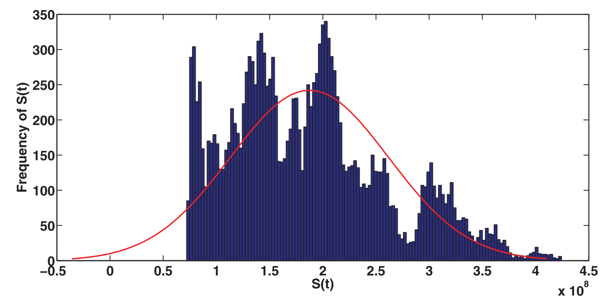


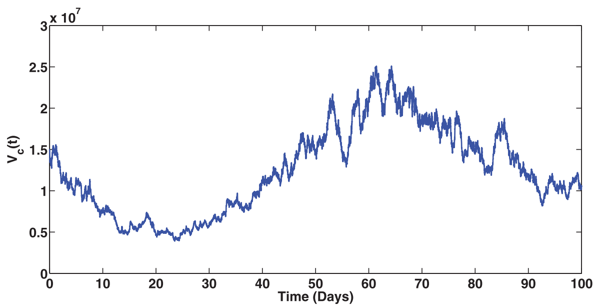
Figure 2: Fitting of the proposed stochastic model to real data



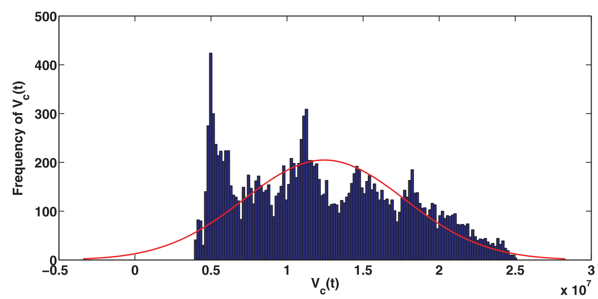
(a) Profile of $S(t)$



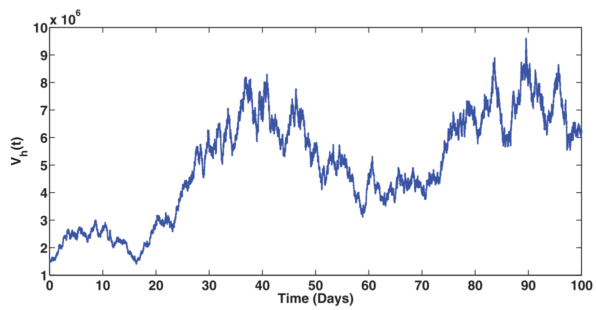
(b) Distribution histogram for $S(t)$



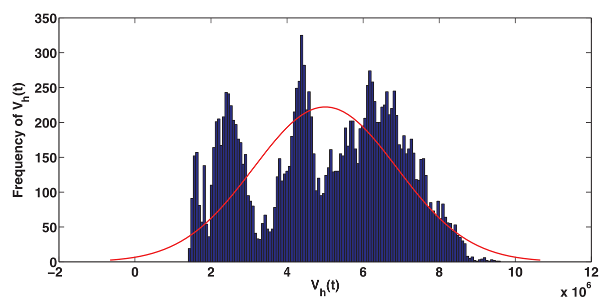
(c) Profile of $V_c(t)$



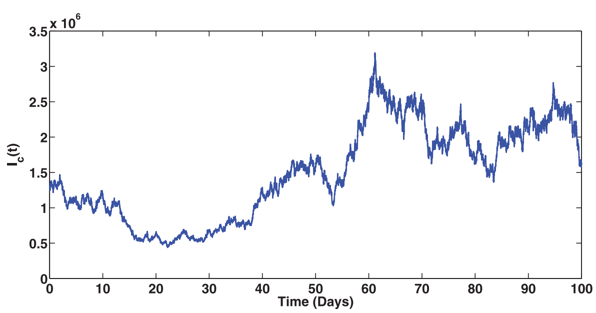
(d) Distribution histogram for $V_c(t)$



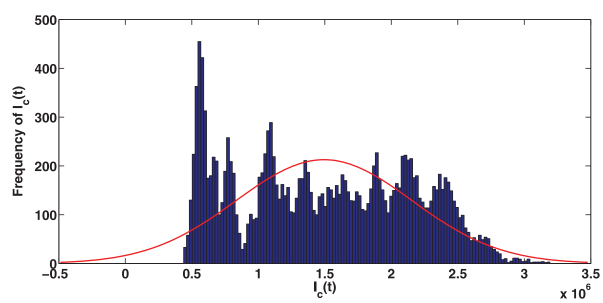
(e) Profile of $V_h(t)$



(f) Distribution histogram for $V_h(t)$

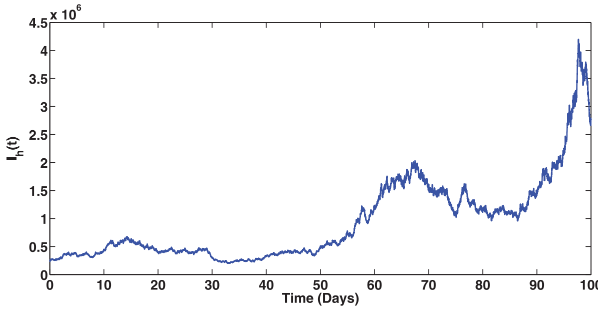


(g) Profile of $I_c(t)$

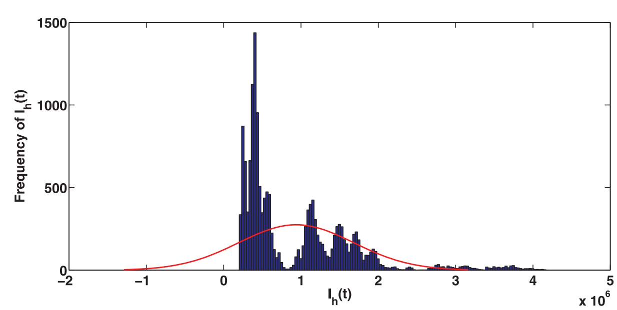


(h) Distribution histogram for $I_c(t)$

Figure 3: Solution profiles for the various compartments when $\alpha_1 = 2.6891 \times 10^{-6}$, $\alpha_2 = 7.3 \times 10^{-7}$, $\beta_a = 5.1024 \times 10^{-9}$, $\beta_b = 1.0 \times 10^{-9}$, so that $\mathscr{R}_0^S = \max\{\mathscr{R}_{0a}^S, \mathscr{R}_{0b}^S\} = \max\{3.0323, 1.5789\} = 3.0323 > 1$

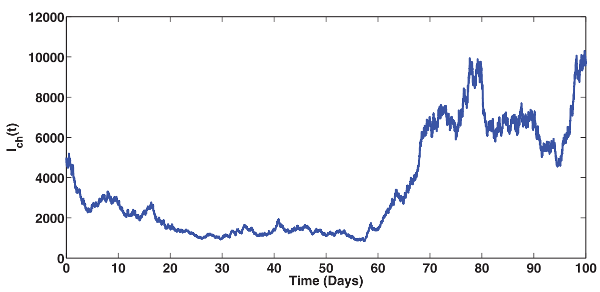


(a) Profile of $I_h(t)$

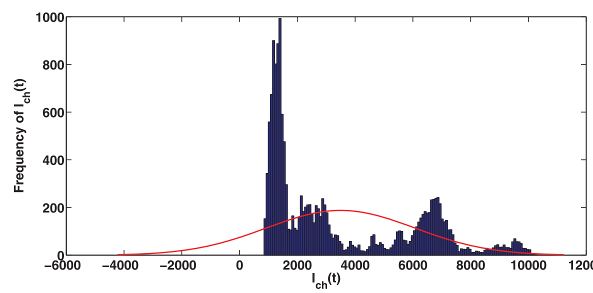


(b) Distribution histogram for $I_h(t)$

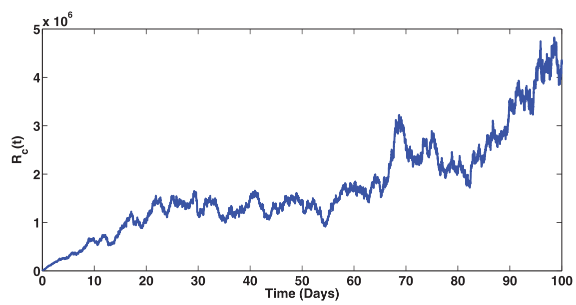
Figure 4: Solution profiles for the various compartments when $\alpha_1 = 2.6891 \times 10^{-6}$, $\alpha_2 = 7.3 \times 10^{-7}$, $\beta_a = 5.1024 \times 10^{-9}$, $\beta_b = 1.0 \times 10^{-9}$, so that $\mathscr{R}_0^S = \max\{\mathscr{R}_{0a}^S, \mathscr{R}_{0b}^S\} = \max\{3.0323, 1.5789\} = 3.0323 > 1$



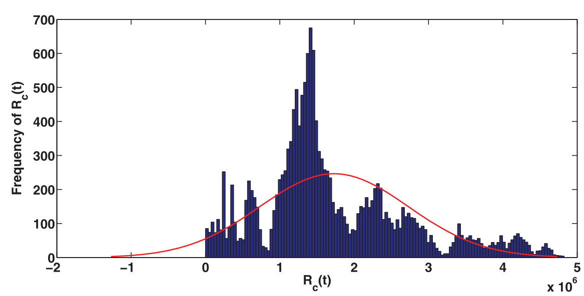
(a) Profile of $I_{ch}(t)$



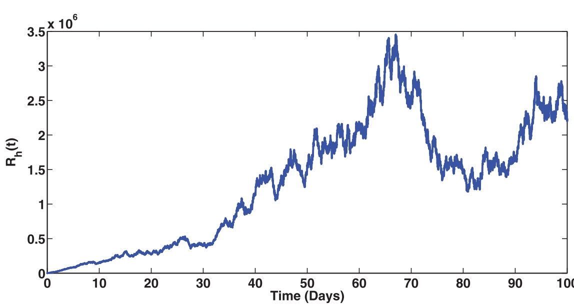
(b) Distribution histogram for $I_{ch}(t)$



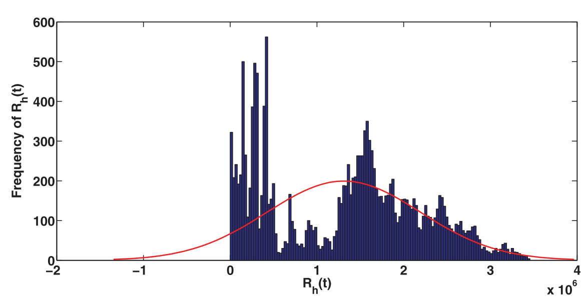
(c) Profile of $R_c(t)$



(d) Distribution histogram for $R_c(t)$

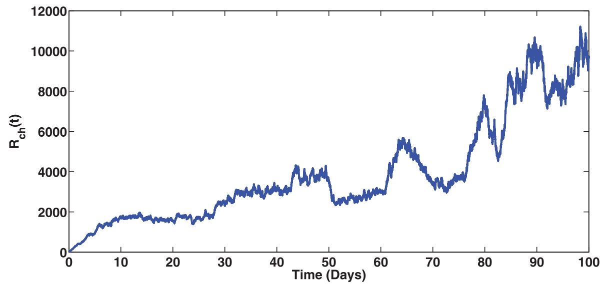


(e) Profile of $R_h(t)$

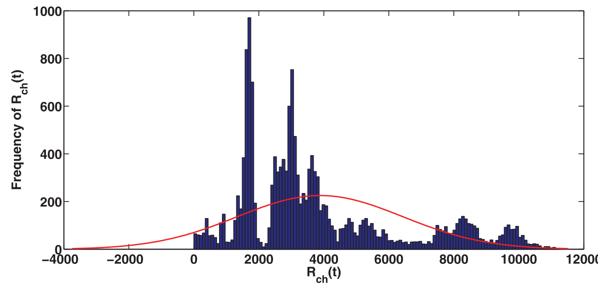


(f) Distribution histogram for $R_h(t)$

Figure 5: (Continued)



(g) Profile of $R_{ch}(t)$



(h) Distribution histogram for $R_{ch}(t)$

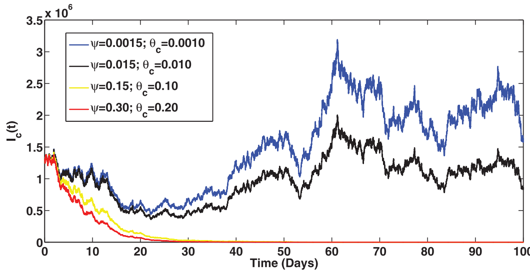
Figure 5: Solution profiles for the various compartments when $\alpha_1 = 2.6891 \times 10^{-6}$, $\alpha_2 = 7.3 \times 10^{-7}$, $\beta_a = 5.1024 \times 10^{-9}$, $\beta_b = 1.0 \times 10^{-9}$, so that $\mathscr{R}_0^S = \max\{\mathscr{R}_{0a}^S, \mathscr{R}_{0b}^S\} = \max\{3.0323, 1.5789\} = 3.0323 > 1$

5.1 Impact of Primary and Booster Vaccination Rates

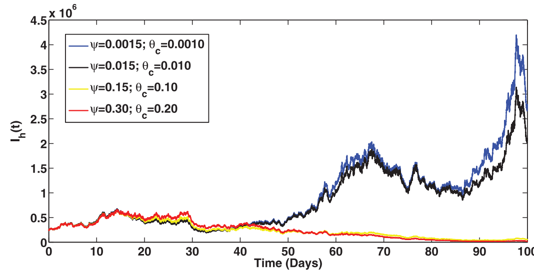
Numerical assessments of the epidemiological impact of COVID-19 and viral hepatitis B vaccination strategies are presented in Figs. 6 and 7, respectively. The solution profiles for infected components at different primary and booster vaccination rates for COVID-19 are shown in Fig. 6. It is observed that increasing primary and booster dose vaccination rates greatly caused reduction in infected classes with COVID-19 (Fig. 6a as expected). This measure also brought about reduction in the infected individuals with viral hepatitis B and the compartment co-infected with both diseases (as can be noted in Figs. 6b and 6c). It is interesting to observe that, stepping up the COVID-19 primary vaccination rate to, $\psi = 0.30$ and booster dose vaccination rate to the level, $\theta_c = 0.20$, the least number of infections is recorded for all the infected components (including those infected with viral hepatitis B). This result also support the epidemiological records from the introduction [10] showing that severe COVID-19 infection can be an important risk factor for viral hepatitis B infection. The solution profiles for the infected components at different primary and booster vaccination rates for viral hepatitis B are shown in Fig. 7. It is observed from this experiment, that increasing primary and booster dose vaccination rates for viral hepatitis B not only caused reduction in infected classes with viral hepatitis B (as seen in Fig. 7a, which is expected), but also brought about noticeable reduction in the infected individuals with COVID-19 and the co-infection of both diseases (as can be seen in Figs. 7b and 7c). It is interesting to note that, keeping the viral hepatitis primary vaccination rate at a high level, $\rho = 0.20$ per day and the booster dose vaccination rate at $\theta_h = 0.15$ per day, the least number of infections is recorded for all the infected components (including singly infected with only viral hepatitis B). This result also support the epidemiological evidences [9] from the introduction section that viral hepatitis B infection could enhance susceptibility to COVID-19 infection.

5.2 Impact of Saturation Effects

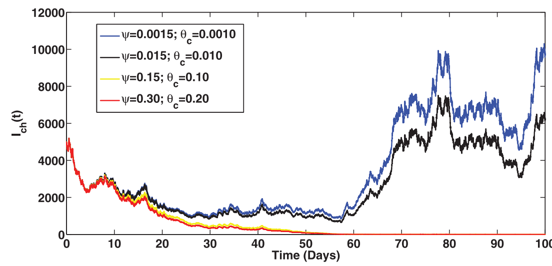
The numerical investigation of the epidemiological impact of saturation effect α_1 on the various components of the model is presented in Figs. 8–11. It is observed that, the saturation effect greatly impacted the different compartments. In particular, while decreasing values for α_1 caused some initial increase in random fluctuations for I_c and I_{ch} compartments (as observed in Figs. 8d and 8f), marginal random fluctuations or impact is observed for the class of individuals infected with viral hepatitis B (as can be noted in Fig. 8e).



(a) Profile of $I_c(t)$

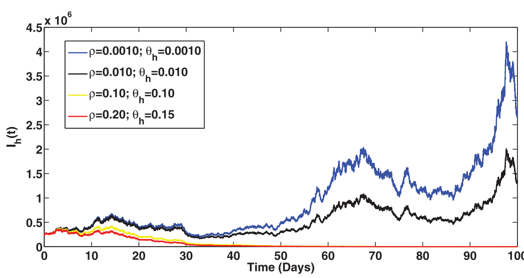


(b) Profile of $I_h(t)$

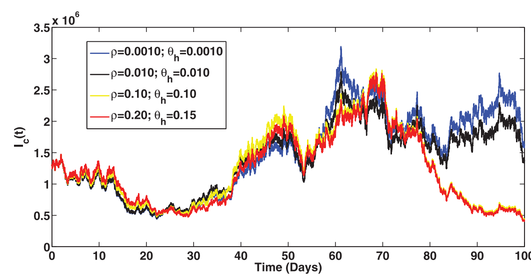


(c) Profile of $I_{ch}(t)$

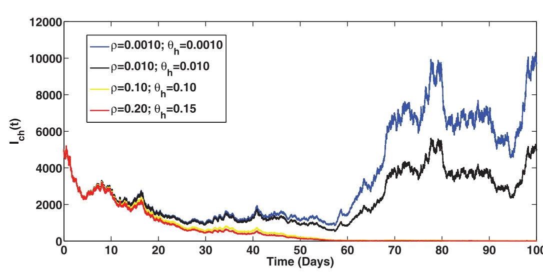
Figure 6: Solution profiles for the infected compartments at different primary and booster vaccination rates for COVID-19, where $\beta_a = 5.1024 \times 10^{-9}$, $\beta_b = 1.0 \times 10^{-9}$, so that $\mathscr{R}_0^S = \max\{\mathscr{R}_{0a}^S, \mathscr{R}_{0b}^S\} = \max\{3.0323, 1.5789\} = 3.0323 > 1$



(a) Profile of $I_h(t)$

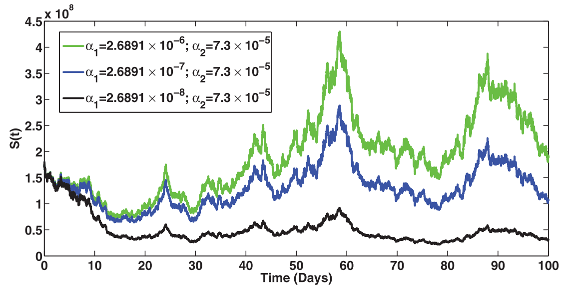


(b) Profile of $I_c(t)$

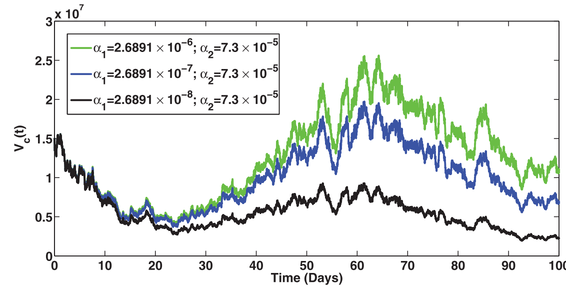


(c) Profile of $I_{ch}(t)$

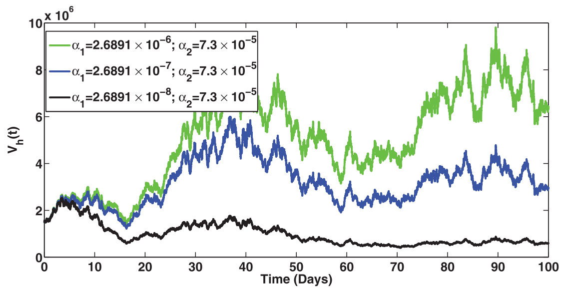
Figure 7: Solution profiles for the infected compartments at different primary and booster vaccination rates for viral hepatitis, where $\beta_a = 5.1024 \times 10^{-9}$, $\beta_b = 1.0 \times 10^{-9}$, so that $\mathscr{R}_0^S = \max\{\mathscr{R}_{0a}^S, \mathscr{R}_{0b}^S\} = \max\{3.0323, 1.5789\} = 3.0323 > 1$



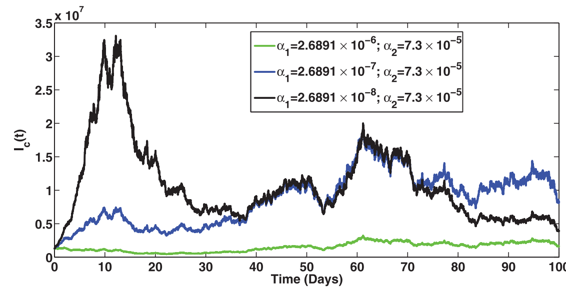
(a) Profile of $S(t)$



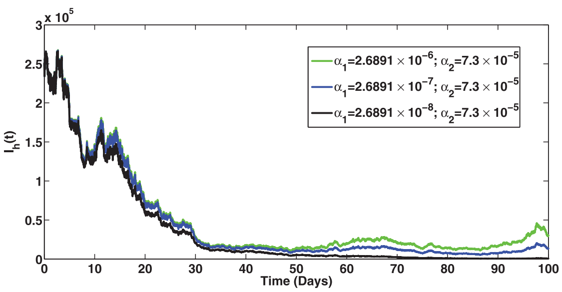
(b) Profile of $V_c(t)$



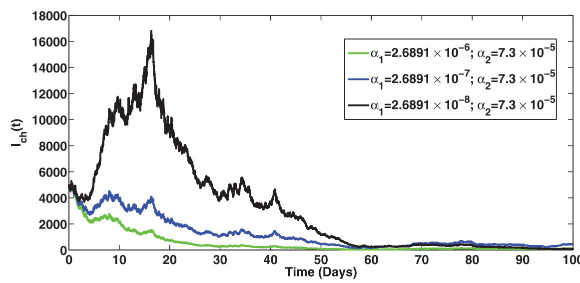
(c) Profile of $V_h(t)$



(d) Profile of $I_c(t)$

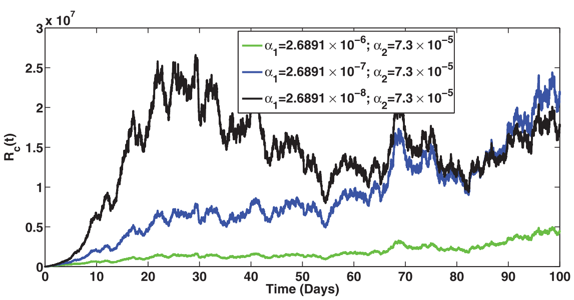


(e) Profile of $I_h(t)$

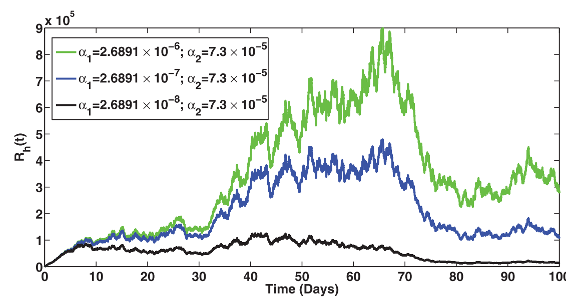


(f) Profile of $I_{ch}(t)$

Figure 8: Solution profiles for the various compartments at different saturation effect, α_1, when $\beta_a = 5.1024 \times 10^{-9}$, $\beta_b = 1.0 \times 10^{-9}$, so that $\mathscr{R}_0^S = \max\{\mathscr{R}_{0a}^S, \mathscr{R}_{0b}^S\} = \max\{3.0323, 1.5789\} = 3.0323 > 1$

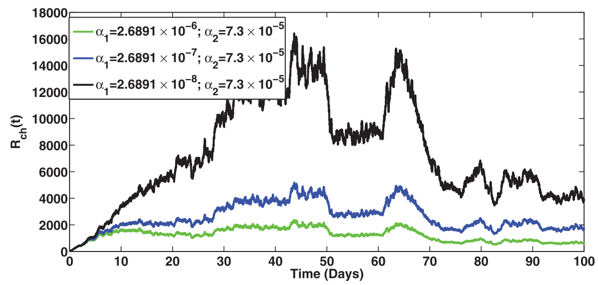


(a) Profile of $R_c(t)$



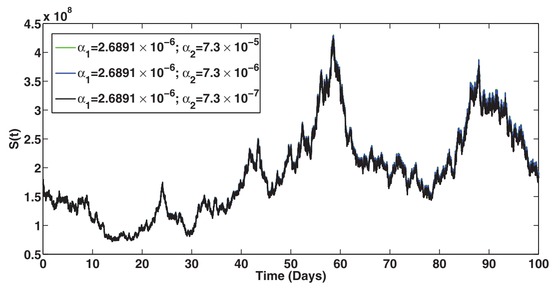
(b) Profile of $R_h(t)$

Figure 9: (Continued)

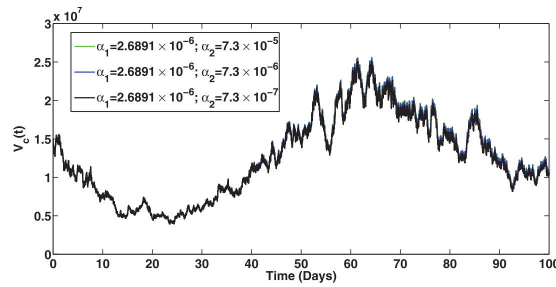


(**c**) Profile of $R_{ch}(t)$

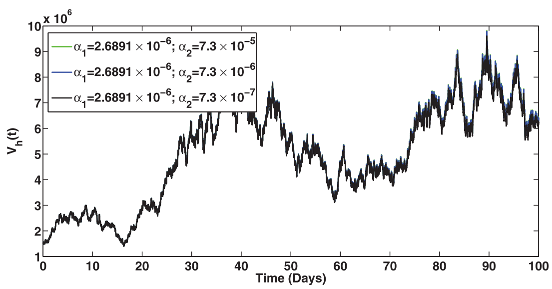
Figure 9: Solution profiles for the various compartments at different saturation effect, α_1, when $\beta_a = 5.1024 \times 10^{-9}$, $\beta_b = 1.0 \times 10^{-9}$, so that $\mathscr{R}_0^S = \max\{\mathscr{R}_{0a}^S, \mathscr{R}_{0b}^S\} = \max\{3.0323, 1.5789\} = 3.0323 > 1$



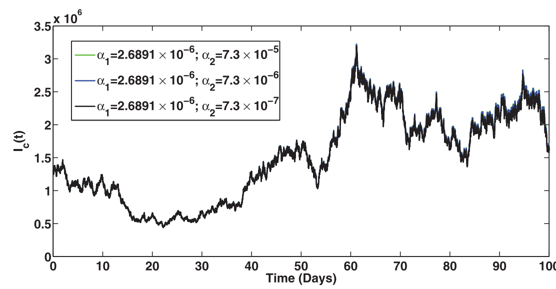
(**a**) Profile of $S(t)$



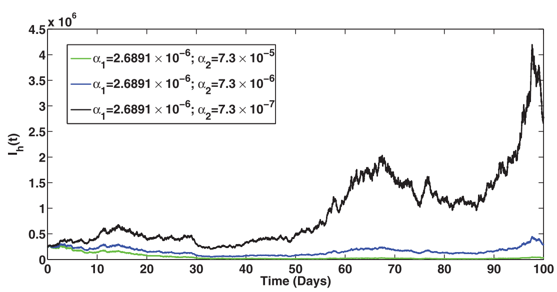
(**b**) Profile of $V_c(t)$



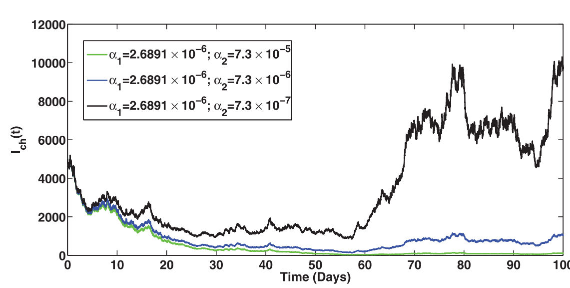
(**c**) Profile of $V_h(t)$



(**d**) Profile of $I_c(t)$

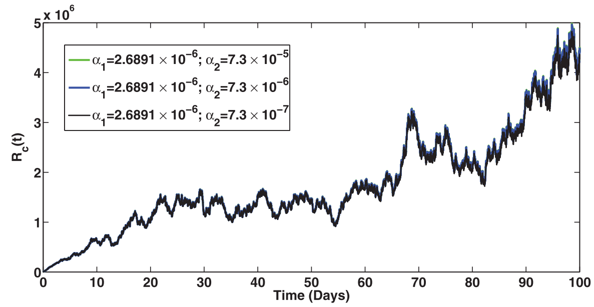


(**e**) Profile of $I_h(t)$

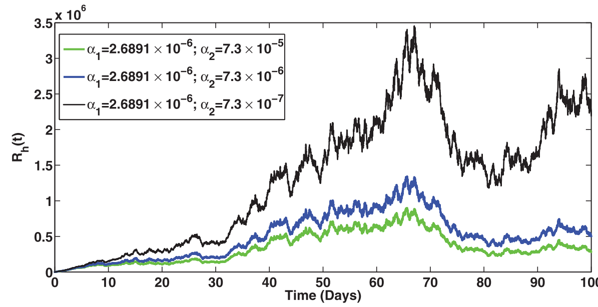


(**f**) Profile of $I_{ch}(t)$

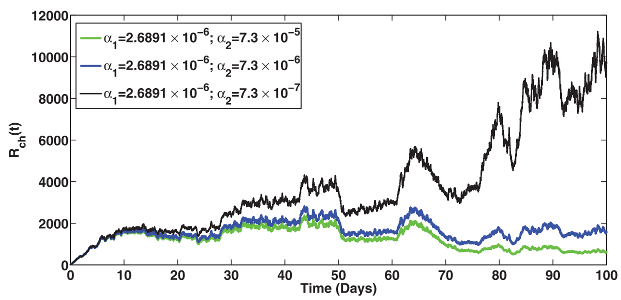
Figure 10: Solution profiles for the various compartments at different saturation effect, α_2, when $\beta_a = 5.1024 \times 10^{-9}$, $\beta_b = 1.0 \times 10^{-9}$, so that $\mathscr{R}_0^S = \max\{\mathscr{R}_{0a}^S, \mathscr{R}_{0b}^S\} = \max\{3.0323, 1.5789\} = 3.0323 > 1$



(a) Profile of $R_c(t)$



(b) Profile of $R_h(t)$



(c) Profile of $R_{ch}(t)$

Figure 11: Solution profiles for the various compartments at different saturation effect, α_2, when $\beta_a = 5.1024 \times 10^{-9}$, $\beta_b = 1.0 \times 10^{-9}$, so that $\mathscr{R}_0^S = \max\{\mathscr{R}_{0a}^S, \mathscr{R}_{0b}^S\} = \max\{3.0323, 1.5789\} = 3.0323 > 1$

6 Conclusion

In this paper, a comprehensive stochastic model was developed to assess the epidemiological effect of vaccine booster doses on the co-dynamics of viral hepatitis B and COVID-19 using the real data from Pakistan. The proposed model incorporates logistic growth and saturated incidence functions. Rigorous analyses employing the tools of stochastic calculus have been carried out to find appropriate conditions required for the existence of unique global solutions, stationary distribution in the sense of ergodicity and disease extinction. The stochastic threshold estimated from the data fitting is given by: $\mathscr{R}_0^S = 3.0651$. Numerical assessments are implemented to illustrate the impact of double dose vaccination and saturated incidence functions on the dynamics of both diseases. The effect of stochastic white noise intensities is also highlighted. Important highlights from the numerical assessment of the perturbed model are presented as follows:

(i) The perturbed system was fitted to the real COVID-19 data from Pakistan (depicted by Fig. 2) with stochastic threshold estimated at $\mathscr{R}_0^S = 3.0651$.

(ii) Increasing the COVID-19 primary vaccination rate to $\psi = 0.30$ and booster dose vaccination rate to the level $\theta_c = 0.20$, the least number of infections is recorded for all the infected components (including those infected with viral hepatitis B and co-infections, as observed in Figs. 6b and 6c).

(iii) It is noted from the experiments that increasing primary and booster dose vaccination rates for viral hepatitis B not only caused reduction in infected classes with viral hepatitis B (as seen in Fig. 7a, which is expected) but also brought about a noticeable reduction in the infected individuals with COVID-19 and the co-infection of both diseases (as can be seen in Figs. 7b and 7c).

However, further investigations to improve the present study with fewer limitations can lead to some new avenues of research. More efficient algorithms can be developed for the proposed model with some more realistic assumptions. The model can also consider time dependent contact rates and time delay. Asymptomatic classes can also be considered for both viruses, and data for both diseases to be used for more accurate model fittings. In the future, we shall consider the impact of quadratic Levy noise and variable diffusion rates on the dynamics of both diseases [57–59].

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Author Contributions: The authors confirm contribution to the paper as follows: study conception and design: A. O., M. A., D. B.; data collection: A. O., M. A., D. B.; analysis and interpretation of results: A. O., M. A., D. B.; draft manuscript preparation: A. O., M. A., D. B. All authors reviewed the results and approved the final version of the manuscript.

Availability of Data and Materials: All data used for the fitting of the model are available at "Pakistan: Coronavirus Pandemic Country Profile. Available online: https://ourworldindata.org/coronavirus/country/pakistan (accessed on 19/02/2023)".

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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