Computer Modeling in Engineering & Sciences

Tech Science Press

DOI: 10.32604/cmes.2023.031538

REVIEW

An Overview of Sequential Approximation in Topology Optimization of Continuum Structure

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Received: 30 June 2023 Accepted: 07 October 2023 Published: 30 December 2023

ABSTRACT

This paper offers an extensive overview of the utilization of sequential approximate optimization approaches in the context of numerically simulated large-scale continuum structures. These structures, commonly encountered in engineering applications, often involve complex objective and constraint functions that cannot be readily expressed as explicit functions of the design variables. As a result, sequential approximation techniques have emerged as the preferred strategy for addressing a wide array of topology optimization challenges. Over the past several decades, topology optimization methods have been advanced remarkably and successfully applied to solve engineering problems incorporating diverse physical backgrounds. In comparison to the large-scale equation solution, sensitivity analysis, graphics post-processing, etc., the progress of the sequential approximation functions and their corresponding optimizers make sluggish progress. Researchers, particularly novices, pay special attention to their difficulties with a particular problem. Thus, this paper provides an overview of sequential approximation functions, related literature on topology optimization methods, and their applications. Starting from optimality criteria and sequential linear programming, the other sequential approximate optimizations are introduced by employing Taylor expansion and intervening variables. In addition, recent advancements have led to the emergence of approaches such as Augmented Lagrange, sequential approximate integer, and non-gradient approximation are also introduced. By highlighting real-world applications and case studies, the paper not only demonstrates the practical relevance of these methods but also underscores the need for continued exploration in this area. Furthermore, to provide a comprehensive overview, this paper offers several novel developments that aim to illuminate potential directions for future research.

KEYWORDS

Topology optimization; sequential approximate optimization; convex linearization; method of moving asymptotes; sequential quadratic programming

Nomenclature

g_0	Objective function
g_j	Constraint functions
x	The design variable vector containing the component x_i
$\boldsymbol{K}$	The global stiffness matrix
$\boldsymbol{F}$	External load vector
$\boldsymbol{U}$	Nodal displacement vector
$\hat{g}_0(\boldsymbol{x})$	Surrogate function of $g_0(\boldsymbol{x})$
$\hat{g}_j(\boldsymbol{x})$	Surrogate function of $g_j(\boldsymbol{x})$
E_i	The ith elemental young's modulus
E_0	Young's modulus of solid material
$E_{\min}$	The minimum stiffness with typical value of $E_0/10^9$
λ	Lagrangian multiplier
$\widetilde{A}_{i,j}$	Approximate terms of Hessian matrix $\widetilde{\boldsymbol{A}}$
$\|\|\cdot\|\|_2$	Euclidean norm
$\widetilde{\boldsymbol{B}}^{(k)}$	Replacement of Hessian matrix
u, v	Penalization factor
V	Total volume in the structural design domain
c	Static compliance
$\overline{c}$	Upper limit of the static compliance
f_E	Interpolation for young's modulus
f_v	Interpolation for volume fraction
t_i	The ith independent topological variable
$\underline{x}_i$, $\overline{x}_i$	Maximum and minimum design variable
$\overline{y}(x)$	Intervening variable
u_i, l_i	Lower and upper bound for moving asymptotes
α_i, β_i, γ	Parameters in MMA algorithm
$\boldsymbol{H}$	Hessian matrix
$\boldsymbol{A}$	Hessian matrix in Lagrangian function

1 Introduction

In a seminal work published in 1988, Bendsøe et al. introduced the homogenization theory into structural design and proposed the so-called homogenization methodology for topology optimization of continuum structures [1]. Since then, numerous approaches have emerged including variable density methods [2,3], evolutionary structural optimization method [4] and its improved version–bidirectional evolutionary structural optimization method (BESO) [5], a level-set method [6,7], moving morphable components framework [8], a feature-driven method [9]. The latest review or monograph on a particular topology optimization approach can be found in the literature for the density-based method [10], for BESO method [11], for level-set method [12], for feature-driven method [13], the floating projection method and its extension [14–16]. The above-mentioned numerical methodologies tend to permeate each other, which sometimes distinguishes their differences strictly [17,18]. Nowadays, with the implementation of topology optimization functions in popular commercial software, such as ANSYS, MSC Nastran, Abaqus, Hyperworks, TOSCA Structure, and so on, topology optimization has achieved widespread success in a variety of engineering applications, particularly in the aircraft and aerospace industry [19–21], building [22], additive manufacturing [23,24], meta-material and

concurrent design [25,26] and energy industry [27–29], where weight, cost, and environment are strictly limited.

A multitude of design variable update schemes are now available in the topology optimization community [17]. Thanks to the educational paper and its open-source codes in all types of methods or one aspect of a specific problem, the beginners started their research without the need to grasp the sequential approximation concept and instead focused on the difficulties of a particular problem [30,31]. The rapid growth of topology optimization is largely attributable to the solid theoretical foundations established by the pioneers' researchers in this area. In contrast to other critical technologies such as sensitivity analysis, filtering techniques, post-processing, etc., the development of sequential approximations and their corresponding optimizers has been relatively slow. The motive of this article is to review the utilization of the sequential approximation in structural optimization studies, especially in topology optimization. Several novel progresses are presented, to irradiate further research.

2 General Mathematical Formulations for Topology Optimization

Let us consider the general topology optimization problem with an objective function g_0 total number of J constraint functions g_j:

$$\begin{aligned} &\text{find}: \ \boldsymbol{x} = \left\{x_1 \quad x_2 \quad \cdots \quad x_I\right\} \\ &\text{minimize}: \ g_0(\boldsymbol{x}) \\ &\text{subject to}: g_j(\boldsymbol{x}) \le 0 \ (j = 1, 2, \cdots, J) \\ &\qquad\qquad \underline{x}_i \le x_i \le \overline{x}_i \quad (i = 1, 2, \cdots, I) \end{aligned} \tag{1}$$

where $\boldsymbol{x}$ is the design variable vector containing the component $\boldsymbol{x}_i$, with the lower and upper bound $\underline{x}_i$ and $\overline{x}_i$.

Assuming linear elasticity for continua, the equilibrium equation can be stated as $\boldsymbol{KU} = \boldsymbol{F}$. Here $\boldsymbol{K}$ represents the global stiffness matrix and $\boldsymbol{F}$ is the external load vector. Solving the equilibrium equation yields a nodal displacement vector $\boldsymbol{U}$. It is virtually impossible to articulate the vector $\boldsymbol{U}$ as an explicit formula with design variables for complex engineering problems. However, the equilibrium equations for $\boldsymbol{U}$ can be solved numerically for any set of design variables.

By writing the displacements as functions of the design variables via solving equilibrium equations, we acquire the nested formulation of the optimization problem as follows:

$$\begin{aligned} &\text{find}: \ \boldsymbol{x} \\ &\text{minimize}: \ \hat{g}_0(\boldsymbol{x}) \\ &\text{subject to}: \hat{g}_j(\boldsymbol{x}) \le 0 \ (j = 1, 2, \cdots, J) \end{aligned} \tag{2}$$

where $\hat{g}_0(\boldsymbol{x})$ and $\hat{g}_j(\boldsymbol{x})$ correspond to the approximate or surrogate functions of $\mathbf{g}_0(\boldsymbol{x})$ and $\mathbf{g}_j(\boldsymbol{x})$. The optimization problem (2) will be solved by solving a sequence of explicit sub-problems. The efficiency of optimization relies heavily on the accuracy of the approximate function. Continuous design variables enable gradient-based optimization algorithms to be utilized in the majority of topology optimization methods. Due to the expense of calculating higher-order derivatives, the first-order derivatives are most prevalent, whereas second or higher-order-based methods are rarely employed [32].

In the following section, we will introduce the most popular optimization solvers in topology optimization, such as optimality criteria (OC), sequential linear programming (SLP), convex linearization (CONLIN), method of moving asymptotes (MMA), sequential quadratic programming (SQP), and others. Topology optimization of continuum structure process explained through a flowchart as shown in Fig. 1, via different steps:

Step 1: Define the design area for optimization

Step 2: Determine the design model-related parameters for topology optimization

Step 3: Implement the interpolation scheme for SIMP (Solid isotropic material with penalization) to predict material distribution within space.

Step 4: Perform sequential approximation (OC, SLP, CONLIN, SQP, MMA) for continuum structure optimization.

Step 5: Determine whether the convergence condition is satisfied, if satisfied then stop the iteration otherwise return to Step 4.

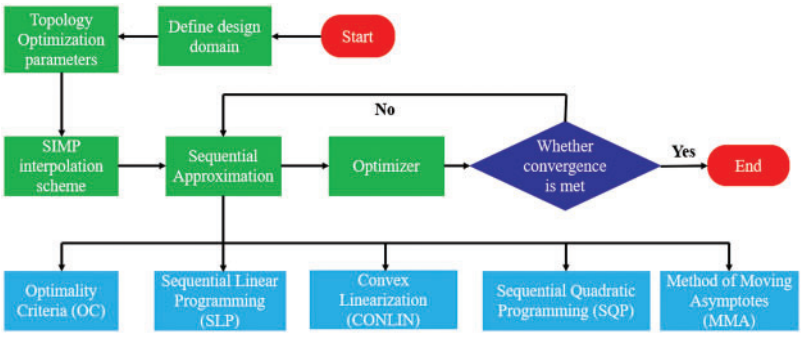


Figure 1: Flowchart for topology optimization of continuum structure using sequential approximation

2.1 Optimality Criteria

The variable density method, especially with the solid isotropic material with penalization (SIMP) interpolation, is undoubtedly one of the most renowned approaches due to its simple concept. The penalization scheme assumes a relationship between the elemental Young's modulus $\boldsymbol{E_i}$ with the relative density $\boldsymbol{\rho_i}$, i.e., $\boldsymbol{E_i} = \boldsymbol{\rho_i^p E_0}$. Here ($p \geq 1$) is the exponent power and $\boldsymbol{E_0}$ refers to Young's modulus of solid material. Applications of thousands of variables typically present optimization solution challenges. Formulated as follows is a heuristic design variable update scheme, also known as optimality criteria.

$$\rho_i^{\text{new}} = \begin{cases} \max\left(\underline{\rho}_i, \rho_i - m\right) & \text{if } \rho_i B_i^{\eta} \leq \max\left(\underline{\rho}_i, \rho_i - m\right) \\ \rho_i B_i^{\eta} & , \text{if } \max\left(\underline{\rho}_i, \rho_i - m\right) < \rho_i B_i^{\eta} < \min\left(1, \rho_i + m\right) \\ \min\left(1, \rho_i + m\right) & \min\left(1, \rho_i + m\right) \leq \rho_i B_i^{\eta} \end{cases} \tag{3}$$

where both the move-limit m and damping parameter η are employed to stabilize the optimization loop. The lower bound $\underline{\boldsymbol{\rho}}_i$ is used to ensure the non-singularity in the finite element analysis.

The term $\boldsymbol{B_i}$ in Eq. (3) can be defined as follows:

$$B_i = -\frac{\partial c}{\partial \rho_i} \Big/ \lambda \frac{\partial V}{\partial \rho_i} \tag{4}$$

where the static compliance c and structural design volume $\boldsymbol{V}$ are adopted as the objective and constraint functions for the static optimization problem. The symbol $\boldsymbol{\lambda}$ denotes the Lagrangian multiplier that can be attained by a bi-sectioning strategy to meet the predefined volume constraint. Due to the prevalence of the 99-line educational program [33], newcomers are more attracted to topology optimization research with the foregoing formulation.

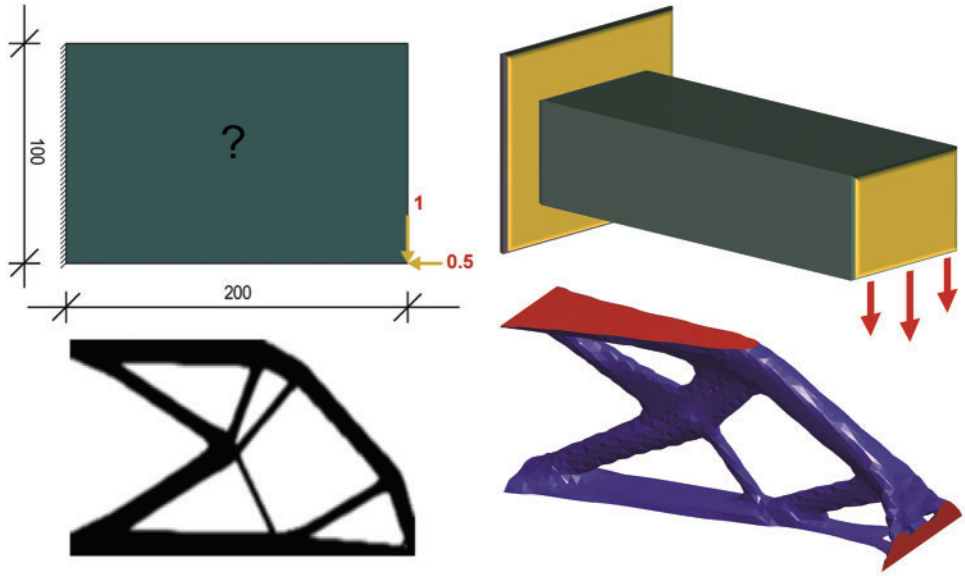
In the density-filter approach [34,35], a modified SIMP interpolation scheme is expressed as follows:

$$E_i = E_{\min} + (E_0 - E_{\min})\,\rho_i^p \tag{5}$$

where $E_{\min}$ has a similar function to the parameter $\underline{\rho_i}$ with the typical value of $\boldsymbol{E_0/10^9}$. Thus, $\boldsymbol{\rho_i}$ is employed in the finite element analysis, which serves as physical density. And it can be zero in the density-filter interpolation scheme [36,37].

In the presented OC algorithm, the Lagrange multiplier associated with a constraint is typically obtained via bisection search. Kumar and Suresh proposed a direct manner for the Lagrange multiplier, which exhibits several benefits including, fewer iterations, robust convergence, and insensitivity to the given material and load [38].

The intuition-based OC scheme permits efficient solutions of the computationally demanding problems in a relatively low number of iterations, especially for the compliance minimization issue with a single volume restriction. As all gradients have negative indicators, removing material will always increase compliance. However, conservative variable update by adjusting parameters allows the OC scheme to be applied to the flexible mechanism design and material design issues with material usage limits [39–41]. In addition, Amir proposed a compliance-contained OC technique employing a similar bisection strategy to resolve the volume minimization issue [42]. For instance, the optimized 2D and 3D cantilever structure obtained from the compliance-contained OC algorithm that is displayed in Fig. 2. Groenwold et al. demonstrated the consistency of the OC and the sequential optimization based on exponential intermediate variables [43,44]. In the following sections, the approximate function based on intermediate variables will be described.



(a) (b)

Figure 2: The optimized results from compliance-contained OC algorithm for: (a) 2D cantilever structure; (b) 3D cantilever structure

2.2 Sequential Linear Programming

For optimization problem (1), the objective function and all constraints can be linear at the design $\boldsymbol{x}^{(k)}$, resulting in the SLP sub-problem during the $\boldsymbol{k}th$ iteration:

$$\begin{aligned} &\text{find}: \ \boldsymbol{x} \\ &\text{minimize}: \ g_0\left(\boldsymbol{x}^{(k)}\right)+\nabla g_0\left(\boldsymbol{x}^{(k)}\right)\left(\boldsymbol{x}-\boldsymbol{x}^{(k)}\right) \\ &\text{subject to}: g_j\left(\boldsymbol{x}^{(k)}\right)+\nabla g_j\left(\boldsymbol{x}^{(k)}\right)\left(\boldsymbol{x}-\boldsymbol{x}^{(k)}\right) \leq 0 \ (j=1,2,\cdots,J) \end{aligned} \tag{6}$$

Eq. (6) is an explicit approximation of the original problem (1) after obtaining $\boldsymbol{g_0}$, $\boldsymbol{g_j}$ and all gradients for objective and constraint functions, which can be efficiently solved by mature algorithms in linear programming. As the linear Taylor expansion approximates locally, it is strongly advised to integrate a move limit into the actual optimization solution.

At present, solving topology optimization problems with SLP is uncommon. It is not surprising because the accuracy of this approximation is inferior to those that will follow in this paper. In the composite optimization problem, manufacturing constraints take the form of linear constraints, which makes the SLP algorithm more efficient at finding a solution [45,46].

2.3 Convex Linearization

Even for seasoned researchers, sequential structural approximation using reciprocal variables can be somewhat confounding. This is demonstrated by introducing an intervening variable $\boldsymbol{y}(\boldsymbol{x})$ for the sake of simplicity. The following expression can be derived from the linear Taylor series expansion:

$$\widetilde{g}\left(\boldsymbol{y}^{(k)}\right) \approx g\left(\boldsymbol{y}^{(k)}\right)+\nabla g\left(\boldsymbol{y}^{(k)}\right)\left(\boldsymbol{y}-\boldsymbol{y}^{(k)}\right)=g\left(\boldsymbol{x}^{(k)}\right)+\nabla g\left(\boldsymbol{y}^{(k)}\right)\left(\boldsymbol{y}-\boldsymbol{y}^{(k)}\right) \tag{7}$$

The partial derivative of g with regard to the intervening variable $\boldsymbol{y}_i$ can be calculated using the chain rule:

$$\frac{\partial g}{\partial y_i}=\frac{\partial g}{\partial x_i}\cdot\frac{dx_i}{dy_i}=\frac{\partial g}{\partial x_i}\frac{1}{\dfrac{dy_i}{dx_i}} \tag{8}$$

When we choose the equations $\boldsymbol{y}_i=\boldsymbol{x}_i$ or $\boldsymbol{y}_i=\boldsymbol{1/x}_i$, Eq. (7) can be reformulated as follows:

$$g^L\left(\boldsymbol{y}^{(k)}\right) \approx g\left(\boldsymbol{x}^{(k)}\right)+\sum_i \frac{\partial g}{\partial x_i}\left(x_i-x_i^{(k)}\right) \tag{9}$$

$$g^R\left(\boldsymbol{y}^{(k)}\right) \approx g\left(\boldsymbol{x}^{(k)}\right)+\sum_i \frac{\partial g}{\partial x_i}\frac{x_i^{(k)}\left(x_i-x_i^{(k)}\right)}{x_i} \tag{10}$$

The Eq. (9) is indeed the linear Taylor expansion, whereas the Eq. (10) is called the reciprocal Taylor expansion. To distinguish between two equations, the superscripts L and R are adopted.

Fleury defined the approximation of $\boldsymbol{g}$ at $\boldsymbol{x}^{(k)}$ by the combination of $\boldsymbol{g}^L$ and $\boldsymbol{g}^R$ in COLIN [47–49]:

$$\widetilde{g}^C\left(\boldsymbol{y}^{(k)}\right) \approx g\left(\boldsymbol{x}^{(k)}\right)+\sum_{i\in S_+} g^L+\sum_{i\in S_-} g^R \tag{11}$$

In Eq. (11), the sets are defined as follows:

$$S^+=\left\{i:\left.\frac{\partial g}{\partial x_i}\right|_{\boldsymbol{x}=\boldsymbol{x}^{(k)}}>0\right\}, \ S^-=\left\{i:\left.\frac{\partial g}{\partial x_i}\right|_{\boldsymbol{x}=\boldsymbol{x}^{(k)}}\leq 0\right\} \tag{12}$$

The COLIN is also known as a conservative approximation [50], i.e., for every possible sets $\boldsymbol{S}^{+}$ and $\boldsymbol{S}^{-}$, the following inequality is true:

$$\widetilde{g}^{C}\left(\boldsymbol{x}^{(k)}\right) \geq g^{L}\left(\boldsymbol{x}^{(k)}\right) \text{ and } \widetilde{g}^{C}\left(\boldsymbol{x}^{(k)}\right) \geq g^{R}\left(\boldsymbol{x}^{(k)}\right) \tag{13}$$

Thus, the sub-problem using CONLIN approximation can be reformulated:

$$\begin{aligned} &\text{find}: \ \boldsymbol{x} \\ &\text{minimize}: \ \widetilde{g}^{C}(\boldsymbol{x}) \\ &\text{subject to}: \widetilde{g}_{j}^{C}(\boldsymbol{x}) \leq 0 \, (j = 1, 2, \cdots, J) \end{aligned} \tag{14}$$

Based on inequality Eq. (13), it can be inferred that the solution of Eq. (14) must be more conservative, i.e., the objective function is larger than that of Eq. (1).

The CONLIN was furthered by multiple researchers. For example, Zhang et al. proposed a modified CONLIN approximation, which strengthens the convexity of the problem by introducing a convex factor [51]. In engineering software, the CONLIN approximation was successfully implemented in the early version of OptiStruct™, even as a milestone of topology optimization in this code [52]. Fig. 3 plots the optimized jacket structure of offshore wind turbine done by the authors' group using OptiStruct™.

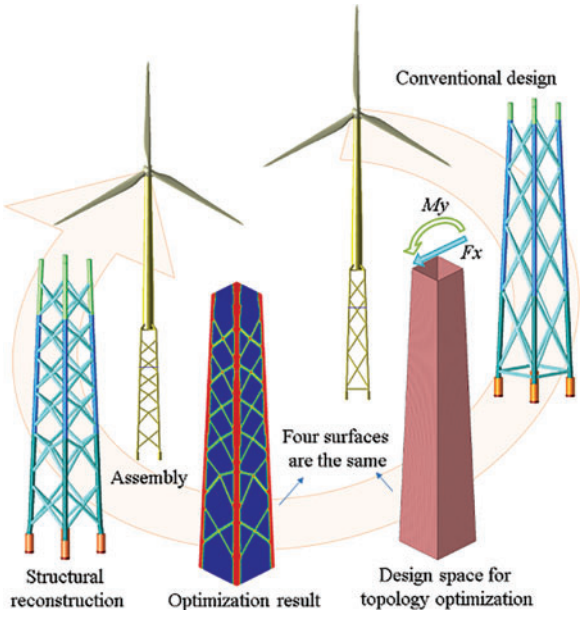


Figure 3: Optimization design procedure of jacket structure for offshore wind turbine

Example I We consider a fourth-order function $g(x) = x + x^2 - \frac{1}{60}x^3 - \frac{1}{50}x^4$ and evaluate the CONLIN approximation at $\boldsymbol{x}_1 = \mathbf{1}$ and $\boldsymbol{x}_2 = \mathbf{6}$.

By differentiating g, we obtain that $\frac{\partial g(x)}{\partial x} = 1 + 2x - \frac{1}{20}x^2 - \frac{2}{25}x^3$ and $\left.\frac{\partial g(x)}{\partial x}\right|_{x_1} = 2.87 > 0$, $\left.\frac{\partial g(x)}{\partial x}\right|_{x_2} = -6.08 < 0$. The CONLIN approximation is the linear and reciprocal approximation at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, respectively. For comparison, we also depict all curves representing $\boldsymbol{g}^{C}$, $\boldsymbol{g}^{L}$, and $\boldsymbol{g}^{R}$ in Fig. 4.

We can observe that the $\boldsymbol{g}^{C}$ is always greater than or equal to $\boldsymbol{g}^{L}$ and $\boldsymbol{g}^{R}$, which is the reason that the CONLIN approximation is also named as a conservative approximation.

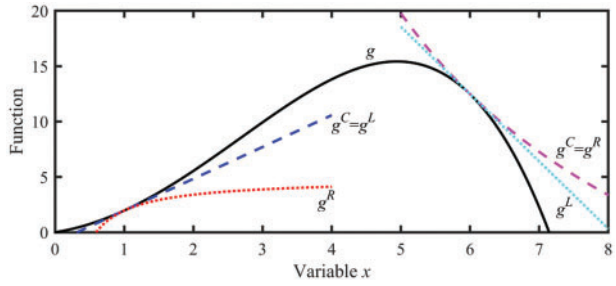


Figure 4: CONLIN approximation of the function g

2.4 The Method of Moving Asymptotes

Despite the fact that COLIN has demonstrated its efficacy for a variety of structural optimization problems, it occasionally converges slowly due to excessively conservative approximations. In contrast, it does not converge at all, indicating that it is insufficiently conservative. To stabilize the optimization process, Svanberg developed a variant of COLIN by constructing artificial asymptotes [53]. The intervening variables in MMA are specified as follows:

$$y_i = \frac{1}{x_i - l_i} \text{ and } y_i = \frac{1}{u_i - x_i} \tag{15}$$

where $\boldsymbol{l_i}$ and $\boldsymbol{u_i}$ are the moving asymptotes. Throughout the whole optimization process, the following in equation will always be satisfied:

$$l_i^{(k)} < x_i^{(k)} < u_i^{(k)} \tag{16}$$

In MMA, the approximating function at $\boldsymbol{x^{(k)}}$ can be expressed as follows:

$$g^M\left(\boldsymbol{x}^{(k)}\right) = \sum_i \left(\frac{\alpha_i^{(k)}}{u_i^{(k)} - x_i} + \frac{\beta_i^{(k)}}{x_i - l_i^{(k)}}\right) + \gamma^{(k)} \tag{17}$$

where

$$\alpha_i^{(k)} = \begin{cases} \left(u_i^{(k)} - x_i^{(k)}\right)^2 \left.\dfrac{\partial g}{\partial x_i}\right|_{\mathbf{x}=\mathbf{x}^{(k)}}, & \text{if } \left.\dfrac{\partial g}{\partial x_i}\right|_{\mathbf{x}=\mathbf{x}^{(k)}} > 0 \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

$$\beta_i^{(k)} = \begin{cases} 0 & \\ -\left(x_i^{(k)} - l_i^{(k)}\right)^2 \left.\dfrac{\partial g}{\partial x_i}\right|_{x=x^{(k)}} & \text{if } \left.\dfrac{\partial g}{\partial x_i}\right|_{x=x^{(k)}} \geq 0 \\ & \text{otherwise} \end{cases} \tag{19}$$

$$\gamma^{(k)} = g\left(\boldsymbol{x}^{(k)}\right) - \sum_i \left(\frac{\alpha_i^{(k)}}{u_i^{(k)} - x_i^{(k)}} + \frac{\beta_i^{(k)}}{x_i^{(k)} - l_i^{(k)}}\right) \tag{20}$$

Thus, the approximate structural optimization problem using MMA can be rewritten:

$$\begin{aligned} &\text{find}: \ \boldsymbol{x} \\ &\text{minimize}: \ \widetilde{g}^M(\boldsymbol{x}) \\ &\text{subject to}: \widetilde{g}_j^M(\boldsymbol{x}) \leq 0 \ (j = 1, 2, \cdots, J) \end{aligned} \tag{21}$$

For the SIMP method, the MMA algorithm is widely regarded as one of the most dependable and efficient optimizers [17]. Followed by MMA, Svanberg proposed a class of globally convergent versions of MMA (GCMMA), also taking into account the optimization efficiency [54]. In GCMMA, the parameter $\boldsymbol{\alpha}_i^{(k)}$ and $\boldsymbol{\beta}_i^{(k)}$ corresponding Eqs. (18) and (19) can be concurrently nonzero, leading to the approximation's non-monotonic behavior.

Example II We choose the same function $\boldsymbol{g}$ as in Example 1, aiming to illustrate the MMA approximation. The upper asymptote is set as 1.5, 4, 20, and 10^4. Fig. 5 plots the MMA approximations for various values of the upper asymptotes.

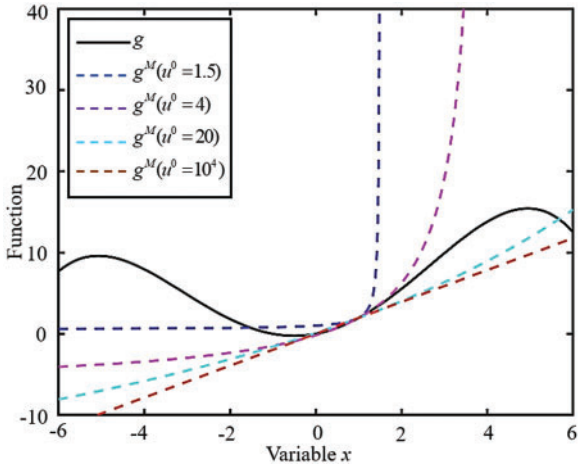


Figure 5: MMA approximation of the function g

We can see that as the upper asymptote approaches infinity from Fig. 5, the MMA approximation is almost linear, which is in agreement with the SLP.

In addition, the MMA algorithm is also used as the optimizer in various topology optimization methods, such as the stiffness spreading method [55–57], parameterized level-set method [58], an approach driven by MMC and moving morphable bars [59–61], series-expansion framework [62–64], the iso-geometric based method [65]. In addition to the compliance minimization problem, MMA is also applied to various non-self-adjoint problems, such as stress-constrained problems [66,67], fiber orientation optimization problems [68], transient excited and geometrically nonlinear structures [69,70], transient heat conduction [71], fail-safe design [72]. Among them, the default parameters may be different, and some numerical skills and experience are required for some particular methodologies or problems. As far as the authors are aware, the sensitivity-based topology optimization solver in commercial software TOSCA Structure™ is the basis of the MMA algorithm. Fig. 6 depicts an optimized mainframe in wind turbine by TOSCA Structure™.

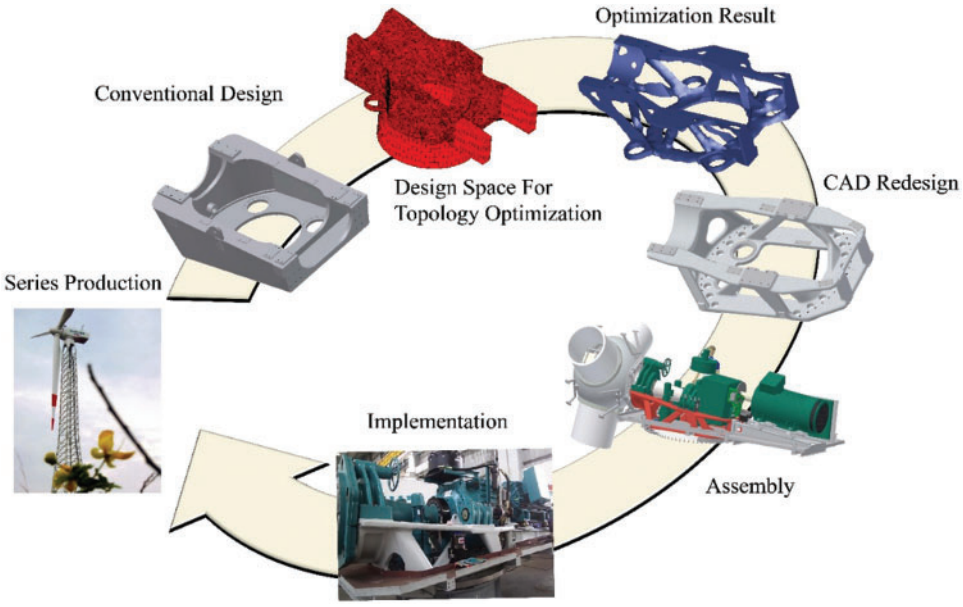


Figure 6: A optimized mainframe in wind turbine by TOSCA Structure™

2.5 Two-Point or Three-Point Approximation

The preceding approximation function is characterized by the first order approximation and makes use of current data. On the basis of previous optimization iterations, it is anticipated that more precise approximations over a broader range can be attained.

Typically, Fadel et al. proposed a two-point approximation with the intervening variables [73]:

$$y_i = x_i^{\mu_i} \tag{22}$$

According to Eq. (7), we can obtain the following approximate function:

$$\widetilde{g}\left(\boldsymbol{x}^{(k)}\right) \approx g\left(\boldsymbol{x}^{(k)}\right) + \sum_i \left.\frac{\partial g}{\partial x_i}\right|_{\boldsymbol{x}=\boldsymbol{x}^{(k)}} \left.\frac{1}{\mu_i x_i^{\mu_i - 1}}\right|_{\boldsymbol{x}=\boldsymbol{x}^{(k-1)}} \left(x_i^{\mu_i} - x_i^{\mu_i}\big|_{x_i = x_i^{(k)}}\right) \tag{23}$$

In Eq. (23), the undetermined parameter $\boldsymbol{\mu_i}$ will be achieved based on the gradient information from the previous iteration which yields:

$$\left(\frac{x_i^{(k-1)}}{x_i^{(k)}}\right)^{\mu_i - 1} = \left.\frac{\partial g}{\partial x_i}\right|_{\boldsymbol{x}=\boldsymbol{x}^{(k-1)}} \Big/ \left.\frac{\partial g}{\partial x_i}\right|_{\boldsymbol{x}=\boldsymbol{x}^{(k)}} \tag{24}$$

It is not surprising that several two-point or three-point approximation functions were proposed to enhance approximate accuracy and expand the approximate range, the majority of which were numerically tested by mathematical problems and truss optimization problems [74–81].

2.6 Sequential Quadratic Programming

When the second-order term is appended in the Taylor expansion of the objective function in SLP, the following SQP-based approximation occurs:

$$\begin{aligned} &\text{find}: \ \boldsymbol{x} \\ &\text{minimize}: \ g_0\left(\boldsymbol{x}^{(k)}\right)+\nabla g_0\left(\boldsymbol{x}^{(k)}\right)\left(\boldsymbol{x}-\boldsymbol{x}^{(k)}\right)+\frac{1}{2}\left(\boldsymbol{x}-\boldsymbol{x}^{(k)}\right)^{\mathrm{T}}\boldsymbol{H}\left(\boldsymbol{x}^{(k)}\right)\left(\boldsymbol{x}-\boldsymbol{x}^{(k)}\right) \\ &\text{subject to}: g_j\left(\boldsymbol{x}^{(k)}\right)+\nabla g_j\left(\boldsymbol{x}^{(k)}\right)\left(\boldsymbol{x}-\boldsymbol{x}^{(k)}\right)\leq 0 \, (j=1,2,\cdots,J) \end{aligned} \tag{25}$$

where the Hessian matrix $\boldsymbol{H}$ can be viewed as the only distinction between the SQP and the SLP. According to matrix $\boldsymbol{H}$, structural optimization problems solved by SQP can be roughly classified into two groups, i.e., SQP with approximate Hessian and exact Hessian. In comparison to the first-order approximation, topology optimization of continuum structure has not been a particularly fruitful domain for quadratic approximations.

2.6.1 SQP with Approximate Hessian

One way to construct the SQP is the utilization of Newton's method to find the stationary point of the Lagrangian function:

$$L\left(\boldsymbol{x},\boldsymbol{\lambda}\right)=g_0\left(\boldsymbol{x}\right)-\sum\lambda_j g_j\left(\boldsymbol{x}\right) \tag{26}$$

where $\boldsymbol{\lambda}_j$ represents the Lagrangian multipliers associated with the constraints $\boldsymbol{g_j(x)}$. Take a second-order Taylor expansion of the Lagrange function at $\boldsymbol{x^{(k)}}$:

$$L=L\left(\boldsymbol{x}^{(k)},\boldsymbol{\lambda}^{(k)}\right)+\nabla L\left(\boldsymbol{x}^{(k)},\boldsymbol{\lambda}^{(k)}\right)\left(\boldsymbol{x}-\boldsymbol{x}^{(k)}\right)+\frac{1}{2}\left(\boldsymbol{x}-\boldsymbol{x}^{(k)}\right)^{\mathrm{T}}\boldsymbol{A}\left(\boldsymbol{x}-\boldsymbol{x}^{(k)}\right) \tag{27}$$

The complete Hessian matrix $\boldsymbol{A}$ of the Lagrangian function in the Lagrange-Newton method can be calculated through Eq. (28).

$$\boldsymbol{A}=\frac{\partial^2 L}{\partial x_i \partial x_k} \tag{28}$$

The computation and storage of Hessian matrix $\boldsymbol{A}$ are burdensome due to the large number of design variables in structural optimization problems. To utilize the second derivative information efficiently in structural optimization problems, Fleury developed the diagonal SQP method and introduced parameters δ_i to control the move limit of design variables [82–84].

$$\widetilde{A}_{i,j}=\begin{cases} A_{i,j}+\delta_i & (j=i) \\ 0 & (j\neq i) \end{cases} \tag{29}$$

where $\widetilde{\boldsymbol{A}}_{\boldsymbol{i,j}}$ are the terms of the approximate Hessian Matrix $\widetilde{\boldsymbol{A}}$. The coupling between design variables is neglected and the Hessian matrix $\widetilde{\boldsymbol{A}}$ is restricted to its diagonal terms. The second derivative information is partially evaluated and employed. However, the diagonal SQP method could obtain superlinear convergence in most cases.

In certain practical situations, the second-order sensitivity information is undesirable. To overcome these difficulties, Grovenwold et al. proposed an incomplete series expansion (ISE) in which the approximate Hessian matrix $\widetilde{\boldsymbol{H}}$ is constructed using first-order gradient information in the current design point and objective function in historic design points [85–88]:

$$\widetilde{\boldsymbol{H}}=\frac{2\left[g_0(\boldsymbol{x}^{(k-1)})-g_0(\boldsymbol{x}^{(k)})-\nabla g_0(\boldsymbol{x}^{(k)})(\boldsymbol{x}^{(k-1)}-\boldsymbol{x}^{(k)})\right]}{||\boldsymbol{x}^{(k-1)}-\boldsymbol{x}^{(k)}||_2^2} \tag{30}$$

where the symbol $||\cdot||_2$ denotes the Euclidean norm. The convexity of ISE approximation can be enforced by restricting the diagonal terms $\widetilde{\boldsymbol{H}}_{\boldsymbol{i,i}}^{(\boldsymbol{k})}$ to be zero or positive. In addition, the approximate

Hessian matrix might contain even higher-order derivative information, deriving a family of approximation functions.

TopSQP is an efficient second-order SQP algorithm developed by Rojas-Labanda and Stople for structural topology optimization [89]. The TopSQP optimization framework concludes with two phases: an inequality quadratic phase (IQP), in which an inequality-constrained convex quadratic subproblem is solved, and an equality-constrained quadratic phase (EQP), in which the active constraints found for the IQP are implemented. Both the IQP and EQP phases utilize the approximate Hessian $\widetilde{\boldsymbol{H}}_1$ of the Lagrangian function:

$$\widetilde{\boldsymbol{H}}_1 = 2\boldsymbol{F}(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{K}^{-1}(\boldsymbol{x}) \ \boldsymbol{F}(\boldsymbol{x}) \tag{31}$$

$$\boldsymbol{F}(x) = \left(\frac{\partial \boldsymbol{K}_1(x_1)}{\partial x_1} \boldsymbol{u}(\boldsymbol{x}) \cdots \frac{\partial \boldsymbol{K}_n(x_n)}{\partial x_n} \boldsymbol{u}(\boldsymbol{x}) \right) \tag{32}$$

Rojas-Labanda and Stople conducted a comprehensive benchmark of topology optimization problems in conjunction with various optimizers, such as OC, MMA, and SQP. They concluded that the second-order information aids in obtaining accurate results and that SQP outperforms all other solutions for classical benchmark solvers [90]. For issues involving Stokes flows, Evgrafov developed a method for minimizing dissipated power that converges locally [91,92].

Recently, Zhang et al. [93] and Yan et al. [94] applied the SQP with approximate Hessian in discrete material optimization. Referring to Powell's work [95], the Hessian matrix is replaced by an approximate matrix $\widetilde{\boldsymbol{B}}^{(k)}$:

$$\widetilde{\boldsymbol{B}}^{(k+1)} = \widetilde{\boldsymbol{B}}^{(k)} + \frac{\boldsymbol{h}^{(k)} (\boldsymbol{h}^{(k)})^{\mathrm{T}}}{(\boldsymbol{h}^{(k)})^{\mathrm{T}} \boldsymbol{s}^{(k)}} - \frac{\widetilde{\boldsymbol{B}}^{(k)} \boldsymbol{s}^{(k)} \left(\widetilde{\boldsymbol{B}}^{(k)} \boldsymbol{s}^{(k)} \right)^{\mathrm{T}}}{(\boldsymbol{s}^{(k)})^{\mathrm{T}} \widetilde{\boldsymbol{B}}^{(k)} \boldsymbol{s}^{(k)}} \tag{33}$$

where $\boldsymbol{s}^{(k)} = \boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}$

$$\boldsymbol{h}^{(k)} = \nabla L(\boldsymbol{x}^{(k+1)}, \boldsymbol{\lambda}^{(k+1)}) - \nabla L(\boldsymbol{x}^{(k)}, \boldsymbol{\lambda}^{(k+1)}) \tag{34}$$

As long as the $\widetilde{\boldsymbol{B}}^{(k)}$ is positive definite, it is possible to ensure the positive definiteness of $\widetilde{\boldsymbol{B}}^{(k+1)}$, and the algorithm will converge globally.

Generally, the SQP with approximate Hessian has garnered the interest of numerous academicians in structural optimization. No matter how the approximate methodology differs, one common pursuit is to obtain faster convergence at the lower calculation cost of sensitivity information.

2.6.2 SQP with Exact Hessian

Different from the aforementioned SQP family algorithms constructed second-order information based on mathematical programming, Sui et al. proposed a novel formulation, also known as the independent continuous mapping (ICM) method in 1996 [96], which can be viewed as an extension of the size optimization problem proposed [97–101]. This method is regarded to achieve topology optimization through material distribution. The description of the first letter "**I**" in the ICM method represents the topological variable of the $\boldsymbol{ith}$ element $\boldsymbol{t_i}$ is independent of the physical parameters such as section area, relative density, and so on. Also, Young's modulus and elemental volume are independently defined:

$$E_i = f_E(t_i) E_0, v_i = f_v(t_i) v_0 \tag{35}$$

where $\boldsymbol{f_E(t_i)}$ and $\boldsymbol{f_v(t_i)}$ relate to Young's modulus and volume of solid material with the topological variable, respectively.

A typical formulation for the function in Eq. (35) can be written as follows:

$$f_E(t_i) = t_i^{\mu}, f_v(t_i) = t_i^{v} \tag{36}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{v}$ are the penalization factor.

The design variables $\boldsymbol{x_i}$ have the form of the reciprocal function of $\boldsymbol{f_E(t_i)}$, i.e.,

$$x_i = \frac{1}{f_E(t_i)} \tag{37}$$

When Eq. (37) is substituted into Eq. (38), it yields:

$$t_i = x_i^{-1/\mu} \tag{38}$$

The elemental volume can be rewritten as follows:

$$f_v(t_i) = t_i^{v} = x_i^{-v/\mu} \tag{39}$$

According to Eq. (40), the first and second order derivatives with respect to $\boldsymbol{x_i}$ can thus be calculated:

$$\frac{\partial f_v(t_i)}{x_i} = -\frac{v}{\mu} x_i^{-(v/\mu+1)}, \frac{\partial f_v^2(t_i)}{\partial {x_i}^2} = \frac{v(v+1)}{\mu^2} x_i^{-(v/\mu+2)} \tag{40}$$

In contrast to prevalent density methods, the ICM method focuses on minimizing the total volume or weight while maintaining constraints on various structural responses. Taking the compliance constraint as an example, the topology optimization formulated can be mathematically stated as follows:

$$\begin{aligned} &\text{find } \boldsymbol{x} \\ &\text{minimize}: V = \textstyle\sum_i v_i \, (i = 1, 2, \cdots, I) \\ &\text{subject to}: c \le \overline{c} \\ &\qquad \underline{x_i} \le x_i \le \overline{x}_i \, (i = 1, 2, \cdots, I) \end{aligned} \tag{41}$$

where $\boldsymbol{V}$ and $\boldsymbol{\overline{c}}$ represent the total volume in the structural design domain and the upper limit of the static compliance, respectively. $\boldsymbol{\underline{x_i}}$ and $\boldsymbol{\overline{x}_i}$ are the minimum and maximum design variables.

The volume function and compliance function can therefore be expressed by first-order and second-order Taylor expansion series. The original topology optimization problem can be converted as a quadratic program with second-order information, which provides another distinguishing feature over the widely used first-order method. From Eq. (41), we can easily obtain the second derivatives in Eq. (40) are always greater than zero. It can be inferred that the Hessian is positive definite and separable, which brings much allowing for efficient solution of the computationally demanding problem in a reasonable number of iterations.

Since the ICM approach was proposed, its applications have undergone tremendous developments with the efforts of their groups, also propelling the industry forward at the breakneck speed [102]. Up to now, the ICM method has been successfully applied to various constraints on structural response, including multiple nodal displacements, natural frequency, buckling, and so on [103–109]. In recent years, Peng et al. conducted systematic research based on the independence of design variables, by introducing the step function into the material property modeling [110–112]. Fig. 7 plots a typical optimized structure obtained from the ICM approach.

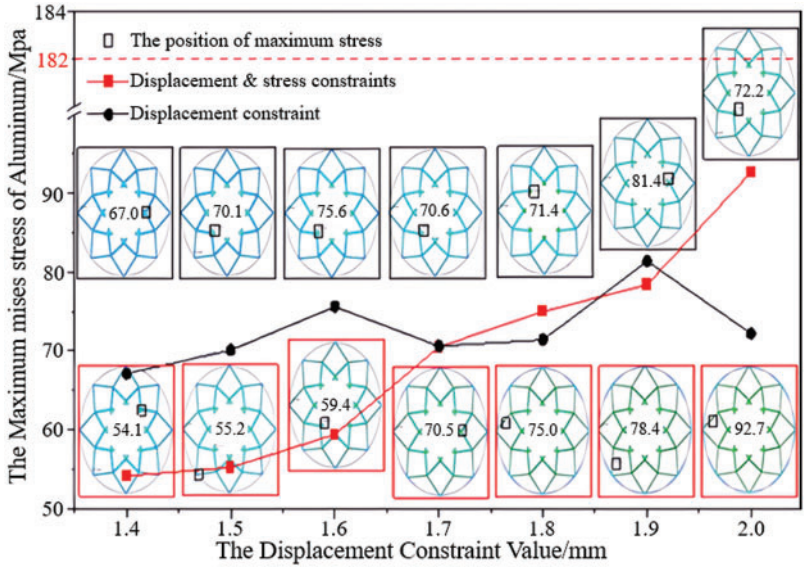


Figure 7: Optimized structures obtained from ICM approach

The ICM method aroused the attention of other scholars. Long et al. extended this method into the framework of meshless analysis [113], the stress-constrained problem for continuum structure subject to harmonic excitation [114], forced vibration structure containing multiple materials [115], transient heat transfer problem [116], concurrent design considering load carrying capabilities and thermal insulation [117], large-scale computing problem resort to reanalysis technique [118], fail-safe design combined with the load uncertainty [119], etc. [120]. Rong et al. introduced a design space expansion strategy to stabilize the ICM optimization process [121–123].

2.7 Augmented Lagrange

In recent years, the AL method has emerged as a viable approach to topology optimization, especially for extensive constraints. The AL method addresses constraints directly by appending them to the objective function as a penalty term with variable parameters. AL is not a novel concept in the field of structural optimization. For instance, the parameterized level set method has been successfully implemented to enforce a sole volume constraint [124].

Utilizing three phase projections including eroded, intermediate, and dilated, da Silva et al. employed the AL function for stress-constrained topology optimization problems while accounting for manufacturing uncertainties [125], which was then extended to the robust design of the compliant mechanism subject to both strength design requirement and manufacturing uncertainty [126]. For mass minimization under local stress constraints, Senhora et al. provided an AL-based topology optimization formulation by combining piecewise vanishing constraints [127]. Later, Giraldo-Londono et al. generalized the AL technique for the transient topology optimization issue by including stress constraints at each time step [128]. An aggregation-free local volume proportion formulation for porous structure was presented by Long et al. [129], which was subsequently developed into a multi-material porous structure [130]. The AL method is also performed to the topology optimization under constraints of multiple nodal displacements, maximum transient responses problem, and fatigue-resistance issue [131–133]. A porous bone structure using local volume constraint generated by the AL method is displayed in Fig. 8.

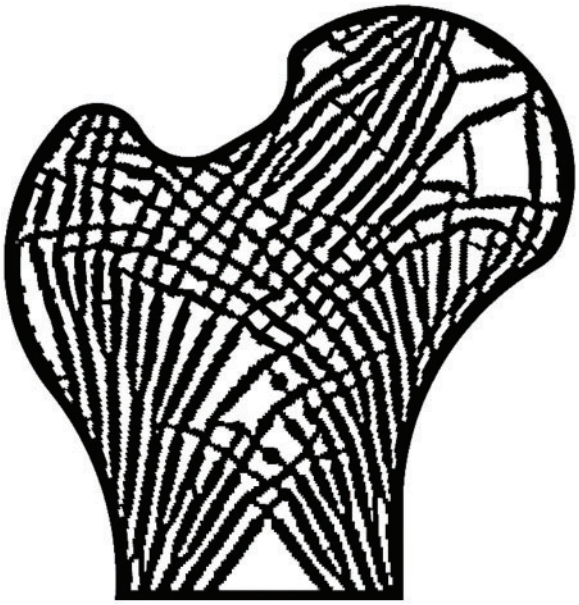


Figure 8: The porous structure generated by the AL method using local volume fraction

2.8 Sequential Approximate Integer Programming

BESO is the predominant discrete variable-based topology optimization approach. More recently, Sivapuram et al. treated topology optimization as a discrete variables-based optimized problem [134]. In their formulation, the initial optimization problem is transformed into SLP, which is then solved by integer linear programming (ILP). They expanded binary structures method into continuum structures subject to fluid structure, fluid flow, and thermal expansion loads via such a fundamental innovation [135–137]. And, they released the open-source code based on Matlab™ for distribution [138].

Liang et al. suggested a sequential approximate integer programming with a trust region framework to restrict the range of discrete design variables by linearizing the non-linear trust region constraint [139]. This provided method was also extended into 3D structures and convective heat transfer problems [140,141].

2.9 Non-Gradient Approximation

The majority of the existing topology optimization method is solved by the gradient-based algorithm, which is due in large part to the efficient sensitivity analysis approach. Sigmund gave a comprehensive analysis of the non-gradient topology optimization from multiple aspects including global solution, discrete designs, simple implementation, and efficiency, particularly for the SIMP method [142].

The two-point or three-point approximation belongs to the mid-range approximation. Since topology optimization requires repeated iterations until convergence, it is a natural choice to construct approximation functions using multi-point information to expand its approximate range. The approximation of this type can also be regarded as the connection of many local approximations, such as response surface and kriging model. Wang et al. presented a Hermite interpolation function using multi-point data generated during the iterative process of optimization [143]. Huang et al. proposed a multi-point approximation by utilizing both the value of an implicit function and its

derivatives [144,145]. Although the multi-point approximation technique has been used in truss optimization, to the authors' best knowledge, the multi-point approximation has not been performed in the topology optimization of continuum structures.

Luo et al. described structural topologies using the material-field series expansion, with the series expansion coefficients serving as the design variables [62]. This method has the added benefit of producing topologies with smooth boundaries. As a significant reduction of design variables, the structural approximation can be constructed on sensitivity or non-gradient data, such as Kriging models [146]. As sensitivity derivation is avoided, the non-gradient approach with few design variables is now effectively applied to large deformation problems, micro-structural design, etc. [147–151].

Recent years have witnessed rapid progress in artificial intelligence and neural networks. Some researchers have focused on topology optimization using these techniques, in an effort to accelerate the optimization iterations or enhance graphics post-processing. AI technology is used to establish the implicit connection between structural response and design variables. Woldseth et al. performed a comprehensive analysis of the combination of artificial neural networks and topology optimization [152]. Consequently, these associated studies fall outside the scope of this article.

2.10 Future Study

Authors are aware that the number of applicable optimizers is relatively limited, particularly for a wide range of multiphysics topology optimization with nontrivial and multiple constraints. For decades, the MMA and its globally convergent variant have been regarded as the most reliable optimizers. The authors conclude that inadequate research has been conducted on the use of contemporary mathematical programming techniques to solve large-scale, complex topology optimization problems. The AL method, ILP, and optimization algorithm based on non-gradient approximation require further development.

3 Conclusion

Sequential approximation, a crucial technique in topology optimization, has attracted a great deal of interest since the beginning of structural optimization. After decades of research advancements in topology optimization, the community has settled on a handful of sequential approximations. MMA and its global convergent version become dominant among them. This allows researchers to concentrate on other essential technologies. This paper provides a comprehensive overview of sequential approximation, its related topology optimization methods, and its applications. The initial section provides a concise introduction to the optimality criteria and sequential linear programming. The subsequent section introduces the intervening variables in order to explore various forms of sequential approximation, including COLIN, MMA, two-point or three-point approximation, and SQP. This paper presents the latest improvements in the field, including AL function, sequential approximate integer programming, and non-gradient approximation, aiming to aid researchers effectively choosing the most suitable approximate form for their studies. It is anticipated that a forthcoming proposal will present a notable advancement in the field of topology optimization, specifically in relation to sequential approximation.

Acknowledgement: We thank Professor Yunkang Sui (Beijing University of Technology) partly for his pioneering research on topology optimization solved by SQP since 1996, and partly for his personality on the first author's scientific career.

Funding Statement: This work was financially supported by the National Key R&D Program (2022YFB4201302), Guang Dong Basic and Applied Basic Research Foundation (2022A1515240057), and the Huaneng Technology Funds (HNKJ20-H88).

Author Contributions: The authors confirm their contribution to the paper as follows: study concept, writing, and interpretation of results: Kai Long, Ayesha Saeed; data collection: Jinhua Zhang, Yara Diaeldin, and Feiyu Lu; analysis and design: Tao Tao, Yuhua Li; draft manuscript preparation: Pengwen Sun, Jinshun Yan. All authors reviewed the results and approved the final version of the manuscript.

Availability of Data and Materials: Data will be provided on request.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

References

1. Bendsøe, M. P., Kikuchi, N. (1988). Generating optimal topologies in structural design using a homogenization method. *Computer Methods in Applied Mechanics and Engineering, 71(2),* 197–224.
2. Bendsøe, M. P. (1989). Optimal shape design as a material distribution problem. *Structural Optimization, 1(4),* 193–202.
3. Zhou, M., Rozvany, G. I. N. (1991). The COC algorithm, part II: Topological, geometry and generalized shape optimization. *Computer Methods in Applied Mechanics and Engineering, 89(1–3),* 309–336.
4. Xie, Y. M., Steven, G. P. (1993). A simple evolutionary procedure for structural optimization. *Computers and Structures, 49(5),* 885–896.
5. Huang, X., Xie, Y. M. (2010). *Evolutionary topology optimization of continuum structures: Methods and applications*. John Wiley & Sons. https://doi.org/10.1002/9780470689486
6. Allaire, G., Jouve, F., Toader, A. M. (2002). A level-set method for shape optimization. *Comptes Rendus Mathematique, 334(12),* 1125–1130.
7. Wang, M. Y., Wang, X., Guo, D. (2003). A level set method for structural topology optimization. *Computer Methods in Applied Mechanics and Engineering, 192(1–2),* 227–246.
8. Guo, X., Zhang, W., Zhong, W. (2014). Doing topology optimization explicitly and geometrically–A new moving morphable components based framework. *Journal of Applied Mechanics, 81(8),* 081009.
9. Zhou, Y., Zhang, W., Zhu, J., Xu, Z. (2016). Feature-driven topology optimization method with signed distance function. *Computer Methods in Applied Mechanics and Engineering, 310(1),* 1–32.
10. Sigmund, O. (2020). EML webinar overview: Topology optimization–status and perspectives. *Extreme Mechanics Letters, 39,* 100855.
11. Xia, L., Xia, Q., Huang, X., Xie, Y. M. (2018). Bi-directional evolutionary structural optimization and advanced structures and materials: A comprehensive review. *Archives of Computational Methods in Engineering, 25,* 437–478.
12. Dijk, N. P., Maute, K., Langellar, M., Keulen, F. (2013). Level-set methods for structural topology optimization: A review. *Structural and Multidisciplinary Optimization, 48,* 437–472.
13. Zhang, W., Zhou, Y. (2022). *The feature-driven method for structural optimization*, pp. 335–340. Elsevier. https://doi.org/10.1016/C2019-0-03253-0
14. Huang, X. (2020). Smooth topological design of structures using the floating projection. *Engineering Structures, 208,* 110330.
15. Fu, Y. F., Rolfe, B., Louis, N. S., Wang, Y., Huang, X. et al. (2020). SEMDOT: Smooth-edged material distribution for optimizing topology algorithm. *Advances in Engineering Software, 150,* 102921.

16. Fu, Y. F., Rolfe, B. (2023). Non-penalization topology optimization for maximizing natural frequency using SEMDOT. *Proceedings of IASS Annual Symposia*, vol. 2023, no. 1, pp. 1–7. International Association for Shell and Spatial Structures (IASS).
17. Sigmund, O., Maute, K. (2013). Topology optimization approaches–A comparative review. *Structural and Multidisciplinary Optimization, 48,* 1031–1055.
18. Sigmund, O. (2022). On benchmarking and good scientific practice in topology optimization. *Structural and Multidisciplinary Optimization, 65,* 315.
19. Zhu, J. H., Zhang, W. H., Xia, L. (2016). Topology optimization in aircraft and aerospace structures. *Archives of Computational Methods in Engineering, 23,* 595–622.
20. Zhu, J. H., Zhou, H., Wang, C., Zhou, L., Yuan, S. et al. (2021). A review of topology optimization for additive manufacturing: Status and challenges. *Chinese Journal of Aeronautics, 34(1),* 91–110.
21. Wu, Y., Qiu, W., Xia, L., Li, W., Feng, K. (2021). Design of an aircraft engine bracket using stress-constrained bi-directional evolutionary structural optimization method. *Structural and Multidisciplinary Optimization, 64,* 4147–4159.
22. Beghini, L. L., Beghini, A., Katz, N., Baker, W. F., Paulino, G. H. (2014). Connecting architecture and engineering through structural topology optimization. *Engineering Structures, 59,* 716–726.
23. Wang, X., Xu, S., Zhou, S., Xu, W., Leary, M. et al. (2016). Topological design and additive manufacturing of porous metal for bone scaffolds and orthopaedic implants: A review. *Biomaterials, 83,* 127–141.
24. Meng, L., Zhang, W., Quan, D., Shi, G., Tang, L. et al. (2020). From topology optimization design to additive manufacturing: Today's success and tomorrow's roadmap. *Archives of Computational Methods in Engineering, 27,* 805–830.
25. Long, K., Yuan, P. F., Xu, S., Xie, Y. M. (2017). Concurrent topological design of composite structures and materials containing multiple phases of distinct poisson's ratios. *Engineering Optimization, 50(4),* 599–614.
26. Da, D., Cui, X. Y., Long, K., Li, G. Y. (2017). Concurrent topological design of composite structures and the underlying multi-phase materials. *Computers and Structures, 179,* 1–14.
27. Zhang, C., Long, K., Zhang, J., Lu, F., Bai, X. et al. (2022). A topology optimization methodology for the offshore wind turbine jacket structure in the concept phase. *Ocean Engineering, 262,* 112974.
28. Lu, F., Long, K., Zhang, C., Zhang, J., Tao, T. (2023). A novel design of the offshore wind turbine tripod structure using topology optimization methodology. *Ocean Engineering, 280,* 114607.
29. Lu, F., Long, K., Diaeldin, Y., Saeed, A., Zhang, J. (2023). A time-domain fatigue damage assessment approach for the tripod structure of off-shore wind turbine. *Sustainable Energy Technologies and Assessments, 60,* 103450.
30. Wang, C., Zhao, Z., Zhou, M., Sigmund, O., Zhang, X. S. (2021). A comprehensive review of educational articles on structural and multidisciplinary optimization. *Structural and Multidisciplinary Optimization, 64,* 2827–2880.
31. Wang, Y., Li, X., Long, K., Wei, P. (2023). Open-source codes of topology optimization: A summary for beginners to start their research. *Computer Modeling in Engineering & Sciences, 137(1),* 1–34. https://doi.org/10.32604/cmes.2023.027603
32. Barthelemy, J. F. M., Haftka, R. T. (1993). Approximation concepts for optimum structural design–A review. *Structural Optimization, 5,* 129–144.
33. Sigmund, O. (2001). A 99 line topology optimization code written in Matlab. *Structural and Multidisciplinary Optimization, 21,* 120–127.
34. Bruns, T. E., Tortorelli, D. A. (2001). Topology optimization of non-linear elastic and compliant mechanisms. *Computer Methods in Applied Mechanics and Engineering, 190(26–27),* 3443–3459.
35. Bourdin, B. (2001). Filters in topology optimization. *International Journal for Numerical Methods in Engineering, 50(9),* 2143–2158.

36. Andreassen, E., Clausen, A., Schevenels, M., Lazarov, B. S. (2011). Efficient topology optimization in Matlab using 88 lines of code. *Structural and Multidisciplinary Optimization, 43,* 1–16.
37. Ferrari, F., Sigmund, O. (2020). A new generation 99 line Matlab code for compliance topology optimization and its extension to 3D. *Structural and Multidisciplinary Optimization, 62,* 2211–2228.
38. Kumar, T., Suresh, K. (2020). Direct lagrange multiplier updates in topology optimization revisited. *Structural and Multidisciplinary Optimization, 63,* 1563–1578.
39. Xia, L., Breitkopf, P. (2015). Design of materials using topology optimization and energy-based homogenization approach in Matlab. *Structural and Multidisciplinary Optimization, 52,* 1229–1241.
40. Long, K., Du, X., Xu, S., Xie, Y. M. (2016). Maximizing the effective Young's modulus of a composite material by exploiting the poisson effect. *Composite Structures, 153,* 593–600.
41. Long, K., Yang, X., Saeed, N., Chen, Z., Xie, Y. M. (2021). Topological design of microstructures of materials containing multiple phases of distinct poisson's ratios. *Computer Modeling in Engineering & Sciences, 126(1),* 293–310. https://doi.org/10.32604/cmes.2021.012734
42. Amir, O. (2015). Revisiting approximate reanalysis in topology optimization: On the advantages of recycled preconditioning in a minimum weight procedure. *Structural and Multidisciplinary Optimization, 51,* 41–57.
43. Grovenwold, A. A., Etman, L. F. P. (2010). On the conditional acceptance of iterates in SAO algorithms based convex separable approximations. *Structural and Multidisciplinary Optimization, 42,* 168–178.
44. Groenwold, A. A., Etman, L. F. P., Wood, D. E. (2010). Approximated approximations for SAO. *Structural and Multidisciplinary Optimization, 41,* 39–56.
45. Duan, Z., Yan, J., Lee, I., Lund, E., Wang, J. (2019). Discrete material selection and structural topology optimization of composite frames for maximum fundamental frequency with manufacturing constraints. *Structural and Multidisciplinary Optimization, 60,* 1741–1758.
46. Duan, Z., Jung, Y., Yan, J., Lee, I. (2020). Reliability-based multi-scale design optimization of composite frames considering structural compliance and manufacturing constraints. *Structural and Multidisciplinary Optimization, 61,* 2401–2421.
47. Fleury, C. (1979). Structural weight optimization by dual method of convex programming. *International Journal for Numerical Methods in Engineering, 14(12),* 1761–1783.
48. Fleury, C., Braibant, V. (1986). Structural optimization: A new dual method using mixed variables. *International Journal for Numerical Methods in Engineering, 23,* 409–428.
49. Fleury, C. (1989). Conlin: An efficient dual optimizer based on convex approximation concepts. *Structural Optimization, 1(2),* 81–89.
50. Haftka, R. T., Stanes, J. H. (1976). Application of a quadratic extended interior penalty function of structural optimization. *AIAA Journal, 14,* 718–724.
51. Zhang, W. H., Fleury, C. (1997). A modification of convex approximation methods for structural optimization. *Computers and Structures, 64,* 89–95.
52. Thomas, H., Zhou, M., Schrramm, U. (2002). Issues of commercial optimization software development. *Structural and Multidisciplinary Optimization, 23,* 97–110.
53. Svanberg, K. (1987). The method of moving asymptotes–A new method for structural optimization. *International Journal for Numerical Methods in Engineering, 24,* 359–373.
54. Svanberg, K. (2002). A class of globally convergent optimization methods based on conservative convex separable approximation. *SIAM Journal on Optimization, 12(2),* 555–573.
55. Wei, P., Ma, H., Wang, M. Y. (2014). The stiffness spreading method for layout optimization of truss structures. *Structural and Multidisciplinary Optimization, 49,* 667–682.
56. Li, Y., Wei, P., Ma, H. (2017). Integrated optimization of heat-transfer systems consisting of discrete thermal conductors and solid material. *International Journal of Heat and Mass Transfer, 113,* 1059–1069.
57. Cao, M., Ma, H., Wei, P. (2018). A modified stiffness spreading method for layout optimization of truss structures. *Acta Mechanica Sinica, 34,* 1072–1083.

58. Li, Y., Li, Z., Wei, P., Wang, W. (2018). Parameterized level-set based topology optimization method considering symmetry and pattern repetition constraints. *Computer Methods in Applied Mechanics and Engineering, 340,* 1079–1101.
59. Zhang, W., Yuan, J., Zhang, J., Guo, X. (2016). A new topology optimization approach based on moving morphable components (MMC) and the ersatz material model. *Structural and Multidisciplinary Optimization, 53,* 1243–1260.
60. Wang, X., Long, K., Hoang, V. N., Hu, P. (2018). An explicit optimization model for integrated layout design of planar multi-component systems using moving morphable bars. *Computer Methods in Applied Mechanics and Engineering, 342,* 46–70.
61. Wang, X., Long, K., Meng, Z., Yu, B., Cheng, C. (2021). Explicit multi-material topology optimization embedded with variable-size movable holes using moving morphable bars. *Engineering Optimization, 53(7),* 1212–1229.
62. Luo, Y., Bao, J. (2019). A Material-field series-expansion method for topology optimization of continuum structures. *Computers and Structures, 225,* 106122.
63. Liu, P., Yan, Y., Zhang, X., Luo, Y., Kang, Z. (2021). Topological design of microstructures using periodic material-field series-expansion and gradient-free optimization algorithm. *Materials and Design, 199,* 109437.
64. Liu, P., Yan, Y., Zhang, X., Luo, Y. (2021). A matlab code for the material-field series-expansion topology optimization method. *Frontiers of Mechanical Engineering, 16,* 607–622.
65. Wang, Y., Xiao, M., Xia, Z., Li, P., Gao, L. (2023). From computer-aided design (CAD) toward human-aided design (HAD): An isogeometric topology optimization approach. *Engineering, 22,* 94–105.
66. Cheng, C., Yang, B., Wang, X., Long, K. (2022). Reliability-based topology optimization using the response surface method for stress-constrained problems considering load uncertainty. *Engineering Optimization, 55(11)*. https://doi.org/10.1080/0305215X.2022.2134357
67. Yang, B., Cheng, C., Wang, X., Bai, S., Long, K. (2023). Robust reliability-based topology optimization for stress-constrained continuum structures using poloynomial chaos expansion. *Structural and Multidisciplinary Optimization, 66,* 88.
68. Wang, X., Meng, Z., Yang, B., Cheng, C., Long, K. et al. (2022). Reliability-based design optimization of material orientation and structural topology of fiber-reinforced composite structures under load uncertainty. *Composite Structures, 291,* 115537.
69. Long, K., Yang, X., Saeed, N., Tian, X., Wen, P. et al. (2021). Topology optimization of transient problem with maximum dynamic response constraint using SOAR scheme. *Frontiers of Mechanical Engineering, 16,* 593–606.
70. Chen, Z., Long, K., Wang, X., Liu, J., Saeed, N. (2021). A new geometrically nonlinear topology optimization formulation for controlling maximum displacement. *Engineering Optimization, 53(8),* 1283–1297.
71. Zhao, Q., Zhang, H., Wang, F., Zhang, T., Li, X. (2021). Topology optimization of non-fourier heat conduction problems considering global thermal dissipation energy minimization. *Structural and Multidisciplinary Optimization, 164,* 1385–1399.
72. Wang, X., Shi, Y., Hoang, V. N., Meng, Z., Long, K. et al. Reliability-based topology optimization of fail-safe structures using moving morhpable bars. *Computer Modelling in Engineering & Sciences, 136(3),* 3173–3195. https://doi.org/10.32604/cmes.2023.025501
73. Fadel, G. M., Riley, M. F., Barthelemy, J. M. (1990). Two point exponential approximation method for structural optimization. *Structural Optimization, 2,* 117–124.
74. Wang, L., Grandhi, R. V. (1995). Improved two-point function approximations for design optimization. *AIAA Journal, 33(9),* 1720–1727.
75. Xu, S., Grandhi, R. V. (1998). Effective two-point function approximations for design optimization. *AIAA Journal, 36(12),* 2269–2275.

76. Guo, X., Yamazaki, K., Cheng, G. D. (2000). A new two-point approximation approach for structural optimization. *Structural and Multidisciplinary Optimization, 20,* 22–28.
77. Guo, X., Yamazaki, K., Cheng, G. D. (2001). A new three-point approximation approach for design optimization problems. *International Journal for Numerical Methods for Numerical Methods in Engineering, 50,* 869–884.
78. Salajegheh, E. (1997). Optimum design of plate structures using three-point approximation. *Structural Optimization, 13,* 142–147.
79. Bruyneel, M., Duysinx, P., Fleury, C. (2002). A family of MMA approximations for structural optimization. *Structural and Multidisciplinary Optimization, 24,* 263–276.
80. Yoon, G. H., Grovenwold, A. A., Choi, D. H. (2014). A globally convergent sequential convex programming using an enhanced two-point diagonal quadratic approximation for structural optimization. *Structural and Multidisciplinary Optimization, 50(5),* 739–753.
81. Lei, L., Khandelwal, K. (2014). Two-point gradient-based MMA (TGMMA) algorithm for topology optimization. *Computers and Structures, 131,* 34–45.
82. Lei, L., Khandelwal, K. (2015). An adaptive quadratic approximation for structural and topology optimization. *Computers and Structures, 151,* 130–147.
83. Fleury, C. (1989). Efficient approximation concepts using second order information. *International Journal for Numerical Methods in Engineering, 28(9),* 2041–2058.
84. Fleury, C. (1989). First and second order convex approximation strategies in structural optimization. *Structural Optimization, 1(1),* 3–10.
85. Grovenwold, A. A., Etman, L. F. P., Snyman, J. A., Rooda, J. E. (2007). Incomplete series expansion for function approximation. *Structural and Multidisciplinary Optimization, 34,* 21–40.
86. Grovenwold, A. A., Etman, L. F. P. (2008). Sequential approximate optimization using dual subproblems based on incomplete series expansions. *Structural and Multidisciplinary Optimization, 36(6),* 547–570.
87. Grovenwold, A. A., Etman, L. F. P. (2008). On the equivalence of optimality criterion and sequential approximate optimization methods in the classical topology layout problem. *International Journal for Numerical Methods in Engineering, 73,* 297–316.
88. Grovenwold, A. A., Etman, L. F. P. (2010). A quadratic approximation for structural topology optimization. *International Journal for Numerical Methods in Engineering, 82(4),* 505–524.
89. Rojas-Labanda, S., Stople, M. (2016). An efficient second-order SQP method for structural topology optimization. *Structural and Multidisciplinary Optimization, 53(6),* 1315–1333.
90. Rojas-Labanda, S., Stople, M. (2015). Benchmarking optimization solvers for structural topology optimization. *Structural and Multidisciplinary Optimization, 52(3),* 527–547.
91. Evgrafov, A. (2014). State space Newton's method for topology optimization. *Computer Methods in Applied Mechanics and Engineering, 278,* 272–290.
92. Evgrafov, A. (2015). On chebyshev's method for topology optimization of stokes flow. *Structural and Multidisciplinary Optimization, 51,* 801–811.
93. Zhang, L., Guo, L., Sun, P., Yan, J., Long, K. (2023). A generalized discrete fiber angle optimization method for composite structures: Bipartite interpolation optimization. *International Journal for Numerical Methods in Engineering, 124(5),* 1211–1229.
94. Yan, J., Sun, P., Zhang, L., Hu, W., Long, K. (2022). SGC–A novel optimization method for the discrete fiber orientation of composites. *Structural and Multidisciplinary Optimization, 65(4),* 1–16.
95. Powell, M. J. D. (1978). Algorithms for nonlinear constraints that use langrangian functions. *Mathematical Programming, 14,* 224–248.
96. Sui, Y., Peng, X. (2018). *Modelling, solving and application for topology optimization of continuum structures*. Tsinghua University Press. https://doi.org/10.1016/C2015-0-04148-X

97. Qian, L. X., Zhong, W. X., Sui, Y. K., Zhang, J. D. (1980). Optimum design of structures with multiple types of element under multiple loading cases and multiple constraints–program system DDDU. *Journal of Dalian Institute of Technology, 19(4),* 1–17.
98. Qian, L. X., Zhong, W. X. (1983). Sequential quadratic programming approach in engineering structural optimization. *Acta Mechanica Solida Sinica, 22(4),* 469–480.
99. Qian, L. X., Zhong, W. X., Cheng, K. T., Sui, Y. K. (1984). An approach to structural optimization–sequential quadratic programming, SQP. *Engineering Optimization, 8(1),* 83–100.
100. Xia, R. W., Liu, P. (1987). Structural optimization based on second-order approximations of functions and dual theory. *Computer Methods in Applied Mechanics and Engineering, 65,* 101–114.
101. Zhou, M., Xia, R. W. (1990). Two-level approximation concept in structural synthesis. *International Journal for Numerical Methods in Engineering, 29(8),* 1681–1699.
102. Sui, Y., Peng, X. (2006). The ICM method with objective function transformed by variable discrete condition for continuum structure. *Acta Mechanica Sinica, 22,* 68–75.
103. Ye, H. L., Chen, N., Sui, Y. K., Tie, J. (2015). Three-dimensional dynamic topology optimization with frequency constraints using composite exponential function and ICM method. *Mathematical Problems with Engineering, 2015,* 491084. https://doi.org/10.1155/2015/491084
104. Ye, H. L., Wang, W. W., Chen, N., Sui, Y. K. (2016). Plate/shell topological optimization subject to linear buckling constraints by adopting composite exponential filtering function. *Acta Mechanica Sinica, 32,* 649–658.
105. Ye, H. L., Dai, Z. J., Wang, W. W., Sui, Y. K. (2019). ICM method for topology optimization of multimaterial continuum structure with displacement constraint. *Acta Mechanica Sinica, 35,* 552–562.
106. Wang, W. W., Ye, H. L., Li, Z. H., Sui, Y. K. (2022). Stiffness and strength topology optimization for bi-disc systems based on dual sequential quadratic programming. *International Journal for Numerical Methods in Engineering, 122(17),* 4073–4093.
107. Ye, H. L., Wang, W. W., Chen, N., Sui, Y. K. (2017). Plate/shell structure topology optimization of orthotropic material for buckling problem based on independent continuous topology variables. *Acta Mechanica Sinica, 33,* 899–911.
108. Wang, W., Ye, H., Sui, Y. (2019). Lightweight topology optimization with buckling and frequency constraints using the independent continuous mapping method. *Acta Mechanica Solida Sinica, 32,* 310–325.
109. Du, J., Guo, Y., Chen, Z., Sui, Y. (2019). Topology optimization of continuum structures considering damage based on independent continuous mapping method. *Acta Mechanica Sinica, 35,* 433–444.
110. Peng, X., Sui, Y. (2021). Lightweight topology optimization with consideration of the fail-safe design principle for continuum structures. *Engineering Optimization, 53(1),* 32–48.
111. Peng, X., Sui, Y. (2018). *Modelling, solving and application for topology optimization of continuum structures: ICM method based on step function*. Cambridge: Elsevier Inc.
112. Sui, Y. K., Peng, X. (2019). Explicit model of dual programming and solving method for a class of separable convex programming problems. *Engineering Optimization, 51(9),* 1604–1625.
113. Long, K., Zuo, Z. X., Xiao, T., Zuberi, R. H. (2010). ICM method combined with meshfree approximation for continuum structure. *Journal of Beijing Institute of Technology, 19(3),* 279–285.
114. Long, K., Wang, X., Liu, H. (2019). Stress-constrained topology optimization of continuum structures subject to harmonic force excitation using sequential quadratic programming. *Structural and Multidisciplinary Optimization, 59,* 1747–1759.
115. Long, K., Wang, X., Gu, X. (2018). Local optimum in multi-material topology optimization and solution by reciprocal variable. *Structural and Multidisciplinary Optimization, 57,* 1283–1295.

116. Long, K., Wang, X., Gu, X. (2018). Multi-material topology optimization for the transient heat conduction problem using a sequential quadratic programming algorithm. *Engineering Optimization, 50(12),* 2091–2107.
117. Long, K., Wang, X., Gu, X. (2018). Concurrent topology optimization for minimization of total mass considering load-carrying capabilities and thermal insulation simultaneously. *Acta Mechanica Sinica, 34,* 315–326.
118. Long, K., Gu, C., Wang, X., Liu, J., Du, Y. et al. (2019). A novel minimum weight formulation of topology optimization implemented with reanalysis approach. *International Journal for Numerical Methods in Engineering, 120,* 567–579.
119. Long, K., Wang, X., Du, Y. (2019). Robust topology optimization formulation including local failure and load uncertainty using sequential quadratic programming. *International Journal of Mechanics and Materials in Design, 15,* 317–332.
120. Saeed, N., Li, X., Long, K., Zhou, H., Saeed, A. et al. (2023). A quadratic approximation for volume minimization topology optimization. *Structures, 53,* 1341–1348.
121. Rong, J. H., Yi, J. H. (2010). A structural topological optimization method for multi-displacement constraints and any initial topology configuration. *Acta Mechanica Sinica, 26,* 735–744.
122. Rong, J. H., Liu, H. X., Yi, L. J., Yi, J. H. (2011). An efficient structural topological optimization method for continuum structures with multiple displacement constraints. *Finite Elements in Analysis and Design, 47,* 913–921.
123. Rong, J. H., Zhao, Z. J., Xie, Y. M., Yi, J. J. (2013). Topology optimization of finite similar periodic continuum structures based on a density exponent interpolation model. *Computer Modelling in Engineering & Sciences, 90(3),* 211–231. https://doi.org/10.1186/1687-2770-2013-86
124. Wei, P., Li, Z., Li, X., Wang, M. Y. (2018). An 88-line MATLAB code for the parameterized level set method based topology optimization using radial basis function. *Structural and Multidisciplinary Optimization, 58,* 831–849.
125. Silva, G. A., Beck, A. T., Sigmund, O. (2019). Stress-constrained topology considering uniform manufacturing uncertainties. *Computer Methods in Applied Mechanics and Engineering, 344(1),* 512–537.
126. Silva, G. A., Bech, A., Sigmund, O. (2019). Topology optimization of compliant mechanisms with stress constraints and manufacturing error robustness. *Computer Methods in Applied Mechanics and Engineering, 354,* 397–421.
127. Senhora, F. V., Giraldo-Londono, O., Menezes, I. F. M., Paulino, G. H. (2020). Topology optimization with local stress constraints: A stress aggregation-free approach. *Structural and Multidisciplinary Optimization, 62,* 1639–1668.
128. Giraldo-Londono, O., Aguilo, M. A., Paulino, G. H. (2021). Local stress constraints in topology optimization of structures subject to arbitrary dynamic loads: A stress aggregation-free approach. *Structural and Multidisciplinary Optimization, 64,* 3287–3309.
129. Long, K., Chen, Z., Zhang, C., Yang, X., Saeed, N. (2021). An aggregation-free local volume fraction formulation for topological design of porous structure. *Material, 14,* 5726.
130. Zhang, C., Long, K., Chen, Z., Yang, X., Lu, F. et al. (2023). Multi-material topology optimization for spatial-varying porous structures. *Computer Modelling in Engineering & Sciences, 138(1),* 369–390. https://doi.org/10.32604/cmes.2023.029876
131. Saeed, N., Long, K., Li, L., Saeed, A., Zhang, C. et al. (2022). An augmented lagrangian method for multiple nodal displacement-constrained topology optimization. *Engineering Optimization,* https://doi.org/10.1080/0305215X.2022.2129628
132. Zhang, C., Long, K., Yang, X., Chen, Z., Saeed, N. et al. (2022). A transient topology optimization with time-varying deformation restriction via augmented Lagrange method. *International Journal of Mechanics and Materials in Design, 18,* 683–700.

133. Chen, Z., Long, K., Zhang, C., Yang, X., Lu, F. et al. (2023). A fatigue-resistance topology optimization formula for continua subject to general loads using rainlfow counting. *Structural and Multidisciplinary Optimization, 66,* 210.
134. Sivapuram, R., Picelli, R. (2018). Topology optimization of binary structures under integer linear programming. *Finite Elements in Analysis and Design, 13,* 49–61.
135. Sivapuram, R., Picelli, R. (2020). Topology design of binary structures subjected to design-dependent thermal expansion and fluid pressure loads. *Structural and Multidisciplinary Optimization, 61,* 1877–1895.
136. Picelli, R., Ranjbarzadeh, S., Sivapuram, R., Gioria, R. S., Silva, E. C. N. (2020). Topology optimization of binary structures under design-dependent fluid-structures interaction loads. *Structural and Multidisciplinary, 62,* 2101–2116.
137. Souza, B. C., Yamabe, P. V. M., Sa, L. F. N., Ranjbarzadeh, S., Picelli, R. et al. (2021). Topology optimization of fluid flow by using integer linear programming. *Structural and Multidisciplinary Optimization, 64,* 1221–1240.
138. Picelli, R., Sivapuram, R., Xie, Y. M. (2021). A 101-line MATLAB code for topology optimization using binary. *Structural and Multidisciplinary Optimization, 63,* 935–954.
139. Liang, Y., Sun, K., Cheng, G. (2020). Discrete variable topology optimization for compliant mechanism design via sequential approximate integer programming with trust region (SAIP-TR). *Structural and Multidisciplinary Optimization, 62,* 2851–2879.
140. Yan, X. Y., Liang, Y., Cheng, G. D. (2021). Discrete variable topology optimization for simplified convective heat transfer via sequential approximate integer programming with trust-region. *International Journal for Numerical Methods in Engineering, 122(20),* 5844–5872.
141. Liang, Y., Yan, X. Y., Cheng, G. (2022). Explicit control of 2D and 3D structural complexity by discrete variable topology optimization method. *Computer Methods in Applied Mechanics and Engineering, 389,* 114302.
142. Sigmund, O. (2011). On the usefulness of non-gradient approaches in topology optimization. *Structural and Multidisciplinary Optimization, 43,* 589–596.
143. Wang, L., Grandhi, R. V., Canfield, R. (2012). Multi-point constraint approximations in structural optimization. *5th Symposium on Multidisciplinary Analysis and Optimization*. https://doi.org/10.2514/6.1994-4280
144. Huang, H., Ke, W., Xia, R. (1998). Numerical accuracy of the multi-point approximation and its application in structural synthesis. *7th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization*, St. Louis, MO, USA.
145. Huang, H., An, H., Ma, H., Chen, S. (2019). An engineering method for complex structural optimization involving both size and topology design variables. *International Journal for Numerical Methods in Engineering, 117,* 291–315.
146. Luo, Y., Xing, J., Kang, Z. (2020). Topology optimization using material-field series expansion and Kriging-based algorithm: An effective non-gradient method. *Computer Methods in Applied Mechanics and Engineering, 364,* 112966.
147. Zhang, X., Xing, J., Liu, P., Luo, Y., Kang, Z. (2021). Realization of full and directional band gap design by non-gradient topology optimization in acoustic metamaterials. *Extreme Mechanics Letters, 42,* 101126.
148. Li, H., Luo, Z., Gao, L., Qin, Q. (2018). Topology optimization for concurrent design of structures with multi-patch microstructures by level sets. *Computer Methods in Applied Mechanics and Engineering, 331,* 536–561.
149. Sun, Z., Wang, Y., Liu, P., Luo, Y. (2022). Topological dimensionality reduction-based machine learning for efficient gradient-free 3D topology optimization. *Materials and Design, 220,* 110885.
150. Xing, J., Luo, Y., Deng, Y., Wu, S., Gai, Y. (2022). Topology optimization design of deformable flexible thermoelectric devices for voltage enhancement. *Engineering Optimization, 55(10)*. https://doi.org/10.1080/0305215X.2022.2108416

151. Zhang, J., Li, J., Liu, P., Luo, Y. (2021). A Gradient-free topology optimization strategy for continuum structures with design-dependent boundary loads. *Symmetry, 13,* 1976.

152. Woldseth, R. V., Aage, N., Bærentzen, J. A., Sigmund, O. (2022). On the use of artificial neural networks in topology optimization. *Structural and Multidisciplinary Optimization, 65,* 294.