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ARTICLE

The Lambert-G Family: Properties, Inference, and Applications

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ABSTRACT

This study proposes a new flexible family of distributions called the Lambert-G family. The Lambert family is very flexible and exhibits desirable properties. Its three-parameter special sub-models provide all significant monotonic and non-monotonic failure rates. A special sub-model of the Lambert family called the Lambert-Lomax (LL) distribution is investigated. General expressions for the LL statistical properties are established. Characterizations of the LL distribution are addressed mathematically based on its hazard function. The estimation of the LL parameters is discussed using six estimation methods. The performance of this estimation method is explored through simulation experiments. The usefulness and flexibility of the LL distribution are demonstrated empirically using two real-life data sets. The LL model better fits the exponentiated Lomax, inverse power Lomax, Lomax-Rayleigh, power Lomax, and Lomax distributions.

KEYWORDS

Lambert function; Lomax distribution; maximum likelihood; hazard function; statistical model; simulation

1 Introduction

Many approaches have been suggested to propose new families of distributions or to generalize some of the classical distributions. These families and generalized distributions provide more flexibility in modeling real-life data in different applied fields. The most common feature of the new families and generalized distributions is represented by having one or more extra shape parameters. Hence, the statistical literature contains many families to generate new distributions by adding one or more shape parameters. Some examples include the Kumaraswamy-G by Cordeiro et al. [1], exponentiated T-X by Alzaghal et al. [2], Weibull-G by Bourguignon et al. [3], odd moment exponential-G by Haq et al. [4], Burr XII-G by Cordeiro et al. [5], generalized odd Burr III-G by Haq et al. [6], generalized odd half-logistic-G by Altun et al. [7], Marshall–Olkin alpha power by Nassar et al. [8], new exponential-X by

Ahmad et al. [9], new extended heavy-tailed family by Aljohani et al. [10], and modified generalized-G by Shama et al. [11].

One of the most notable approaches to generating new distributions is constructed by using the Lambert-W (LW) function which is also known as the product logarithm function. This approach is discussed by Corless [12]. It is defined (for $z \in [0, \infty)$) as the roots of the following function:

$$W(z)\, e^{W(z)} = z; \qquad z \in \mathbb{C}. \tag{1}$$

The above equation contains only one real-valued solution. Recently, the LW function has been used in the distribution of prime numbers as discussed by Visser [13]. Goerg [14] adopted the LW function to introduce new families of distributions in the context of random variable transformations. Iriarte et al. [15] generated the Lambert-F class as an alternative family for positive data analysis.

This study introduces a new wider class based on the LW function called the Lambert-G (LG) family, in which the baseline distribution is a continuous distribution with positive support. The proposed approach applies a transformation to a baseline cumulative distribution function (cdf), as illustrated in Definition 1. The newly generated cdf of the LG family, with two extra shape parameters, has the quantile function (qf), which is expressed in a closed form in terms of the LW function; hence, the proposed generator is called the LG family.

The LG family has some desirable properties and it can be justified as follows. (i) The three-parameter special sub-models of the LG family are capable of modeling all important hazard rate (hr) shapes including increasing, decreasing, unimodal, J-shape, reversed J-shape, bathtub, and modified-bathtub failure rates; (ii) Moreover, the densities of its sub-models accommodate reversed J shaped, right-skewed, symmetric, left-skewed, decreasing-increasing-decreasing densities; (iii) The LG special sub-models generalize some well-known distributions in the distribution theory literature such as the modified Weibull model [16]; (iv) The LG special models provide better fit than other generalized models under the same baseline distribution as shown in case of the Lambert-Lomax (LL) model.

The paper is organized in the following sections. In Section 2, the LG family is presented. In Section 3, we provide three special sub-models of the LG family. The properties of the LL distribution along with its analytical shapes are explored in Section 4. In Section 5, the parameters of the LL distribution are estimated via six classical estimation methods. Section 6 presents simulation results to address the behavior of different estimators. To show the empirical importance of the LL distribution, two real-life data sets are analyzed in Section 7. Final remarks are given in Section 8.

2 The LG Family

Definition 1. A random variable X is said to follow the LG family, denoted by $X \backsim \mathrm{LG}(x; \upsilon, \omega, \boldsymbol{\xi})$, if its cdf is given by

$$F(x; \upsilon, \omega, \boldsymbol{\xi}) = 1 - \exp\left(-\left\{-\ln\left[\overline{G}(x; \boldsymbol{\xi})\right]\right\}^{\upsilon} \exp\left\{-\omega \ln\left[\overline{G}(x; \boldsymbol{\xi})\right]\right\}\right), \ x > 0, \ \upsilon, \omega, \boldsymbol{\xi} > 0, \tag{2}$$

where υ and ω are additional shape parameters and $\overline{G}(x; \boldsymbol{\xi})$ is a baseline survival function (sf) with a vector of unknown parameters $\boldsymbol{\xi}$.

The probability density function (pdf) corresponding to Eq. (2) reduces to

$$f(x; \upsilon, \omega, \boldsymbol{\xi}) = \frac{\left\{\upsilon - \omega \ln\left[\overline{G}(x; \boldsymbol{\xi})\right]\right\} h(x; \boldsymbol{\xi})}{\left\{-\ln\left[\overline{G}(x; \boldsymbol{\xi})\right]\right\}^{1-\upsilon}} \exp\left\{-\omega \ln\left[\overline{G}(x; \boldsymbol{\xi})\right]\right\} \overline{F}(x; \upsilon, \omega, \boldsymbol{\xi}), \tag{3}$$

where $\overline{F}(x;\upsilon,\omega,\boldsymbol{\xi})$ is the sf of the LG family and $h(x;\boldsymbol{\xi})$ is the hr function (hrf) of a baseline distribution.

The hrf of the LG family becomes

$$\varphi(x;\upsilon,\omega,\boldsymbol{\xi}) = \left\{\upsilon - \omega \ln\left(\overline{G}(x;\boldsymbol{\xi})\right)\right\} h(x;\boldsymbol{\xi}) \left\{-\ln\left[\overline{G}(x;\boldsymbol{\xi})\right]\right\}^{\upsilon-1} \exp\left\{-\omega \ln\left[\overline{G}(x;\boldsymbol{\xi})\right]\right\}. \tag{4}$$

According to Eq. (4), the hrf of the LG family has a flexible property because it depends on the value of the extra parameter υ and the baseline hrf. Furthermore, the importance of the proposed family follows from its ability to generate new flexible distributions without adding new extra parameters by letting $\upsilon = \omega = 1$ in Eq. (2). Then, Eq. (2) reduces to

$$F(x;\boldsymbol{\xi}) = 1 - \exp\left\{-\overline{G}(x;\boldsymbol{\xi}) \ln\left[\overline{G}(x;\boldsymbol{\xi})\right]\right\} = 1 - \left[\overline{G}(x;\boldsymbol{\xi})\right]^{\frac{1}{\overline{G}(x;\boldsymbol{\xi})}}. \tag{5}$$

Eq. (5) is called the reduced Lambert-G (RLG) family.

The pdf and hrf of the RLG are given, respectively, by

$$f(x;\boldsymbol{\xi}) = h(x;\boldsymbol{\xi}) \left\{1 - \ln\left[\overline{G}(x;\boldsymbol{\xi})\right]\right\} \left[\overline{G}(x;\boldsymbol{\xi})\right]^{-\overline{F}(x;\boldsymbol{\xi})} \tag{6}$$

and

$$\varphi(x;\boldsymbol{\xi}) = h(x;\boldsymbol{\xi}) \frac{1 - \ln\left[\overline{G}(x;\boldsymbol{\xi})\right]}{\overline{G}(x;\boldsymbol{\xi})}. \tag{7}$$

3 Special Models of the LG Family

In this section, we provide three specific models of the LG family. These special distributions provide modified flexible forms of some standard distributions namely the exponential, Pareto, and Lomax distributions. The special sub-models of the LG family are called the Lambert-exponential (LE), Lambert-Pareto (LP), and LL distributions. These special models are capable of modeling all important hrf shapes including increasing, decreasing, unimodal, J-shape, reversed J-shape, bathtub, and modified bathtub failure rates. Moreover, the densities of these sub-models can also provide reversed J-shaped, right-skewed, symmetric, left-skewed, decreasing-increasing-decreasing densities.

3.1 The LE Distribution

The LE cdf follows from Eq. (2) by setting $\overline{G}(x;\theta) = \exp(-\theta x)$. Then, the LE cdf becomes

$$F(x;\upsilon,\alpha,\lambda) = 1 - \exp\left(-\alpha x^{\upsilon} e^{\lambda x}\right), \ x > 0, \ \upsilon,\alpha,\lambda > 0, \tag{8}$$

where $\alpha = \theta^{\upsilon}$ and $\lambda = \omega\theta$ are scale parameters and υ is a shape parameter.

The corresponding pdf and hrf of the LE distribution take the forms

$$f(x;\upsilon,\alpha,\lambda) = \alpha\left[\upsilon + \lambda x\right] x^{\upsilon-1} \exp\left(\lambda x - \alpha x^{\upsilon} e^{\lambda x}\right) \tag{9}$$

and

$$\varphi(x;\upsilon,\alpha,\lambda) = \alpha\left[\upsilon + \lambda x\right] x^{\upsilon-1} \exp(\lambda x). \tag{10}$$

The hrf shapes of the LE distribution depend only on the value of υ and can be increasing or bathtub shaped. The LE distribution is also known in the literature as the modified Weibull distribution (see Lai et al. [16]). Hence, the LE model reduces to type I extreme-value distribution for $\upsilon = 0$ and reduces to the Weibull distribution for $\lambda = 0$.

3.2 The LP Distribution

Consider the sf of the Pareto distribution $\overline{G}(x;\alpha) = (1/x)^{\alpha}$, $x \geq 1$ and $\alpha > 0$. Substituting $\overline{G}(x;\alpha) = (1/x)^{\alpha}$ in (2) yields the cdf of the LP distribution

$$F(x;\upsilon,\lambda,\beta) = 1 - \exp\left(-\lambda x^{\beta}\left[\ln(x)\right]^{\upsilon}\right), \quad x > 1, \upsilon, \lambda, \beta > 0, \tag{11}$$

where $\lambda = \alpha^{\upsilon}$ is a scale parameters and $\beta = \omega\alpha$ and υ are shape parameters. The corresponding pdf and hrf are given by

$$f(x;\upsilon,\lambda,\beta) = \lambda\left[\upsilon + \beta\ln(x)\right]x^{\beta-1}\left[\ln(x)\right]^{\upsilon-1}\exp\left(-\lambda x^{\beta}\left[\ln(x)\right]^{\upsilon}\right) \tag{12}$$

and

$$\varphi(x;\upsilon,\lambda,\beta) = \lambda\left[\upsilon + \beta\ln(x)\right]x^{\beta-1}\left[\ln(x)\right]^{\upsilon-1}. \tag{13}$$

The shape of the LP hrf depends on the values of β and υ and it can provide increasing or bathtub shapes. The Weibull and Pareto distributions are special cases from the LP distribution. The Weibull distribution follows when $\upsilon = 0$ and the Pareto model is obtained for $\upsilon = 1$ and $\beta = 0$. Fig. 1 provides some possible shapes of the density and hazard functions of the LE and LP distributions.

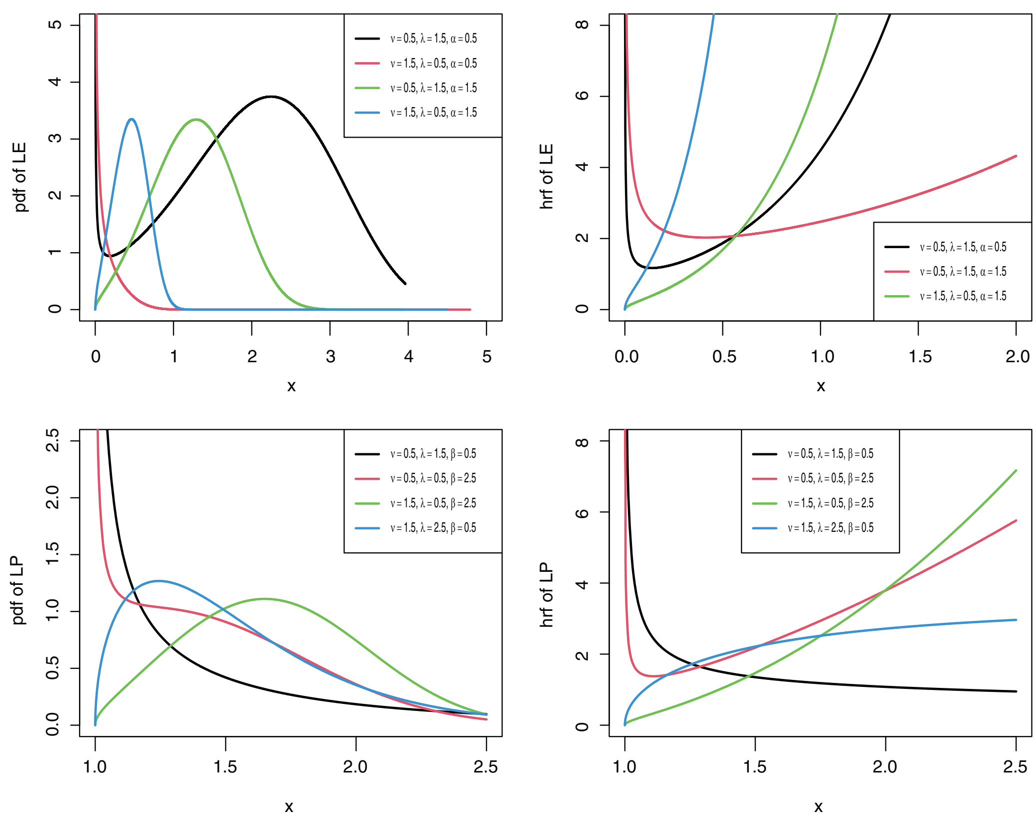

Figure 1: Possible shapes for the density and hazard functions of the LE and LP distributions

3.3 *The LL Distribution*

By taking the sf of the Lomax distribution, $\overline{G}(x;\alpha) = \left(1+\frac{x}{\alpha}\right)^{-1}$, $\alpha > 0$, as a baseline sf in (2). The cdf of the LL distribution reduces to

$$F(x;\upsilon,\omega,\alpha) = 1 - \exp\left\{-\left(1+\frac{x}{\alpha}\right)^{\omega}\left[\ln\left(1+\frac{x}{\alpha}\right)\right]^{\upsilon}\right\},\ x > 0,\ \upsilon,\omega,\alpha > 0, \tag{14}$$

where $\upsilon > 0$ and $\omega > 0$ are two extra shape parameters.

The corresponding pdf and hrf are

$$f(x;\upsilon,\omega,\alpha) = \frac{1}{\alpha}\left[\upsilon + \omega\ln\left(1+\frac{x}{\alpha}\right)\right]\left(1+\frac{x}{\alpha}\right)^{\omega-1}\left[\ln\left(1+\frac{x}{\alpha}\right)\right]^{\upsilon-1}\overline{F}(x;\upsilon,\omega,\alpha) \tag{15}$$

and

$$\varphi(x;\upsilon,\omega,\alpha) = \frac{1}{\alpha}\left[\upsilon + \omega\ln\left(1+\frac{x}{\alpha}\right)\right]\left[1+\frac{x}{\alpha}\right]^{\omega-1}\left[\ln\left(1+\frac{x}{\alpha}\right)\right]^{\upsilon-1}. \tag{16}$$

The Lomax distribution follows as a special case of the LL distribution with $\upsilon = 1$ and $\omega = 0$. The behavior of the LL density is plotted in Fig. 2. The hrf plots of the LL model are shown in Fig. 3. These figures show the strong effects of the two shape parameters ω and υ on the shapes of the pdf and hrf of the LL distribution.

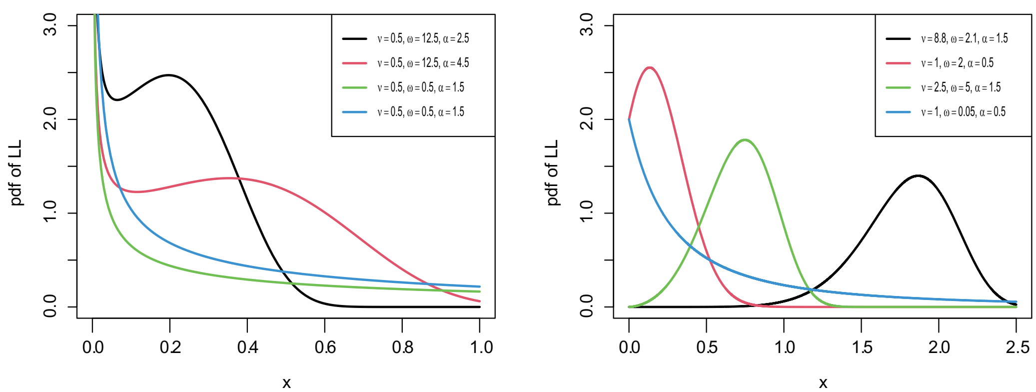

Figure 2: Possible shapes for the pdf of the LL distribution

4 Properties of the LL Distribution

In this section, we provide some basic statistical properties of the LL distribution.

4.1 *Behavior of the Density and Hazard Rate Functions*

The pdf limits of the LL distribution as $x \to 0$ and as $x \to \infty$ are

$$\lim_{x\to 0} f(x;\upsilon,\omega,\alpha) = \begin{cases} \infty & \upsilon < 1, \\ 1/\alpha & \upsilon = 1, \text{ and } \\ 0 & \upsilon > 1 \end{cases} \quad \lim_{x\to\infty} f(x;\upsilon,\omega,\alpha) = 0.$$

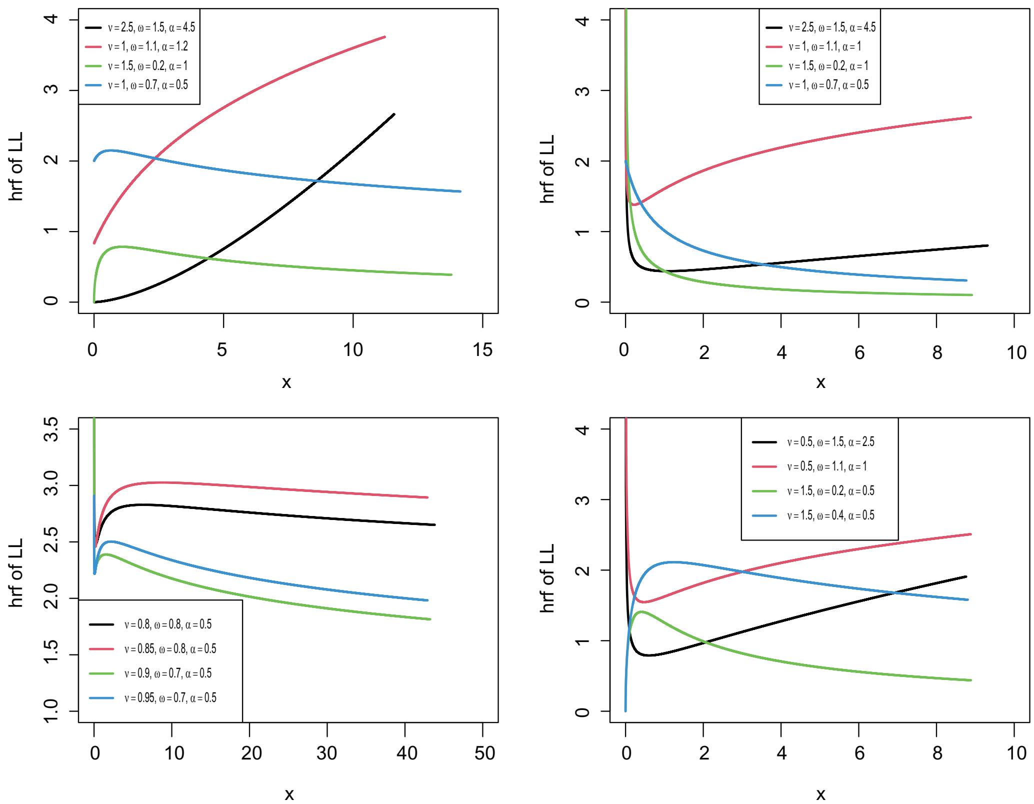

Figure 3: Possible shapes for the hrf of the LL distribution

Theorem 4.1. The graph of the pdf of the LL distribution is log-concave if $\upsilon \geq 1$ for all x.

Proof. Setting $y = \ln\left(1 + \frac{x}{\alpha}\right)$ in the LL pdf (15) and taking the logarithm, we have

$$\psi\left(y\right) = \frac{1}{\alpha}\left(\upsilon + \omega y\right) y^{\upsilon-1} \exp\left(\left(\omega - 1\right) y - y^{\upsilon} \mathrm{e}^{\omega y}\right).$$

Differentiating twice concerning y, we have

$$\psi''\left(y\right) = -\frac{\omega^2}{\left(\upsilon + \omega y\right)^2} - \frac{\left(\upsilon - 1\right)\left(1 + \upsilon y^{\upsilon} \mathrm{e}^{\omega y}\right)}{y^2}.$$

Note that $y = \ln\left(1 + \frac{x}{\alpha}\right)$ implies that $y > 0$. So, we can conclude that for all values of ω and $\upsilon \geq 1$, $\psi''\left(y\right) < 0$. Hence, the pdf of the LL distribution is log-concave for all x.

The hrf limits of the LL distribution as $x \to 0$ and as $x \to \infty$ are

$$\lim_{x\to 0} \varphi\left(x;\upsilon,\omega,\alpha\right) = \begin{cases} \infty & \upsilon < 1, \\ 1/\alpha & \upsilon = 1, \\ 0 & \upsilon > 1 \end{cases} \quad \text{and} \quad \lim_{x\to\infty} \varphi\left(x;\upsilon,\omega,\alpha\right) = \begin{cases} 0 & \omega < 1, \\ \infty & \omega \geq 1. \end{cases}$$

Theorem 4.2. The hrf of the LL distribution is

- Increasing for $\upsilon \geq 1$ and $\omega \geq 1$.

- Decreasing for $\upsilon \leq 1$ and $\omega < 0.5$ or for $\upsilon < 1$, $0.5 < \omega < 1$ and $\upsilon < 4\,\omega\,(1-\omega)$.
- Unimodal for $\upsilon \geq 1$ and $0.5 < \omega < 1$.
- Bathtub for $\upsilon < 1$ and $\omega \geq 1$.
- Decreasing-increasing-decreasing for $\upsilon < 1$, $0.5 < \omega < 1$ and $\upsilon \geq 4\,\omega\,(1-\omega)$.

Proof. From Eq. (16), we have

$$\ln\left[\varphi\left(x;\upsilon,\omega,\alpha\right)\right] = -\ln\left(\alpha\right) + \ln\left[\upsilon + \omega\ln\left(1+\frac{x}{\alpha}\right)\right] + \left(\omega-1\right)\ln\left[1+\frac{x}{\alpha}\right] + \left(\upsilon-1\right)\ln\left[\ln\left(1+\frac{x}{\alpha}\right)\right].$$

The derivative of $\ln\left[\varphi\left(x;\upsilon,\omega,\alpha\right)\right]$ follows as

$$\frac{d}{dx}\ln\left[\varphi\left(x;\upsilon,\omega,\alpha\right)\right] = \frac{\Psi\left(\ln\left(1+\frac{x}{\alpha}\right)\right)}{x+\alpha},$$

where

$$\Psi\left(y\right) = \omega\left(\omega-1\right)y^2 + \upsilon\left(2\omega-1\right)y + \upsilon\left(\upsilon-1\right), \quad y > 0,$$

where $y = \ln\left(1+\frac{x}{\alpha}\right)$.

Clearly, both the hrf of the LL distribution and $\Psi\left(y\right)$ have the same sign, hence the quantity $\Psi\left(y\right)$ has the following cases:

Case 1: For $\omega = 1$, $\Psi\left(y\right)$ reduces to $y+\upsilon-1 = 0$, hence the hrf of the LL distribution is increasing in x if $\upsilon \geq 1$. Also, if $\upsilon < 1$, the hrf is a bathtub shape with a minimum value at the point $y_1 = 1-\upsilon$ since $\Psi'\left(y_1\right) > 0$.

Case 2: If $\upsilon = 1$, $\Psi\left(y\right)$ reduce to $\Psi\left(y\right) = \omega\left(\omega-1\right)y^2 + \left(2\omega-1\right)y$. Hence, the LL hrf is increasing in x for $\omega \geq 1$. The LL hrf is decreasing in x for $\omega \leq 0.5$. Moreover, the LL hrf is unimodal with maximum value at the point $y_1 = \frac{2\omega-1}{\omega\left(\omega-1\right)}$ for $0.5 < \omega < 1$ since $\Psi'\left(y_1\right) < 0$.

Case 3: If $\upsilon > 1$ and $\omega > 1$, then $\Psi\left(y\right)$ is positive. Hence, the LL hrf is increasing in x.

Case 4: If $\upsilon < 1$ and $\omega \leq 0.5$, hence $\Psi\left(y\right)$ is negative. Then, the LL hrf is decreasing in x.

To discuss other cases, we have to get two critical values of $\Psi\left(y\right)$ which can be written as

$$y_1 = \frac{\upsilon\left(1-2\omega\right) - \sqrt{4\upsilon\omega\left(\omega-1\right)+\upsilon^2}}{2\omega\left(\omega-1\right)}$$

and

$$y_2 = \frac{\upsilon\left(1-2\omega\right) + \sqrt{\upsilon\left[4\omega\left(\omega-1\right)+\upsilon\right]}}{2\omega\left(\omega-1\right)}.$$

Case 5: For $\upsilon > 1$ and $\omega < 1$, $\Psi\left(y\right)$ has a critical value at the point y_2 which changes the sign from positive to negative. Thus, the LL hrf is unimodal.

Case 6: The function $\Psi\left(y\right)$ has two critical values y_1 and y_2 where the sign is negative on $\left(0,y_1\right)\bigcup\left(y_2,\infty\right)$ and positive on $\left(y_1,y_2\right)$ if $4\omega\left(1-\omega\right) < \upsilon < 1$ and $0.5 < \omega < 1$. Hence, the LL hrf is decreasing-increasing-decreasing. Additionally, for $\upsilon < 4\omega\left(1-\omega\right)$, the sign is always negative, thus the LL hrf is decreasing.

Case 7: For $\upsilon < 1$ and $\omega > 1$, $\Psi\left(y\right)$ has a critical value at the point y_2 which changes the sign from negative to positive. Thus, the LL hrf is the bathtub.

4.2 Moments

Theorem 4.3. The r^{th} raw moments of the LL distribution can be obtained as

$$E\left(X^{r}\right)=\alpha^{r}\sum_{i=0}^{r}\sum_{j=0}^{\infty}\binom{r}{i}\frac{i^{j}\left(-1\right)^{r-i}A_{k_1,\ldots,k_j}}{j!}\Gamma\left(S_j+1\right). \tag{17}$$

Proof. The r^{th} raw moments of the LL distribution can be defined as

$$E\left(X^{r}\right)=\int_{0}^{\infty}\frac{x^{r}}{\alpha}f\left(x;\upsilon,\omega,\alpha\right)dx.$$

Let $Y=\ln\left(1+\dfrac{X}{\alpha}\right)$, $y>0$, then $E\left(X^{r}\right)$ becomes

$$E\left(X^{r}\right)=\alpha^{r}\int_{0}^{\infty}\left(e^{y}-1\right)^{r}\left[\upsilon+\omega y\right]e^{\omega y}y^{\upsilon-1}\exp\left(-y^{\upsilon}e^{\omega y}\right)dy.$$

By using the binomial and series expansions, the r^{th} raw moments take the form

$$E\left(X^{r}\right)=\alpha^{r}\int_{0}^{\infty}\sum_{i=0}^{r}\sum_{j=0}^{\infty}\binom{r}{i}\frac{i^{j}\left(-1\right)^{r-i}}{j!}\left[\upsilon+\omega y\right]e^{\omega y}y^{j+\upsilon-1}\exp\left(-y^{\upsilon}e^{\omega y}\right)dy.$$

Let $z=y^{\upsilon}e^{\omega y}$, hence y in terms in z becomes $y=\sum_{k=1}^{\infty}a_k z^{k/\upsilon}$, where $a_k=\dfrac{\left(-1\right)^{k+1}k^{j-2}\left(\omega/\upsilon\right)^{k-1}}{\left(k-1\right)!}$. Then, the above integral reduces to

$$E\left(X^{r}\right)=\alpha^{r}\int_{0}^{\infty}\sum_{i=0}^{r}\sum_{j=0}^{\infty}\binom{r}{i}\frac{i^{j}\left(-1\right)^{r-i}}{j!}\left(\sum_{k=1}^{\infty}a_k z^{k/\upsilon}\right)^{j}\exp\left(-z\right)dz. \tag{18}$$

Let $\left(\sum_{k=1}^{\infty}a_k z^{k/\upsilon}\right)^{j}=\sum_{k_1,\ldots,k_j=1}^{\infty}A_{k_1,\ldots,k_j}z^{S_j}$, where $A_{k_1,\ldots,k_j}=a_{k_1}\ldots a_{k_j}$ and $S_j=j_1+\ldots+j_j$. Hence, the integral in (18) gives

$$E\left(X^{r}\right)=\alpha^{r}\sum_{i=0}^{r}\sum_{j=0}^{\infty}\binom{r}{i}\frac{i^{j}\left(-1\right)^{r-i}A_{k_1,\ldots,k_j}}{j!}\int_{0}^{\infty}z^{S_j}\exp\left(-z\right)dz.$$

Finally, we have

$$E\left(X^{r}\right)=\alpha^{r}\sum_{i=0}^{r}\sum_{j=0}^{\infty}\binom{r}{i}\frac{i^{j}\left(-1\right)^{r-i}A_{k_1,\ldots,k_j}}{j!}\Gamma\left(S_j+1\right),$$

which completes the proof.

As well as the measures of skewness, kurtosis, and asymmetry of the LL distribution are obtained by the following relations:

$$\beta_1=\frac{\left(\mu_3'-3\mu_2'\mu+2\mu^3\right)^2}{\left(\mu_2'-\mu^2\right)^3},\ \beta_2=\frac{\mu_4'-4\mu_3'\mu+6\mu_2'\mu^2-3\mu^4}{\left(\mu_2'-\mu^2\right)^2}\text{ and }\beta_3=\frac{\mu_3'-3\mu_2'\mu+2\mu^3}{\left(\mu_2'-\mu^2\right)^{3/2}}.$$

Table 1 provides some important LL measures for various parametric values. The numerical values show that the LL distribution can be right skewed for different values of υ and ω.

Table 1: Some LL measures for several parametric combinations of its parameters

Actual values			Mean	Variance	Skewness	Kurtosis	Asymmetry	CV
υ	ω	α						
0.5	0.50	0.50	0.5353	0.7821	16.1629	31.3442	4.0203	1.6522
		1.00	1.0706	3.1285	16.1629	31.3442	4.0203	1.6522
		1.50	1.6059	7.0391	16.1629	31.3442	4.0203	1.6522
	1.00	0.50	0.2775	0.1085	4.0298	8.4889	2.0074	1.1871
		1.00	0.5551	0.4342	4.0298	8.4889	2.0074	1.1871
		1.50	0.8326	0.9769	4.0298	8.4889	2.0074	1.1871
	1.50	0.50	0.1986	0.0427	2.2224	5.5215	1.4908	1.0399
		1.00	0.3973	0.1706	2.2224	5.5215	1.4908	1.0399
		1.50	0.5959	0.3839	2.2224	5.5215	1.4908	1.0399
1.00	0.50	0.50	0.5280	0.3170	5.7621	12.4969	2.4004	1.0665
		1.00	1.0559	1.2681	5.7621	12.4969	2.4004	1.0665
		1.50	1.5839	2.8532	5.7621	12.4969	2.4004	1.0665
	1.00	0.50	0.3521	0.0856	1.8671	5.4262	1.3664	0.8311
		1.00	0.7042	0.3425	1.8671	5.4262	1.3664	0.8311
		1.50	1.0563	0.7707	1.8671	5.4262	1.3664	0.8311
	1.50	0.50	0.2758	0.0409	0.9781	3.9475	0.9890	0.7330
		1.00	0.5516	0.1635	0.9781	3.9475	0.9890	0.7330
		1.50	0.8275	0.3679	0.9781	3.9475	0.9890	0.7330
1.50	0.50	0.50	0.5499	0.1879	2.5495	6.9434	1.5967	0.7882
		1.00	1.0998	0.7515	2.5495	6.9434	1.5967	0.7882
		1.50	1.6497	1.6909	2.5495	6.9434	1.5967	0.7882
	1.00	0.50	0.4093	0.0706	0.9585	4.1124	0.9790	0.6491
		1.00	0.8187	0.2824	0.9585	4.1124	0.9790	0.6491
		1.50	1.2280	0.6354	0.9585	4.1124	0.9790	0.6491
	1.50	0.50	0.3358	0.0378	0.4804	3.3091	0.6931	0.5793
		1.00	0.6716	0.1513	0.4804	3.3091	0.6931	0.5793
		1.50	1.0074	0.3405	0.4804	3.3091	0.6931	0.5793

4.3 Quantile Function

The qf of the LL distribution follows, by inverting its cdf (15), as

$$x = Q(u) = \alpha \left\{ \exp \left[e^{k(u)} \right] - 1 \right\}, \tag{19}$$

where

$$k(u) = \frac{1}{\upsilon}\left(\{\ln[-\ln(1-u)]\} - \upsilon W\left(\frac{\omega[-\ln(1-u)]^{1/\upsilon}}{\upsilon}\right)\right)$$

and $W(.)$ is the LW function. Then, if U has a uniform distribution in (0, 1), the solution of nonlinear equation $x = Q(u)$ has the LL distribution. Setting $u = 0.5$ in (19) gives the median (M) of the LL distribution. Additionally, by setting $u = 0.25$ and $u = 0.75$, one can obtain the lower and higher quartiles, respectively. The qf is calculated by using the Maple software.

4.4 Order Statistics

The order statistic for the LL distribution will be discussed in this section. It will also be useful to derive the pdf of the k^{th} order statistic $X_{(k)}$ of the ordered sample $X_{(1)}, X_{(2)}, ..., X_{(n)}$ drawn from the LL with parameters υ, ω and α. The pdf $f_{X_{(k)}}(x)$ of $X_{(k)}$ is given by

$$f_{X_{(k)}}(x) = n\binom{n-1}{k-1} f(x) F(x)^{k-1}[1-F(x)]^{(n-k)}, \quad k = 1, 2, ..., n. \tag{20}$$

We have

$$F(x;\upsilon,\omega,\alpha)^{k-1} = \left(1 - e^{-\left(1+\frac{x}{\alpha}\right)^{\omega}\left[\ln\left(1+\frac{x}{\alpha}\right)\right]^{\upsilon}}\right)^{k-1} = \sum_{l=0}^{k-1}\binom{k-1}{l}(-1)^{l} e^{-l\left(1+\frac{x}{\alpha}\right)^{\omega}\left[\ln\left(1+\frac{x}{\alpha}\right)\right]^{\upsilon}} \tag{21}$$

and

$$[1-F(x;\upsilon,\omega,\alpha)]^{(n-k)} = e^{-(n-k)\left(1+\frac{x}{\alpha}\right)^{\omega}\left[\ln\left(1+\frac{x}{\alpha}\right)\right]^{\upsilon}}. \tag{22}$$

Substituting (21) and (22) in (20), one can write

$$\begin{aligned} f_{X_{(k)}}(x;\upsilon,\omega,\alpha) = {} & n\binom{n-1}{k-1}\sum_{l=0}^{k-1}\binom{k-1}{l}(-1)^{l}\left(\frac{1}{\alpha}\right)\left(1+\frac{x}{\alpha}\right)^{\omega-1}\left[\ln\left(1+\frac{x}{\alpha}\right)\right]^{\upsilon-1} \\ & \times\left(\upsilon+\omega\ln\left(1+\frac{x}{\alpha}\right)\right)\exp\left(-(n-k+l+1)\left(1+\frac{x}{\alpha}\right)^{\omega}\left[\ln\left(1+\frac{x}{\alpha}\right)\right]^{\upsilon}\right). \end{aligned} \tag{23}$$

Hence, the largest order statistic density follows as

$$\begin{aligned} f_{X_{(n)}}(x;\upsilon,\omega,\alpha) = {} & n\sum_{l=0}^{n-1}\binom{k-1}{l}(-1)^{l}\left(\frac{1}{\alpha}\right)\left(1+\frac{x}{\alpha}\right)^{\omega-1}\left[\ln\left(1+\frac{x}{\alpha}\right)\right]^{\upsilon-1}\left(\upsilon+\omega\ln\left(1+\frac{x}{\alpha}\right)\right) \\ & \exp\left(-(l+1)\left(1+\frac{x}{\alpha}\right)^{\omega}\left[\ln\left(1+\frac{x}{\alpha}\right)\right]^{\upsilon}\right). \end{aligned} \tag{24}$$

The smallest order statistic pdf reduces to

$$\begin{aligned} f_{X_{(1)}}(x;\upsilon,\omega,\alpha) = {} & n\left(\frac{1}{\alpha}\right)\left(1+\frac{x}{\alpha}\right)^{\omega-1}\left[\ln\left(1+\frac{x}{\alpha}\right)\right]^{\upsilon-1}\left(\upsilon+\omega\ln\left(1+\frac{x}{\alpha}\right)\right) \\ & \exp\left(-(n+l)\left(1+\frac{x}{\alpha}\right)^{\omega}\left[\ln\left(1+\frac{x}{\alpha}\right)\right]^{\upsilon}\right). \end{aligned} \tag{25}$$

5 Estimation of the LL Parameters

In this section, different techniques are used to estimate the LL parameters.

5.1 Maximum Likelihood

The LL parameters are estimated by the maximum likelihood (ML). Consider a random sample from the LL distribution denoted by $x_1,\ x_2, ..., x_n$. Hence, the log-likelihood function follows as

$$L(\upsilon, \omega, \alpha) = n\log[\alpha] + (\upsilon - 1)\sum_{i=1}^{n}\log[\log(\delta_i)] + \sum_{i=1}^{n}\log[\upsilon + \omega\log(\delta_i)] + (\omega - 1)\sum_{i=1}^{n}\log(\delta_i) - \sum_{i=1}^{n}(\delta_i)^{\omega}\log(\delta_i)^{\upsilon}, \tag{26}$$

where $\delta_i = \left(1 + \frac{x_i}{\alpha}\right)$.

The ML estimators (MLE) of υ, ω and α can be obtained by simultaneously solving the following non-linear system:

$$\frac{\partial}{\partial\upsilon}L(\upsilon, \omega, \alpha) = \sum_{i=1}^{n}\log[\log(\delta_i)] - \sum_{i=1}^{n}(\delta_i)^{\omega}\log(\delta_i)^{\upsilon}\log[\log(\delta_i)] + \sum_{i=1}^{n}[\upsilon + \omega\log(\delta_i)]^{-1}, \tag{27}$$

$$\frac{\partial}{\partial\omega}L(\upsilon, \omega, \alpha) = \sum_{i=1}^{n}\left\{\omega + v[\log(\delta_i)]^{-1}\right\}^{-1} - \sum_{i=1}^{n}\log(\delta_i)^{\upsilon+1}(\delta_i)^{\omega} + \sum_{i=1}^{n}\log(\delta_i) \tag{28}$$

and

$$\frac{\partial}{\partial\alpha}L(\upsilon, \omega, \alpha) = -\frac{n}{\alpha} - \frac{\omega - 1}{\alpha}\sum_{i=1}^{n}\frac{x_i}{(\alpha + x_i)} - \frac{\upsilon + \omega - 1}{\alpha^2}\sum_{i=1}^{n}x_i\,[(\delta_i)\log(\delta_i)]^{-1} - \frac{1}{\alpha^2}\sum_{i=1}^{n}\left[x_i\,(\delta_i)^{\omega-1}\log(\delta_i)^{\upsilon}\left(-\upsilon\log(\delta_i)^{-1} - \omega\right)\right]. \tag{29}$$

5.2 Least-Squares and Weighted Least-Squares

The least squares (LS) and the weighted LS (WLS) methods are introduced by Swain et al. [17]. The LS estimators (LSE) of the LL parameters are calculated by minimizing the following function concerning υ, ω and α:

$$B(\upsilon, \omega, \alpha) = \sum_{i=1}^{n}\left(1 - \exp\left[-\left(\delta_{(i)}\right)^{\omega}\left(\ln\delta_{(i)}\right)^{\upsilon}\right] - \frac{i}{n+i}\right)^2, \tag{30}$$

where $\delta_{(i)} = \left(1 + \frac{x_{(i)}}{\alpha}\right)$. Moreover, the LSE is obtained by simultaneously solving the following non-linear system:

$$\frac{\partial B(\upsilon, \omega, \alpha)}{\partial\upsilon} = \sum_{i=1}^{n}\left(1 - \exp\left[-\left(\delta_{(i)}\right)^{\omega}\left(\ln\delta_{(i)}\right)^{\upsilon}\right] - \frac{i}{n+i}\right)F'_{\upsilon}\left(x_{(i)}\right) = 0, \tag{31}$$

$$\frac{\partial B(\upsilon, \omega, \alpha)}{\partial\omega} = \sum_{i=1}^{n}\left(1 - \exp\left[-\left(\delta_{(i)}\right)^{\omega}\left(\ln\delta_{(i)}\right)^{\upsilon}\right] - \frac{i}{n+i}\right)F'_{\omega}\left(x_{(i)}\right) = 0 \tag{32}$$

and

$$\frac{\partial B(\upsilon,\omega,\alpha)}{\partial\alpha}=\sum_{i=1}^{n}\left(1-\exp\left[-\left(\delta_{(i)}\right)^{\omega}\left(\ln\delta_{(i)}\right)^{\upsilon}\right]-\frac{i}{n+i}\right)F'_{\alpha}\left(x_{(i)}\right)=0, \tag{33}$$

where

$$F'_{\upsilon}\left(x_{(i)}\right)=\left(\delta_{(i)}\right)^{\omega}\log\left(\delta_{(i)}\right)^{\upsilon}\log\left(\log\delta_{(i)}\right)\exp\left[-\left(\delta_{(i)}\right)^{\omega}\log\left(\delta_{(i)}\right)^{\upsilon}\right], \tag{34}$$

$$F'_{\omega}\left(x_{(i)}\right)=\left(\delta_{(i)}\right)^{\omega}\log\left(\delta_{(i)}\right)^{\upsilon+1}\exp\left[-\left(\delta_{(i)}\right)^{\omega}\log\left(\delta_{(i)}\right)^{\upsilon}\right] \tag{35}$$

and

$$F'_{\alpha}\left(x_{(i)}\right)=-\frac{x_{(i)}}{\alpha}\left(\delta_{(i)}\right)^{\omega-1}\log\left(\delta_{(i)}\right)^{\upsilon-1}f\left(x_{(i)};\upsilon,\omega,\alpha\right). \tag{36}$$

The WLS estimators (WLSE) of the LL parameters follow by minimizing the function

$$W(\upsilon,\omega,\alpha)=\sum_{i=1}^{n}\varphi(i,n)\left(1-\exp\left[-\left(\delta_{(i)}\right)^{\omega}\left(\ln\delta_{(i)}\right)^{\upsilon}\right]-\frac{i}{n+i}\right)^{2}, \tag{37}$$

where $\varphi(i,n)=(n+1)^{2}\,(n+2)/i\,(n-i+1)$.

Also, these estimators are determined by solving the following non-linear system:

$$\sum_{i=1}^{n}\varphi(i,n)\left(1-\exp\left[-\left(\delta_{(i)}\right)^{\omega}\left(\ln\delta_{(i)}\right)^{\upsilon}\right]-\frac{i}{n+i}\right)F'_{k}\left(x_{(i)}\right)=0, \tag{38}$$

where $F'_{k}\left(x_{(i)}\right)=0$ are given in Eqs. (34)–(36) for $k=\upsilon,\omega,\alpha$.

5.3 Cramér–Von Mises

The Cramér–von Mises (CM) method was introduced by Choi et al. [18] depending on the CM statistics (Boos [19]). Then, the CM estimators (CME) of the LL parameters minimize the following function:

$$C(\upsilon,\omega,\alpha)=\frac{1}{12n}+\sum_{i=1}^{n}\left(1-\exp\left[-\left(\delta_{(i)}\right)^{\omega}\left(\ln\delta_{(i)}\right)^{\upsilon}\right]-\frac{2i-1}{2n}\right)^{2}, \tag{39}$$

with respect to υ, α and ω. They also are obtained by solving the following non-linear system:

$$\sum_{i=1}^{n}\left(1-\exp\left[-\left(\delta_{(i)}\right)^{\omega}\left(\ln\delta_{(i)}\right)^{\upsilon}\right]-\frac{2i-1}{2n}\right)F'_{k}\left(x_{(i)}\right)=0, \tag{40}$$

where $F'_{k}\left(x_{(i)}\right)=0$ are given in Eqs. (34)–(36) for $k=\upsilon,\omega,\alpha$.

5.4 Anderson–Darling and Right-Tail Anderson–Darling

Depending on the Anderson–Darling (AD) statistic, the AD method was proposed by Anderson et al. [20] and [21]. Hence, the AD estimators (ADE) of the LL parameters are calculated by minimizing the function

$$A\left(\upsilon,\omega,\alpha\right)=-n-\frac{1}{n}\sum_{i=1}^{n}\left(2i-1\right)\left\{\log\left(1-\exp\left[-\left(\delta_{(i)}\right)^{\omega}\left(\ln\delta_{(i)}\right)^{\upsilon}\right]\right)+\log\left(\exp\left[-\left(\delta_{(i)}\right)^{\omega}\left(\ln\delta_{(i)}\right)^{\upsilon}\right]\right)\right\}. \quad (41)$$

Therefore, the ADE is also obtained by solving the following non-linear system:

$$\sum_{i=1}^{n+1}\left(2i-1\right)\left(\frac{F_k'\left(x_{(i)}\right)}{F\left(x_{(i)}\right)}-\frac{F_k'\left(x_{(n+1-i)}\right)}{1-F\left(x_{(n+1-i)}\right)}\right)=0,\ k=\upsilon,\omega,\alpha, \quad (42)$$

where $F_k'\left(x_{(i)}\right)=0$ are given in Eqs. (34)–(36).

Luceno [22] applied some motivations to the AD statistic called the right-tail AD (RAD) statistic. Hence, to obtain the RAD estimators (RADE) of the LL parameters, we minimize the following function:

$$RA\left(\upsilon,\omega,\alpha\right)=\frac{n}{2}-2\sum_{i=1}^{n}F\left(x_{(i)}\right)-\frac{1}{n}\sum_{i=1}^{n}\left(2i-1\right)\log\left(1-F\left(x_{(n+1-i)}\right)\right). \quad (43)$$

Additionally, the RADE is obtained by solving the non-linear system

$$\frac{\partial RA\left(\upsilon,\omega,\alpha\right)}{\partial\upsilon}=-n\sum_{i=0}^{n}\frac{F_k'\left(x_{(i)}\right)}{F\left(x_{(i)}\right)}+\frac{1}{n}\sum_{i=1}^{n+1}\left(2i-1\right)\frac{F_k'\left(x_{(n-i+1)}\right)}{1-F\left(x_{(n-i+1)}\right)}=0,\ k=\upsilon,\omega,\alpha, \quad (44)$$

where $F_k'\left(x_{(i)}\right)=0$ are given in Eqs. (34)–(36).

All above mentioned non-linear systems of equations have no exact solutions, so the optim and nlminb functions in R software can be adopted for this purpose.

6 Simulation Analysis

This section presents numerical simulation results to explore the efficiency and performance of different estimators for the LL parameters. The following algorithm is adopted to evaluate different estimators:

1. Set different initial values of sample size n and the parameters υ, ω and α.
2. Generate several random samples of size n from the LL distribution using its qf.
3. The outcomes in the previous step are used to calculate the parameter estimates, $\widehat{\Theta}=(\widehat{\upsilon},\widehat{\omega},\widehat{\alpha})$, by using the MLE, LSE, WLSE, CME, ADE and RADE.
4. The three above steps are repeated 6,000 times.
5. Based on $\widehat{\Theta}$ and Θ, the absolute bias (AB) and root-mean-squared error (RMSE) are determined using the following formulae:

$$AB\left(\widehat{\Theta}\right)=\frac{1}{6000}\sum_{i=1}^{6000}\left|\widehat{\Theta}-\Theta\right|$$

and

$$RMSE\left(\widehat{\Theta}\right)=\sqrt{\frac{1}{6000}\sum_{i=1}^{6000}\left(\widehat{\Theta}-\Theta\right)^{2}}.$$

From the LL distribution 6,000 samples are generated for $n = \{20, 50, 100, 200\}$ and several parametric values for $\upsilon = (0.6, 0.4, 1.5)$, $\omega = (0.8, 1.5, 0.4)$ and $\alpha = (1.5, 0.8, 0.6)$ using different estimation methods.

The AB and RMSE of the MLE, CME, LSE, ADE, WLSE, and RADE are presented in Tables 2–4. Moreover, the partial and overall ranks of the estimators are calculated in Table 5. From the values in these tables, one can note that

1. All estimates show the property of consistency, i.e., the AB and RMSE decrease as sample size increases for all parametric combinations.
2. According to AB and RMSE, the ordering of performance of estimators (from best to worst) for all parameters is the MLE, ADE, WLSE, RADE, LSE and CME.

Table 2: The AB and RMSE of different estimators for $\upsilon = 0.6$, $\omega = 0.8$ and $\alpha = 1.5$

n	Est.	Est. Par.	MLE	LSE	WLSE	CME	ADE	RADE
20	AB	$\widehat{\upsilon}$	0.0264	0.0581	0.1284	0.1578	0.1051	0.1754
		$\widehat{\omega}$	0.3019	1.4720	1.3029	1.7125	0.6682	1.1226
		$\widehat{\alpha}$	0.6472	3.1034	2.2290	2.5726	1.2439	2.0723
	RMSE	$\widehat{\upsilon}$	0.0653	0.0961	0.1917	0.2471	0.1763	0.2772
		$\widehat{\omega}$	2.4905	3.2428	3.0630	3.8694	1.6943	2.5228
		$\widehat{\alpha}$	4.3459	5.8860	4.5391	5.1254	2.7913	4.2717
50	AB	$\widehat{\upsilon}$	0.0030	0.0330	0.0721	0.0843	0.0427	0.0919
		$\widehat{\omega}$	0.0091	0.6705	0.4907	0.6270	0.2050	0.4111
		$\widehat{\alpha}$	0.0243	1.5432	0.9413	1.0911	0.4128	0.8535
	RMSE	$\widehat{\upsilon}$	0.0185	0.0441	0.0976	0.1166	0.0778	0.1308
		$\widehat{\omega}$	0.0728	1.4914	1.0426	1.3586	0.4923	0.7953
		$\widehat{\alpha}$	0.1906	3.1088	1.7544	2.1046	0.8887	1.5772
100	AB	$\widehat{\upsilon}$	0.0002	0.0235	0.0495	0.0560	0.0187	0.0601
		$\widehat{\omega}$	0.0003	0.3304	0.2686	0.3341	0.0797	0.2492
		$\widehat{\alpha}$	0.0008	0.7874	0.5549	0.6240	0.1636	0.5225
	RMSE	$\widehat{\upsilon}$	0.0040	0.0305	0.0641	0.0736	0.0424	0.0794
		$\widehat{\omega}$	0.0080	0.5500	0.4027	0.5182	0.1830	0.3524
		$\widehat{\alpha}$	0.0231	1.2495	0.8125	0.9439	0.3792	0.7404
200	AB	$\widehat{\upsilon}$	1.824E-5	0.0173	0.0337	0.0389	0.0042	0.0409
		$\widehat{\omega}$	3.647E-5	0.2194	0.1714	0.2127	0.0178	0.1677
		$\widehat{\alpha}$	4.895E-5	0.5296	0.3555	0.4061	0.0361	0.3589
	RMSE	$\widehat{\upsilon}$	0.0014	0.0222	0.0434	0.0497	0.0166	0.0525
		$\widehat{\omega}$	0.0028	0.2969	0.2324	0.2994	0.0700	0.2277
		$\widehat{\alpha}$	0.0038	0.7261	0.4757	0.5632	0.1452	0.4852

Table 3: The AB and RMSE of different estimators for $\upsilon = 0.4$, $\omega = 1.5$ and $\alpha = 0.8$

n	Est.	Est. Par.	MLE	LSE	WLSE	CME	ADE	RADE
20	AB	$\widehat{\upsilon}$	0.1031	0.1550	0.1364	0.1744	0.1113	0.2030
		$\widehat{\omega}$	1.5750	3.4991	3.0284	3.5275	1.6103	2.8384
		$\widehat{\alpha}$	0.7193	1.6832	1.4733	1.5649	0.7893	1.4113
	RMSE	$\widehat{\upsilon}$	0.1861	0.2483	0.2167	0.2836	0.1976	0.3421
		$\widehat{\omega}$	3.7365	6.6284	5.9463	6.6117	3.8827	5.7166
		$\widehat{\alpha}$	1.6107	3.2113	2.8603	2.9109	1.8347	2.8283
50	AB	$\widehat{\upsilon}$	0.0421	0.0898	0.0781	0.0918	0.0443	0.1018
		$\widehat{\omega}$	0.3909	1.6592	1.2153	1.6029	0.4151	1.1408
		$\widehat{\alpha}$	0.2039	0.8158	0.6186	0.7613	0.2209	0.6092
	RMSE	$\widehat{\upsilon}$	0.0812	0.1207	0.1060	0.1297	0.0867	0.1455
		$\widehat{\omega}$	0.9886	3.6132	2.6606	3.3993	0.9859	2.4690
		$\widehat{\alpha}$	0.4764	1.6838	1.2597	1.5159	0.5045	1.2460
100	AB	$\widehat{\upsilon}$	0.0164	0.0606	0.0529	0.0621	0.0153	0.0649
		$\widehat{\omega}$	0.1142	0.8530	0.6218	0.8429	0.1220	0.6206
		$\widehat{\alpha}$	0.0624	0.4337	0.3358	0.4209	0.0668	0.3467
	RMSE	$\widehat{\upsilon}$	0.0435	0.0791	0.0694	0.0816	0.0410	0.085
		$\widehat{\omega}$	0.3236	1.5718	1.0518	1.6204	0.3309	1.0746
		$\widehat{\alpha}$	0.1719	0.7609	0.5323	0.7661	0.1812	0.5814
200	AB	$\widehat{\upsilon}$	0.0036	0.0424	0.0368	0.0422	0.0031	0.0452
		$\widehat{\omega}$	0.0201	0.5118	0.3919	0.5163	0.0221	0.4035
		$\widehat{\alpha}$	0.0114	0.2665	0.21587	0.2662	0.0121	0.2285
	RMSE	$\widehat{\upsilon}$	0.0179	0.0542	0.0472	0.0541	0.0157	0.0584
		$\widehat{\omega}$	0.0991	0.7735	0.5505	0.7676	0.1079	0.5725
		$\widehat{\alpha}$	0.0559	0.3948	0.3019	0.3894	0.0582	0.3225

Table 4: The AB and RMSE of different estimators for $\upsilon = 1.5$, $\omega = 0.4$ and $\alpha = 0.6$

n	Est.	Est. Par.	MLE	LSE	WLSE	CME	ADE	RADE
20	AB	$\widehat{\upsilon}$	0.4811	0.5322	0.5088	0.5657	0.4728	0.5389
		$\widehat{\omega}$	1.1040	2.0602	1.757	2.0698	1.1153	1.6573
		$\widehat{\alpha}$	0.8345	1.6631	1.3916	1.5393	0.8873	1.4196
	RMSE	$\widehat{\upsilon}$	0.5817	0.6417	0.6108	0.6962	0.5707	0.6441
		$\widehat{\omega}$	2.574	4.9099	4.3821	4.8676	2.8235	4.16114
		$\widehat{\alpha}$	1.9135	3.9267	3.3962	3.5889	2.2378	3.6226

(Continued)

Table 4 (continued)

n	Est.	Est. Par.	MLE	LSE	WLSE	CME	ADE	RADE
50	AB	$\widehat{\upsilon}$	0.33430	0.3964	0.3667	0.4070	0.3472	0.4116
		$\widehat{\omega}$	0.4330	0.7562	0.5572	0.7544	0.4472	0.5720
		$\widehat{\alpha}$	0.3337	0.5948	0.4391	0.5740	0.3485	0.4710
	RMSE	$\widehat{\upsilon}$	0.4040	0.4713	0.4364	0.4875	0.4181	0.4846
		$\widehat{\omega}$	0.7342	1.7734	1.0560	1.7807	0.6869	1.1838
		$\widehat{\alpha}$	0.5925	1.3937	0.8317	1.3576	0.5711	1.0537
100	AB	$\widehat{\upsilon}$	0.2501	0.3121	0.2808	0.3158	0.2718	0.3284
		$\widehat{\omega}$	0.2798	0.4248	0.3332	0.4202	0.3032	0.3554
		$\widehat{\alpha}$	0.2105	0.3244	0.2520	0.3109	0.2300	0.2768
	RMSE	$\widehat{\upsilon}$	0.3084	0.3726	0.3368	0.3785	0.3310	0.3911
		$\widehat{\omega}$	0.3727	0.6582	0.4588	0.7003	0.4066	0.5013
		$\widehat{\alpha}$	0.2946	0.5332	0.3667	0.5300	0.3248	0.4301
200	AB	$\widehat{\upsilon}$	0.1842	0.2391	0.2058	0.2452	0.2025	0.2500
		$\widehat{\omega}$	0.1898	0.2877	0.2254	0.2934	0.2195	0.2535
		$\widehat{\alpha}$	0.1421	0.2098	0.1652	0.2108	0.1618	0.1903
	RMSE	$\widehat{\upsilon}$	0.2313	0.2897	0.2558	0.2966	0.2514	0.3054
		$\widehat{\omega}$	0.2444	0.3778	0.2884	0.3877	0.2819	0.3252
		$\widehat{\alpha}$	0.1879	0.2893	0.2186	0.2915	0.2157	0.2587

Table 5: Partial and overall ranks of all estimation methods for various combination of υ, ω and α

Est. Par.	Initial values	MLE	LSE	WLSE	CME	ADE	RADE
$\widehat{\upsilon}$	First	8	19	32	40	21	48
	Second	12	34	24	38	12	48
	Third	11	46	28	48	17	32
$\widehat{\omega}$	First	10	48	35	47	15	26
	Second	11	53	36	51	17	35
	Third	11	46	28	48	17	32
$\widehat{\alpha}$	First	11	54	35	45	17	30
	Second	8	49	29	43	16	29
	Third	9	46	24	41	15	33
Sum		90	381	267	396	144	326
Overall rank		1	5	3	6	2	4

7 Real-Life Applications

In this section, we analyze two real-life data sets to demonstrate the performance of the LL distribution in practice. Two real-life data sets are fitted to compare the proposed LL model with other five known competitors, namely:

1. The Exponentiated Lomax (EL) distribution [23] with pdf

$$g(x;\beta,\lambda,\alpha)=\frac{\lambda\beta}{\alpha}\left(1+\frac{x}{\alpha}\right)^{-\lambda-1}\left(1-\left(1+\frac{x}{\alpha}\right)^{-\lambda}\right)^{\beta-1},\qquad x>0,\ \beta,\lambda,\alpha>0.$$

2. The Poisson–Lomax (PoL) distribution [24] with pdf

$$g(x;\beta,\lambda,\alpha)=\frac{\lambda\,\alpha\,\beta}{1-\exp(-\lambda)}\,(1+\beta x)^{-\alpha-1}\exp\left[-\lambda(1+\beta x)^{-\alpha}\right],\qquad x>0,\ \beta,\lambda,\alpha>0.$$

3. The Lomax-Rayleigh (LR) distribution [25] with pdf

$$g(x;\lambda,\alpha)=\frac{2\lambda}{\alpha}x\left(1+\frac{x^2}{\alpha}\right)^{-\lambda-1},\qquad x>0,\ \beta,\lambda,\alpha>0.$$

4. The power Lomax (PL) distribution [26] with pdf

$$g(x;\beta,\lambda,\alpha)=\frac{\lambda\beta}{\alpha}x^{\beta-1}\left(1+\frac{x^{\beta}}{\alpha}\right)^{-\lambda-1},\qquad x>0,\ \beta,\lambda,\alpha>0.$$

5. The Lomax (L) distribution [27] with pdf

$$g(x;\lambda,\alpha)=\frac{\lambda}{\alpha}\left(1+\frac{x}{\alpha}\right)^{-\lambda-1},\qquad x>0,\ \lambda,\alpha>0.$$

The first data represent 63 service times (thousand hours) of aircraft windshield (unit in thousand hours) as reported in Murthy et al. [28]. The data are as follows:

0.046	1.436	2.592	0.140	1.492	2.600	0.150	1.580
2.670	0.248	1.719	2.717	0.280	1.794	2.819	0.313
1.915	2.820	0.389	1.920	2.878	0.487	1.963	2.950
0.622	1.978	3.003	0.900	2.053	3.102	0.952	2.065
3.304	0.996	2.117	3.483	1.003	2.137	3.500	1.010
2.141	3.622	1.085	2.163	3.665	1.092	2.183	3.695
1.152	2.240	4.015	1.183	2.341	4.628	1.244	2.435
4.806	1.249	2.464	4.881	1.262	2.543	5.140	

The second data represents 63 strengths of 1.5 cm glass fibers which are measured by the National Physical Laboratory, in England as reported in Smith et al. [29]. The data are as follows:

0.55	1.64	1.39	1.82	1.60	1.13	1.70	1.55
0.93	1.68	1.49	2.01	1.62	1.29	1.77	1.61
1.25	1.73	1.53	0.77	1.66	1.48	1.84	1.63
1.36	1.81	1.59	1.11	1.69	1.50	0.84	1.67
1.49	2.00	1.61	1.28	1.76	1.55	1.24	1.70
1.52	0.74	1.66	1.42	1.84	1.61	1.30	1.78
1.58	1.04	1.68	1.50	2.24	1.62	1.48	1.89
1.61	1.27	1.76	1.54	0.81	1.66	1.51	

Table 6 provides a brief summary for both data sets.

Table 6: Summary of the aircraft windshield and glass fibers data

Data	Min	Q1	median	Mean	Q3	SD	Skewness	Kurtosis	Max
Aircraft windshield	0.0460	1.1220	2.0650	2.0850	2.8200	1.2452	0.4292	–0.3535	5.1400
Glass fibers	0.5500	1.3750	1.5900	1.5070	1.6850	0.3241	0.0000	0.00000	2.2400

The parameters of the fitted distributions are estimated using the ML method and some discrimination measures are calculated to explore the efficiency of the competing distributions. These measures include the Akaike information criterion (AIC), Bayesian IC (BIC), corrected AIC (CAIC), Hannan–Quinn IC (HQIC), and $-\ell$, where ℓ is the maximized log-likelihood. Additionally, goodness-of-fit statistics such as Anderson–Darling (An), Cramér–von Mises (Cr), and Kolmogorov–Smirnov (K-S) with its corresponding p-value (K-S p-value) are also calculated. More details about the goodness-of-fit statistics can be explored by Shama et al. [30].

Tables 7 and 8 present the MLEs of the parameters of the fitted distributions along with their discrimination measures for both data sets, respectively. Based on Tables 7 and 8, we conclude that the LL distribution is the best model as compared to other Lomax extensions. Tables 9 and 10 provide the values of goodness-of-fit measures of the LL model and other models for the two data sets. These results also indicate that the LL model provides a better fit to aircraft windshield and glass fibers data as compared to other Lomax models.

Table 7: The MLEs and discrimination measures for aircraft windshield data

Models	Estimates			$-\ell$	AIC	BIC	CAIC	HQIC
LL $(\upsilon, \omega, \alpha)$	0.9211	3.9175	7.2869	98.12831	202.2566	208.6861	211.6861	204.7853
EL (β, λ, α)	1.9012	228722.8	329853.6	103.5468	213.0936	219.5232	222.5232	215.6224
PoL (α, β, λ)	216.4421	0.0041	3.3768	100.4224	206.8449	213.2743	207.2516	209.3736
LR (λ, α)	15.99717	88.1644	—	102.4106	208.8212	213.1075	215.1075	210.5071
PL (β, λ, α)	1.8978	79935.96	115504.9	103.5469	213.0937	219.5231	213.5005	215.6224
L (λ, α)	17638.99	36780.66	—	109.2997	222.5995	226.8858	228.8858	224.2853

Table 8: The MLEs and discrimination measures for glass fibers data

Models	Estimates			$-\ell$	AIC	BIC	CAIC	HQIC
LL $(\upsilon, \omega, \alpha)$	2.1658	29.8357	10.7791	14.3206	34.6413	41.0707	35.0481	37.1700
EL (β, λ, α)	31.3556	23822.5418	9120.6519	31.3852	68.7704	75.1998	69.1772	71.2991
PoL (α, β, λ)	331.0121	0.0081	34.8844	30.6526	67.3052	73.7347	67.7120	69.8340
LR (λ, α)	2687.635	6379.087	—	49.8010	103.6019	107.8882	103.8019	105.2878
PL (β, λ, α)	31.3664	7007.7084	2682.2111	31.3893	68.7786	75.2081	69.1854	71.3074
L (λ, α)	26613.03	40103.90	—	88.8314	181.6629	185.9492	181.8629	183.3487

Table 9: Goodness-of-fit statistics of the competing models for aircraft windshield data

Models	Cr	An	K-S	K-S *p*-value
LL	0.0347	0.2403	0.0667	0.9241
EL	0.2341	1.3196	0.1442	0.1321
PoL	0.0992	0.6039	0.1062	0.4451
LR	0.0757	1.1182	0.0849	0.7217
PL	0.2035	1.2315	0.1438	0.1340
L	0.7790	3.8821	0.2078	0.0073

Table 10: Goodness-of-fit statistics of the competing models for glass fibers data

Models	Cr	An	K-S	K-S *p*-value
LL	0.1710	0.9615	0.1358	0.1953
EL	0.7862	4.2873	0.2290	0.0027
PoL	0.7565	4.1374	0.2224	0.0039
LR	0.4656	2.5544	0.3339	0.0000
PL	0.7863	4.2878	0.2290	0.0027
L	18.5583	121.9004	0.7739	0.0000

The probability-probability (P-P) plots of the fitted distributions for both data sets are provided in Figs. 4 and 5. The fitted density, cdf, sf, and P-P plots of the LL distribution for both data sets are displayed in Figs. 6 and 7. The plots support the results in Tables 7–10 and illustrate that the LL distribution provides a better approximation between the theoretical and empirical curves. Furthermore, Figs. 8 and 9 present the histograms of aircraft windshield and glass fibers data along with the fitted densities of the LL model and other studied distributions. All plots provide evidence that the LL distribution is the most well-adjusted model for aircraft windshield and glass fibers data.

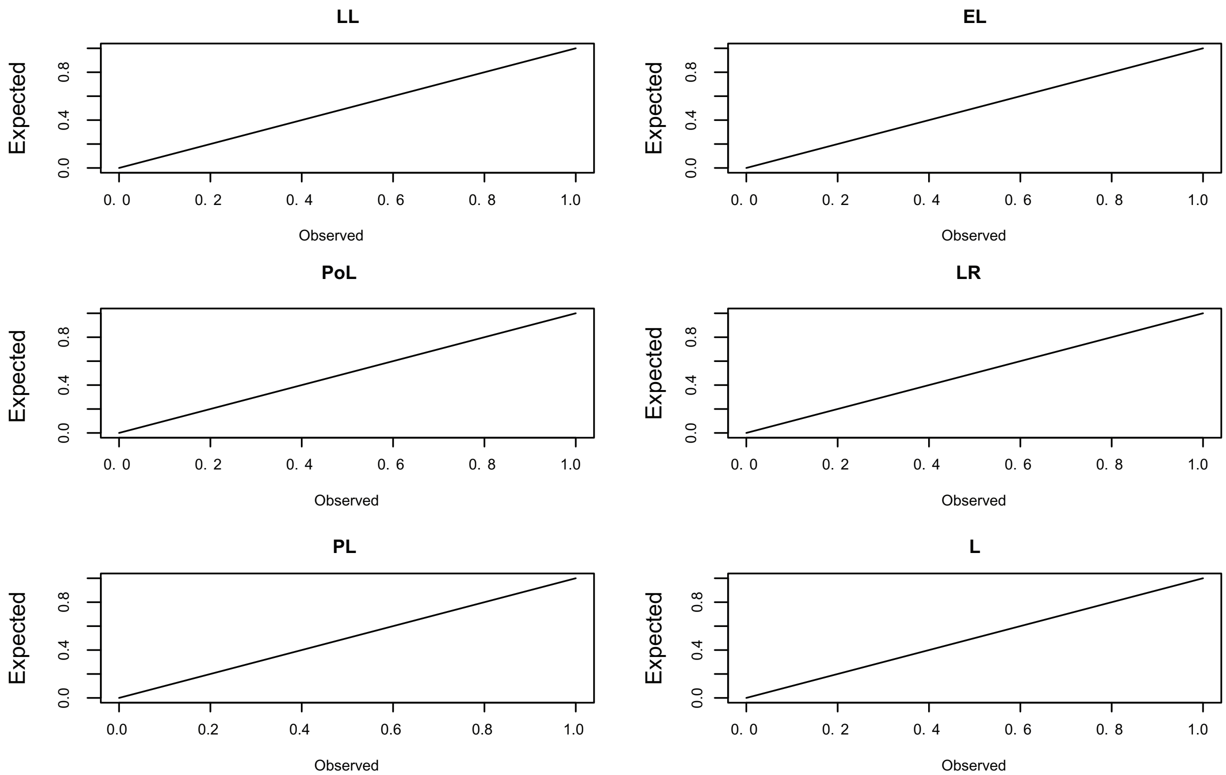

Figure 4: P-P plots for aircraft windshield data

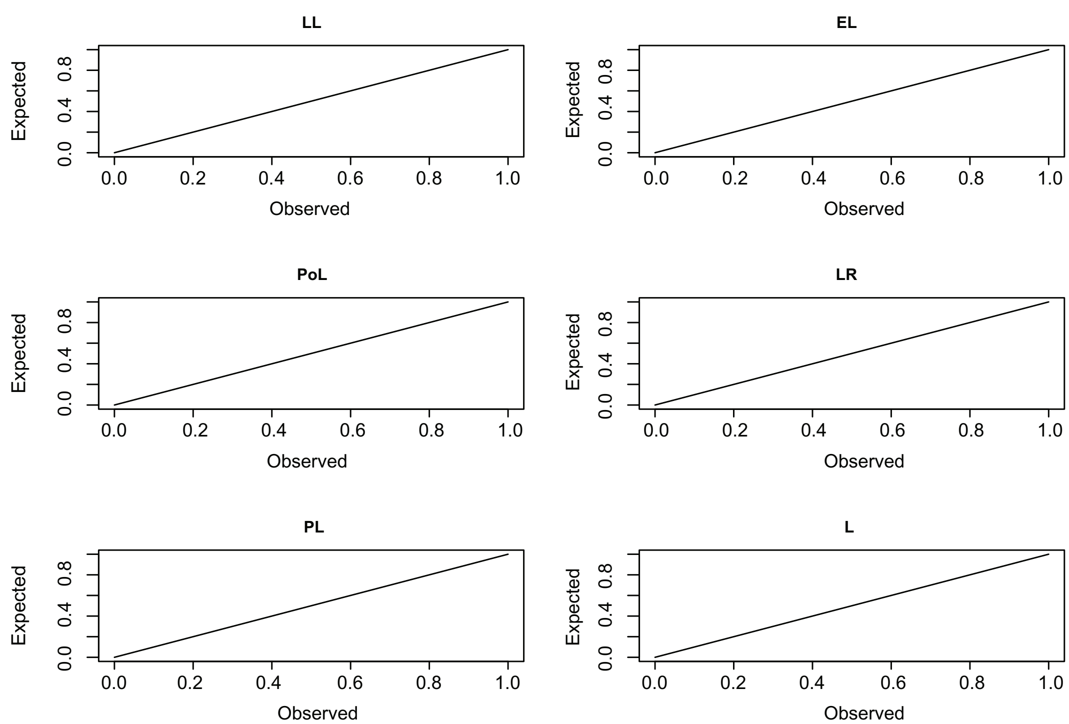

Figure 5: P-P plots for glass fibers data

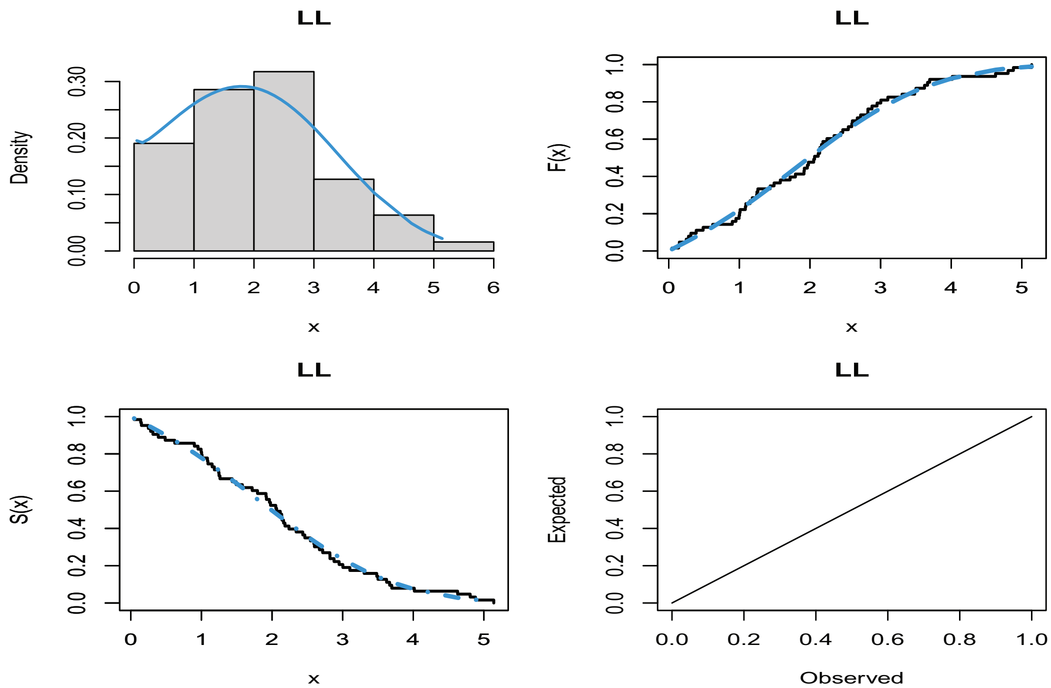

Figure 6: Plots of the fitted functions of the LL model for aircraft windshield data

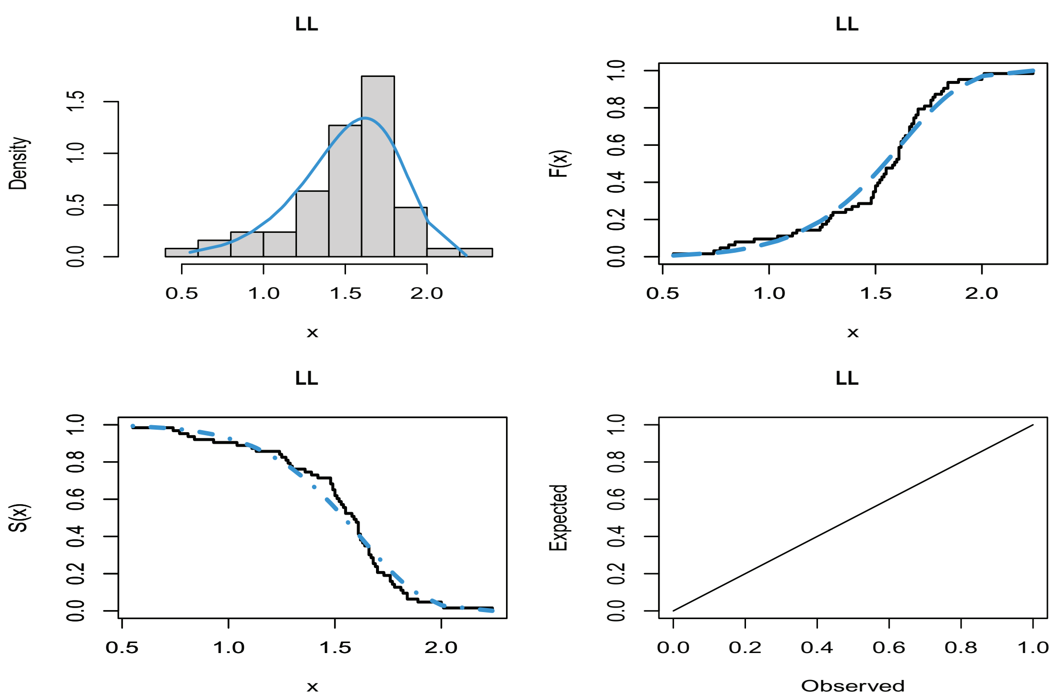

Figure 7: Plots of the fitted functions of the LL model for glass fibers data

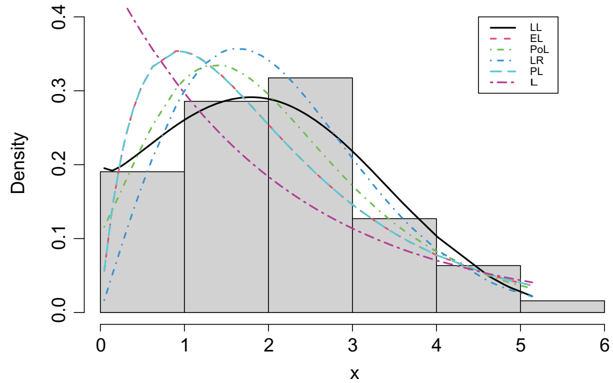

Figure 8: Histogram of aircraft windshield data along with the estimated densities

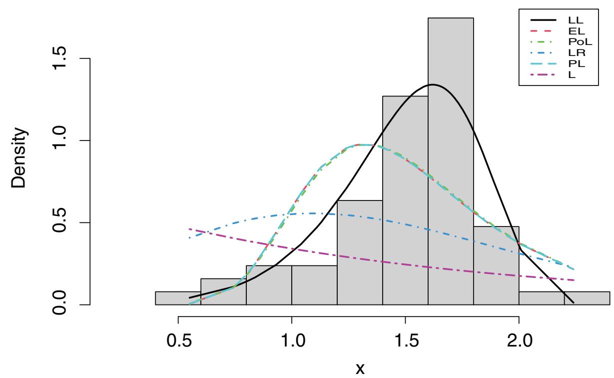

Figure 9: Histogram of glass fibers data along with the estimated densities

8 Conclusions

This study introduces a new flexible family called the LG family. Its special sub-models can represent various shapes of aging failure criteria, including monotonic and non-monotonic failure rates. The densities of the sub-models of the LG family can be reversed-J shaped, right-skewed, symmetric, left-skewed, decreasing-increasing-decreasing densities. One of its special models, namely the LL, is studied in detail. The failure rate shapes of the LL distribution are derived and proved mathematically. In addition, various statistical properties of the LL distribution are investigated. Six estimation methods are employed to estimate the LL parameters, and their performance is explored via simulation results. The numerical experiments illustrate the accuracy of the maximum likelihood; hence, they are recommended for estimating the LL parameters. Two real-life datasets are analyzed, indicating that the LL distribution can provide a better fit for modeling actual data compared to some competing Lomax models.

The perspectives of this study can include the development of a bivariate LL distribution and the construction of a discrete version of the LL model.

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Availability of Data and Materials: This work is mainly a methodological development and has been applied on secondary data which are provided in the manuscript.

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