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ARTICLE

Bayesian and Non-Bayesian Analysis for the Sine Generalized Linear Exponential Model under Progressively Censored Data

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ABSTRACT

This article introduces a novel variant of the generalized linear exponential (GLE) distribution, known as the sine generalized linear exponential (SGLE) distribution. The SGLE distribution utilizes the sine transformation to enhance its capabilities. The updated distribution is very adaptable and may be efficiently used in the modeling of survival data and dependability issues. The suggested model incorporates a hazard rate function (HRF) that may display a rising, J-shaped, or bathtub form, depending on its unique characteristics. This model includes many well-known lifespan distributions as separate sub-models. The suggested model is accompanied with a range of statistical features. The model parameters are examined using the techniques of maximum likelihood and Bayesian estimation using progressively censored data. In order to evaluate the effectiveness of these techniques, we provide a set of simulated data for testing purposes. The relevance of the newly presented model is shown via two real-world dataset applications, highlighting its superiority over other respected similar models.

KEYWORDS

Sine G family; generalized linear failure rate; progressively censored data; moments; maximum likelihood estimation; Bayesian estimation; simulation

1 Introduction

In various practical fields like medicine, engineering, and finance, among others, it is essential to model and analyze data related to the lifespan of objects or processes. Various lifetime distributions have been applied to describe such data. For example, the exponential and Rayleigh distributions and their variations. Each distribution possesses unique features determined by the behavior of the failure rate function, which can either steadily decrease or increase, remain constant, exhibit non-monotonic patterns, have a bathtub-shaped curve, or even follow an unimodal trend. In [1], Sarhan et al. proposed the generalized linear failure rate (GLFR) distribution which has another name the GLE distribution. It has a decreasing or unimodal probability density function (PDF) and its HRF can be increasing, decreasing, and bathtub-shaped. The GLE distribution has many applications in applied statistics and reliability analysis. The GLFR distribution is very flexible and has more special cases, as linear failure rate (linear exponential) (LFR), generalized exponential, generalized Rayleigh, exponential and Rayleigh a very well-known distribution for modeling lifetime data in reliability and medical studies. Lifetime data is frequently analyzed using the exponential, Rayleigh, linear failure rate, or exponentiated exponential distributions. It is well known that an exponential distribution can only have a constant HRF, whereas Rayleigh, linear failure rate, and generalized exponential distributions can only have monotone HRFs (increasing in the case of Rayleigh or LFR and increasing/decreasing in the case of the generalized exponential distribution). However, in practice, non-monotonic functions like bathtub-shaped HRFs must also be considered. However, the GLE distribution has a bathtub-shaped HRF and generalizes several well-known distributions, including the traditional LFR distribution. The previous elements motivate us to introduce a new extension of the GLE distribution. The cumulative distribution function (CDF) and the PDF for GLE are as follows:

$$H(x;\alpha,\theta,\lambda) = \left(1 - e^{-\alpha x - \frac{\theta}{2}x^2}\right)^{\lambda}, x > 0, \tag{1}$$

and

$$h(x;\alpha,\theta,\lambda) = \lambda\,(\alpha + \theta x)\,e^{-\left(\alpha x + \frac{\theta}{2}x^2\right)}\left(1 - e^{-\alpha x - \frac{\theta}{2}x^2}\right)^{\lambda-1}, \tag{2}$$

respectively, where $\lambda > 0$ is a shape parameter, $\alpha > 0$ and $\theta > 0$ are scale parameters. Many researchers constructed generalizations of the GLE distribution. For instance, various univariate extensions of the GLFR distribution have been introduced, including the generalized linear exponential (GLE) [2], beta linear failure rate [3], exponentiated GLFR [4], generalized exponential LFR [5], odd generalized exponential GLFR [6], inverted GLFR [7], Marshall-Olkin extended GLFR [8], truncated Cauchy power LFR [9] and modified beta GLFR [10] distributions.

Over recent years, numerous methods for augmenting parameters in distributions have been put forth and examined. These expanded distributions offer versatility in specific applications, including but not limited to economics, engineering, biological studies, and environmental sciences. Some well-known families are the Marshall-Olkin-G by [11], the beta-G by [12], the Kumaraswamy-G by [13], the logistic-G by [14], exponentiated generalized-G by [15], the Weibull-G by [16], the logistic-X family by [17], generalized inverted kumaraswamy by [18], marshall-olkin odd Burr III-G family by [19], type II exponentiated half logistic generated family by [20], odd generalized N-H generated family by [21], new truncated muth generated family by [22], exponentiated generalized Weibull exponential by [23], and new inverse Rayleigh distribution by [24].

Recently, there has been a growing focus on developing families of distributions based on trigonometric functions. These families offer a balance between simplicity in their definitions, enabling a clear understanding of their mathematical properties, and a high degree of applicability for modeling various real-world datasets. This balance is achieved through the effective utilization of flexible trigonometric functions. As far as we are aware, the sine-G family of distributions is one of the earliest examples of such trigonometric distribution families. In [25], Kumar et al. introduced a novel approach for generating new probability distributions by modifying trigonometric functions. They modified the sine function to create a unique statistical distribution known as the sine-G family, with the CDF and

PDF defined as follows:

$$F_S(x;\kappa) = \sin\left(\frac{\pi}{2}H(x;\kappa)\right), \quad x \in R, \tag{3}$$

$$f_S(x;\kappa) = \frac{\pi}{2}h(x;\kappa)\cos\left(\frac{\pi}{2}H(x;\kappa)\right), \tag{4}$$

respectively. The HRF is given by

$$\xi_S(x;\kappa) = \frac{\pi}{2}h(x;\kappa)\tan\left(\frac{\pi}{4}(1 + H(x;\kappa))\right), \tag{5}$$

where $H(x;\kappa)$ and $h(x;\kappa)$ are the CDF and PDF of a certain continuous distribution with parameters vector denoted by $\kappa = (\alpha, \theta, \lambda)$. This family has many advantages like, it is simple form, the two cumulative functions $F(x;\kappa)$ and $H(x;\kappa)$ have the same number of parameters; there is no additional parameter, avoiding any problem of over-parametrization, In addition to, $F(x;\kappa)$ has the ability to increase the flexibility of $H(x;\kappa)$, providing new flexible models. These distributions are linked to a predefined reference distribution, a choice made by the practitioner according to the specific study's context. It has been confirmed that the S-G family (i) provides an appealing alternative to the reference family, as it satisfies the inequality $H(x;\kappa) \leq F_S(x;\kappa)$ for any $x \in R$, (ii) maintains an acceptable level of mathematical complexity without introducing additional parameters, and (iii) offers the flexibility to construct diverse statistical models capable of handling data with varying characteristics.

Other trigonometric families of distributions have been developed. See, for instance, beta trigonometric distribution by [26], hyperbolic cosine-F family [27], odd hyperbolic cosine family of lifetime distributions by [28], odd hyperbolic cosine exponential-exponential distribution by [29], transmuted arcsine distribution by [26], the arcsine exponentiated-X family by [30].

The failure of components and units, which make up the majority of operational systems in the fields of industrial and mechanical engineering, has been extensively studied by statisticians. Their research focuses on tracking the functioning units until they fail, recording their lifespans, using statistical inference methods to analyze the data gathered, and then calculating the reliability and hazard functions for the entire system using the data gathered. However, some experimental units are pricey and very reliable; therefore, in this case, the number of experimental units and the length of the lifetime experiment of these units must be reduced. The progressively Type-II censoring strategy satisfies the lifespan experiment's requirements for good estimators while preventing certain experimental units from failing.

The main objectives of this study are to contribute to the statistical literature and address some issues about the failure of units and components for various applications of the extension model of the trigonometric family. The following reasons are sufficient justification for doing so:

- Introducing the sine generalized linear exponential distribution as a novel three-parameter model based on the sine-G family of distributions.
- The PDF can exhibit several features, such as being unimodal, declining, right-skewed, or heavy-tailed. Similarly, the HRF might display growing, J-shaped. These properties are desired in a range of applications, such as survival analysis, reliability, and uncertainty modeling.
- There is a closed-form expression for the equivalent quantile.
- The new suggested model is very flexible and it has five sub-models.
- It is possible to compute several statistical features, including the quantiles, Bowley's skewness, Moor's kurtosis, moments, moment generating function, incomplete moments, conditional

moments, Lorenz and Bonferroni curves, residual life and inverted residual life functions, and so on.

- Using progressively Type-II censoring schemes to prevent certain experimental units from failing.
- The parameters of the SGLE distribution can be estimated utilizing by Bayesian and non-Bayesian estimation methods.
- For illustrative purposes, this study examines SGLE distribution distinct datasets in the actual world. We demonstrate, by highlighting its functionalities, that the SGLE distribution may serve as a more viable alternative to formidable competitors.

This article's remaining sections are organized as follows. The sine generalized linear exponential distribution and its sub-models are represented in Section 2. Section 3 introduces a linear representation of the SGLE density function. Section 4 provides information on the statistical characteristics of the SGLE distribution, such as quantiles, Bowley's skewness, Moor's kurtosis, moments, moment generating function, incomplete moments, conditional moments, Lorenz and Bonferroni curves, residual life and inverted residual life functions. In Section 5, the progressively Type-II censoring scheme is carried out. In Section 6, the model parameters' Bayesian and non-Bayesian inference is carried out. In Section 7, two real datasets show the applicability and flexibility of the SGLE distribution. Section 8 delves into the results of the simulation. Furthermore, the conclusion is presented in Section 9, which is located at the end of the paper.

2 Sine Generalized Linear Exponential Model

In this section, we construct a new flexible model called the sine generalized linear exponential model by inserting (1) into (3), we obtain the CDF as follows:

$$F_{SGLE}(x;\kappa) = \sin\left[\frac{\pi}{2}\left(1 - e^{-\alpha x - \frac{\theta}{2}x^2}\right)^{\lambda}\right], \quad x > 0, \tag{6}$$

and the corresponding PDF is

$$f_{SGLE}(x;\kappa) = \frac{\pi}{2}\lambda\,(\alpha + \theta x)\,e^{-\left(\alpha x + \frac{\theta}{2}x^2\right)}\left(1 - e^{-\alpha x - \frac{\theta}{2}x^2}\right)^{\lambda-1}\cos\left[\frac{\pi}{2}\left(1 - e^{-\alpha x - \frac{\theta}{2}x^2}\right)^{\lambda}\right], \tag{7}$$

where $\kappa = (\alpha, \theta, \lambda)$. The survival function and HRF for the SGLE are, respectively, given by

$$\overline{F}_{SGLE}(x;\kappa) = 1 - \sin\left[\frac{\pi}{2}\left(1 - e^{-\alpha x - \frac{\theta}{2}x^2}\right)^{\lambda}\right], \tag{8}$$

and

$$\xi_{SGLE}(x;\kappa) = \frac{\pi\lambda}{2}\,(\alpha + \theta x)\,e^{-\left(\alpha x + \frac{\theta}{2}x^2\right)}\left(1 - e^{-\alpha x - \frac{\theta}{2}x^2}\right)^{\lambda-1} \times \tan\left[\frac{\pi}{4}(1 + \left(1 - e^{-\alpha x - \frac{\theta}{2}x^2}\right)^{\lambda}\right]. \tag{9}$$

Fig. 1 discussed density and hazard rate for the SGLE distribution with different values of parameters.

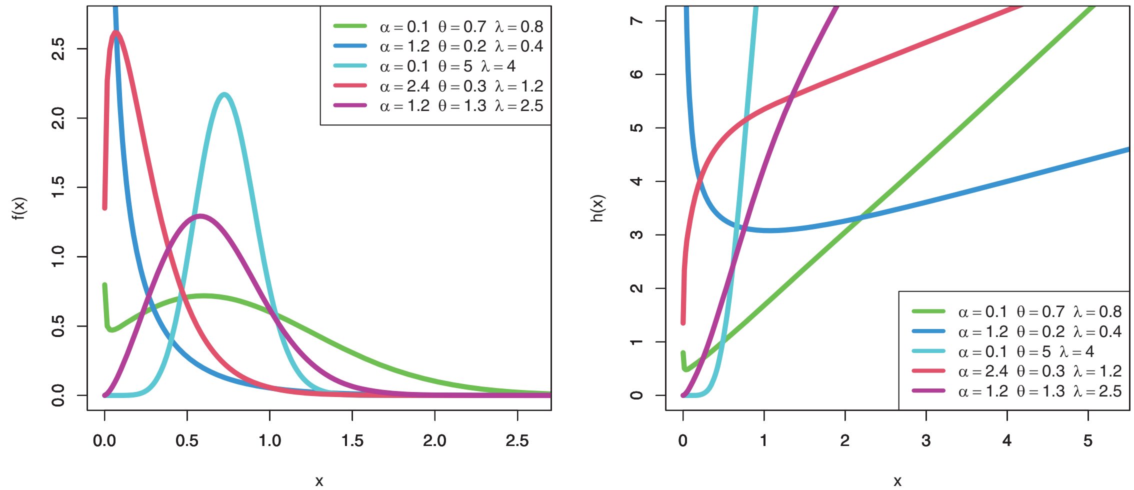

Figure 1: Density and hazard rate for the SGLE distribution

2.1 Some Special Models of the SGLE Model

The SGLE model contains five sub-models:

1. At $\lambda = 1$ the SGLE model reduces to the sine LE model.
2. At $\theta = 0$ the SGLE model reduces to the sine generalized exponential model.
3. At $\alpha = 0$ the SGLE model reduces to the sine generalized Rayleigh model.
4. At $\lambda = 1$, $\theta = 0$ the SGLE model reduces to the sine exponential model.
5. At $\lambda = 1$, $\alpha = 0$ the SGLE model reduces to the sine Rayleigh model.

3 Linear Representation of the SGLE Density Function

In this section, we derived the density expansion of the SGLE distribution. Using the Taylor series expansion of the cosine function,

$$\cos\left[\frac{\pi}{2}G(x)\right] = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!}\left(\frac{\pi}{2}G(x)\right)^{2i},$$

we have

$$\cos\left[\frac{\pi}{2}\left(1 - e^{-\alpha x - \frac{\theta}{2}x^2}\right)^{\lambda}\right] = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!}\left(\frac{\pi}{2}\right)^{2i}\left(1 - e^{-\alpha x - \frac{\theta}{2}x^2}\right)^{2\lambda i}, \tag{10}$$

inserting (10) in (7), the SGLE density function reduces to

$$f_{SGLE}(x;\kappa) = \lambda \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!}\left(\frac{\pi}{2}\right)^{2i+1}(\alpha + \theta x)\, e^{-\left(\alpha x + \frac{\theta}{2}x^2\right)}\left(1 - e^{-\alpha x - \frac{\theta}{2}x^2}\right)^{\lambda(2i+1)-1}. \tag{11}$$

But

$$\left(1 - e^{-\alpha x - \frac{\theta}{2}x^2}\right)^{\lambda(2i+1)-1} = \sum_{j=0}^{\infty}(-1)^j\binom{\lambda(2i+1)-1}{j}e^{-j\left(\alpha x + \frac{\theta}{2}x^2\right)}, \tag{12}$$

applying (12) in (11), we obtain

$$f_{SGLE}(x;\kappa) = \lambda \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{(2i)!} \left(\frac{\pi}{2}\right)^{2i} \binom{\lambda(2i+1)-1}{j} (\alpha + \theta x)\, e^{-(j+1)\left(\alpha x + \frac{\theta}{2}x^2\right)}. \tag{13}$$

Expanding $e^{-(j+1)\frac{\theta}{2}x^2}$ in power series as

$$e^{-(j+1)\frac{\theta}{2}x^2} = \sum_{k=0}^{\infty} \frac{(-1)^k (j+1)^k}{k!} \left(\frac{\theta}{2}\right)^k x^{2k}, \tag{14}$$

inserting (14) in (13) the SGLE density function can be written as

$$f_{SGLE}(x;\kappa) = \sum_{i,j,k=0}^{\infty} \omega_{i,j,k} \left(\alpha x^{2k} + \theta x^{2k+1}\right) e^{-(j+1)\alpha x}, \tag{15}$$

where

$$\omega_{i,j,k} = \lambda \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j+k}(j+1)^k}{(2i)!\,k!} \left(\frac{\pi}{2}\right)^{2i+1} \left(\frac{\theta}{2}\right)^k \binom{\lambda(2i+1)-1}{j}.$$

4 Statistical Properties

In this section, we studied some important mathematical and statistical properties of the SGLE distribution, specifically quantile function, ordinary moments, incomplete moments, Lorenz and Bonferroni curves, and moments of the residual life and reversed residual lives.

4.1 Quantile Function

Quantile functions find utility in theoretical, statistical, and Monte Carlo scenarios. In Monte Carlo simulations, these functions are utilized to generate simulated random variables for both traditional and contemporary continuous distributions. To derive the quantile function $Q(u)$ for the SGLE distribution, represented as $x = Q(u)$, we can obtain it by reversing the process described in Eq. (6) as follows:

$$Q(u;\kappa) = F^{-1}_{SGLE}(u;\kappa) = \frac{-\alpha + \sqrt{\alpha^2 - 2\theta \ln\left[1 - \left(\frac{2}{\pi}\arcsin(u)\right)^{\frac{1}{\lambda}}\right]}}{\theta}, u \in (0,1), \tag{16}$$

The median is given by

$$Median = \frac{-\alpha + \sqrt{\alpha^2 - 2\theta \ln\left[1 - \left(\frac{2}{\pi}\arcsin(0.5)\right)^{\frac{1}{\lambda}}\right]}}{\theta}.$$

One of the initial proposals for a skewness measure is the Bowley skewness, introduced by Kenney and Keeping in 1962, and it is defined as follows:

$$SK = \frac{Q(\frac{3}{4}) + Q(\frac{1}{4}) - 2Q(\frac{1}{2})}{Q(\frac{3}{4}) - Q(\frac{1}{4})}.$$

Conversely, the Moors kurtosis, as introduced by Moors in 1988 and calculated using quantiles, is expressed as

$$KU = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{8})}.$$

In this context, $Q(.)$ denotes the quantile function. The metrics SK and KU exhibit reduced sensitivity to extreme data points, and they are applicable to distributions that may not possess moments. In the case of symmetric unimodal distributions, a positive kurtosis value suggests that the distribution has heavier tails and is more peaked compared to a normal distribution, while a negative kurtosis value indicates lighter tails and a flatter shape. Fig. 2 discusses SK and KT for the SGLE distribution with different values of parameters.

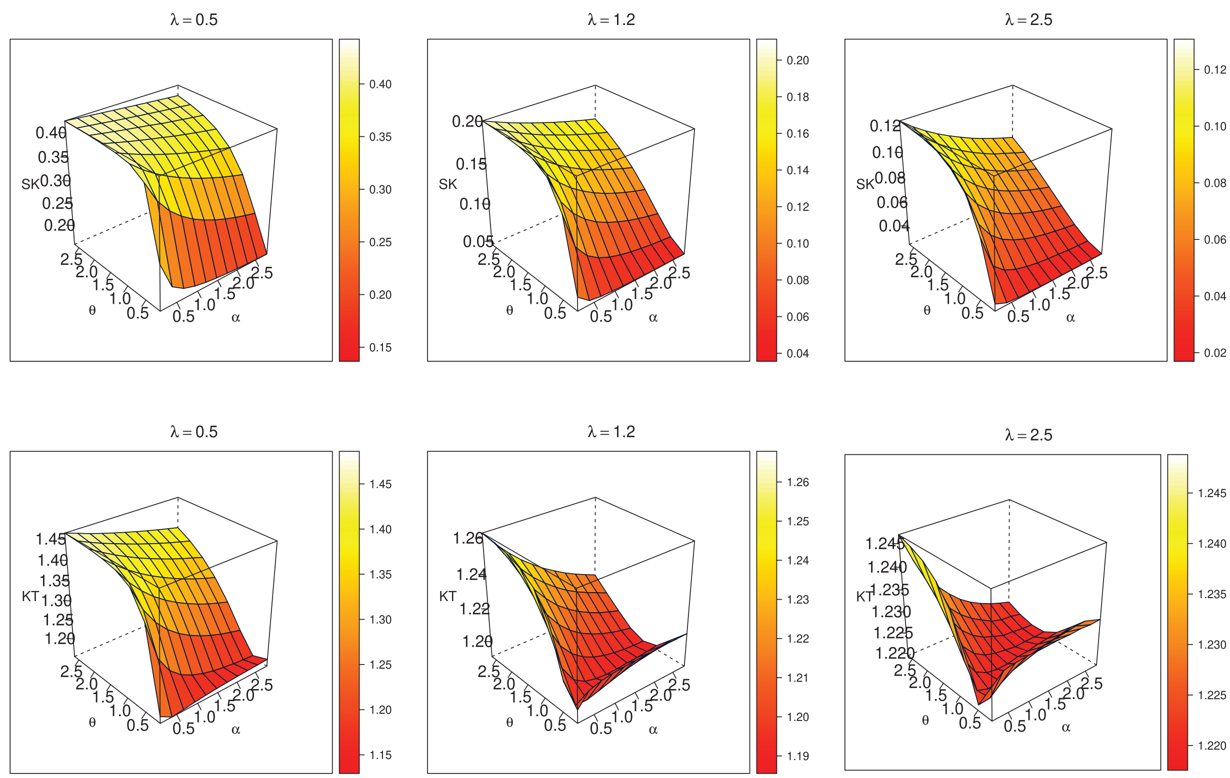

Figure 2: Bowley skewness and Moors kurtosis for the SGLE distribution

4.2 Moments and Moment Generating Functions

In this particular section, we will establish the formulas for both the typical and moment-generating functions of the SGLE distribution. These moment calculations for various orders are essential for estimating the device's expected lifespan, as well as assessing the spread, skewness, and kurtosis of data sets encountered in reliability-related situations.

4.2.1 Moments

The r_{th} moment of the SGLE distribution can be derived using Eq. (7).

$$\mu_r^{/} = E(X^r) = \int_0^{\infty} x^r f_{SGLE}(x;\kappa)dx$$

$$= \sum_{i,j,k=0}^{\infty} \omega_{i,j,k} \int_0^{\infty} \left(\alpha x^{2k+r} + \theta x^{2k+r+1}\right) e^{-(j+1)\alpha x} dx.$$

After a series of transformations, which involve substituting a new variable $z = (j+1)\alpha x$ and introducing the gamma function, we arrive at the following result:

$$\mu_r^{/} = \sum_{i,j,k=0}^{\infty} \omega_{i,j,k} \left(\alpha + \frac{\theta(2k+r+1)}{\alpha(j+1)}\right) \frac{\Gamma(2k+r+1)}{[\alpha(j+1)]^{2k+r+1}}. \tag{17}$$

Table 1 shows some numerical values of moments for the SGLE distribution.

Table 1: Mean, variance, SK and KT for different values of parameters

λ	α	θ	Mean	Variance	SK	KT
0.5	0.1	0.3	0.8080	0.5545	0.1898	1.1519
		1	0.4942	0.1807	0.1620	1.1638
		1.7	0.3919	0.1084	0.1561	1.1691
		2.4	0.3358	0.0775	0.1536	1.1717
	0.9	0.3	0.2604	0.1153	0.4031	1.3923
		1	0.2235	0.0716	0.3653	1.3002
		1.7	0.2020	0.0534	0.3387	1.2538
		2.4	0.1869	0.0431	0.3182	1.2256
	1.7	0.3	0.1484	0.0417	0.4180	1.4427
		1	0.1387	0.0329	0.4041	1.3959
		1.7	0.1315	0.0277	0.3918	1.3609
		2.4	0.1258	0.0241	0.3809	1.3337
	2.5	0.3	0.1029	0.0207	0.4218	1.4562
		1	0.0991	0.0181	0.4149	1.4307
		1.7	0.0961	0.0162	0.4085	1.4092
		2.4	0.0934	0.0148	0.4024	1.3900
1.3	0.1	0.3	1.6960	0.7007	0.0462	1.2106
		1	0.9983	0.2161	0.0429	1.2133
		1.7	0.7817	0.1278	0.0424	1.2138
		2.4	0.6651	0.0907	0.0421	1.2139
	0.9	0.3	0.6704	0.2359	0.1485	1.2162
		1	0.5475	0.1270	0.1063	1.1963
		1.7	0.4818	0.0884	0.0881	1.1948
		2.4	0.4380	0.0682	0.0780	1.1960
	1.7	0.3	0.3934	0.0940	0.1763	1.2424

(Continued)

Table 1 (continued)

λ	α	θ	Mean	Variance	SK	KT
		1	0.3578	0.0679	0.1506	1.2179
		1.7	0.3330	0.0540	0.1333	1.2065
		2.4	0.3140	0.0451	0.1208	1.2005
	2.5	0.3	0.2750	0.0481	0.1840	1.2524
		1	0.2609	0.0397	0.1691	1.2347
		1.7	0.2496	0.0342	0.1573	1.2233
		2.4	0.2403	0.0301	0.1475	1.2152
3	0.1	0.3	2.5250	0.6094	0.0232	1.2325
		1	1.4575	0.1849	0.0222	1.2331
		1.7	1.1346	0.1090	0.0222	1.2330
		2.4	0.9623	0.0773	0.0220	1.2331
	0.9	0.3	1.1687	0.2904	0.0725	1.2260
		1	0.9096	0.1360	0.0463	1.2249
		1.7	0.7823	0.0895	0.0380	1.2268
		2.4	0.7008	0.0668	0.0339	1.2279
	1.7	0.3	0.7091	0.1294	0.0976	1.2362
		1	0.6251	0.0842	0.0739	1.2264
		1.7	0.5706	0.0630	0.0617	1.2244
		2.4	0.5305	0.0506	0.0541	1.2241
	2.5	0.3	0.5009	0.0689	0.1065	1.2420
		1	0.4660	0.0530	0.0906	1.2325
		1.7	0.4396	0.0435	0.0795	1.2281
		2.4	0.4184	0.0369	0.0715	1.2257

4.2.2 Moment Generating Function

The moment-generating function of the SGLE distribution is

$$\begin{aligned} M(t) = E(e^{tX}) &= \int_0^\infty e^{tx} f_{SGLE}(x;\kappa)dx \\ &= \sum_{i,j,k=0}^{\infty} \omega_{i,j,k} \int_0^\infty \left(\alpha x^{2k} + \theta x^{2k+1}\right) e^{-[(j+1)\alpha - t]x} dx \\ &= \sum_{i,j,k=0}^{\infty} \omega_{i,j,k} \left(\alpha + \frac{\theta(2k+1)}{\alpha(j+1)-t}\right) \frac{\Gamma(2k+1)}{[\alpha(j+1)-t]^{2k+1}}. \end{aligned} \tag{18}$$

4.3 Incomplete Moments

The s^{th} incomplete moment of the SGLE distribution is given by

$$\xi_s(t) = E(x^s \mid X < t) = \int_0^t x^s f_{SGLE}(x;\kappa)dx$$
$$= \sum_{i,j,k=0}^{\infty} \omega_{i,j,k} \int_0^t \left(\alpha x^{2k+s} + \theta x^{2k+s+1}\right) e^{-(j+1)\alpha x} dx$$
$$= \sum_{i,j,k=0}^{\infty} \omega_{i,j,k} \left[\frac{\alpha\gamma(2k+s+1,(j+1)\alpha t)}{[(j+1)\alpha]^{2k+s+1}} + \frac{\theta\gamma(2(k+1)+s,(j+1)\alpha t)}{[(j+1)\alpha]^{2(k+1)+s}}\right], \quad (19)$$

where $\gamma(n,x) = \int_0^x t^{n-1}e^{-t}dt$ denotes the lower incomplete gamma function.

4.4 Conditional Moments

The conditional moment of the SGLE distribution can be written as

$$\Omega_s(t) = E(x^s \mid X > t) = \frac{1}{\overline{F}(t)}\psi_s(t)$$

where

$$\psi_s(t) = \int_t^{\infty} x^s f_{SGLE}(x;\kappa)dx = \sum_{i,j,k=0}^{\infty} \omega_{i,j,k} \int_t^{\infty} \left(\alpha x^{2k+s} + \theta x^{2k+s+1}\right) e^{-(j+1)\alpha x} dx$$
$$= \sum_{i,j,k=0}^{\infty} \omega_{i,j,k} \left[\frac{\alpha\Gamma(2k+s+1,(j+1)\alpha t)}{[(j+1)\alpha]^{2k+s+1}} + \frac{\theta\Gamma(2(k+1)+s,(j+1)\alpha t)}{[(j+1)\alpha]^{2(k+1)+s}}\right], \quad (20)$$

where $\Gamma(n,x) = \int_x^{\infty} t^{n-1}e^{-t}dt$ denotes the upper incomplete gamma function.

4.5 Lorenz and Bonferroni Curves

The Lorenz curve was first introduced by Lorenz in the year 1905, and the Bonferroni curve. These curves have found applications in various fields, including economics, where they are used to analyze income distribution and poverty. Additionally, they serve as tools for quantifying the inequality within the distribution of a variable and apply to a wide range of disciplines, such as reliability, demography, medicine, and insurance. For a positive random variable X, both the Lorenz and Bonferroni curves, at a specified probability p, can be expressed as follows:

$$L(p) = \frac{1}{\mu}\int_0^q xf(x)dx = \frac{1}{\mu}\sum_{i,j,k=0}^{\infty} \omega_{i,j,k} \left[\frac{\alpha\gamma(2(k+1),(j+1)\alpha q)}{[(j+1)\alpha]^{2(k+1)}} + \frac{\theta\gamma(2(k+1)+1,(j+1)\alpha q)}{[(j+1)\alpha]^{2(k+1)+1}}\right], \quad (21)$$

and

$$B(p) = \frac{1}{\mu p}\int_0^q xf(x)dx = \frac{1}{\mu p}\sum_{i,j,k=0}^{\infty} \omega_{i,j,k} \left[\frac{\alpha\gamma(2(k+1),(j+1)\alpha q)}{[(j+1)\alpha]^{2(k+1)}} + \frac{\theta\gamma(2(k+1)+1,(j+1)\alpha q)}{[(j+1)\alpha]^{2(k+1)+1}}\right], \quad (22)$$

respectively, where $\mu = E(X)$, and $q = Q(p)$ is the quantile function of X at p.

4.6 Residual Life and Reversed Residual Life Functions

Assume that a component remains operational until time $t \geq 0$. The residual life is the duration from time t until the point of failure, and it is described by the conditional random variable denoted as $X - t|X > t$. The r^{th}-order moment of the residual life is

$$\mu_r(t) = E((X-t)^r \mid X > t) = \frac{1}{\overline{F}(t)} \int_t^{\infty} (x-t)^r f(x)dx, r \geq 1.$$

For SGLE distribution, we get

$$\mu_r(t) = \frac{1}{\overline{F}(t)} \sum_{i,j,k=0}^{\infty} \sum_{h=0}^{r} \omega_{i,j,k} \binom{r}{h} (-t)^{r-h} \int_t^{\infty} x^r f_{SGLE}(x;\kappa)dx$$
$$= \frac{1}{\overline{F}(t)} \sum_{i,j,k=0}^{\infty} \sum_{h=0}^{r} \omega_{i,j,k} \binom{r}{h} (-t)^{r-h} \left[\frac{\alpha\Gamma(2k+r+1,(j+1)\alpha t)}{[(j+1)\alpha]^{2k+r+1}} + \frac{\theta\Gamma(2(k+1)+r,(j+1)\alpha t)}{[(j+1)\alpha]^{2(k+1)+r}} \right]. \quad (23)$$

The average remaining lifespan (also known as the life expectancy at time t) signifies the anticipated additional life duration for a component or device that is still functioning at age t. To calculate the mean residual life (MRL) for the SGLE distribution, you can set $r = 1$ in Eq. (23), which is defined as

$$\mu(t) = E(X_t) = E(X \mid X > t).$$

In the realm of reliability theory, the extra time a component can continue operating after it has already failed by time t is referred to as the reversed residual life function (RRL). It represents the duration of time the component remains inactive. The conditional random variable $X_{(t)} = t - X|X < t$ denotes the time that has passed since the failure of X, given that it failed at or before time t. The r^{th} order moment of the reversed residual life, also known as the inactivity time, can be calculated using a commonly known formula.

$$m_r(t) = E((t-X)^r \mid X \leq t) = \frac{1}{F(t)} \int_0^t (t-x)^r f\ (x)dx, r \geq 1$$
$$= \frac{1}{F(t)} \sum_{i,j,k=0}^{\infty} \sum_{h=0}^{r} \omega_{i,j,k} \binom{r}{h} (-t)^{r-h} \int_0^t x^r f_{SGLE}(x;\kappa)dx$$
$$= \frac{1}{F(t)} \sum_{i,j,k=0}^{\infty} \sum_{h=0}^{r} \omega_{i,j,k} \binom{r}{h} (-t)^{r-h} \left[\frac{\alpha\gamma(2k+r+1,(j+1)\alpha t)}{[(j+1)\alpha]^{2k+r+1}} + \frac{\theta\gamma(2(k+1)+r,(j+1)\alpha t)}{[(j+1)\alpha]^{2(k+1)+r}} \right].$$

5 Progressively Type-II Censoring Schemes

Progressively censored samples are those that are removed from further analysis at different phases of an experiment, while not all of the remaining specimens are. Sample specimens that are still present after each censorship stage are kept under observation until they fail or until the next censoring stage.

Under progressively Type-II censored samples: Firstly, the experimenter adds n independent, identical units to the measure of life. Secondly, the experimenter determines m observation of the censored sample. Thirdly, the experimenter chooses optimal scheme R by the experience of the experimenter but the following constrains must be met as: $R_i \geq 0; n-m+\sum_{i=1}^{m} R_i = 0$. The remaining $n-1$ surviving units get R_1 units randomly removed from them after the first failure occurs, let us say at time $x_{1:m:n}$. The remaining $n-R_1-2$ surviving units get R_2 units randomly removed from them when the second failure occurs at time $x_{2:m:n}$. When the m^{th} failure occurs at time $x_{m:m:n}$, the experiment is over, and the $R_m = n-m-\sum_{i=1}^{m-1} R_i$ surviving units are taken out of the test. The progressive Type-II censoring method is denoted by $R = (R_1, R_2, \ldots, R_m)$. progressive Type-II censoring, with a predetermined R censoring scheme.

Assume that n independent units are put through a life test with the associated failure times of $x_{1:m:n} < x_{2:m:n} < \ldots < x_{m:m:n}$, and m, respectively. Additionally, assume that the progressive Type-II censoring scheme is $R_1, R_2, \ldots, R_m$ and that the pre-fixed number of failures to be seen is m. The failure times that have been m entirely observed will be shown as $x_{i:m:n}; i = 1, 2, \ldots, m$. The likelihood function is then as follows:

$$L(x;\kappa) = C\prod_{i=1}^{m} f(x_{i:m:n};\kappa)\left[1-F(x_{i:m:n};\kappa)\right]^{R_i}, \tag{24}$$

where C may be a constant defined as $C = n(n-R_1-1)\cdots(n-\sum_{i=1}^{m-1}(R_i+1))$ (see [31] for details) and see Fig. 3.

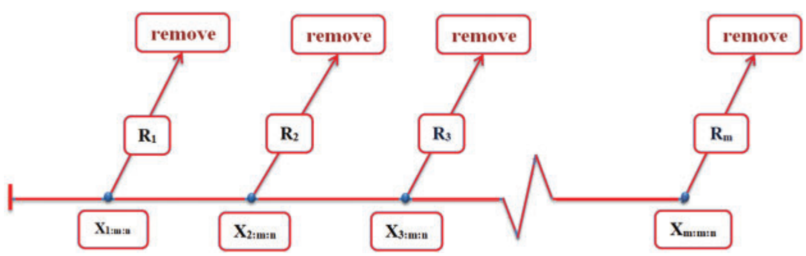

Figure 3: A diagram showing Type-II progressive censorship

Based on the SGLE distribution by Eqs. (6), (7) and (24), the likelihood function of the SGLE based on progressive Type-II censored sample is then as follows:

$$\begin{aligned} L(x;\kappa) = {} & \left(\frac{\pi}{2}\right)^{n} \lambda^{n} e^{-\sum_{i=1}^{m}\left(\alpha x_{i:m:n}+\frac{\theta}{2}x^{2}\right)} \prod_{i=1}^{m}\left(\alpha+\theta x_{i:m:n}\right)\left(1-e^{-\alpha x_{i:m:n}-\frac{\theta}{2}x_{i:m:n}^{2}}\right)^{\lambda-1} \\ & \times \prod_{i=1}^{m}\cos\left[\frac{\pi}{2}\left(1-e^{-\alpha x_{i:m:n}-\frac{\theta}{2}x_{i:m:n}^{2}}\right)^{\lambda}\right]. \end{aligned} \tag{25}$$

Based on Eq. (25) and Fig. 3, we note that there is more than one special case, such as the following:

- Complete sample when $m = n$ and $R_i = 0;\; i = 1, \ldots, m$.
- Type-II censored sample when $m < n$ and $R_m = n-m$ and $R_i = 0;\; i = 1, \ldots, m-1$.

More information on the increasingly progressive censored samples may be found in Balakrishnan et al. [32] and Balakrishnan et al. [31]. Aggarwala et al. [33] have discussed the differences in the situation of progressive Type-II censoring where lifespan distributions are Weibull, log-normal, and exponential. For more information and examples, see [34–37].

6 Inference and Estimation Methods

In this section, Bayesian and non-Bayesian inference have been discussed for parameters of SGLE distribution.

6.1 Maximum Likelihood Estimation

The maximum likelihood estimates (MLEs) possess favorable characteristics and find utility in constructing confidence intervals, regions, and test statistics. We calculate the MLEs for the parameters of the SGLE distribution using complete samples exclusively, see [38,39]. Consider a random sample of size n, denoted as $x_1, \ldots, x_n$, drawn from the SGLE distribution as defined in Eq. (7). Let $U_n(\kappa) = \left(\frac{\partial L_n}{\partial \alpha}, \frac{\partial L_n}{\partial \theta}, \frac{\partial L_n}{\partial \lambda}\right)^T$ be $q \times 1$ vector of parameters. The log-likelihood function is given by

$$\begin{aligned} L_n =& n\log\left(\frac{\pi}{2}\right) + n\log(\lambda) + \sum_{i=1}^{n}\log\left(\alpha + \theta x_{i:m:n}\right) - \alpha\sum_{i=1}^{n} x_{i:m:n} - \frac{\theta}{2}\sum_{i=1}^{n} x_{i:m:n}^2 \\ &+ (\lambda-1)\sum_{i=1}^{n}\log\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}\right) + \sum_{i=1}^{n}\log\cos\left[\frac{\pi}{2}\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}\right)^{\lambda}\right] \\ &+ \sum_{i=1}^{m} R_i \log\left\{1 - \sin\left[\frac{\pi}{2}\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}\right)^{\lambda}\right]\right\}. \end{aligned} \tag{26}$$

The log-likelihood can be maximized through direct utilization of the SAS program or R-language, or by solving the nonlinear likelihood equations derived from differentiating (26).

$$\begin{aligned} \frac{\partial L_n}{\partial \alpha} =& \sum_{i=1}^{n}\frac{1}{(\alpha + \theta x_{i:m:n})} - \sum_{i=1}^{n} x_{i:m:n} + (\lambda-1)\sum_{i=1}^{n}\frac{x_{i:m:n}e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}}{1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}} \\ &- \frac{\pi}{2}\sum_{i=1}^{n} x_{i:m:n}e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}\tan\left[\frac{\pi}{2}\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}\right)^{\lambda}\right] \\ &- \lambda\frac{\pi}{2}\sum_{i=1}^{m} R_i \frac{x_{i:m:n}e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}\right)^{\lambda-1}\cos\left[\frac{\pi}{2}\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}\right)^{\lambda}\right]}{1 - \sin\left[\frac{\pi}{2}\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}\right)^{\lambda}\right]}, \end{aligned} \tag{27}$$

$$\begin{aligned} \frac{\partial L_n}{\partial \theta} =& \sum_{i=1}^{n}\frac{x_{i:m:n}}{(\alpha + \theta x_{i:m:n})} - \frac{1}{2}\sum_{i=1}^{n} x_{i:m:n}^2 + (\lambda-1)\sum_{i=1}^{n}\frac{\frac{x_{i:m:n}^2}{2}e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}}{1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}} \\ &- \frac{\pi}{4}\sum_{i=1}^{n} x_{i:m:n}^2 e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}\tan\left[\frac{\pi}{2}\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}\right)^{\lambda}\right] \\ &- \lambda\frac{\pi}{2}\sum_{i=1}^{m} R_i \frac{\frac{x_{i:m:n}^2}{2}e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}\right)^{\lambda-1}\cos\left[\frac{\pi}{2}\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}\right)^{\lambda}\right]}{1 - \sin\left[\frac{\pi}{2}\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}\right)^{\lambda}\right]}, \end{aligned} \tag{28}$$

and

$$\frac{\partial L_n}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \log\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2} x_{i:m:n}^2}\right) - \frac{\pi}{2} \sum_{i=1}^{n} \left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2} x_{i:m:n}^2}\right)^{\lambda}$$
$$\times \log\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2} x_{i:m:n}^2}\right) \tan\left[\frac{\pi}{2}\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2} x_{i:m:n}^2}\right)^{\lambda}\right]$$
$$+ \frac{\pi}{2} \sum_{i=1}^{m} R_i \frac{\log\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2} x_{i:m:n}^2}\right)\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2} x_{i:m:n}^2}\right)^{\lambda} \cos\left[\frac{\pi}{2}\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2} x_{i:m:n}^2}\right)^{\lambda}\right]}{1 - \sin\left[\frac{\pi}{2}\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2} x_{i:m:n}^2}\right)^{\lambda}\right]}. \tag{29}$$

The maximum likelihood estimation (MLE) of parameters is obtained by setting $\frac{\partial L_n}{\partial \alpha} = \frac{\partial L_n}{\partial \theta} = \frac{\partial L_n}{\partial \lambda} = 0$ and solving these equations simultaneously to get the $MLE(\widehat{\kappa})$. These equations cannot be solved analytically, and statistical software can be used to solve them numerically via iterative methods. Since the closed-form solutions to Eqs. (27)–(29) do not exist based on progressive Type-II censored samples, the Newton-Raphson (NR) iteration method is used to obtain the estimations. In the reference [40], the algorithm is described with the (maxLik) package which implements the NR iteration of maximization.

It is standard that under some regularity conditions, $\hat{\alpha}$, $\hat{\theta}$ and $\hat{\lambda}$ are approximately distributed as multivariate normal with mean α, θ and λ covariance matrix $I^{-1}(\alpha, \theta, \lambda)$. Then, the $100(1-\gamma)\%$ two sided confidence interval of α, θ and λ, can be given by

$$\widehat{\alpha} \pm Z_{\frac{\gamma}{2}} \sqrt{Var\left(\hat{\alpha}\right)},\ \& \ \widehat{\theta} \pm Z_{\frac{\gamma}{2}} \sqrt{Var\left(\hat{\theta}\right)},\ \& \ \widehat{\lambda} \pm Z_{\frac{\gamma}{2}} \sqrt{Var\left(\hat{\lambda}\right)}, \tag{30}$$

where $Z_{\frac{\gamma}{2}}$ is that the percentile of the standard normal distribution with right-tail probability $\frac{\gamma}{2}$.

6.2 Bayesian Estimation Method

In this subsection, we establish Bayesian estimates that treat the parameter uncertainty as being represented by a joint prior distribution that was created prior to the failure data being gathered. Because it allows for the inclusion of prior knowledge in the study, the Bayesian technique is very helpful in reliability analysis. Based on the square error loss function (SELF), Bayesian estimates of the unknown parameters α, θ, and λ are derived. The parameters α, θ, and λ are assumed to be independent and to follow the following gamma prior distributions:

$$\begin{cases} \pi_1(\alpha) \propto \alpha^{q_1 - 1} e^{-w_1 \alpha}, & \alpha > 0, \\ \pi_2(\theta) \propto \theta^{q_2 - 1} e^{-w_2 \theta}, & \theta > 0, \\ \pi_3(\lambda) \propto \lambda^{q_3 - 1} e^{-w_3 \lambda}, & \lambda > 0, \end{cases} \tag{31}$$

where it is assumed that all of the hyper-parameters q_i and w_i have non-negative values and are known.

The updated distribution of the parameters α, θ, and λ, represented as $\pi^*(\alpha, \theta, \lambda|\underline{\mathrm{x}})$, can be computed by integrating the likelihood function from Eq. (25) with the prior distributions from Eq. (31).

$$\pi^*\left(\alpha, \theta, \lambda \mid \underline{\mathrm{x}}\right) = \frac{\pi_1(\alpha)\,\pi_2(\theta)\,\pi_3(\lambda)\,L(bx;\kappa)}{\int_0^\infty \int_0^\infty \int_0^\infty \pi_1(\alpha)\,\pi_2(\theta)\,\pi_3(\lambda)\,L(bx;\kappa)d\alpha d\theta d\lambda}. \tag{32}$$

The square error loss function (SELF), a symmetrical loss function that attributes equal losses to overestimation and underestimation, is frequently employed. If an estimator $\hat{\kappa}$ is to estimate the parameter κ, then the SELF is defined as

$$L\left(\kappa, \hat{\kappa}\right) = \left(\hat{\kappa} - \kappa\right)^2.$$

The Bayes estimate of any function of alpha, theta, and lambda, such as $g(\alpha, \theta, \lambda)$ under the SELF, can therefore be calculated as

$$\hat{g}_{SELF}\left(\alpha, \theta, \lambda \mid \underline{\mathrm{x}}\right) = E_{\alpha,\theta,\lambda|x}\left(g\left(\alpha, \theta, \lambda\right)\right). \tag{33}$$

When many integrals can be used to solve the expectation in Eq. (33), but it is not possible to acquire these multiple integrals mathematically. Therefore, samples from the joint posterior density function in Eq. (32) can be produced using the MCMC method. To employ the Markov Chain Monte Carlo (MCMC) method, we incorporate the Gibbs sampling step within the Metropolis-Hastings (M-H) sampler procedure. In statistics, two highly effective MCMC techniques frequently used are the Metropolis-Hastings and Gibbs sampling methods.

The following equation yields the joint posterior density function of α, θ, and λ:

$$\begin{aligned}\Pi(\kappa|x) \propto\ & \lambda^{n+q_3-1} e^{-\sum_{i=1}^{m}\left(\alpha x_{i:m:n} + \frac{\theta}{2}x^2\right)} \prod_{i=1}^{m}\left(\alpha + \theta x_{i:m:n}\right)\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}\right)^{\lambda-1} \\ & \times \alpha^{q_1-1} e^{-w_1\alpha}\theta^{q_2-1} e^{-w_2\theta} e^{-w_3\lambda} \prod_{i=1}^{m}\cos\left[\frac{\pi}{2}\left(1 - e^{-\alpha x_{i:m:n} - \frac{\theta}{2}x_{i:m:n}^2}\right)^{\lambda}\right].\end{aligned} \tag{34}$$

It is clear that the joint posterior of θ in Eq. (34) does not exhibit typical forms, making the use of the Metropolis-Hasting sampler necessary for the implementation of the MCMC approach. The "coda" package in R 4.3.0 software can be used to implement the Metropolis-Hastings algorithm within Gibbs sampling.

7 Applications

The SGLE model seeks to be employed in practical settings, such as the fit of real-world data, thanks to its desirable flexible qualities. We discuss this finding after taking into account the two well-cited real-world data sets below. Nine more effective models that have two or three tuning parameters and are expanded or modified versions of the exponential model are also taken into account for comparison. Namely, we consider generalized failure rate distribution (GFR), exponentiated Weibull-H exponential (SEWHE) [41], distribution, sine exponential (SEx) [42] distribution, alpha-sine Weibull (ASW) [43], sine-inverse Weibull (SIW) [44], sine-Burr XII (SBXII) [45], exponentiated Weibull (EW) [46], alpha power inverse Weibull (APIW) [47], alpha power Weibull (APW) [48], generalized inverse Weibull (GIW) [49], extended odd Weibull Rayleigh (EOWR) [50] distributions.

Akaike's (A), Bayesian (B), Consistent Akaike's (CA), and Hannan-Quinn (HQ) model selection information criteria are all used to demonstrate the utility of the SGLE distribution in contrast to competing models. To evaluate the validity of the SGRF model in contrast to other competing models, three additional goodness-of-fit statistics are also used: "Anderson-Darling (ADG), Cramer-von Mises (CVMG), and Kolmogorov-Smirnov (KSD) (with its *p*-value (PVKS))". We used the R software along with the "AdequacyModel" package to estimate all unknown parameters through the maximum likelihood method. The standard errors (StEr) for these parameters were also computed and are reported in Tables 2 and 3. Based on these computations, the optimal distribution corresponds to the lowest values of A, B, CA, HQ, ADG, CVMG, and KSD statistics, along with the highest *p*-value. However, the estimated values of these goodness-of-fit measures for the various datasets are presented in Tables 2 and 3.

Table 2: Different measures for each model by MLE: data I

Models		Estimates	StEr	A	B	CA	HQ	KSD	PVKS	CVMGG	ADG
SGLE	α	0.1735	0.3458	104.9843	111.6866	105.3535	107.6433	0.0534	0.9894	0.0352	0.2797
	θ	0.7560	0.1811								
	λ	2.4422	1.1195								
GFR	α	0.1655	0.3673	107.4824	107.8516	110.1414	114.1847	0.0723	0.8630	0.0652	0.4779
	θ	1.2506	0.2451								
	λ	2.6331	1.1074								
SEWHE	α	14.2450	37.1414	105.2991	114.2356	105.9241	108.8445	0.0403	0.9999	0.0172	0.1557
	β	0.2028	0.4361								
	θ	1.1062	1.7527								
	λ	2.9008	4.4375								
SEx	α	0.3958	0.0439	183.9069	186.1410	183.9666	184.7933	0.3544	0.0000	0.1119	0.7921
SIW	α	1.6488	0.1493	142.3868	142.5687	146.8550	144.1595	0.1589	0.0613	0.5383	3.4100
	β	1.5045	0.1149								
SBXII	α	4.9359	0.6941	115.7887	110.2569	115.9705	117.5614	0.1750	0.0293	0.3426	2.1790
	β	0.2804	0.0455								
APIW	α	303.6005	402.5094	140.2925	146.9948	140.6617	142.9515	0.0534	0.9894	0.4948	3.1242
	β	2.4982	0.1935								
	θ	0.2775	0.0692								
GIW	α	303.6005	402.5094	140.2925	146.9948	140.6617	142.9515	0.0534	0.9894	0.4948	3.1242
	β	2.4982	0.1935								
	θ	0.2775	0.0692								
EOWR	α	1.4106	0.2233	144.8535	151.5558	145.2227	147.5125	0.0539	0.9825	0.0392	0.2814
	β	0.3753	0.3172								
	θ	0.2904	0.0379								

Table 3: Different measures for each model by MLE: data II

Models		Estimates	StEr	A	B	CA	HQ	KSD	PVKS	CVMGG	ADG
SGLE	α	0.4010	0.2300	113.0032	117.8360	113.7305	114.7070	0.0794	0.9737	0.0292	0.1810
	θ	0.0358	0.0897								
	λ	1.4151	0.4800								
GFR	α	0.5980	0.3513	113.9516	117.9679	114.6554	117.7844	0.0816	0.9661	0.0298	0.1833
	θ	0.1158	0.1587								
	λ	1.4727	0.5657								
SEWHE	α	18.9511	686.6481	114.7737	121.2174	116.0237	117.0454	0.0826	0.9622	0.0329	0.1988
	β	0.2121	3.1465								

(Continued)

Table 3 (continued)

Models		Estimates	StEr	A	B	CA	HQ	KSD	PVKS	CVMGG	ADG
	θ	2.6809	27.8125								
	λ	0.6078	5.5706								
SEx	α	0.3318	0.0510	113.4639	118.0749	113.9578	114.0319	0.1457	0.4118	0.0308	0.1893
SIW	α	1.2331	0.1583	123.6408	123.9937	126.8626	124.7766	0.1652	0.2646	0.2041	1.2683
	β	0.7742	0.0853								
SBXII	α	2.0316	0.3323	115.0093	118.2311	116.3622	119.1451	0.1645	0.2692	0.0995	0.6194
	β	0.4466	0.0836								
EW	α	0.2971	0.0380	115.5108	120.3435	116.2380	117.2145	0.0983	0.8673	0.0584	0.3735
	β	0.0096	0.0096								
	θ	0.7951	0.5305								
APIW	α	64.7848	108.6950	125.6909	130.5237	126.4182	127.3947	0.0794	0.9737	0.2035	1.2612
	β	1.2576	0.1435								
	θ	0.2272	0.0961								
APW	α	375.4554	68.1517	114.3618	119.1946	115.0891	116.0656	0.0794	0.9737	0.0370	0.2276
	β	0.6930	0.3480								
	θ	1.7175	1.0715								
GIW	α	0.6432	0.4198	132.3845	137.2173	133.1118	134.0883	0.1668	0.2544	0.2969	1.7983
	β	1.1904	0.5173								
	θ	0.9736	0.1097								
EOWR	1	0.8012	0.2273	129.4340	134.2667	130.1612	131.1377	0.0938	0.9006	0.0416	0.2488
	2	2.1965	2.3254								
	3	0.4552	0.3683								

The first set of data: The first data set has been obtained by reference [51] as its source. It includes the single carbon fibre tensile strength (in GPa). Data I: "0.312, 0.314, 0.479, 0.552, 0.700, 0.803, 0.861, 0.865, 0.944, 0.958, 0.966, 0.997, 1.006, 1.021, 1.027, 1.055, 1.063, 1.098, 1.140, 1.179, 1.224, 1.240, 1.253, 1.270, 1.272, 1.274, 1.301, 1.301, 1.359, 1.382, 1.382, 1.426, 1.434, 1.435, 1.478, 1.490, 1.511, 1.514, 1.535, 1.554, 1.566, 1.570, 1.586, 1.629, 1.633, 1.642, 1.648, 1.684, 1.697, 1.726, 1.770, 1.773, 1.800, 1.809, 1.818, 1.821, 1.848, 1.880, 1.954, 2.012, 2.067, 2.084, 2.090, 2.096, 2.128, 2.233, 2.433, 2.585, 2.585". The result of this application has baen presented by Table 2 and Figs. 4–8.

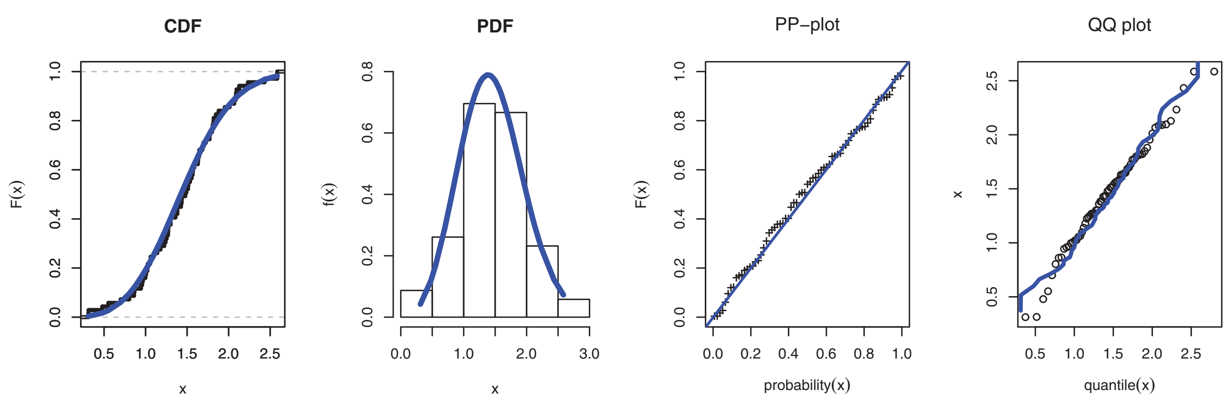

Figure 4: Fitted application for the SGLE distribution of data I with different graph

To check the estimators by MLE for the SGLE parameters of data set I, Figs. 5, and 6 have been plotted to check these estimators have maximum and uniqueness values of MLE. Also, to check the estimators by Bayesian, Figs. 7, and 8 have been plotted to check these estimators have convergence and normality shapes of Bayesian estimates for the SGLE parameters.

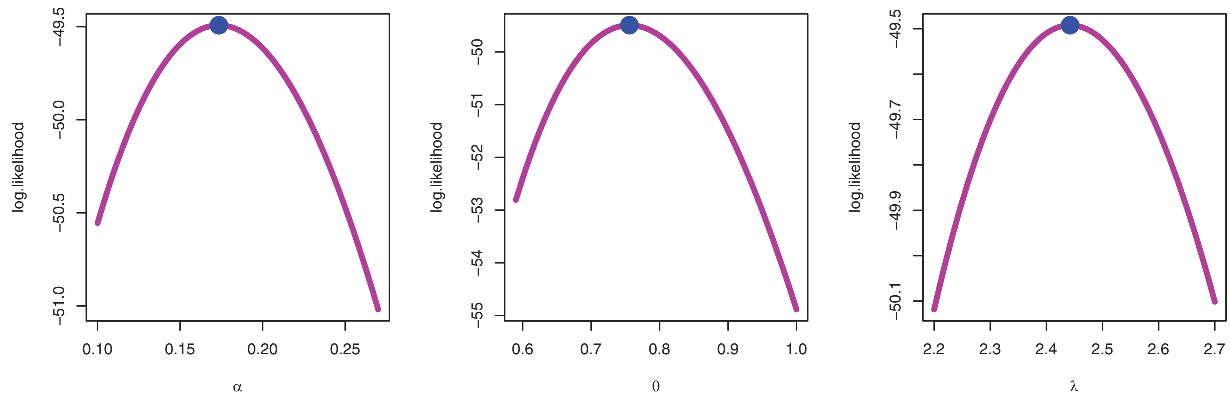

Figure 5: Profile MLE for for the SGLE parameters of data I

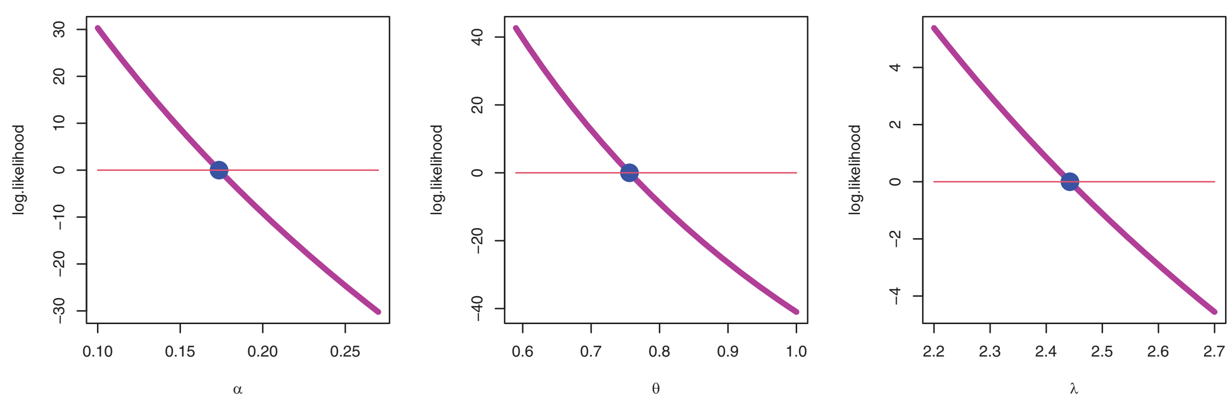

Figure 6: Uniqueness property MLE for the SGLE parameters of data I

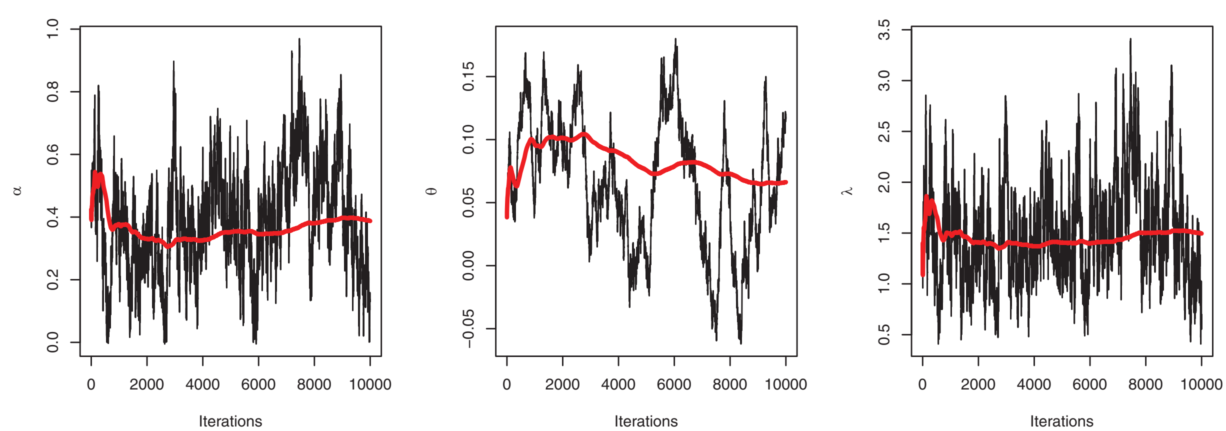

Figure 7: Trace plot of Bayesian estimators for the SGLE parameters: data I

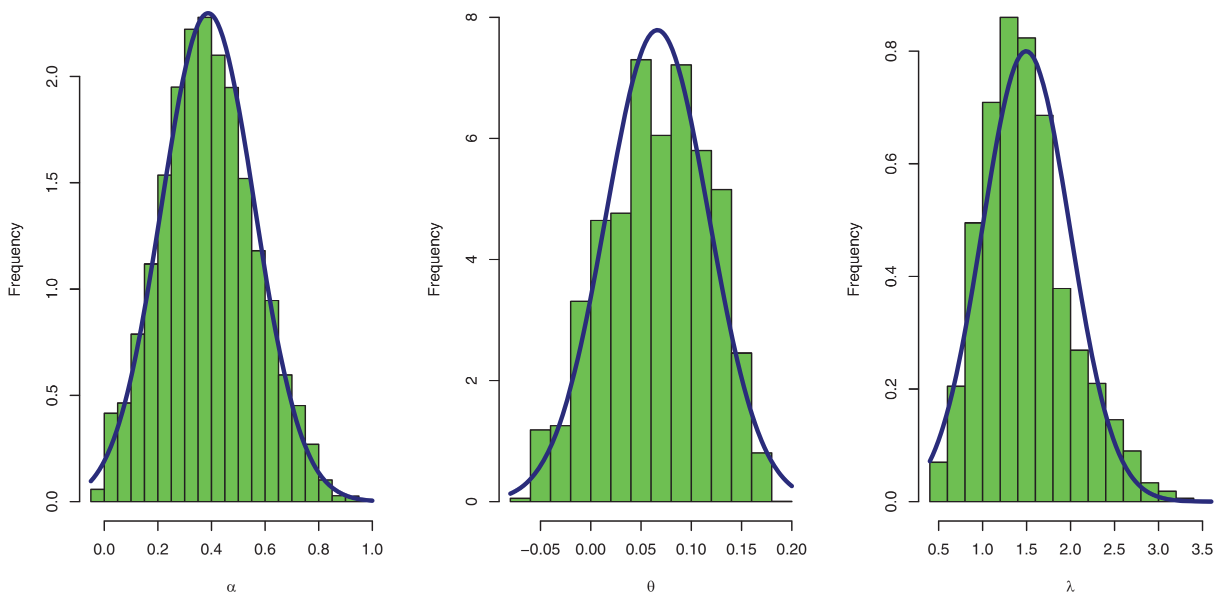

Figure 8: Histogram plot with normal curve of Bayesian estimators for the SGLE parameters: data I

The second data set, which can be accessible on June 30 2022, (see https://dataverse.harvard.edu/) shows the TFP growth in agricultural production for 37 African nations between 2001 and 2010. Data II: "4.6, 0.9, 1.8, 1.4, 0.2, 3.9, 1.8, 0.8, 2.0, 0.8, 1.6, 0.8, 2.0, 1.6, 0.5, 0.1, 2.5, 2.4, 0.6, 1.1, 0.7, 1.7, 1.0, 1.7, 2.5, 3.5, 0.3, 0.9, 2.3, 0.5, 1.5, 5.1, 0.2, 1.5, 3.3, 1.4, 3.3". The result of this application has been presented by Table 3 and Figs. 9–13.

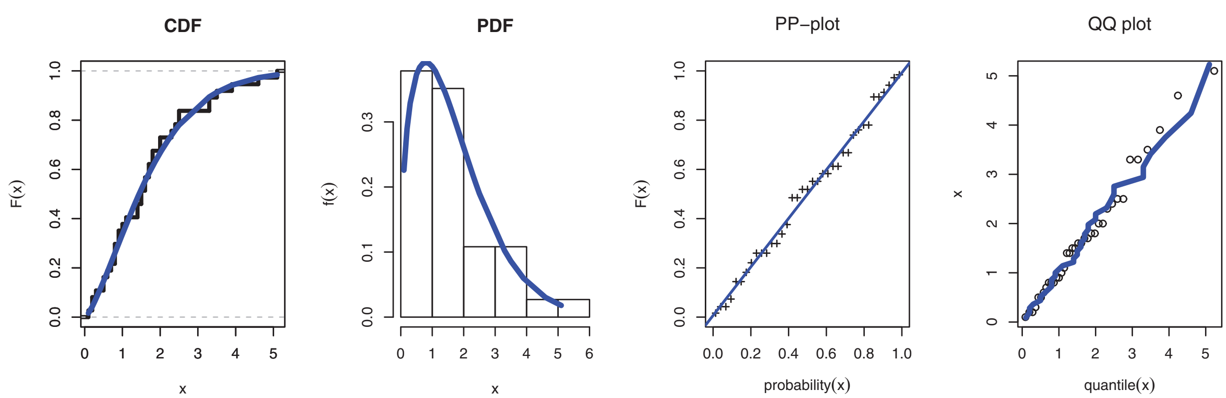

Figure 9: Fitted application for the SGLE distribution of data II with different graph

To check the estimators by MLE for the SGLE parameters of data set II, Figs. 10, and 11 have been plotted to check these estimators have maximum and uniqueness values of MLE. Also, to check the estimators by Bayesian for data set II, Figs. 12, and 13 have been plotted to check these estimators have convergence and normality shapes of Bayesian estimates for the SGLE parameters.

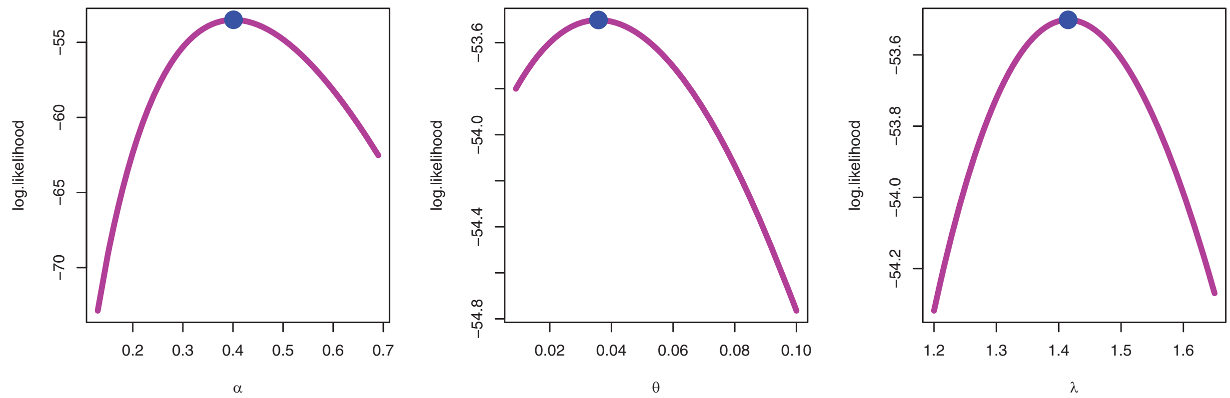

Figure 10: Profile MLE for the SGLE parameters of data II

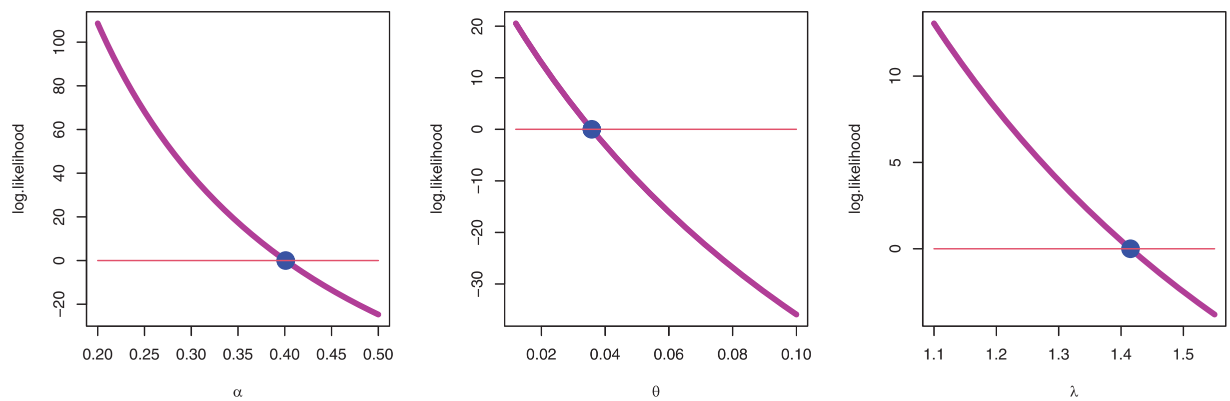

Figure 11: Uniqueness property MLE for the SGLE parameters of data II

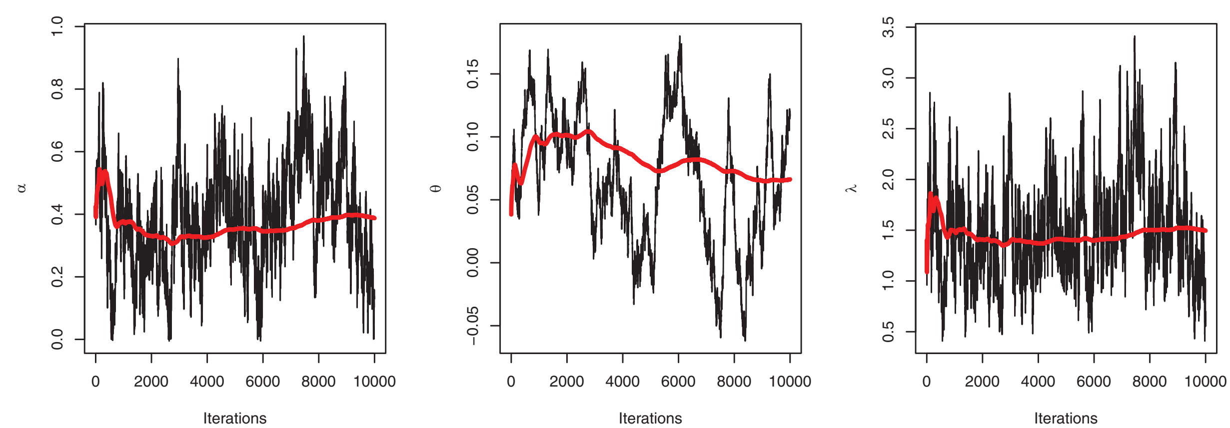

Figure 12: Trace plot of Bayesian estimators for the SGLE parameters: data II

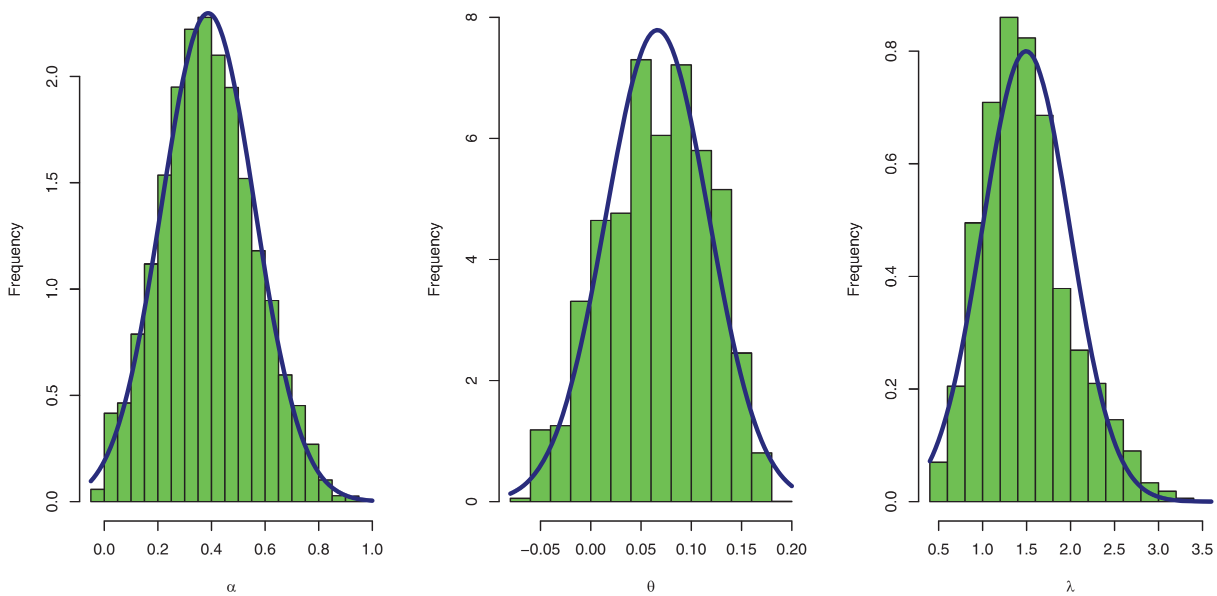

Figure 13: Histogram plot with normal curve of Bayesian estimators for the SGLE parameters: data II

It is obvious that, when compared to the other distributions under the different data sets, the SGLE distribution is the best distribution. On the basis of the same data, we also create a fitted/empirical CDF, histogram, fitted density, PP plot, and quantile-quantile plots of the SGLE distribution (see Figs. 4 and 9). The results in Tables 2 and 3 show that the SGLE distribution is the most effective model to fit the various data when compared to all other given distributions indicated in Tables 2 and 3. Graphical representations in Figs. 4 and 9 support these findings.

8 Simulation

This section compares MLE and Bayesian estimates of the SGLE distribution parameter using Monte Carlo simulations using progressive Type-II censored samples. The simulation results are run in order to investigate and output in terms of mean square error ($\omega 2$), relative absolute bias ($\omega 1$), length confidence interval ($\omega 3$), and coverage probability of a confidence interval ($\omega 4$). We generate ten thousand random samples from the SGLE distribution for numerous individual parameters. For various sample sizes, $n = 40, 70, 100, 150$, and 200, failure censored sample m and different scheme, including:

Scheme (R) I: $R_m = n - m$, $R_i = 0$; $i = 1, \ldots, m - 1$.

Scheme (R) II: $R_1 = n - m$, $R_i = 0$; $i = 2, \ldots, m$.

Complete: $m = n$, where $R_i = 0; i = 1, \ldots, m$.

For the random variables generating, the values of the parameters α, θ, and λ are chosen as follows:

Case 1: $\alpha = 0.5, \theta = 0.6, \lambda = 0.5$.

Case 2: $\alpha = 0.5, \theta = 0.6, \lambda = 2.2$.

Case 3: $\alpha = 0.5, \theta = 1.8, \lambda = 0.5$.

Case 4: $\alpha = 1.1, \theta = 0.8, \lambda = 0.9$.

All necessary calculations were conducted utilizing R 4.3.0 software, employing three beneficial packages: the ‘coda’ package (MCMC by M-H algorithm) to make some Bayesian inference, the (maxLik) package (Newton-Raphson algorithm) to obtain likelihood inference, and the spread, skewness, and kurtosis of data sets encountercensored’ package to generate censored samples. Selecting initial parameter values involves options like leveraging domain knowledge, employing guess-and-check techniques, initiating random values within a defined range, executing grid searches in discrete parameter spaces, utilizing optimization algorithms for value generation, conducting sensitivity analyses for robustness, and referencing values from prior studies. The chosen method depends on the problem context, optimization algorithm, and parameter specifics, often prompting the exploration of multiple approaches. In our simulation study, we employed the guess-and-check method alongside optimization algorithms, specifically utilizing the “nlminb” function for generating initial values.

The following is a summary of Tables 4–7 included in the observations that follow:

- As the sample size grows, the $\omega 2$, $\omega 1$, and LCI drop.
- As the number of steps (m) rises, the $\omega 1$, $\omega 2$, and LCI drop.
- For the majority of analysed cases of the SGLE distribution under progressively Type-II censored data, the Bayesian estimates are more effective than alternative approaches.

Table 4: $\omega 1$, $\omega 2$, $\omega 3$, and $\omega 4$ for parameters SGLE distribution by MLE and Bayesian: case 1

$\alpha = 0.5, \theta = 0.6, \lambda = 0.5$				MLE				Bayesain			
R	n	m		$\omega 1$	$\omega 2$	$\omega 3$	$\omega 4$	$\omega 1$	$\omega 2$	$\omega 3$	$\omega 4$
I	40	28	α	0.1953	0.3723	2.3623	94.9%	0.1066	0.0414	0.7405	97.30%
			θ	0.0090	2.9581	6.7455	93.9%	0.1687	0.0698	0.8739	98.00%
			λ	0.0203	0.0291	0.6683	96.1%	0.0204	0.0076	0.3331	97.50%
		36	α	0.0522	0.1845	1.6817	95.5%	0.0504	0.0141	0.4396	99.40%
			θ	0.1270	0.6235	3.0824	94.2%	0.1110	0.0264	0.5032	99.10%
			λ	0.0123	0.0214	0.5731	96.2%	0.0075	0.0041	0.2418	99.00%
	100	70	α	0.1609	0.1556	1.5147	94.9%	0.0361	0.0287	0.6061	98.00%
			θ	0.1187	0.6089	2.9682	93.9%	0.1643	0.0588	0.7960	98.10%
			λ	0.0868	0.0166	0.4751	97.1%	0.0358	0.0055	0.2680	96.10%
		90	α	0.1341	0.1244	1.3583	95.2%	0.0202	0.0140	0.4479	99.50%
			θ	0.1096	0.2593	1.9805	94.5%	0.0892	0.0216	0.5122	99.10%
			λ	0.0691	0.0155	0.4691	97.6%	0.0225	0.0038	0.2306	96.60%
II	40	28	α	0.0517	0.1786	1.6545	95.8%	0.0742	0.0314	0.6062	98.20%
			θ	0.2084	0.4287	2.5205	95.8%	0.1535	0.0551	0.7682	98.20%
			λ	0.0090	0.0247	0.6159	96.6%	0.0071	0.0084	0.3327	98.00%
		36	α	0.0722	0.1637	1.5803	95.1%	0.0745	0.0151	0.4615	99.80%
			θ	0.1883	0.4016	2.4457	95.5%	0.0960	0.0239	0.5132	99.40%
			λ	0.0085	0.0216	0.5763	96.7%	0.0132	0.0045	0.2581	98.60%
	100	70	α	0.1973	0.1193	1.2979	95.4%	0.0327	0.0274	0.6124	97.50%
			θ	0.2017	0.1760	1.5756	95.3%	0.1254	0.0473	0.7326	98.00%
			λ	0.0910	0.0174	0.4849	97.6%	0.0326	0.0063	0.2865	94.60%
		90	α	0.1986	0.1040	1.2032	96.6%	0.0276	0.0148	0.4567	99.00%
			θ	0.1878	0.1454	1.4287	95.7%	0.0859	0.0212	0.5066	99.60%
			λ	0.0867	0.0149	0.4474	97.5%	0.0271	0.0039	0.2340	96.60%
Complete n = 40			α	0.0031	0.1341	1.4362	95.7%	0.0789	0.0318	0.6328	94.73%
			θ	0.1777	0.2885	2.0646	95.8%	0.1316	0.0514	0.7616	94.74%
			λ	0.0075	0.0189	0.5393	96.3%	0.0203	0.0068	0.3193	94.73%

(Continued)

Table 4 (continued)

$\alpha = 0.5, \theta = 0.6, \lambda = 0.5$				MLE				Bayesain			
R	n	m		$\omega1$	$\omega2$	$\omega3$	$\omega4$	$\omega1$	$\omega2$	$\omega3$	$\omega4$
Complete n = 70			α	0.1107	0.1113	1.2906	95.8%	0.0196	0.0145	0.4469	94.71%
			θ	0.1857	0.1837	1.6232	95.5%	0.0895	0.0202	0.5104	94.71%
			λ	0.0546	0.0159	0.4823	97.4%	0.0095	0.0040	0.2406	94.70%
Complete n = 100			α	0.1782	0.1039	1.2146	96.2%	0.0138	0.0093	0.3765	94.66%
			θ	0.1638	0.1331	1.3781	95.5%	0.0629	0.0120	0.3876	94.67%
			λ	0.0826	0.0142	0.4391	97.2%	0.0225	0.0027	0.1899	94.68%
Complete n = 150			α	0.2398	0.0812	1.0137	97.0%	0.0343	0.0067	0.3039	94.60%
			θ	0.1868	0.1050	1.1925	94.8%	0.0488	0.0085	0.3312	94.63%
			λ	0.0991	0.0115	0.3738	97.1%	0.0293	0.0021	0.1688	94.64%
Complete n = 200			α	0.2810	0.0789	0.9538	96.2%	0.0535	0.0059	0.2697	94.56%
			θ	0.2151	0.0948	1.0966	95.0%	0.0459	0.0066	0.2937	94.55%
			λ	0.1117	0.0109	0.3455	97.3%	0.0348	0.0018	0.1476	94.60%

Table 5: $\omega1$, $\omega2$, $\omega3$, and $\omega4$ for parameters SGLE distribution by MLE and Bayesian: case 2

$\alpha = 0.5, \theta = 0.6, \lambda = 2.2$				MLE				Bayesain			
R	n	m		$\omega1$	$\omega2$	$\omega3$	$\omega4$	$\omega1$	$\omega2$	$\omega3$	$\omega4$
I	40	28	α	0.6481	0.8067	3.2853	94.6%	0.0091	0.0109	0.3993	94.66%
			θ	0.4101	0.5819	2.8319	94.7%	0.0997	0.0342	0.6541	94.65%
			λ	0.5118	10.0287	11.6086	96.0%	0.0081	0.0023	0.1742	94.68%
		36	α	0.4029	0.4768	2.5904	95.5%	0.0043	0.0072	0.3329	94.58%
			θ	0.2179	0.2615	1.9390	95.5%	0.0616	0.0161	0.4427	94.59%
			λ	0.3220	4.8543	8.1822	96.1%	0.0041	0.0008	0.1027	94.64%
	100	70	α	0.4091	0.3254	2.0883	95.3%	0.0018	0.0082	0.3474	94.60%
			θ	0.2654	0.2232	1.7443	95.3%	0.0395	0.0227	0.5686	94.58%
			λ	0.2467	1.9750	5.0840	95.6%	0.0066	0.0022	0.1620	94.61%
		90	α	0.3388	0.2392	1.7993	95.0%	0.0088	0.0050	0.2687	94.53%
			θ	0.1913	0.1216	1.2913	95.1%	0.0358	0.0113	0.4026	94.51%
			λ	0.2200	1.6129	4.6051	95.5%	0.0035	0.0006	0.1002	94.57%
II	40	28	α	0.3312	0.4025	2.4020	96.1%	0.0053	0.0137	0.4403	94.66%
			θ	0.1373	0.1883	1.6709	94.8%	0.0895	0.0325	0.6257	94.67%
			λ	0.2966	4.5424	7.9574	96.8%	0.0079	0.0026	0.1843	94.67%
		36	α	0.3040	0.4018	2.3777	95.5%	0.0114	0.0076	0.3313	94.58%
			θ	0.1292	0.1809	1.6463	95.0%	0.0464	0.0152	0.4528	94.60%
			λ	0.3309	4.0023	7.3082	95.4%	0.0039	0.0008	0.1015	94.64%
	100	70	α	0.2236	0.1974	1.6862	96.2%	0.0027	0.0089	0.3601	94.61%
			θ	0.1106	0.0896	1.1451	95.5%	0.0406	0.0175	0.5013	94.62%
			λ	0.1592	1.1968	4.0648	94.9%	0.0073	0.0025	0.1720	94.64%
		90	α	0.2164	0.1841	1.6286	96.9%	0.0005	0.0055	0.2819	94.52%
			θ	0.1169	0.0807	1.0796	96.4%	0.0277	0.0114	0.4044	94.52%
			λ	0.1481	1.1544	4.0155	95.8%	0.0031	0.0007	0.1001	94.59%
Complete n = 40			α	0.3081	0.3609	2.2775	95.8%	0.0051	0.0108	0.3937	94.65%
			θ	0.1599	0.1630	1.5383	95.3%	0.0589	0.0256	0.6042	94.66%
			λ	0.2724	3.5646	7.0218	95.1%	0.0066	0.0026	0.1912	94.68%
Complete n = 70			α	0.2726	0.2376	1.8356	96.0%	0.0016	0.0062	0.2969	94.54%
			θ	0.1377	0.1020	1.2097	94.9%	0.0376	0.0125	0.4158	94.53%

(Continued)

Table 5 (continued)

$\alpha = 0.5, \theta = 0.6, \lambda = 2.2$				MLE				Bayesain			
R	n	m		$\omega 1$	$\omega 2$	$\omega 3$	$\omega 4$	$\omega 1$	$\omega 2$	$\omega 3$	$\omega 4$
			λ	0.2077	1.9030	5.1049	95.1%	0.0036	0.0009	0.1119	94.61%
Complete n = 100			α	0.1639	0.1695	1.5824	95.8%	0.0038	0.0037	0.2390	94.42%
			θ	0.0936	0.0731	1.0371	96.2%	0.0205	0.0072	0.3159	94.44%
			λ	0.1255	1.0890	3.9469	95.1%	0.0018	0.0005	0.0880	94.55%
Complete n = 150			α	0.1698	0.1474	1.4682	96.7%	0.0004	0.0029	0.2063	94.31%
			θ	0.0964	0.0616	0.9467	95.9%	0.0187	0.0057	0.2828	94.30%
			λ	0.1130	0.8056	3.3823	95.7%	0.0018	0.0004	0.0740	94.49%
Complete n = 200			α	0.0639	0.1022	1.2478	96.4%	0.0015	0.0022	0.1876	94.17%
			θ	0.0358	0.0414	0.7936	95.7%	0.0146	0.0042	0.2407	94.18%
			λ	0.0571	0.5372	2.8320	95.4%	0.0014	0.0003	0.0637	94.38%

Table 6: $\omega 1$, $\omega 2$, $\omega 3$, and $\omega 4$ for parameters SGLE distribution by MLE and Bayesian: case 3

$\alpha = 0.5, \theta = 1.8, \lambda = 0.5$				MLE				Bayesain			
R	n	m		$\omega 1$	$\omega 2$	$\omega 3$	$\omega 4$	$\omega 1$	$\omega 2$	$\omega 3$	$\omega 4$
I	40	28	α	0.4150	0.5204	2.7097	95.0%	0.1237	0.0443	0.7168	94.71%
			θ	0.1546	6.4644	9.9117	93.6%	0.0288	0.0147	0.4229	94.73%
			λ	0.0613	0.0335	0.7082	96.3%	0.0195	0.0068	0.3056	94.73%
		36	α	0.3122	0.4113	2.4396	94.8%	0.0781	0.0183	0.4715	94.70%
			θ	0.0937	2.1927	5.7697	93.6%	0.0139	0.0058	0.2686	94.71%
			λ	0.0409	0.0293	0.6665	96.0%	0.0067	0.0043	0.2562	94.71%
	100	70	α	0.0121	0.2481	1.9533	94.7%	0.0029	0.0352	0.6743	94.62%
			θ	0.0863	2.0403	5.5689	93.0%	0.0270	0.0139	0.4926	94.66%
			λ	0.0523	0.0191	0.5327	97.0%	0.0254	0.0054	0.2760	94.66%
		90	α	0.0178	0.2098	1.7961	95.3%	0.0082	0.0172	0.4879	94.63%
			θ	0.0503	0.7836	3.4535	93.4%	0.0061	0.0058	0.2316	94.65%
			λ	0.0512	0.0178	0.5139	96.8%	0.0220	0.0034	0.2195	94.65%
II	40	28	α	0.3140	0.3506	2.2391	94.4%	0.1334	0.0425	0.6984	94.72%
			θ	0.0082	1.4898	4.7866	94.9%	0.0283	0.0189	0.4904	94.75%
			λ	0.0518	0.0296	0.6670	96.2%	0.0255	0.0080	0.3366	94.74%
		36	α	0.2281	0.2926	2.0740	95.0%	0.0743	0.0160	0.4387	94.70%
			θ	0.0072	1.3308	4.5240	95.1%	0.0158	0.0060	0.2768	94.71%
			λ	0.0321	0.0261	0.6306	96.5%	0.0041	0.0042	0.2415	94.72%
	100	70	α	0.0644	0.1813	1.6650	94.7%	0.0062	0.0299	0.6166	94.65%
			θ	0.0113	0.6247	3.0988	94.9%	0.0200	0.0242	0.5894	94.68%
			λ	0.0615	0.0180	0.5120	97.2%	0.0294	0.0056	0.2692	94.69%
		90	α	0.0464	0.1710	1.6191	96.1%	0.0149	0.0166	0.4895	94.63%
			θ	0.0037	0.4868	2.7363	94.6%	0.0126	0.0079	0.3267	94.64%
			λ	0.0524	0.0164	0.4922	96.6%	0.0208	0.0035	0.2170	94.65%
Complete n = 40			α	0.2200	0.3007	2.1068	94.8%	0.0947	0.0367	0.6871	94.71%
			θ	0.0012	1.1855	4.2703	95.1%	0.0286	0.0208	0.5222	94.72%
			λ	0.0285	0.0251	0.6187	96.1%	0.0092	0.0064	0.3055	94.72%
Complete n = 70			α	0.0254	0.1881	1.7003	95.4%	0.0216	0.0157	0.4708	94.66%
			θ	0.0135	0.6533	3.1685	94.7%	0.0152	0.0079	0.3268	94.67%
			λ	0.0230	0.0181	0.5263	96.8%	0.0097	0.0038	0.2324	94.66%
Complete n = 100			α	0.0787	0.1638	1.5797	96.4%	0.0068	0.0097	0.3725	94.62%

(Continued)

Table 6 (continued)

$\alpha = 0.5, \theta = 1.8, \lambda = 0.5$				MLE				Bayesain			
R	n	m		$\omega 1$	$\omega 2$	$\omega 3$	$\omega 4$	$\omega 1$	$\omega 2$	$\omega 3$	$\omega 4$
			θ	0.0049	0.4591	2.6573	93.5%	0.0090	0.0046	0.2498	94.63%
			λ	0.0581	0.0159	0.4812	96.9%	0.0213	0.0027	0.1977	94.63%
Complete n = 150			α	0.2375	0.1339	1.3573	95.1%	0.0359	0.0073	0.3258	94.53%
			θ	0.0337	0.3053	2.1541	94.7%	0.0067	0.0035	0.2220	94.57%
			λ	0.0987	0.0137	0.4168	96.5%	0.0294	0.0020	0.1603	94.58%
Complete n = 200			α	0.2822	0.1264	1.2795	95.3%	0.0363	0.0066	0.2939	94.46%
			θ	0.0603	0.2663	1.9787	94.7%	0.0080	0.0027	0.1957	94.53%
			λ	0.1093	0.0127	0.3864	96.2%	0.0289	0.0017	0.1429	94.53%

Table 7: $\omega 1$, $\omega 2$, $\omega 3$, and $\omega 4$ for parameters SGLE distribution by MLE and Bayesian: case 4

$\alpha = 1.1, \theta = 0.8, \lambda = 0.9$				MLE				Bayesain			
R	n	m		$\omega 1$	$\omega 2$	$\omega 3$	$\omega 4$	$\omega 1$	$\omega 2$	$\omega 3$	$\omega 4$
I	40	28	α	0.1093	0.9154	3.7227	94.4%	0.0402	0.0196	0.5213	94.73%
			θ	0.0392	6.9566	10.3436	94.5%	0.1380	0.0634	0.7851	94.74%
			λ	0.0625	0.1188	1.3336	95.4%	0.0258	0.0070	0.3061	94.73%
		36	α	0.0460	0.5524	2.9082	95.7%	0.0211	0.0088	0.3583	94.69%
			θ	0.1808	2.0335	5.5639	95.3%	0.0833	0.0205	0.4804	94.69%
			λ	0.0504	0.0990	1.2213	95.4%	0.0155	0.0035	0.2241	94.70%
	100	70	α	0.0684	0.4942	2.7412	96.1%	0.0268	0.0190	0.5145	94.68%
			θ	0.1615	3.1563	6.9493	94.6%	0.1139	0.0527	0.7770	94.67%
			λ	0.0283	0.0568	0.9297	94.2%	0.0217	0.0060	0.2986	94.67%
		90	α	0.0049	0.3074	2.1744	96.9%	0.0118	0.0084	0.3396	94.62%
			θ	0.0751	0.9841	3.8835	94.8%	0.0640	0.0176	0.4614	94.64%
			λ	0.0073	0.0431	0.8138	93.8%	0.0133	0.0032	0.2141	94.65%
II	40	28	α	0.0443	0.4814	2.7145	96.3%	0.0321	0.0187	0.5151	94.74%
			θ	0.4326	1.5757	4.7323	94.2%	0.1111	0.0474	0.7315	94.74%
			λ	0.0147	0.0960	1.2144	95.3%	0.0248	0.0081	0.3285	94.72%
		36	α	0.0045	0.4258	2.5593	96.3%	0.0260	0.0097	0.3651	94.69%
			θ	0.2842	1.1870	4.1789	95.6%	0.0713	0.0176	0.4601	94.69%
			λ	0.0238	0.0758	1.0766	93.8%	0.0124	0.0033	0.2140	94.70%
	100	70	α	0.0412	0.2799	2.0671	97.4%	0.0220	0.0175	0.5042	94.69%
			θ	0.2335	0.7215	3.2498	96.0%	0.1064	0.0473	0.7523	94.69%
			λ	0.0061	0.0429	0.8120	94.0%	0.0197	0.0058	0.2961	94.69%
		90	α	0.0196	0.2375	1.9096	93.7%	0.0136	0.0082	0.3437	94.65%
			θ	0.1393	0.5469	2.8673	95.5%	0.0603	0.0172	0.4643	94.64%
			λ	0.0043	0.0372	0.7559	93.9%	0.0137	0.0032	0.2197	94.66%
Complete n = 40			α	0.0080	0.3920	2.4554	96.9%	0.0327	0.0205	0.5392	94.72%
			θ	0.1901	1.0183	3.9125	95.3%	0.0935	0.0452	0.7647	94.72%
			λ	0.0263	0.0700	1.0334	96.1%	0.0257	0.0064	0.2952	94.71%
Complete n = 70			α	0.0239	0.2718	2.0421	95.0%	0.0184	0.0083	0.3449	94.65%
			θ	0.2144	0.7154	3.2483	95.8%	0.0679	0.0178	0.4691	94.66%
			λ	0.0050	0.0446	0.8285	93.7%	0.0129	0.0032	0.2121	94.67%
Complete n = 100			α	0.0397	0.2209	1.8355	92.6%	0.0074	0.0052	0.2833	94.61%
			θ	0.1282	0.4950	2.7299	95.9%	0.0376	0.0090	0.3527	94.60%
			λ	0.0138	0.0331	0.7118	92.8%	0.0049	0.0020	0.1692	94.62%
Complete n = 150			α	0.0524	0.1717	1.6096	92.6%	0.0064	0.0038	0.2277	94.54%

(Continued)

Table 7 (continued)

$\alpha = 1.1, \theta = 0.8, \lambda = 0.9$				MLE				Bayesain			
R	n	m		$\omega1$	$\omega2$	$\omega3$	$\omega4$	$\omega1$	$\omega2$	$\omega3$	$\omega4$
			θ	0.1629	0.3943	2.4092	96.0%	0.0352	0.0067	0.2918	94.59%
			λ	0.0164	0.0260	0.6302	92.6%	0.0054	0.0015	0.1515	94.49%
Complete n = 200			α	0.0476	0.1450	1.4793	93.6%	0.0069	0.0032	0.2141	94.46%
			θ	0.1410	0.3165	2.1615	95.3%	0.0268	0.0050	0.2610	94.53%
			λ	0.0193	0.0214	0.5696	93.5%	0.0027	0.0011	0.1274	94.53%

9 Conclusion

In this paper, we suggest a novel modification of the generalized linear exponential distribution termed the sine generalized linear exponential distribution, which makes use of the sine transformation's features. The new distribution is extremely versatile and may be used to simulate survival data and reliability difficulties successfully. Depending on its settings, the new proposed model may have a rising, J-shaped HRF. As special sub-models, it incorporates various well-known lifespan distributions. The suggested model's statistical features are described in detail. Under progressively censored data, the model parameters are addressed using maximum likelihood and Bayesian estimate approaches. We give simulated data to put these strategies to the test. Two real-world dataset applications highlight the importance of the newly presented model when compared to numerous regarded comparable models.

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References

1. Sarhan, A. M., Kundu, D. (2009). Generalized linear failure rate distribution. *Communications in Statistics-Theory and Methods, 38(5),* 642–660. https://doi.org/10.1080/03610920802272414
2. Mahmoud, M., Alam, F. M. A. (2010). The generalized linear exponential distribution. *Statistics & Probability Letters, 80(11),* 1005–1014.
3. Jafari, A. A., Mahmoudi, E. (2022). Beta-linear failure rate distribution and its applications. *Journal of the Iranian Statistical Society, 14(1),* 89–105.

4. Sarhan, A. M., Ahmad, A. E. B. A., Alasbahi, I. A. (2013). Exponentiated generalized linear exponential distribution. *Applied Mathematical Modelling, 37(5),* 2838–2849. https://doi.org/10.1016/j.apm.2012.06.019
5. El-Damcese, M. A., Mustafa, A., El-Desouky, B. S., Mustafa, M. E. (2016). The odd generalized exponential linear failure rate distribution. *Journal of Statistics Applications and Probability, 5,* 299–309.
6. Luguterah, A., Nasiru, S. (2017). The odd generalized exponential generalized linear exponential distribution. *Journal of Statistics Applications and Probability, 6,* 139–148. https://doi.org/10.18576/jsap/060112
7. Mahmoud, M. A. W., Ghazal, M. G. M., Radwan, H. M. M. (2017). Inverted generalized linear exponential distribution as a lifetime model. *Applied Mathematics & Information Sciences, 11,* 1747–1765.
8. Okasha, H. M., Kayid, M. (2016). A new family of marshall-olkin extended generalized linear exponential distribution. *Journal of Computational and Applied Mathematics, 296,* 576–592. https://doi.org/10.1016/j.cam.2015.10.017
9. Alsadat, N., Elgarhy, M., Hassan, A. S., Ahmad, H., El-Hamid Eisa, A. (2023). A new extension of linear failure rate distribution with estimation, simulation, and applications. *AIP Advances, 13(10),* 105019. https://doi.org/10.1063/5.0170297
10. Jamal, F., Elbatal, I., Chesneau, C., Elgarhy, M., Hassan, A. (2019). Modified beta generalized linear failure rate distribution: Theory and applications. *Journal of Prime Research in Mathematics, 15(1),* 21–48.
11. Marshall, A. W., Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and weibull families. *Biometrika, 84(3),* 641–652. https://doi.org/10.1093/biomet/84.3.641
12. Nicholas Eugene, C. L., Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics-Theory and Methods, 31(4),* 497–512. https://doi.org/10.1081/STA-120003130
13. Cordeiro, G. M., de Castro, M., (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation, 81(7),* 883–898. https://doi.org/10.1080/00949650903530745
14. Torabi, H., Montazeri, N. H. (2014). The logistic-uniform distribution and its applications. *Communications in Statistics-Simulation and Computation, 43(10),* 2551–2569. https://doi.org/10.1080/03610918.2012.737491
15. Cordeiro, G. M., Ortega, E. M. M., da Cunha, D. C. C. (2022). The exponentiated generalized class of distributions. *Journal of Data Science, 11(1),* 1–27.
16. Bourguignon, M., Silva, R. B., Cordeiro, G. M. (2022). The weibull-G family of probability distributions. *Journal of Data Science, 12(1),* 53–68.
17. Tahir, M. H., Cordeiro, G. M., Alzaatreh, A., Mansoor, M., Zubair, M. (2016). The logistic-X family of distributions and its applications. *Communications in Statistics-Theory and Methods, 45(24),* 7326–7349. https://doi.org/10.1080/03610926.2014.980516
18. Jamal, F., Nasir, M. A., Ozel, G., Elgarhy, M., Khan, N. M. (2019). Generalized inverted Kumaraswamy generated family of distributions: Theory and applications. *Journal of Applied Statistics, 46(16),* 2927–2944. https://doi.org/10.1080/02664763.2019.1623867
19. Afify, A. Z., Cordeiro, G. M., Ibrahim, N. A., Jamal, F., Elgarhy, M. et al. (2021). The Marshall-Olkin Odd Burr III-G family: Theory, estimation, and engineering applications. *IEEE Access, 9,* 4376–4387. https://doi.org/10.1109/ACCESS.2020.3044156
20. Al-Mofleh, H., Elgarhy, M., Afify, Z. A., Zannon, M. (2020). Type II exponentiated half logistic generated family of distributions with applications. *Electronic Journal of Applied Statistical Analysis, 13(2),* 536–561.
21. Ahmad, Z., Elgarhy, M., Hamedani, G., Butt, N. S. (2020). Odd generalized N-H generated family of distributions with application to exponential model. *Pakistan Journal of Statistics and Operation Research, 16(1),* 53–71.
22. Almarashi, A., Jamal, F., Chesneau, C., Elgarhy, M. (2021). A new truncated muth generated family of distributions with applications. *Complexity, 2021,* 1211526. https://doi.org/10.1155/2021/1211526

23. Abonongo, A. I. L., Abonongo, J. (2024). Exponentiated generalized weibull exponential distribution: Properties, estimation and applications. *Computational Journal of Mathematical and Statistical Sciences, 3(1),* 57–84. https://doi.org/10.21608/cjmss.2023.243845.1023
24. El-Sherpieny, E. S. A., Muhammed, H. Z., Almetwally, E. M. (2023). A new inverse rayleigh distribution with applications of COVID-19 data: Properties, estimation methods and censored sample. *Electronic Journal of Applied Statistical Analysis, 16(2),* 449–472.
25. Kumar, D., Singh, U., Singh, S. K. (2015). A new distribution using sine function its application to bladder cancer patients data. *Journal of Statistics Applications & Probability, 4(3),* 417–427.
26. Nadarajah, S., Kotz, S. (2006). Beta trigonometric distribution. *Portuguese Economic Journal, 5(3),* 207–224. https://doi.org/10.1007/s10258-006-0013-6
27. Kharazmi, O., Saadatinik, A. (2016). Hyperbolic cosine-F family of distributions with an application to exponential distribution. *Gazi University Journal of Science, 29(4),* 811–829.
28. Kharazmi, O., Saadatinik, A., Alizadeh, M., Hamedani, G. G. (2019). Odd hyperbolic cosine-FG family of lifetime distributions. *Journal of Statistical Theory and Applications, 18,* 387–401. https://doi.org/10.2991/jsta.d.191112.003
29. Kharazmi, O., Saadatinik, A., Jahangard, S. (2019). Odd hyperbolic cosine exponential-exponential (OHC-EE) distribution. *Annals of Data Science, 6,* 765–785. https://doi.org/10.1007/s40745-019-00200-z
30. He, W., Ahmad, Z., Afify, A. Z., Goual, H. (2020). The arcsine exponentiated-X family: Validation and insurance application. *Complexity, 2020,* 8394815. https://doi.org/10.1155/2020/8394815
31. Balakrishnan, N., Aggarwala, R. (2000). *Progressive censoring: theory, methods, and applications*, pp. 11–29. Boston: Birkhauser.
32. Balakrishnan, N., Sandhu, R. A. (1995). A simple simulational algorithm for generating progressive Type-II censored samples. *The American Statistician, 49(2),* 229–230. https://doi.org/10.1080/00031305.1995.10476150
33. Aggarwala, R., Balakrishnan, N. (1996). Recurrence relations for single and product moments of progressive type-II right censored order statistics from exponential and truncated exponential distributions. *Annals of the Institute of Statistical Mathematics, 48(4),* 757–771. https://doi.org/10.1007/BF00052331
34. Aggarwala, R., Balakrishnan, N. (1998). Some properties of progressive censored order statistics from arbitrary and uniform distributions with applications to inference and simulation. *Journal of Statistical Planning and Inference, 70(1),* 35–49. https://doi.org/10.1016/S0378-3758(97)00173-0
35. Balakrishnan, N., Kannan, N., Lin, C., Ng, H. (2003). Point and interval estimation for gaussian distribution, based on progressively Type-II censored samples. *IEEE Transactions on Reliability, 52(1),* 90–95. https://doi.org/10.1109/TR.2002.805786
36. Maiti, K., Kayal, S. (2021). Estimation of parameters and reliability characteristics for a generalized rayleigh distribution under progressive Type-II censored sample. *Communications in Statistics-Simulation and Computation, 50(11),* 3669–3698. https://doi.org/10.1080/03610918.2019.1630431
37. Muhammed, H. Z., Almetwally, E. M. (2024). Bayesian and non-bayesian estimation for the shape parameters of new versions of bivariate inverse weibull distribution based on progressive Type II censoring. *Computational Journal of Mathematical and Statistical Sciences, 3(1),* 85–111. https://doi.org/10.21608/cjmss.2023.250678.1028
38. Okasha, H., Nassar, M. (2023). On a new version of weibull model: Statistical properties, parameter estimation and applications. *Computer Modeling in Engineering & Sciences, 137(3),* 2219–2241. https://doi.org/10.32604/cmes.2023.028783
39. Algarni, A., Almarashi, A. M. (2022). A new rayleigh distribution: Properties and estimation based on progressive Type-II censored data with an application. *Computer Modeling in Engineering & Sciences, 130(1),* 379–396. https://doi.org/10.32604/cmes.2022.017714

40. Henningsen, A., Toomet, O. (2011). maxLik: A package for maximum likelihood estimation in R. *Computational Statistics, 26,* 443–458. https://doi.org/10.1007/s00180-010-0217-1

41. Alyami, S. A., Elbatal, I., Alotaibi, N., Almetwally, E. M., Elgarhy, M. (2022). Modeling to factor productivity of the United Kingdom food chain: Using a new lifetime-generated family of distributions. *Sustainability, 14(14),* 8942. https://doi.org/10.3390/su14148942

42. Park, J. C., Lim, H. Y., Kang, D. S. (2021). Predicting rebar endpoints using sin exponential regression model. http://arxiv.org/abs/2110.08955

43. Benchiha, S., Sapkota, L. P., Al Mutairi, A., Kumar, V., Khashab, R. H. et al. (2023). A new sine family of generalized distributions: Statistical inference with applications. *Mathematical and Computational Applications, 28(4),* 83. https://doi.org/10.3390/mca28040083

44. Souza, L., Junior, W., de Brito, C., Chesneau, C., Ferreira, T. et al. (2019). On the Sin-G class of distributions: Theory, model and application. *Journal of Mathematical Modeling, 7(3),* 357–379.

45. Isa, A. M., Ali, B. A., Zannah, U. (2022). Sine burr xii distribution: Properties and application to real data sets. *Arid-Zone Journal of Basic & Applied Research, 1,* 48–58.

46. Pal, M., Ali, M. M., Woo, J. (2006). Exponentiated weibull distribution. *Statistica, 66(2),* 139–147.

47. Basheer, A. M. (2019). Alpha power inverse weibull distribution with reliability application. *Journal of Taibah University for Science, 13(1),* 423–432. https://doi.org/10.1080/16583655.2019.1588488

48. Nassar, M., Alzaatreh, A., Mead, M., Abo-Kasem, O. (2017). Alpha power weibull distribution: Properties and applications. *Communications in Statistics-Theory and Methods, 46(20),* 10236–10252. https://doi.org/10.1080/03610926.2016.1231816

49. de Gusmão, F., Ortega, E., Cordeiro, G. (2011). The generalized inverse weibull distribution. *Statistical Papers, 52,* 591–619. https://doi.org/10.1007/s00362-009-0271-3

50. Almetwally, E. M. (2021). Extended odd weibull inverse rayleigh distribution with application on carbon fibres. *Mathematical Sciences Letters, 10(1),* 5–14. https://doi.org/10.18576/msl

51. Mohammad, Z., Raqab, M. T. M., Kundu, D. (2008). Estimation of $P(Y < X)$ for the three-parameter generalized exponential distribution. *Communications in Statistics-Theory and Methods, 37(18),* 2854–2864.