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ARTICLE

Composite Fractional Trapezoidal Rule with Romberg Integration

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ABSTRACT

The aim of this research is to demonstrate a novel scheme for approximating the Riemann-Liouville fractional integral operator. This would be achieved by first establishing a fractional-order version of the 2-point Trapezoidal rule and then by proposing another fractional-order version of the $(n + 1)$-composite Trapezoidal rule. In particular, the so-called divided-difference formula is typically employed to derive the 2-point Trapezoidal rule, which has accordingly been used to derive a more accurate fractional-order formula called the $(n + 1)$-composite Trapezoidal rule. Additionally, in order to increase the accuracy of the proposed approximations by reducing the true errors, we incorporate the so-called Romberg integration, which is an extrapolation formula of the Trapezoidal rule for integration, into our proposed approaches. Several numerical examples are provided and compared with a modern definition of the Riemann-Liouville fractional integral operator to illustrate the efficacy of our scheme.

KEYWORDS

Composite fractional Trapezoidal rule; Romberg integration

1 Introduction

Mathematical modeling of technical problems and chemical, physical, and economic phenomena [1,2] can all benefit from the utilization of fractional calculus [3]. As a result, a variety of definitions of fractional-order operators have been established recently. Such operators have played crucial roles in advancing various subjects and practical applications in the field of applied mathematics (see [4,5]). Researchers have studied numerous modern developments in fractional calculus theory, exploring their potential applications in a variety of scientific and engineering fields. The fundamentals of fractional calculus have been particularly recognized as significant mathematical tools for describing numerous actual applications [6]. As a consequence, numerous researchers have developed and approved a variety of fractional-order differentiators and integrators, see [7–10].

It is essential to emphasize that fractional-order differentiators have two primary operators, the Riemann-Liouville derivative operator and the Caputo derivative operator, which are both methods that relate to the local derivative of a function [11]. The Caputo differentiator has been shown to be superior to the Riemann-Liouville differentiator for a number of real-world applications [12,13], despite the divergent views of many mathematicians [14,15]. This is because it can be used with certain supposed initial conditions once taking the derivatives of fractional-order [16,17].

The Riemann-Liouville integrator is regarded as an inverse operator for both former operators, regardless of which one is better. This is because the Caputo differentiator is an enhanced formula from the Riemann-Liouville differentiator [11,12]. The fractional-order integrator assumes that various elements cannot be coordinated and are continuously identical. Typically, we use the fractional-order integrator to represent an indefinite integral. Previous attempts to come up with a more general version of the fundamental theorem in fractional calculus were limited, with only two successful attempts [18,19]. However, [18] did come up with a number of different versions of this theorem. The researchers in [19] presented a further coordinated approach, but they did not address the fractional definite integral definition. Similarly, the formulations in [20,21] also lacked a clear definition of the fractional definite integral.

Apart from the research done by Ortigueira et al. and Machado et al. [22], which introduced a definition of the fractional definite integral and analyzed the major theorem of fractional calculus, no other formulations have been made [23]. Given this background and following the approach in classical numerical analysis, this paper aims to establish a novel formulation, referred to as the composite fractional formula, for providing a good approximation for the definite fractional integral, specifically the Riemann-Liouville fractional integrator. Such an operator will be approximated using a 3-point central fractional formula, which will be derived by employing the definite fractional integral definition proposed by O. Manuel and J. Machado's in conjunction with the generalized Taylor's formula. Subsequently, the aforementioned formula will be further extended to $(n + 1)$-composite points, leading to the main result of the research. Also, in order to obtain accurate approximations, we incorporate the so-called Romberg integration into our approaches.

We arrange the subsequent portions of this article as follows: Section 2 offers a concise overview of the fundamental concepts and definitions of fractional calculus. Section 3 talks about the main results of this study and how the $(n + 1)$-point composite fractional formula was made. This formula is used to get close to the Riemann-Liouville fractional integrator. In the same regard and to get a more accurate results of such a formula, we employ Romberg integration in Section 4, whereas Section 5 states several illustrative examples to endorse the generated findings, followed by a concluding part that summarizes the key notes of the whole study.

2 Preliminaries

In this portion, certain fundamental definitions and vital preliminaries regarding fractional calculus are reviewed. This really prepares to our principal results later on.

Definition 1. *[24,25] The Riemann-Liouville fractional integrator of a function $h(u)$ might be declared as*

$$J_a^{\mu} h(u) = \frac{1}{\Gamma(\mu)} \int_0^u h(v)(u - v)^{\mu-1} dv, \tag{1}$$

where $u > 0$ and $\mu > 0$.

In the forthcoming content, we recollect specific features of Riemann-Liouville fractional integrator for the sake of completeness:

$$J_a^0 h(u) = h(u). \tag{2}$$

$$J_a^\mu u^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\mu+\alpha+1)} u^{\mu+\alpha}, \quad \alpha \geq -1. \tag{3}$$

$$J_a^\mu J_a^\gamma h(u) = J_a^\gamma J_a^\mu h(u) \quad \mu, \gamma \geq 0. \tag{4}$$

$$J_a^\mu J_a^\gamma h(u) = J_a^{\mu+\gamma} h(u) \quad \mu, \gamma \geq 0. \tag{5}$$

Definition 2. *[24,25] The Caputo fractional differentiator of a function $h(u)$ be declared as*

$$D_*^\mu h(u) = \frac{1}{\Gamma(n-\mu)} \int_0^u (u-s)^{n-\mu-1} h^{(n)}(s)\, ds, \tag{6}$$

where $n-1 < \mu \leq n$ in which $n \in \mathbb{N}$ and $u > 0$.

Some of Caputo differentiator properties are outlined in the following content for the sake of completeness [24,25]:

$$D_*^\mu c = 0, \text{ where } c \text{ is constant.} \tag{7}$$

$$D_*^\mu u^\rho = \frac{\Gamma(\rho+1)}{\Gamma(\rho-\mu+1)} u^{\rho-\mu}, \text{ for } \rho > \mu - 1. \tag{8}$$

$$D_*^\mu (\mu h(u) + \omega g(u)) = \mu D_*^\mu (h(u)) + \omega D_*^\mu (g(u)), \text{ where } \mu \text{ and } \omega \text{ are constants.} \tag{9}$$

Furthermore, we present below additional properties associated with the composition of the prior two operators [24,25]:

$$D_*^\beta J_0^\beta h(u) = h(u), \tag{10}$$

and

$$J_a^\beta D_*^\beta h(u) = h(u) - \sum_{i=1}^{m} h^i(0^+) \frac{u^i}{i!}, \ u > 0, \tag{11}$$

where $m-1 < \beta \leq m$ in which $m \in \mathbb{N}$.

Definition 3. [24,25] *By means of Riemann-Liouville fractional integrator, the Caputo fractional differentiator of a function $h(u)$ might be declared as*

$$D_*^\mu h(u) = D_*^n \left[J_a^\rho h(u) \right], \tag{12}$$

where $\mu > 0$, $\rho = n - \mu$, $0 < \rho < 1$ and n is the smallest integral greater than μ.

In the following content, we remember two highly significant results that would play a crucial role in establishing the primary outcomes of this study. The first one is attributed to M. Ortigueira and J. Machado, who developed an accurate formula for determining exact values of definite fractional integrals [23]. On the other hand, the second result pertains to Odibat and Momani, who provided a generalisation of the renowned Taylor theorem [25].

Definition 4. *[23] The definite fractional integral of the function g can be outlined by*

$$J_a^\beta g(t) = \int_a^b g^{(-\beta+1)}(t).dt = \int_a^b D_*^{-\beta+1} g(t)dt, \tag{13}$$

where $\beta - 1 < m \leq \beta$ in which $m \in \mathbb{N}$, and $-\infty < a < b < \infty$.

Theorem 1. *[26] (Divided-difference formula) Assume that $P_n(x)$ represents the n^{th}-Lagrange polynomial that matchs with the function f at different points $x_0, x_1, \ldots, x_n$. The divided differences of f in relation to that points can be utilized to outline $P_n(x)$ as follows:*

$$P_{0,1,\ldots,n}(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, x_2, \cdots, x_k](x - x_0)(x - x_1)\ldots(x - x_{k-1}), \tag{14}$$

where

$$f[x_i, x_{i+1}, \cdots, x_{i+k-1}, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \cdots, x_{i+k}] - f[x_i, x_{i+1}, \cdots, x_{i+k-1}]}{x_{i+k} - x_i}, \tag{15}$$

for $i = 0, 1, 2, \ldots, n-1$.

3 Main Results

In this section, we intend first to establish a fraction-order version of the 2-point Trapezoidal rule for providing an approximation of the Riemann–Liouville fractional integral operator. This would pave the way next to establish another fraction-order version called the $(n+1)$-point composite fractional Trapezoidal rule for approximating the same operator. As a consequence, several corollaries are provided as well.

Theorem 2. *(The 2-point fractional Trapezoidal Rule) Let $f \in C^2[a, b]$ and $a = x_0 < x_1 = x_0 + h = b$ with $h = \frac{b-a}{n}$ for n subintervals of $[a, b]$. Then the 2-point fractional Trapezoidal rule for approximating the Riemann–Liouville fractional integral operator is given by*

$$\begin{aligned} J_a^\alpha f(x) = hf(x_0) + \frac{h^\alpha}{\Gamma(\alpha+2)}(f(x_1) - f(x_0)) \\ + \left(\frac{x_1^{\alpha+2} - x_0^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{(x_0 + x_1)(x_1^{\alpha+1} - x_0^{\alpha+1})}{2\Gamma(\alpha+2)} + \frac{1}{2}hx_0x_1\right)f''(\xi), \end{aligned} \tag{16}$$

where $\xi \in (a, b)$.

Proof. To show this result, we apply directly on the divided-difference formula reported in Theorem 1 to get

$$f(x) = P_{0,1}(x) + Error, \tag{17}$$

or

$$f(x) = f[x_0] + f[x_0, x_1](x - x_0) + \frac{f''(\xi)}{2!}(x - x_0)(x - x_1), \tag{18}$$

for $\xi \in (a, b)$. By operating Riemann–Liouville fractional integrator to the both sides of the above equality, we obtain

$$J_a^\alpha f(x) = J_a^\alpha f[x_0] + f[x_0, x_1]J_a^\alpha (x - x_0) + \frac{f''(\xi)}{2!}J_a^\alpha (x^2 - (x_0 + x_1)x - x_0x_1). \tag{19}$$

Accordingly, by utilizing the definite fractional integral reported in Eq. (13), we obtain

$$\begin{aligned} J_a^{\alpha} f(x) = f[x_0] \int_a^b D_*^{-\alpha+1}\, dx + f[x_0, x_1] \int_a^b D_*^{-\alpha+1}(x - x_0) dx \\ + \frac{f''(\xi)}{2} \int_a^b \left(D_*^{-\alpha+1} x^2 - (x_0 + x_1) D_*^{-\alpha+1} x + D_*^{-\alpha+1}(x_0 x_1) \right) dx. \end{aligned} \tag{20}$$

Considering the sense of divided-difference coupled with assuming $\rho = -\alpha + 1$ yield

$$J_a^{\alpha} f(x) = hf(x_0) + \left(\frac{f(x_1) - f(x_0)}{h} \right) \int_a^b D_*^{\rho}(x - x_0) dx + \frac{f''(\xi)}{2} \int_a^b \left(D_*^{\rho} x^2 - (x_0 - x_1) D_*^{\rho} x + h x_0 x_1 \right) dx, \tag{21}$$

or

$$\begin{aligned} J_a^{\alpha} f(x) = hf(x_0) + \left(\frac{f(x_1) - f(x_0)}{h} \right) \int_{x_0}^{x_1} \frac{1}{\Gamma(2-\rho)} (x - x_0)^{1-\rho} dx \\ + \frac{f''(\xi)}{2} \left\{ \int_{x_0}^{x_1} \left(\frac{2}{\Gamma(3-\rho)} x^{2-\rho} - \frac{1}{\Gamma(2-\rho)} (x_0 + x_1) x^{1-\rho} dx \right) + h x_0 x_1 \right\}. \end{aligned} \tag{22}$$

This consequently implies

$$\begin{aligned} J_a^{\alpha} f(x) = hf(x_0) + \left(\frac{f(x) - f(x_0)}{h} \right) \left(\frac{h^{2-\rho}}{\Gamma(3-\rho)} \right) \\ + \frac{f''(\xi)}{2} \left(\frac{2}{\Gamma(4-\rho)} (x_1^{3-\rho} - x_0^{3-\rho}) - \frac{(x_0 + x_1)}{\Gamma(3-\rho)} (x_1^{2-\rho} - x_0^{2-\rho}) + h x_0 x_1 \right), \end{aligned} \tag{23}$$

or

$$J_a^{\alpha} f(x) = hf(x_0) + \frac{h^{1-\rho}(f(x) - f(x_0))}{\Gamma(3-\rho)} + \left(\frac{(x_1^{3-\rho} - x_0^{3-\rho})}{\Gamma(4-\rho)} - \frac{(x_0 + x_1)(x_1^{2-\rho} - x_0^{2-\rho})}{2\Gamma(3-\rho)} + \frac{1}{2} h x_0 x_1 \right) f''(\xi). \tag{24}$$

Hence, by replacing back ρ with $-\alpha + 1$, we get the desired result. ■

Corollary 1. *Under the same assumptions of Theorem 2 with* $\alpha = 1$, *we obtain the* 2*-point classical Trapezoidal rule for approximating the definite integral* $\int_a^b f(x)dx$, *i.e., we have*

$$\int_a^b f(x) dx = \frac{h}{2}(f(x_0) + f(x_1)) - \frac{h^3}{12} f''(\xi), \tag{25}$$

where $\xi \in (a, b)$.

Proof. Note that supposing $\alpha = 1$ in Eq. (16) immediately yields

$$\int_a^b f(x) dx = hf(x_0) + \frac{h}{2}(f(x_1) - f(x_0)) + \left(\frac{x_1^3 - x_0^3}{6} - \frac{(x_0 + x_1)(x_1^2 - x_0^2)}{4} + \frac{1}{2} h x_0 x_1 \right) f''(\xi), \tag{26}$$

i.e.,

$$\int_a^b f(x)dx = \frac{h}{2}(f(x_0) + f(x_1)) + \left(\frac{h(x_1^2 + x_0x_1 + x_0^2)}{6} - \frac{h(x_1 + x_0)^2}{4} + \frac{1}{2}hx_0x_1\right)f''(\xi). \tag{27}$$

After some calculations, we obtain

$$\int_a^b f(x)dx = \frac{h}{2}(f(x_0) + f(x_1)) + \frac{h}{2}f''(\xi)\left(-\frac{1}{6}x_1^2 + \frac{x_0x_1}{3} - \frac{1}{6}x_0^2\right), \tag{28}$$

or

$$\int_a^b f(x)dx = \frac{h}{2}(f(x_0) + f(x_1)) - \frac{h}{12}f''(\xi)(x_1 - x_0)^2, \tag{29}$$

which consequently implies the desired result. ■

Theorem 2. *(The $(n+1)$-point composite fractional Trapezoidal Rule) Let $f \in C^2[a,b]$ and $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ with $h = \frac{b-a}{n}$ for n subintervals of $[a,b]$. Then the $(n+1)$-point composite fractional Trapezoidal rule for approximating the Riemann–Liouville fractional integral operator is given by*

$$J_a^\alpha f(x) \approx \frac{h^\alpha}{\Gamma(\alpha+2)}(f(x_n) - f(x_0)) + h\sum_{i=0}^{n} f(x_i), \tag{30}$$

where $0 < \alpha \leq 1$.

Proof. To prove this result, we use again the definite fractional integral reported in Eq. (13) to obtain

$$J_a^\alpha f(x) = \int_a^b D_*^{-\alpha+1}f(x)dx = \int_{x_0}^{x_1} D_*^{-\alpha+1}f(x)dx + \int_{x_1}^{x_2} D_*^{-\alpha+1}f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} D_*^{-\alpha+1}f(x)dx. \tag{31}$$

Now, with the use of the 2-point fractional Trapezoidal Rule reported in Eq. (16), we get

$$\begin{aligned} J_a^\alpha f(x) \approx{} & hf(x_0) + \frac{h^\alpha}{\Gamma(\alpha+2)}(f(x_1) - f(x_0)) + hf(x_1) + \frac{h^\alpha}{\Gamma(\alpha+2)}(f(x_2) - f(x_1)) \\ & + hf(x_2) + \frac{h^\alpha}{\Gamma(\alpha+2)}(f(x_3) - f(x_2)) + \cdots + hf(x_n) + \frac{h^\alpha}{\Gamma(\alpha+2)}(f(x_n) - f(x_{n-1}). \end{aligned} \tag{32}$$

This immediately implies

$$\begin{aligned} J_a^\alpha f(x) \approx{} & h\Big(f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_n)\Big) + \frac{h^\alpha}{\Gamma(\alpha+2)} \\ & \times \Big(f(x_1) - f(x_0) + f(x_2) - f(x_1) + f(x_3) - f(x_2) + \cdots + f(x_{n-1}) - f(x_{n-2}) + f(x_n) - f(x_{n-1})\Big), \end{aligned} \tag{33}$$

or

$$J_a^\alpha f(x) \approx h\Big(f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_n)\Big) + \frac{h^\alpha}{\Gamma(\alpha+2)}\Big(f(x_n) - f(x_0)\Big), \tag{34}$$

which is equivalent to the desired result. ■

Corollary 2. *Under the same assumptions of Theorem 2 with* $\alpha = 1$, *we obtain the* $(n+1)$*-point classical Trapezoidal rule for approximating the definite integral* $\int_a^b f(x)dx$, *i.e., we have*

$$\int_a^b f(t)dt \approx \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n)\right). \tag{35}$$

Proof. Omitted. ■

The $(n+1)$-composite fractional Trapezoidal formula requires splitting the interval $[a, b]$ into n subintervals, but the 2-point fractional Trapezoidal formula is a straightforward approximation technique constructed by connecting two points. The number of points (or subintervals) used for approximation is where the main differences lie. The $(n+1)$-composite Trapezoidal formula approximates using $n+1$ points, whereas the 2-point Trapezoidal formula employs only two points-the interval's ends. As n rises, the $(n+1)$-composite Trapezoidal formula will sure offer a more precise and computationally efficient approximation.

4 Romberg Integration

The Romberg integration method utilizes the composite Trapezoidal rule to obtain initial approximations and subsequently implements the Richardson extrapolation procedure to enhance these approximations. It is worth noting that Richardson extrapolation might be applied to any approach that has the form

$$K - N(h) = M_1 h + M_2 h^2 + \cdots + M_n h^n, \tag{36}$$

where $N(h)$ is an approximation of the unknown value K and $M_1, M_2, \ldots, M_n$ are constants. The given formula in Eq. (36) represents an approximation procedure of K. The truncation error in the above equation is primarily influenced by $M_1 h$ when h is small, resulting in approximations of $O(h)$. Richardson's extrapolation, on the other hand, employs an averaging scheme to generate formulas with truncation error of higher-order [27]. Without loss of generality and by assuming $\alpha = 1$, we intend to consider the 2-point composite Trapezoidal formula, which has the form

$$J_a f(x) = \frac{h}{2}(f(x) + f(x+h)) - \frac{b-a}{12} h^2 f''(\xi), \tag{37}$$

where $\xi \in (a, b)$. For simplification purposes, we rewrite the above formula as follows:

$$TV = (AV)_h + ch^2, \tag{38}$$

where

$$TV = \text{True Value } = J_a f(x), \tag{39}$$

$$(AV)_h = \text{Approximate Value in terms of } h = \frac{h}{2}(f(x) + f(x+h)), \tag{40}$$

and

$$c = -\frac{b-a}{12} f''(\xi). \tag{41}$$

In a similar manner, we can state the 3-point composite Trapezoidal rule in terms of $\frac{h}{2}$ as follows:

$$J_a f(x) = \frac{h}{4}\left(f(x) + 2f\left(x+\frac{h}{2}\right) + f(x+h)\right) - \frac{b-a}{12}\left(\frac{h}{2}\right)^2 f''(\xi), \tag{42}$$

where $\xi \in (a,b)$. Again, for simplicity, we rewrite the above formula as follows:

$$TV = (AV)_{\frac{h}{2}} + \left(\frac{ch^2}{4}\right), \tag{43}$$

where TV and c are reported respectively in Eqs. (39) and (41), and where

$$(AV)_{\frac{h}{2}} = \text{Approximate Value in terms of } \frac{h}{2} = \frac{h}{4}\left(f(x) + 2f\left(x+\frac{h}{2}\right) + f(x+h)\right). \tag{44}$$

Solving Eqs. (38) and (43) yields

$$(TV) = (AV)_{\frac{h}{2}} + \frac{(AV)_{\frac{h}{2}} - (AV)_h}{3}. \tag{45}$$

Now, supposing $R_{2,2} = TV$, $R_{1,1} = (AV)_h$ and $R_{2,1} = (AV)_{\frac{h}{2}}$ yields

$$R_{2,2} = R_{2,1} + \frac{R_{2,1} + R_{1,1}}{3}, \tag{46}$$

which is called Romberg integration for approximating $J_a f(x)$. If we continue in this manner, we can generate Table 1 that represents the whole approximations within Romberg integration.

Table 1: Approximations within Romberg integration

k	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$\cdots$
1	$R_{1,1}$				
2	$R_{2,1}$	$R_{2,2}$			
3	$R_{3,1}$	$R_{3,2}$	$R_{3,3}$		
4	$R_{4,1}$	$R_{4,2}$	$R_{4,3}$	$R_{4,4}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

All approximations within Romberg integration reported in Table 1 are listed below for completeness:

$$R_{3,2} = R_{3,1} + \frac{R_{3,1} - R_{2,1}}{3},$$

$$R_{4,2} = R_{4,1} + \frac{R_{4,1} - R_{3,1}}{3},$$

$$R_{3,3} = R_{3,2} + \frac{R_{3,2} - R_{2,2}}{15},$$

$$R_{4,3} = R_{4,2} + \frac{R_{4,2} - R_{3,2}}{15},$$

$$R_{4,4} = R_{4,3} + \frac{R_{4,3} - R_{3,3}}{63},$$
$$\vdots$$

where the denominator can be found via $\left(4^{k-1} - 1\right)$ for $k = 2, 3, 4, \ldots$

The challenges that we encountered in adapting Romberg integration to the fractional calculus context may be summarized as follows:

1. For some combined functions resulting from the computational operations, Romberg integration causes more computational complexity. Actually, to avoid this, we handled some functions that lacked this issue, and we are looking to improve our current work to be adapted to all functions, even if they are complex.
2. The discontinuities of functions within the interval pose challenges for Romberg integration, and to handle such a situation, we have adapted the step size chosen in our approach.

Based on the previous discussion, we believe that the approximation of fractional integrals can be improved by the development of the composite fractional Trapezoidal rule with Romberg integration. In particular, the composite fractional Trapezoidal rule with Romberg integration provides a robust numerical method for computing the fractional integrals, allowing researchers to analyze and model complex systems exhibiting fractional dynamics, which have applications in various scientific and engineering fields. Also, the composite fractional Trapezoidal rule with Romberg integration can be integrated with other numerical techniques for solving fractional differential equations, integral equations, or even optimization problems.

5 Numerical Examples

This section aims to confirm the adequacy of our last proposed formula, the composite fractional Trapezoidal rule with Romberg integration, for approximating the Riemann–Liouville fractional integral operator for several functions. Tables are used to show and compare about the acquired discoveries.

In the following content, we discuss two numerical examples which would confirm the validity of the proposed scheme in comparison with the results gained by Ortigueira et al. [23].

Example 1. *Let us consider the main function is* $f(x) = 3x^3 - 5x^2 + 7x$. *In this regard, we assume that we want to find an approximate value for* ${}_aJ^{\alpha}f(x)$ *when* $\alpha = 0.85$ *and* $\alpha = 1$ *over the interval* $[0, 1]$ *by choosing subintervals' number as* $n = 8$. *Observe that the values of* ${}_aJ^{\alpha}f(x)$ *when* $\alpha = 0.85$ *and* $\alpha = 1$ *that could be obtained using Ortigueira et al. [23] or formula (13) can be viewed in Table 2.*

Table 2: The values of ${}_aJ^{\alpha}f(x)$ using formula (13)

α	${}_aJ^{\alpha}f(x)$
0.85	2.933434
1	2.583333

In what follows, we depict respectively two tables (Tables 3 and 4) that include approximate values for ${}_aJ^{\alpha}f(x)$ *when* $\alpha = 0.85$ *and* $\alpha = 1$, *which are generated with the use of the composite fractional Trapezoidal rule with Romberg integration.*

Table 3: Approximation of $J_a^\alpha f(x)$ using composite fractional Trapezoidal rule with Romberg integration for $\alpha = 0.85$

n	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$	$O(h^{14})$	$O(h^{16})$
1	5.000000							
2	6.670654	7.227539						
3	6.217862	6.066932	5.989558					
4	5.281600	4.969512	4.896351	4.878999				
5	4.420836	4.133915	4.078208	4.065222	4.062031			
6	3.772925	3.556955	3.518491	3.509607	3.507428	3.506886		
7	3.328363	3.180175	3.155056	3.149288	3.147875	3.147523	3.147435	
8	3.039189	2.942798	2.926972	2.923352	2.922466	2.922246	2.922191	2.922177

Table 4: Approximation of $J_a^\alpha f(x)$ using composite fractional Trapezoidal rule with Romberg integration for $\alpha = 1$

n	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$	$O(h^{14})$	$O(h^{16})$
1	5.000000							
2	6.312500	6.750000						
3	5.703125	5.500000	5.416666					
4	4.769531	4.458333	4.388888	4.372574				
5	3.989257	3.729166	3.680555	3.669312	3.666554			
6	3.442626	3.260416	3.229166	3.222001	3.220247	3.219811		
7	3.091125	2.973958	2.954861	2.950507	2.949442	2.949177	2.949111	
8	2.586249	2.586249	2.586249	2.586249	2.586249	2.586249	2.586249	2.586249

It should be noticed here that the last two approximated values generated in Tables 3 and 4 using the proposed composite fractional Trapezoidal rule with Romberg integration are very close to the values of $J_a^\alpha f(x)$ obtained using formula (13) reported in Table 2. In particular, one can find the absolute values of the errors gained from the two approaches to be as 0.011257 and 0.002916 for $\alpha = 0.85$ and $\alpha = 1$, respectively. This consequently confirms the validity of our proposed approach, especially when considering $\alpha = 1$.

Example 2. *Let us consider here the function is $f(x) = sin(x^2)$. Herein, we want to find an approximate value for $J_a^\alpha f(x)$ when $\alpha = 0.75$ and $\alpha = 1$ over the interval $[0, \pi]$ by considering $n = 7$. The values of $J_a^\alpha f(x)$ when $\alpha = 0.75$ and $\alpha = 1$ obtained using formula (13) can be summarized in Table 5.*

Table 5: The values of $J_a^\alpha f(x)$ using formula (13)

α	$J_a^\alpha f(x)$
0.75	0.585096
1	0.772651

In Tables 6 and 7 listed below, we provide respectively the gained approximate results for $J_a^\alpha f(x)$ when $\alpha = 0.75$ and $\alpha = 1$ in which these results are generated by using the composite fractional Trapezoidal rule with Romberg integration.

Table 6: Approximation of $J_a^\alpha f(x)$ using composite fractional Trapezoidal rule with Romberg integration for $\alpha = 0.75$

n	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$	$O(h^{14})$
1	−1.351831						
2	−0.326644	0.015084					
3	−0.609035	−0.703166	−0.751049				
4	0.037105	0.252485	0.3161950	0.333136			
5	0.294583	0.380409	0.3889370	0.390092	0.390315		
6	0.459484	0.514451	0.523387	0.525522	0.526053	0.526185	
7	0.533625	0.543387	0.553420	0.554849	0.561519	0.567528	0.570297

Table 7: Approximation of $J_a^\alpha f(x)$ using composite fractional Trapezoidal rule with Romberg integration for $\alpha = 1$

n	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$	$O(h^{14})$
1	−0.371236						
2	−0.593902	−0.668124					
3	0.098899	0.329833	0.396364				
4	0.373710	0.465314	0.474346	0.475584			
5	0.535718	0.589721	0.598015	0.599978	0.600466		
6	0.771175	0.771175	0.771175	0.771175	0.771175	0.771175	
7	0.771863	0.771863	0.771863	0.771863	0.771863	0.771863	0.771863

In light of the two approximated values provided at the end of Tables 6 and 7 obtained by using the proposed composite fractional Trapezoidal rule with Romberg integration, we can assert that those values are very close to the values of $J_a^\alpha f(x)$ that are obtained by using formula (13) reported in Table 5. For instance, the absolute values of the errors generated by the two considered approaches are 0.014799 and 0.000788 for $\alpha = 0.85$ and $\alpha = 1$, respectively. Thus, the validation of our proposed approach is achieved here as well, especially when considering $\alpha = 1$.

In the subsequent content, we provide further numerical examples with the aim of illustrating the success of our proposed method against the failure of formula (13) in approximating $J_a^\alpha f(x)$. Such a failure has been detected when we have run a MATLAB code prepared according to formula (13) for some given functions. This code has succussed for some functions, as in Examples 1 and 2 provided previously, but it has failed with some others as it gave us "NaN" when $0 < \alpha < 1$!

On the other hand, our proposed composite fractional Trapezoidal formula with Romberg integration have worked appropriately in approximating $J_a^\alpha f(x)$. The indicator of the right work of our purposed formula can be represented by several comparisons between the gained results of our approach, and the exact results obtained by considering the classical case of finding the integration using the well-known build function "$int(f(x), x, a, b)$". To see this, we list below the following examples.

Example 3. *Consider here the main function is $f(x) = e^{x^2}$ defined on $\left[\frac{1}{2}, \frac{3}{4}\right]$. Assume $\alpha = 0.5$, $\alpha = 1$ and $n = 8$. The values of $J_a^\alpha f(x)$ when $\alpha = 1$ obtained by using formula (13) and the build function "$int(f(x), x, a, b)$" are listed in Table 8.*

Table 8 : The values of $J_a^\alpha f(x) = \int_{\frac{1}{2}}^{\frac{3}{4}} f(x).dx$

α	Using $int(f(x), x, a, b)$	Using formula (13)
1	0.372928	0.122928

Based on Table 8, one might easily observe the big error that can be gained from using formula (13) in comparison with the value offered by "$int(f(x), x, a, b)$". This confirms the failure of using formula (13) in approximating $J_a^\alpha f(x)$ when $\alpha = 1$. To address this point, we use our proposed composite fractional formula for approximating $J_a^\alpha f(x)$ when $\alpha = 1$ and $\alpha = 0.5$. The results of using this method are listed below in Tables 9 and 10, respectively.

Table 9: Approximation of $J_a^\alpha f(x)$ using composite fractional Trapezoidal rule with Romberg integration for $\alpha = 1$

n	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$	$O(h^{14})$	$O(h^{16})$
1	0.527217							
2	0.464683	0.443838						
3	0.426138	0.413290	0.411253					
4	0.403206	0.395562	0.394381	0.394113				
5	0.389906	0.385472	0.384799	0.384647	0.384610			
6	0.382337	0.379814	0.379436	0.379351	0.379330	0.379325		
7	0.378092	0.376678	0.376469	0.376421	0.376410	0.376407	0.376406	
8	0.375740	0.374956	0.374842	0.374816	0.374809	0.374808	0.374808	0.374807

Table 10: Approximation of $J_a^\alpha f(x)$ using composite fractional Trapezoidal rule with Romberg integration for $\alpha = 0.5$

n	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$	$O(h^{14})$	$O(h^{16})$
1	0.759770							
2	0.741789	0.735796						
3	0.682197	0.662333	0.657436					
4	0.616036	0.593983	0.589426	0.588347				
5	0.557093	0.537445	0.533676	0.532791	0.532573			
6	0.509295	0.493363	0.490424	0.489738	0.489569	0.489527		
7	0.472429	0.460140	0.457925	0.457410	0.457283	0.457251	0.457243	
8	0.444824	0.435622	0.433988	0.433608	0.433515	0.433491	0.433486	0.433484

In view of the previous numerical results, one can obviously notice that the last approximate value in Table 9 is very close to the value of $J_a^\alpha f(x)$ that is obtained by using $int(f(x), x, a, b)$ reported in Table 8. Herein, we find that the absolute value of the error generated by the two considered method is 0.001879 for $\alpha = 1$. This confirms the validation of our proposed approach as well.

Example 4. *Consider $f(x) = \frac{x^2-1}{\sqrt{x^2+1}}$ defined on $[1, 4]$. Suppose $\alpha = 0.95, 1$ and $n = 8$. The values of $J_a^\alpha f(x)$ when $\alpha = 1$ obtained by using formula (13) and the build function "$int(f(x), x, a, b)$" are listed in Table 11.*

Table 11: The values of $J_a^\alpha f(x) = \int_1^4 f(x).dx$

α	Using $int(f(x), x, a, b)$	Using formula (13)
1	5.719096	8.719096

On the basis of 11, we notice that there is a big error from using formula (13) in comparison with the value offered by "$int(f(x), x, a, b)$". For the purpose of dealing with this issue, we use our proposed fractional formula to approximate $J_a^\alpha f(x)$ when $\alpha = 1$ and $\alpha = 0.95$. The results of using this method are listed below in Tables 12 and 13, respectively.

Table 12: Approximation of $J_a^\alpha f(x)$ using composite fractional Trapezoidal rule with Romberg integration for $\alpha = 1$

n	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$	$O(h^{14})$	$O(h^{16})$
1	10.914103							
2	13.838804	14.813705						
3	12.524540	12.086451	11.904634					
4	10.490121	9.811981	9.660350	9.624726				
5	5.741309	5.741309	5.741309	5.741309	5.741309			
6	5.737081	5.731974	5.731234	5.731068	5.731027	5.731017		
7	5.731015	5.731014	5.731014	5.731014	5.731014	5.731014	5.731014	
8	5.725461	5.725460	5.725460	5.725460	5.725460	5.725460	5.725460	5.725460

Table 13: Approximation of $J_a^\alpha f(x)$ using composite fractional Trapezoidal rule with Romberg integration for $\alpha = 0.95$

n	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$	$O(h^{14})$	$O(h^{16})$
1	10.914103							
2	13.788352	14.746435						
3	12.569419	12.163108	11.990886					
4	10.596960	9.939473	9.791231	9.756316				
5	8.908882	8.346189	8.239970	8.215347	8.209304			

(Continued)

Table 13 (continued)

n	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$	$O(h^{14})$	$O(h^{16})$
6	7.702390	7.300227	7.230496	7.214473	7.210548	7.209571		
7	6.008927	6.005432	6.004578	6.004365	6.004312	6.004299	6.004296	
8	5.882029	5.880015	5.879523	5.879401	5.879370	5.879362	5.879361	5.879360

In light of the previous discussion, it can be clearly observed that the approximate value located at the end of Table 12 is very close to the value of $J_a^\alpha f(x)$ that can be obtained using $int(f(x), x, a, b)$ reported in Table 11. Particulary, we find that the absolute value of the error generated by the two considered method is 0.006364 for $\alpha = 1$.

Example 5. *Consider $f(x) = \log(x^2)$ defined on $[1, 10]$. Assume $\alpha = 0.65, 1$ and $n = 8$. The values of $J_a^\alpha f(x)$ when $\alpha = 1$ obtained by using formula (13) and the build function "$int(f(x), x, a, b)$" are listed in Table 14.*

Table 14: The values of $J_a^\alpha f(x) = \int_1^{10} f(x).dx$

α	Using $int(f(x), x, a, b)$	Using formula (13)
1	28.051701	NaN

In consideration of 14, we notice that there is no value from using formula (13) in comparison with the value 28.051701 offered by "$int(f(x), x, a, b)$". Therefore, we use our proposed fractional formula to approximate $J_a^\alpha f(x)$ when $\alpha = 1$ and $\alpha = 0.65$. The results of using this method are listed below in Tables 15 and 16, respectively.

Table 15: Approximation of $J_a^\alpha f(x)$ using composite fractional Trapezoidal rule with Romberg integration for $\alpha = 1$

n	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$	$O(h^{14})$	$O(h^{16})$
1	41.446531							
2	56.789264	61.903508						
3	53.274830	52.103352	51.450008					
4	37.548220	37.146279	37.053221	37.030390				
5	31.131408	31.095317	31.086491	31.084297	31.083749			
6	28.136059	28.136058	28.136057	28.136057	28.136057	28.136057		
7	28.096962	28.096961	28.096961	28.096961	28.096961	28.096961	28.096961	
8	28.075873	28.075872	28.075872	28.075872	28.075872	28.075872	28.075872	28.075872

From the previous discussion, it can be clearly observed that the approximate value located at the end of Table 15 is very close to the value of $J_a^\alpha f(x)$ that can be obtained using $int(f(x), x, a, b)$ reported in Table 14. Particulary, we find that the absolute value of the error generated by the two considered method is 0.024171 for $\alpha = 1$.

Table 16: Approximation of $J_a^\alpha f(x)$ using composite fractional Trapezoidal rule with Romberg integration for $\alpha = 0.65$

n	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$	$O(h^{14})$	$O(h^{16})$
1	41.446531							
2	48.999630	51.517330						
3	47.260729	46.681095	46.358680					
4	37.579194	37.306576	37.242644	37.226911				
5	32.256805	32.223701	32.215577	32.213556	32.213051			
6	30.829617	30.806963	30.801407	30.800025	30.799680	30.799593		
7	29.871055	29.855839	29.852109	29.851181	29.850950	29.850892	29.850877	
8	29.235661	29.225575	29.223103	29.222488	29.222335	29.222297	29.222287	29.222285

A final remark that should be stated here is that the changes in the fractional order have significant effects on the performance of the composite fractional Trapezoidal rule with Romberg integration. As one can see from the given numerical examples, the closer the value of the fractional order is to 1 for a given function, the closer we are to the classical integral of that function. Also, we believe that the composite fractional Trapezoidal rule with Romberg integration may be used as an alternative numerical method for solving integral equations, fractional differential equations, or even optimization problems that arise in physics, control theory, modeling biological systems, fractional optimal control problems, modeling electrochemical processes, fractional partial differential equations, and many others.

6 Conclusion

This paper has successfully established a novel approach for finding a good approximation for the Riemann-Liouville fractional integrator. Specifically, we have successfully derived the so-called 2-point fractional Trapezoidal formula and the $(n + 1)$-composite fractional Trapezoidal formula to approximate the considered operator. We have also employed the method of Romberg integration to obtain accurate approximations. We have performed several numerical comparisons using a modern definition of the Riemann-Liouville fractional integral operator. These comparisons have revealed the superiority of our method over others. In the future, we will use the proposed method to solve more nonlinear problems of different kinds [28,29], and we suggest the readers use the new approach to solve partial integro-differential equations and fractional differential equations with physical applications such as [30,31].

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