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ARTICLE

A Collocation Technique via Pell-Lucas Polynomials to Solve Fractional Differential Equation Model for HIV/AIDS with Treatment Compartment

Gamze Yıldırım[1,2] and Şuayip Yüzbaşı[3,*]

[1]Department of Mathematics, Faculty of Science, Akdeniz University, Antalya, 07058, Turkey

[2]Department of Mathematics, Faculty of Basic Science, Gebze Technical University, Kocaeli, 41400, Turkey

[3]Department of Mathematics, Faculty of Science, Bartın University, Bartın, 74100, Turkey

*Corresponding Author: Şuayip Yüzbaşı. Email: suayipyuzbasi@bartin.edu.tr, suayipyuzbasi@gmail.com

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ABSTRACT

In this study, a numerical method based on the Pell-Lucas polynomials (PLPs) is developed to solve the fractional order HIV/AIDS epidemic model with a treatment compartment. The HIV/AIDS mathematical model with a treatment compartment is divided into five classes, namely, susceptible patients (S), HIV-positive individuals (I), individuals with full-blown AIDS but not receiving ARV treatment (A), individuals being treated (T), and individuals who have changed their sexual habits sufficiently (R). According to the method, by utilizing the PLPs and the collocation points, we convert the fractional order HIV/AIDS epidemic model with a treatment compartment into a nonlinear system of the algebraic equations. Also, the error analysis is presented for the Pell-Lucas approximation method. The aim of this study is to observe the behavior of five populations after 200 days when drug treatment is applied to HIV-infectious and full-blown AIDS people. To demonstrate the usefulness of this method, the applications are made on the numerical example with the help of MATLAB. In addition, four cases of the fractional order derivative ($p = 1$, $p = 0.95$, $p = 0.9$, $p = 0.85$) are examined in the range $[0, 200]$. Owing to applications, we figured out that the outcomes have quite decent errors. Also, we understand that the errors decrease when the value of N increases. The figures in this study are created in MATLAB. The outcomes indicate that the presented method is reasonably sufficient and correct.

KEYWORDS

Collocation method; fractional differential equations; HIV/AIDS epidemic model; Pell-Lucas polynomials

1 Introduction

Fractional differential equations are quite popular among scientists in order to model various stable physical phenomena. Recently, fractional order derivatives have attracted great attention due to their numerous applications in nonlinear complex systems arising in fluid mechanics, damping laws, electrical networks, signal processing, diffusion-reaction processes, relaxation processes, electrochemistry, mathematical biology, physics and various important phenomena in other branches of science and engineering. Fractional derivatives provide more precise models of real-life problems than integer order derivatives. Because some systems exhibit memory, history, or non-local effects that may be difficult to model with the help of integer order derivatives, many natural phenomenon problems are modeled utilizing fractional calculus. In recent years, different types of powerful techniques such

as the generalized differential transformation method [1], the Adomian decomposition method [2], the homotopy analysis method [3], the homotopy analysis transformation method [4], the modified Laplace transform method [5] and the homotopy perturbation transform method [6,7] have been introduced to find the approximate solution of the fractional model of this type of differential equations.

Fractional differential equations are frequently used in mathematical modeling. Mathematical modeling can provide an understanding of the mechanisms underlying the transmission and spread of disease, help identify key factors in the disease transmission process, recommend effective control and preventive measures, and provide an estimate of the severity and potential scale of the epidemic. Mathematical biology is a multidisciplinary field of study. Mathematical biology is also used as an important tool in medicine, as well as in differential equations, basic sciences, and many fields of engineering such as genetic engineering, biomedical engineering, and clinical engineering. Lately, mathematical biology has been successfully applied to several important fields in medicine including epidemiology, genetics, drug design and discovery, biofluids, cardiovascular diseases, immunology, microbiology, neuroscience, oncology, virology, and more. Thus, medical phenomena are better understood, and practical action paths are found. The improvements in healthcare and quality of life are achieved because there are important contributions such as early diagnosis, effective medicines, and control of epidemics. From past to present, many models such as the logistic equation and the Lotka-Volterra equations based on prey and predator populations [8], HIV (human immunodeficiency virus) infection model [9–12], SIR model (Susceptible-Infected-Removed) and COVID-19 model [13], etc., have been developed in the field of medicine using mathematical biology.

Recently, the studies on mathematical modeling of human immunodeficiency virus (HIV) have become very popular. HIV, which causes the acquired immunodeficiency syndrome (AIDS), devastates the human body's skill to fight infections. This disease is very hazardous and can be mortal if left untreated. The first AIDS case was detected in 1981 [14]. In 2017, the US Center for Disease Control and Prevention (CDC) [15] declared that if HIV/AIDS is not treated with antiretroviral drugs, HIV infection progresses in several phases. According to some official reports, it is observed that between 2000 and 2016, HIV-related deaths in Africa decreased by one-third thanks to ART (Antiretroviral Therapy) [16]. There are many methods such as the exponential collocation method (ECM) [17], the Bessel collocation method (BCM) [18], the differential transform method (DTM) [19], the optimization method [20], the Galerkin-like method [21], Laplace Adomian decomposition method (LADM) [22], homotopy perturbation method (HPM) [23], variational iteration method (VIM) [24] for solving the HIV infection model. In addition, there are many methods such as homotopy perturbation method, variational iteration method and the Adomian decomposition method (ADM) [25], the fourth kind Chebyshev wavelet method [26], shifted Legendre collocation method [27], the homotopy analysis method [28], LADM [29,30], the Legendre wavelet approach [31], the septic B-spline scheme [32], the fractional approximation method [33] for solving fractional order HIV models. On the other hand, there are some studies conducted for the HIV/AIDS epidemic model with antiretroviral therapy (ART). Luo et al. [34] gave the global stability of disease-free equilibrium and the endemic equilibrium, investigated the long-time stochastic dynamic of the model, and gave some numerical simulations. Wang et al. [35] presented the optimal control problem and conducted numerical simulations. Chen et al. [36] developed a type-2 fuzzy logic controller for antiretroviral therapy of HIV infection and performed simulations for two strategies. Ali et al. [37] constructed the existence and uniqueness conditions of the HIV/AIDS model by utilizing Schaefer and Banachtype fixed point theorems. The model's qualitative analysis investigated, determined the existence/uniqueness of the solution to the HIV/AIDS model [38], established the stability results

for the system by incorporating the Ulam-Hyers method, and applied Newton's polynomial and the Toufik-Atangana numerical method.

Spectral methods are one of the common methods used for solving mathematical models because they are extremely sensitive. Inasmuch as it uses linear combinations of the orthogonal polynomials as basis functions, these methods lead to accurate approximate solutions [39,40]. Basically, three types of these methods can be identified. These are Tau [41], collocation [42,43] and Galerkin [44] and these are based on orthogonal systems such as Bernstein polynomials, Bessel polynomials, Chebyshev polynomials, Hermite polynomials, Legendre polynomials, Laguerre polynomials, Pell-Lucas polynomials, etc. Compared to other approximation methods, the presented method requires fewer calculations. With the presented method, an approximate solution can be achieved with small N values selected for problems that do not have an exact solution. However, in some methods in the literature, more iterations are required to obtain approximate solutions for systems of the nonlinear equation or even nonlinear equations [45].

On the other hand, there are many methods based on the Pell-Lucas polynomials (PLPs) on the solutions of Fredholm-type delay integro-differential equations [46], functional differential equations [47], population models [8], Fredholm-Volterra integro-differential equations [48], parabolic-type partial integro-differential equations [49], nonlinear Lane-Emden pantograph differential equations [50] and an SIR model on the spread of the novel coronavirus (nCoV-2019) pandemic [13] and very effective results have been obtained from these methods. However, there is no method based on the Pell-Lucas polynomial solutions of the fractional order HIV/AIDS epidemic model with a treatment compartment (FHEMTC) in the literature. This reveals the importance and novelty of this study. For this reason, this study is also important as it is a new application of the method by developing it for fractional problems. In this paper, we consider the following FHEMTC [14,15]:

$$
\begin{aligned}
&{}^{C}D_{t}^{(\rho)}S(t) = \Lambda - \beta S(t)I(t) - \mu_1 S(t) - dS(t)\\
&{}^{C}D_{t}^{(\rho)}I(t) = \beta S(t)I(t) + \alpha_1 T(t) - dI(t) - k_1 I(t) - k_2 I(t)\\
&{}^{C}D_{t}^{(\rho)}A(t) = k_1 I(t) - (\delta_1 + d)A(t) + \alpha_2 T(t)\\
&{}^{C}D_{t}^{(\rho)}T(t) = k_2 I(t) - \alpha_1 T(t) - (d + \delta_2 + \alpha_2)T(t)\\
&{}^{C}D_{t}^{(\rho)}R(t) = \mu_1 S(t) - dR(t)\\
&S(0) = S_0, \quad I(0) = I_0, \quad A(0) = A_0, \quad T(0) = T_0, \quad R(0) = R_0.
\end{aligned}
\tag{1}
$$

Representations of parameters and variables in (1) are given in Table 1.

Table 1: Representations of the parameters and the variables in model (1)

Parameter/ Variable	Explanation
$S(t)$	The number of susceptible patients
$I(t)$	The number of HIV-positive individuals who are infectious (i.e., who are not receiving antiretroviral ARV treatment or for whom the treatment is not effective)
$A(t)$	The number of individuals with full-blown AIDS who are not receiving ARV treatment or for whom the treatment is not effective

(Continued)

Table 1 (continued)

Parameter/ Variable	Explanation
$T(t)$	The total number of individuals being treated with ARV and for whom the treatment is effective
$R(t)$	The number of individuals who have changed their sexual habits sufficiently so that they are immune to HIV infection by sexual contact
$P(t)$	Total population and $P = S(t) + I(t) + A(t) + T(t) + R(t)$
Λ	The recruitment rate of susceptible people into the population
β	The contact rate between the susceptible and the infectious people
μ_1	The rate at which susceptible individuals change their sexual habits per unit time
d	the natural death rate
α_1	The rate at which treated individuals leave compartment $T(t)$ and return to the infectious compartment
k_1	The rate at which individuals leave the infectious class and become individuals with full-blown AIDS
k_1	The rate at which individuals with HIV receive treatment
δ_1	The disease-induced death rates for individuals in compartments $A(t)$
δ_2	The disease-induced death rates for individuals in compartments $I(t)$
α_2	The rate at which treated individuals leave the treated class and enter the AIDS compartment $A(t)$
$0 < p \leq 1$	fractional order derivative

In this study, we examine the approximate solutions based on PLPs of (1):

$$\begin{aligned} S_N(t) &= \sum_{j=0}^{N} c_{1,j} Q_j(t), \\ I_N(t) &= \sum_{j=0}^{N} c_{2,j} Q_j(t), \\ A_N(t) &= \sum_{j=0}^{N} c_{3,j} Q_j(t), \\ T_N(t) &= \sum_{j=0}^{N} c_{4,j} Q_j(t), \\ R_N(t) &= \sum_{j=0}^{N} c_{5,j} Q_j(t). \end{aligned} \tag{2}$$

Here, $c_{i,j}$ $(i = 1, 2, 3, 4, 5)$ are the unknown coefficients, $N > 0$ and $Q_j(t)$ are PLPs and these polynomials are defined by [51,52]

$$Q_j(t) = \sum_{p=0}^{[\![i/2]\!]} 2^{j-2p} \frac{j}{j-p} \binom{j-p}{p} t^{j-2p}. \tag{3}$$

Please see [51,52] to learn more about PLPs.

The flow of this paper is created as follows: In Section 2, the required matrix relations of the method are presented. In Section 3, the numerical method for FHEMTC is presented. In Section 4, the error analysis is given. In Section 5, the parameters and initial conditions in the HIV/AIDS epidemic

model are determined and Pell-Lucas collocation method (PLCM) is applied to the obtained model. In Section 6, a brief conclusion of the article is presented.

2 Preliminaries and Matrix Relations

In this section, we define the fundamental facts on fractional derivatives and then we express the terms of FHEMTC (1) in matrix form.

Lemma 2.1. PLPs (3) are written the following matrix form [50]:

$$\mathbf{Q}_N(t) = \mathbf{F}_N(t)\mathbf{K}_N, \tag{4}$$

where, $\mathbf{Q}_N(t) = \begin{bmatrix} Q_0(t) & Q_1(t) & \cdots & Q_N(t) \end{bmatrix}$, $\mathbf{F}_N(t) = \begin{bmatrix} 1 & t & t^2 & \cdots & t^N \end{bmatrix}$. When N is even and odd, we can, respectively, write the matrix $\mathbf{K}_N^T$ as follows:

$$\mathbf{K}_N^T = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 2^1 \frac{1}{1}\binom{1}{0} & 0 & \cdots & 0 \\ 2^0 \frac{2}{1}\binom{1}{1} & 0 & 2^2 \frac{2}{2}\binom{2}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^0 \frac{N}{\frac{N}{2}}\binom{\frac{N}{2}}{\frac{N}{2}} & 0 & 2^2 \frac{N}{\frac{N+2}{2}}\binom{\frac{N+2}{2}}{\frac{N-2}{2}} & \cdots & 2^N \frac{N}{N}\binom{N}{0} \end{bmatrix},$$

and

$$\mathbf{K}_N^T = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 2^1 \frac{1}{1}\binom{1}{0} & 0 & \cdots & 0 \\ 2^0 \frac{2}{1}\binom{1}{1} & 0 & 2^2 \frac{2}{2}\binom{2}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 2^1 \frac{N}{\frac{N+1}{2}}\binom{\frac{N+1}{2}}{\frac{N-1}{2}} & 0 & \cdots & 2^N \frac{N}{N}\binom{N}{0} \end{bmatrix}.$$

Proof. If the vector $\mathbf{F}_N(t)$ is multiplied by $\mathbf{K}_N$ from the right side, then (4) is obtained. ■

Lemma 2.2. The assumed solution forms (2) can also be written in matrix form as

$$\begin{aligned} S_N(t) &= \mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{1,N}, \\ I_N(t) &= \mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{2,N}, \\ A_N(t) &= \mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{3,N}, \\ T_N(t) &= \mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{4,N}, \\ R_N(t) &= \mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{5,N}, \end{aligned} \tag{5}$$

where, $\mathbf{C}_{k,N} = \begin{bmatrix} c_{k,0} & c_{k,1} & \cdots & c_{k,N} \end{bmatrix}^T$ $(k = 1, 2, 3, 4, 5)$ and other matrices are as in Lemma 2.1.

Proof. If $\mathbf{Q}_N(t) = \mathbf{F}_N(t)\mathbf{K}_N$ is multiplied, separately, by $\mathbf{C}_{1,N}$, $\mathbf{C}_{2,N}$, $\mathbf{C}_{3,N}$, $\mathbf{C}_{4,N}$ and $\mathbf{C}_{5,N}$ from the right, we have (5). ■

Definition 2.1. Using [53], we describe the fractional derivative of $h(t)$ in the Caputo sense as

$${}^{C}D_t^{(p)}h(t) = J^{m-p}D^m h(t) = \frac{1}{\Gamma(m-p)}\int_0^t (t-x)^{m-p-1}h^{(m)}(x)dx,$$

for $m-1 < p < m$, $m \in \mathbb{N}$, $t > 0$, $h \in C_{-1}^m$ where $D = \dfrac{d}{dx}$. When $h(t)$ equals t^n, the Caputo fractional derivative of $h(t)$ can be written as [54]

$$
{}^{C}D_{t}^{(p)}t^{\eta} = \begin{cases} 0, & \text{for} \quad \eta \in \mathbb{N}_0 \quad \text{and} \quad \eta < \lceil p \rceil \\ \dfrac{\Gamma(\eta+1)}{\Gamma(\eta+1-p)}t^{\eta-p}, & \text{for} \quad \eta \in \mathbb{N}_0 \quad \text{and} \quad \eta \geq \lceil p \rceil \quad \text{or} \quad \eta \notin \mathbb{N}_0 \quad \text{and} \quad \eta > \lfloor p \rfloor \end{cases} \tag{6}
$$

Here, if c is constant, then ${}^{C}D_{t}^{(p)}c = 0$.

Lemma 2.3. The matrix forms of the *p-th order* fractional derivatives of the assumed solution forms (2) in (1) become

$$
\begin{aligned}
S_N^{(p)}(t) &= \mathbf{F}_N^{(p)}(t)\mathbf{K}_N\mathbf{C}_{1,N}, \\
I_N^{(p)}(t) &= \mathbf{F}_N^{(p)}(t)\mathbf{K}_N\mathbf{C}_{2,N}, \\
A_N^{(p)}(t) &= \mathbf{F}_N^{(p)}(t)\mathbf{K}_N\mathbf{C}_{3,N}, \\
T_N^{(p)}(t) &= \mathbf{F}_N^{(p)}(t)\mathbf{K}_N\mathbf{C}_{4,N}, \\
R_N^{(p)}(t) &= \mathbf{F}_N^{(p)}(t)\mathbf{K}_N\mathbf{C}_{5,N}.
\end{aligned} \tag{7}
$$

Here,

$$
\mathbf{F}_N^{(p)}(t) = \left[0 \quad \frac{\Gamma(2)}{\Gamma(2-p)}t^{1-p} \quad \frac{\Gamma(3)}{\Gamma(3-p)}t^{2-p} \quad \cdots \quad \frac{\Gamma(N+1)}{\Gamma(N+1-p)}t^{N-p} \right] \tag{8}
$$

and other matrices are given in Lemma 2.2.

Proof. The *p-th order* fractional derivative of $\mathbf{F}_N(t)$ can be written as in (8) with the help of (6). Then, the *p-th order* fractional derivative of the assumed solution form (2) is taken. The proof is completed by using (8) in the derived terms. ■

Lemma 2.4. The nonlinear term in (1) can be expressed the following matrix form:

$$
S_N(t)I_N(t) = \left(\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{1,N}\right)\left(\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{2,N}\right). \tag{9}
$$

Here, all matrices are as in Lemma 2.2.

Proof. Through Lemma 2.2, we write $S_N(t) = \mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{1,N}$ and $I_N(t) = \mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{2,N}$. Consequently, the matrix multiplication of these two terms gives us the proof of Lemma. ■

Lemma 2.5. The initial conditions in (1) can be written with matrix relations

$$
\begin{aligned}
\mathbf{L}_N\mathbf{C}_{1,N} &= S_0, \\
\mathbf{L}_N\mathbf{C}_{2,N} &= I_0, \\
\mathbf{L}_N\mathbf{C}_{3,N} &= A_0, \\
\mathbf{L}_N\mathbf{C}_{4,N} &= T_0, \\
\mathbf{L}_N\mathbf{C}_{5,N} &= R_0.
\end{aligned} \tag{10}
$$

Here, $\mathbf{L}_N = \mathbf{F}_N(0)\mathbf{K}_N$. Also, other matrices are given in Lemma 2.2.

Proof. By substituting $t \to 0$ in (5), the proof is completed. Let's note that we denote the matrix product $\mathbf{F}_N(0)\mathbf{K}_N$ by $\mathbf{L}_N$. ■

3 Numerical Method

In this part, PLCM is given for the approximate solution of FHEMTC (1) by utilizing collocation points.

Theorem 3.1. Supposed that the approximate solutions of the HIV/AIDS epidemic model (1) are in form (2), then, the following matrix relations are obtained:

$$\begin{aligned}
\mathbf{F}_N^{(p)}(t)\mathbf{K}_N\mathbf{C}_{1,N} &= \Lambda - \beta\left(\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{1,N}\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{2,N}\right) - \mu_1\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{1,N} - d\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{1,N}\\
\mathbf{F}_N^{(p)}(t)\mathbf{K}_N\mathbf{C}_{2,N} &= \beta\left(\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{1,N}\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{2,N}\right) + \alpha_1\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{4,N} - d\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{2,N} - k_1\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{2,N}\\
&\quad -k_2\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{2,N}\\
\mathbf{F}_N^{(p)}(t)\mathbf{K}_N\mathbf{C}_{3,N} &= k_1\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{2,N} - (\delta_1+d)\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{3,N} + \alpha_2\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{4,N}\\
\mathbf{F}_N^{(p)}(t)\mathbf{K}_N\mathbf{C}_{4,N} &= k_2\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{2,N} - \alpha_1\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{4,N} - (d+\delta_2+\alpha_2)\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{4,N}\\
\mathbf{F}_N^{(p)}(t)\mathbf{K}_N\mathbf{C}_{5,N} &= \mu_1\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{1,N} - d\mathbf{F}_N(t)\mathbf{K}_N\mathbf{C}_{5,N}.
\end{aligned} \tag{11}$$

Here, all matrices are given in Lemmas 2.2–2.4.

Proof. Firstly, Eq. (7) are written instead of fractional-order derivative terms in model (1), Eq. (9) is written instead of nonlinear term in model (1) and Eq. (5) are written instead of solution forms in the model (1). Hence, the system (11) is obtained, so the proof is completed. ■

Definition 3.1. The collocation points in interval $[0, h]$ are defined by

$$t_k = \frac{h}{N}k, \quad k = 0, 1, \ldots, N. \tag{12}$$

Theorem 3.2. Assume that the approximate solutions of (1) are given in (5). In that case, FHEMTC (1) becomes as follows:

$$\begin{aligned}
&\mathbf{G}_0\mathbf{C}_{1,N} + \mathbf{D}_{1,0}\mathbf{C}_{2,N} + (\mu_1+d)\mathbf{M}_0\mathbf{C}_{1,N} = \Lambda,\\
&\mathbf{G}_0\mathbf{C}_{2,N} + \mathbf{D}_{2,0}\mathbf{C}_{2,N} - \alpha_1\mathbf{M}_0\mathbf{C}_{4,N} = 0,\\
&\mathbf{G}_0\mathbf{C}_{3,N} - k_1\mathbf{M}_0\mathbf{C}_{2,N} + (\delta_1+d)\mathbf{M}_0\mathbf{C}_{3,N} - \alpha_2\mathbf{M}_0\mathbf{C}_{4,N} = 0,\\
&\mathbf{G}_0\mathbf{C}_{4,N} - k_2\mathbf{M}_0\mathbf{C}_{2,N} + (\alpha_1+d+\delta_2+\alpha_2)\mathbf{M}_0\mathbf{C}_{4,N} = 0,\\
&\mathbf{G}_0\mathbf{C}_{5,N} - \mu_1\mathbf{M}_0\mathbf{C}_{1,N} + d\mathbf{M}_0\mathbf{C}_{5,N} = 0,\\
&\mathbf{G}_1\mathbf{C}_{1,N} + \mathbf{D}_{1,1}\mathbf{C}_{2,N} + (\mu_1+d)\mathbf{M}_1\mathbf{C}_{1,N} = \Lambda,\\
&\mathbf{G}_1\mathbf{C}_{2,N} + \mathbf{D}_{2,1}\mathbf{C}_{2,N} - \alpha_1\mathbf{M}_1\mathbf{C}_{4,N} = 0,\\
&\mathbf{G}_1\mathbf{C}_{3,N} - k_1\mathbf{M}_1\mathbf{C}_{2,N} + (\delta_1+d)\mathbf{M}_1\mathbf{C}_{3,N} - \alpha_2\mathbf{M}_1\mathbf{C}_{4,N} = 0,\\
&\mathbf{G}_1\mathbf{C}_{4,N} - k_2\mathbf{M}_1\mathbf{C}_{2,N} + (\alpha_1+d+\delta_2+\alpha_2)\mathbf{M}_1\mathbf{C}_{4,N} = 0,\\
&\mathbf{G}_1\mathbf{C}_{5,N} - \mu_1\mathbf{M}_1\mathbf{C}_{1,N} + d\mathbf{M}_1\mathbf{C}_{5,N} = 0,\\
&\qquad\vdots\\
&\mathbf{G}_N\mathbf{C}_{1,N} + \mathbf{D}_{1,N}\mathbf{C}_{2,N} + (\mu_1+d)\mathbf{M}_N\mathbf{C}_{1,N} = \Lambda,\\
&\mathbf{G}_N\mathbf{C}_{2,N} + \mathbf{D}_{2,N}\mathbf{C}_{2,N} - \alpha_1\mathbf{M}_N\mathbf{C}_{4,N} = 0,\\
&\mathbf{G}_N\mathbf{C}_{3,N} - k_1\mathbf{M}_N\mathbf{C}_{2,N} + (\delta_1+d)\mathbf{M}_N\mathbf{C}_{3,N} - \alpha_2\mathbf{M}_N\mathbf{C}_{4,N} = 0,\\
&\mathbf{G}_N\mathbf{C}_{4,N} - k_2\mathbf{M}_N\mathbf{C}_{2,N} + (\alpha_1+d+\delta_2+\alpha_2)\mathbf{M}_N\mathbf{C}_{4,N} = 0,\\
&\mathbf{G}_N\mathbf{C}_{5,N} - \mu_1\mathbf{M}_N\mathbf{C}_{1,N} + d\mathbf{M}_N\mathbf{C}_{5,N} = 0,\\
&\mathbf{L}_N\mathbf{C}_{1,N} = S_0,\\
&\mathbf{L}_N\mathbf{C}_{2,N} = I_0,\\
&\mathbf{L}_N\mathbf{C}_{3,N} = A_0,\\
&\mathbf{L}_N\mathbf{C}_{4,N} = T_0,\\
&\mathbf{L}_N\mathbf{C}_{5,N} = R_0
\end{aligned} \tag{13}$$

where,

$$\begin{aligned}
&\mathbf{G}_i = \mathbf{F}_N^{(p)}(t_i)\mathbf{K}_N,\\
&\mathbf{D}_{1,i} = \beta\mathbf{F}_N(t_i)\mathbf{K}_N\mathbf{C}_{1,N}\mathbf{F}_N(t_i)\mathbf{K}_N,\\
&\mathbf{D}_{2,i} = -\beta\mathbf{F}_N(t_i)\mathbf{K}_N\mathbf{C}_{1,N}\mathbf{F}_N(t_i)\mathbf{K}_N + (d+k_1+k_2)\mathbf{F}_N(t_i)\mathbf{K}_N,\\
&\mathbf{M}_i = \mathbf{F}_N(t_i)\mathbf{K}_N.
\end{aligned} \tag{14}$$

Also, other matrices are given in Theorem 3.1. and Lemma 2.5.

Proof. The collocation points (12) are substituted into (11) and thus the following system of algebraic equations is obtained:

$$\begin{aligned}
\mathbf{F}_N^{(p)}(t_0)\mathbf{K}_N\mathbf{C}_{1,N} &= \Lambda - \beta\left(\mathbf{F}_N(t_0)\mathbf{K}_N\mathbf{C}_{1,N}\mathbf{F}_N(t_0)\mathbf{K}_N\mathbf{C}_{2,N}\right) - \mu_1\mathbf{F}_N(t_0)\mathbf{K}_N\mathbf{C}_{1,N} - d\mathbf{F}_N(t_0)\mathbf{K}_N\mathbf{C}_{1,N}\\
\mathbf{F}_N^{(p)}(t_0)\mathbf{K}_N\mathbf{C}_{2,N} &= \beta\left(\mathbf{F}_N(t_0)\mathbf{K}_N\mathbf{C}_{1,N}\mathbf{F}_N(t_0)\mathbf{K}_N\mathbf{C}_{2,N}\right) + \alpha_1\mathbf{F}_N(t_0)\mathbf{K}_N\mathbf{C}_{4,N} - d\mathbf{F}_N(t_0)\mathbf{K}_N\mathbf{C}_{2,N} - k_1\mathbf{F}_N(t_0)\mathbf{K}_N\mathbf{C}_{2,N}\\
&\quad -k_2\mathbf{F}_N(t_0)\mathbf{K}_N\mathbf{C}_{2,N}\\
\mathbf{F}_N^{(p)}(t_0)\mathbf{K}_N\mathbf{C}_{3,N} &= k_1\mathbf{F}_N(t_0)\mathbf{K}_N\mathbf{C}_{2,N} - (\delta_1+d)\mathbf{F}_N(t_0)\mathbf{K}_N\mathbf{C}_{3,N} + \alpha_2\mathbf{F}_N(t_0)\mathbf{K}_N\mathbf{C}_{4,N}\\
\mathbf{F}_N^{(p)}(t_0)\mathbf{K}_N\mathbf{C}_{4,N} &= k_2\mathbf{F}_N(t_0)\mathbf{K}_N\mathbf{C}_{2,N} - \alpha_1\mathbf{F}_N(t_0)\mathbf{K}_N\mathbf{C}_{4,N} - (d+\delta_2+\alpha_2)\mathbf{F}_N(t_0)\mathbf{K}_N\mathbf{C}_{4,N}\\
\mathbf{F}_N^{(p)}(t_0)\mathbf{K}_N\mathbf{C}_{5,N} &= \mu_1\mathbf{F}_N(t_0)\mathbf{K}_N\mathbf{C}_{1,N} - d\mathbf{F}_N(t_0)\mathbf{K}_N\mathbf{C}_{5,N}\\
\mathbf{F}_N^{(p)}(t_1)\mathbf{K}_N\mathbf{C}_{1,N} &= \Lambda - \beta\left(\mathbf{F}_N(t_1)\mathbf{K}_N\mathbf{C}_{1,N}\mathbf{F}_N(t_1)\mathbf{K}_N\mathbf{C}_{2,N}\right) - \mu_1\mathbf{F}_N(t_1)\mathbf{K}_N\mathbf{C}_{1,N} - d\mathbf{F}_N(t_1)\mathbf{K}_N\mathbf{C}_{1,N}\\
\mathbf{F}_N^{(p)}(t_1)\mathbf{K}_N\mathbf{C}_{2,N} &= \beta\left(\mathbf{F}_N(t_1)\mathbf{K}_N\mathbf{C}_{1,N}\mathbf{F}_N(t_1)\mathbf{K}_N\mathbf{C}_{2,N}\right) + \alpha_1\mathbf{F}_N(t_1)\mathbf{K}_N\mathbf{C}_{4,N} - d\mathbf{F}_N(t_1)\mathbf{K}_N\mathbf{C}_{2,N} - k_1\mathbf{F}_N(t_1)\mathbf{K}_N\mathbf{C}_{2,N}\\
&\quad -k_2\mathbf{F}_N(t_1)\mathbf{K}_N\mathbf{C}_{2,N}\\
\mathbf{F}_N^{(p)}(t_1)\mathbf{K}_N\mathbf{C}_{3,N} &= k_1\mathbf{F}_N(t_1)\mathbf{K}_N\mathbf{C}_{2,N} - (\delta_1+d)\mathbf{F}_N(t_1)\mathbf{K}_N\mathbf{C}_{3,N} + \alpha_2\mathbf{F}_N(t_1)\mathbf{K}_N\mathbf{C}_{4,N}\\
\mathbf{F}_N^{(p)}(t_1)\mathbf{K}_N\mathbf{C}_{4,N} &= k_2\mathbf{F}_N(t_1)\mathbf{K}_N\mathbf{C}_{2,N} - \alpha_1\mathbf{F}_N(t_1)\mathbf{K}_N\mathbf{C}_{4,N} - (d+\delta_2+\alpha_2)\mathbf{F}_N(t_1)\mathbf{K}_N\mathbf{C}_{4,N}\\
\mathbf{F}_N^{(p)}(t_1)\mathbf{K}_N\mathbf{C}_{5,N} &= \mu_1\mathbf{F}_N(t_1)\mathbf{K}_N\mathbf{C}_{1,N} - d\mathbf{F}_N(t_1)\mathbf{K}_N\mathbf{C}_{5,N}\\
&\vdots\\
\mathbf{F}_N^{(p)}(t_N)\mathbf{K}_N\mathbf{C}_{1,N} &= \Lambda - \beta\left(\mathbf{F}_N(t_N)\mathbf{K}_N\mathbf{C}_{1,N}\mathbf{F}_N(t_N)\mathbf{K}_N\mathbf{C}_{2,N}\right) - \mu_1\mathbf{F}_N(t_N)\mathbf{K}_N\mathbf{C}_{1,N} - d\mathbf{F}_N(t_N)\mathbf{K}_N\mathbf{C}_{1,N}\\
\mathbf{F}_N^{(p)}(t_N)\mathbf{K}_N\mathbf{C}_{2,N} &= \beta\left(\mathbf{F}_N(t_N)\mathbf{K}_N\mathbf{C}_{1,N}\mathbf{F}_N(t_N)\mathbf{K}_N\mathbf{C}_{2,N}\right) + \alpha_1\mathbf{F}_N(t_N)\mathbf{K}_N\mathbf{C}_{4,N} - d\mathbf{F}_N(t_N)\mathbf{K}_N\mathbf{C}_{2,N} - k_1\mathbf{F}_N(t_N)\mathbf{K}_N\mathbf{C}_{2,N}\\
&\quad -k_2\mathbf{F}_N(t_N)\mathbf{K}_N\mathbf{C}_{2,N}\\
\mathbf{F}_N^{(p)}(t_N)\mathbf{K}_N\mathbf{C}_{3,N} &= k_1\mathbf{F}_N(t_N)\mathbf{K}_N\mathbf{C}_{2,N} - (\delta_1+d)\mathbf{F}_N(t_N)\mathbf{K}_N\mathbf{C}_{3,N} + \alpha_2\mathbf{F}_N(t_N)\mathbf{K}_N\mathbf{C}_{4,N}\\
\mathbf{F}_N^{(p)}(t_N)\mathbf{K}_N\mathbf{C}_{4,N} &= k_2\mathbf{F}_N(t_N)\mathbf{K}_N\mathbf{C}_{2,N} - \alpha_1\mathbf{F}_N(t_N)\mathbf{K}_N\mathbf{C}_{4,N} - (d+\delta_2+\alpha_2)\mathbf{F}_N(t_N)\mathbf{K}_N\mathbf{C}_{4,N}\\
\mathbf{F}_N^{(p)}(t_N)\mathbf{K}_N\mathbf{C}_{5,N} &= \mu_1\mathbf{F}_N(t_N)\mathbf{K}_N\mathbf{C}_{1,N} - d\mathbf{F}_N(t_N)\mathbf{K}_N\mathbf{C}_{5,N}.
\end{aligned} \tag{15}$$

Now, the relations in (14) are used in the system (15). Then, by writing this system and conditions (10) as a single system, we obtain (13), which completes proof. ■

Corollary 3.1. When nonlinear algebraic system (13) is solved through MATLAB, the unknown coefficients $\mathbf{C}_{i,N}$ $(i = 1, 2, 3, 4, 5)$ in (5) are calculated. Thus, the approximate solutions of FHEMTC (1) are obtained.

4 Error Analysis

This section has two purposes. The first purpose is to get an error bound for the presented method. The second purpose is to create an error estimation method. Let $S(t)$, $I(t)$, $A(t)$, $T(t)$, $R(t)$ be the exact solutions and $S_N(t)$, $I_N(t)$, $A_N(t)$, $T_N(t)$, $R_N(t)$ be the approximate solutions based on PLPs of FHEMTC (1).

Theorem 4.1. (Upper Boundary of Errors) Assume that the generalized Maclaurin series are represented by $S_N^{Mac}(t) = \mathbf{F}_N(t)\widetilde{\mathbf{C}}_{1,N}$, $I_N^{Mac}(t) = \mathbf{F}_N(t)\widetilde{\mathbf{C}}_{2,N}$, $A_N^{Mac}(t) = \mathbf{F}_N(t)\widetilde{\mathbf{C}}_{3,N}$, $T_N^{Mac}(t) = \mathbf{F}_N(t)\widetilde{\mathbf{C}}_{4,N}$, $R_N^{Mac}(t) = \mathbf{F}_N(t)\widetilde{\mathbf{C}}_{5,N}$. Then, the errors of the approximate solutions based on PLPs of (1) are bounded by

$$
\begin{aligned}
&\|S(t) - S_N(t)\|_\infty \le \lambda_N(\|\widetilde{\mathbf{C}}_{1,N}\|_\infty + \|\mathbf{K}_N\|_\infty\|\mathbf{C}_{1,N}\|_\infty) + \frac{h^{N+1}}{(N+1)!}\|S^{(N+1)}(c_t)\|_\infty,\\
&\|I(t) - I_N(t)\|_\infty \le \lambda_N(\|\widetilde{\mathbf{C}}_{2,N}\|_\infty + \|\mathbf{K}_N\|_\infty\|\mathbf{C}_{2,N}\|_\infty) + \frac{h^{N+1}}{(N+1)!}\|I^{(N+1)}(c_t)\|_\infty,\\
&\|A(t) - A_N(t)\|_\infty \le \lambda_N(\|\widetilde{\mathbf{C}}_{3,N}\|_\infty + \|\mathbf{K}_N\|_\infty\|\mathbf{C}_{3,N}\|_\infty) + \frac{h^{N+1}}{(N+1)!}\|A^{(N+1)}(c_t)\|_\infty,\\
&\|T(t) - T_N(t)\|_\infty \le \lambda_N(\|\widetilde{\mathbf{C}}_{4,N}\|_\infty + \|\mathbf{K}_N\|_\infty\|\mathbf{C}_{4,N}\|_\infty) + \frac{h^{N+1}}{(N+1)!}\|T^{(N+1)}(c_t)\|_\infty,\\
&\|R(t) - R_N(t)\|_\infty \le \lambda_N(\|\widetilde{\mathbf{C}}_{5,N}\|_\infty + \|\mathbf{K}_N\|_\infty\|\mathbf{C}_{5,N}\|_\infty) + \frac{h^{N+1}}{(N+1)!}\|R^{(N+1)}(c_t)\|_\infty.
\end{aligned}
\tag{16}
$$

Here, $\|\mathbf{F}_N(t)\|_\infty \le \max\{h^N, 1\} := \lambda_N$, $\triangle\mathbf{C}_{i,N} = \|\mathbf{C}_{i,N+1}\|_\infty - \|\mathbf{C}_{i,N}\|_\infty$ (i = 1, 2, 3, 4, 5). Also, the coefficient matrices of $S_N^{Mac}(t)$, $I_N^{Mac}(t)$, $A_N^{Mac}(t)$, $T_N^{Mac}(t)$, $R_N^{Mac}(t)$ are indicated by $\widetilde{\mathbf{C}}_{i,N}$ (i = 1, 2, 3, 4, 5), respectively.

Proof. Firstly, using the triangle inequality, we can write

$$
\begin{aligned}
&\|S(t) - S_N(t)\|_\infty \le \|S(t) - S_N^{Mac}(t)\|_\infty + \|S_N^{Mac}(t) - S_N(t)\|_\infty,\\
&\|I(t) - I_N(t)\|_\infty \le \|I(t) - I_N^{Mac}(t)\|_\infty + \|I_N^{Mac}(t) - I_N(t)\|_\infty,\\
&\|A(t) - A_N(t)\|_\infty \le \|A(t) - A_N^{Mac}(t)\|_\infty + \|A_N^{Mac}(t) - A_N(t)\|_\infty,\\
&\|T(t) - T_N(t)\|_\infty \le \|T(t) - T_N^{Mac}(t)\|_\infty + \|T_N^{Mac}(t) - T_N(t)\|_\infty,\\
&\|R(t) - R_N(t)\|_\infty \le \|R(t) - R_N^{Mac}(t)\|_\infty + \|R_N^{Mac}(t) - R_N(t)\|_\infty.
\end{aligned}
\tag{17}
$$

Secondly, using Lemma 2.2, we know the matrix representations of the Pell-Lucas polynomial solutions from Lemma 2.2. Therefore, we have

$$
\begin{aligned}
&\|S_N^{Mac}(t) - S_N(t)\|_\infty = \|\mathbf{F}_N(t)(\widetilde{\mathbf{C}}_{1,N} - \mathbf{K}_N\mathbf{C}_{1,N})\|_\infty \le \|\mathbf{F}_N(t)\|_\infty\left(\|\widetilde{\mathbf{C}}_{1,N}\|_\infty + \|\mathbf{K}_N\|_\infty\|\mathbf{C}_{1,N}\|_\infty\right),\\
&\|I_N^{Mac}(t) - I_N(t)\|_\infty = \|\mathbf{F}_N(t)(\widetilde{\mathbf{C}}_{2,N} - \mathbf{K}_N\mathbf{C}_{2,N})\|_\infty \le \|\mathbf{F}_N(t)\|_\infty\left(\|\widetilde{\mathbf{C}}_{2,N}\|_\infty + \|\mathbf{K}_N\|_\infty\|\mathbf{C}_{2,N}\|_\infty\right),\\
&\|A_N^{Mac}(t) - A_N(t)\|_\infty = \|\mathbf{F}_N(t)(\widetilde{\mathbf{C}}_{3,N} - \mathbf{K}_N\mathbf{C}_{3,N})\|_\infty \le \|\mathbf{F}_N(t)\|_\infty\left(\|\widetilde{\mathbf{C}}_{3,N}\|_\infty + \|\mathbf{K}_N\|_\infty\|\mathbf{C}_{3,N}\|_\infty\right),\\
&\|T_N^{Mac}(t) - T_N(t)\|_\infty = \|\mathbf{F}_N(t)(\widetilde{\mathbf{C}}_{4,N} - \mathbf{K}_N\mathbf{C}_{4,N})\|_\infty \le \|\mathbf{F}_N(t)\|_\infty\left(\|\widetilde{\mathbf{C}}_{4,N}\|_\infty + \|\mathbf{K}_N\|_\infty\|\mathbf{C}_{4,N}\|_\infty\right),\\
&\|R_N^{Mac}(t) - R_N(t)\|_\infty = \|\mathbf{F}_N(t)(\widetilde{\mathbf{C}}_{5,N} - \mathbf{K}_N\mathbf{C}_{5,N})\|_\infty \le \|\mathbf{F}_N(t)\|_\infty\left(\|\widetilde{\mathbf{C}}_{5,N}\|_\infty + \|\mathbf{K}_N\|_\infty\|\mathbf{C}_{5,N}\|_\infty\right).
\end{aligned}
\tag{18}
$$

Owing to $0 \le t \le h$, the term $\|\mathbf{F}_N(t)\|_\infty$ can be written as follows:

$$
\|\mathbf{F}_N(t)\|_\infty \le \max\{h^N, 1\} := \lambda_N. \tag{19}
$$

Substituting (19) into (18), the following inequalities are obtained:

$$\begin{aligned}
&\|S_N^{Mac}(t) - S_N(t)\|_\infty \le \lambda_N \left(\|\widetilde{\mathbf{C}}_{1,N}\|_\infty + \|\mathbf{K}_N\|_\infty \|\mathbf{C}_{1,N}\|_\infty\right), \\
&\|I_N^{Mac}(t) - I_N(t)\|_\infty \le \lambda_N \left(\|\widetilde{\mathbf{C}}_{2,N}\|_\infty + \|\mathbf{K}_N\|_\infty \|\mathbf{C}_{2,N}\|_\infty\right), \\
&\|A_N^{Mac}(t) - A_N(t)\|_\infty \le \lambda_N \left(\|\widetilde{\mathbf{C}}_{3,N}\|_\infty + \|\mathbf{K}_N\|_\infty \|\mathbf{C}_{3,N}\|_\infty\right), \\
&\|T_N^{Mac}(t) - T_N(t)\|_\infty \le \lambda_N \left(\|\widetilde{\mathbf{C}}_{4,N}\|_\infty + \|\mathbf{K}_N\|_\infty \|\mathbf{C}_{4,N}\|_\infty\right), \\
&\|R_N^{Mac}(t) - R_N(t)\|_\infty \le \lambda_N \left(\|\widetilde{\mathbf{C}}_{5,N}\|_\infty + \|\mathbf{K}_N\|_\infty \|\mathbf{C}_{5,N}\|_\infty\right).
\end{aligned} \tag{20}$$

On the other hand, we can write the remainder terms of the Maclaurin series for $0 \le t \le h$

$$\begin{aligned}
&\frac{t^{N+1}}{(N+1)!} S^{(N+1)}(c_t), \\
&\frac{t^{N+1}}{(N+1)!} I^{(N+1)}(c_t), \\
&\frac{t^{N+1}}{(N+1)!} A^{(N+1)}(c_t), \\
&\frac{t^{N+1}}{(N+1)!} T^{(N+1)}(c_t), \\
&\frac{t^{N+1}}{(N+1)!} R^{(N+1)}(c_t).
\end{aligned} \tag{21}$$

Hence, the following inequalites can be written:

$$\begin{aligned}
&\|S(t) - S_N^{Mac}(t)\|_\infty \le \frac{h^{N+1}}{(N+1)!} \|S^{(N+1)}(c_t)\|_\infty, \\
&\|I(t) - I_N^{Mac}(t)\|_\infty \le \frac{h^{N+1}}{(N+1)!} \|I^{(N+1)}(c_t)\|_\infty, \\
&\|A(t) - A_N^{Mac}(t)\|_\infty \le \frac{h^{N+1}}{(N+1)!} \|A^{(N+1)}(c_t)\|_\infty, \\
&\|T(t) - T_N^{Mac}(t)\|_\infty \le \frac{h^{N+1}}{(N+1)!} \|T^{(N+1)}(c_t)\|_\infty, \\
&\|R(t) - R_N^{Mac}(t)\|_\infty \le \frac{h^{N+1}}{(N+1)!} \|R^{(N+1)}(c_t)\|_\infty.
\end{aligned} \tag{22}$$

Combining the inequalities (20) and (22) in (17), the inequalities (16) are obtained. So, the proof is completed. ■

Theorem 4.2. (Error Estimation) Assume that the residual functions of FHEMTC (1) for the approximate solutions based on PLPs (2) are represented by $Re_{i,N}(t)$ (i = 1, 2, 3, 4, 5). According to this, the error problem is obtained as

$$
\begin{aligned}
&{}^{C}D_{t}^{(p)}e_{1,N}(t) - \Lambda + \beta\big(e_{1,N}(t)e_{2,N}(t) + I_N(t)e_{1,N}(t) + S_N(t)e_{2,N}(t)\big) + \mu_1 e_{1,N}(t) + de_{1,N}(t) = -Re_{1,N}(t)\\
&{}^{C}D_{t}^{(p)}e_{2,N}(t) - \beta\big(e_{1,N}(t)e_{2,N}(t) + I_N(t)e_{1,N}(t) + S_N(t)e_{2,N}(t)\big) - \alpha_1 e_{4,N}(t) + de_{2,N}(t) + k_1 e_{2,N}(t) + k_2 e_{2,N}(t)\\
&\qquad = -Re_{2,N}(t)\\
&{}^{C}D_{t}^{(p)}e_{3,N}(t) - k_1 e_{2,N}(t) + (\delta_1 + d)e_{3,N}(t) - \alpha_2 e_{4,N}(t) = -Re_{3,N}(t)\\
&{}^{C}D_{t}^{(p)}e_{4,N}(t) - k_2 e_{2,N}(t) + \alpha_1 e_{4,N}(t) + (d + \delta_2 + \alpha_2)e_{4,N}(t) = -Re_{4,N}(t)\\
&{}^{C}D_{t}^{(p)}e_{5,N}(t) - \mu_1 e_{1,N}(t) + de_{5,N}(t) = -Re_{5,N}(t)\\
&e_{i,N}(0) = 0, \quad (i = 1, 2, 3, 4, 5)
\end{aligned}
\tag{23}
$$

where,

$$
\begin{aligned}
&e_{1,N}(t) = S(t) - S_N(t)\\
&e_{2,N}(t) = I(t) - I_N(t)\\
&e_{3,N}(t) = A(t) - A_N(t)\\
&e_{4,N}(t) = T(t) - T_N(t)\\
&e_{5,N}(t) = R(t) - R_N(t).
\end{aligned}
\tag{24}
$$

Proof. Inasmuch as the approximate solutions based on PLPs (2) satisfy FHEMTC (1), the proof begins by writing

$$
\begin{aligned}
&{}^{C}D_{t}^{(p)}S_N(t) - \Lambda + \beta S_N(t)I_N(t) + \mu_1 S_N(t) + dS_N(t) = Re_{1,N}(t)\\
&{}^{C}D_{t}^{(p)}I_N(t) - \beta S_N(t)I_N(t) - \alpha_1 T_N(t) + dI_N(t) + k_1 I_N(t) + k_2 I_N(t) = Re_{2,N}(t)\\
&{}^{C}D_{t}^{(p)}A_N(t) - k_1 I_N(t) + (\delta_1 + d)A_N(t) - \alpha_2 T_N(t) = Re_{3,N}(t)\\
&{}^{C}D_{t}^{(p)}T_N(t) - k_2 I_N(t) + \alpha_1 T_N(t) + (d + \delta_2 + \alpha_2)T_N(t) = Re_{4,N}(t)\\
&{}^{C}D_{t}^{(p)}R_N(t) - \mu_1 S_N(t) + dR_N(t) = Re_{5,N}(t)\\
&S_N(0) = S_0, \quad I_N(0) = I_0, \quad A_N(0) = A_0, \quad T_N(0) = T_0, \quad R_N(0) = R_0.
\end{aligned}
\tag{25}
$$

When system (25) is subtracted from model (1) and the relations in (24) are used, we gain (23). Hence, the proof is completed. ■

Corollary 4.1. If the problem (23) is solved using PLCM in Section 3 for $M > 0$, then the estimated error functions are calculated.

5 Application Results

The aim of this section is to show the accuracy and efficiency of the proposed matrix approaches. For this purpose, some numerical computations for FHEMTC (1) are performed. The parameters and initial conditions are determined according to references in [15,37,38] for simulations and these values are given in Table 2. The figures in this study are created in MATLAB.

Table 2: The values of variables of FHEMTC (1)

Parameters	Values	References
Λ	0.55	[15]
β	0.03	[15]
d	0.0196	[15]
k_1	0.15	[15]
k_2	0.35	[15]
α_1	0.08	[15]
α_2	0.03	[15]
δ_1	0.0909	[15]
δ_2	0.0667	[15]
μ_1	0.03	[15]
Initial conditions	**Values**	**References**
S_0	35	[15]
I_0	24	[15]
A_0	15	[15]
T_0	8	[15]
R_0	0	[15]

According to the selected parameters, FHEMTC (1) becomes

$$\begin{aligned}
&{}^{C}D_t^{(p)}S(t) = 0.55 - 0.03S(t)I(t) - 0.03S(t) - 0.0196S(t)\\
&{}^{C}D_t^{(p)}I(t) = 0.03S(t)I(t) + 0.08T(t) - 0.0196I(t) - 0.15I(t) - 0.35I(t)\\
&{}^{C}D_t^{(p)}A(t) = 0.15I(t) - (0.0909 + 0.0196)A(t) + 0.03T(t)\\
&{}^{C}D_t^{(p)}T(t) = 0.35I(t) - 0.08T(t) - (0.0196 + 0.0667 + 0.03)T(t)\\
&{}^{C}D_t^{(p)}R(t) = 0.03S(t) - 0.0196R(t)\\
&S(0) = 35, \quad I(0) = 24, \quad A(0) = 15, \quad T(0) = 8, \quad R(0) = 0.
\end{aligned} \tag{26}$$

When PLCM is applied to the model (26) for $N = 4$, the solution forms are written as

$$\begin{aligned}
&S_4(t) = \textstyle\sum_{n=0}^{4} c_{1,n}Q_n(t),\\
&I_4(t) = \textstyle\sum_{n=0}^{4} c_{2,n}Q_n(t),\\
&A_4(t) = \textstyle\sum_{n=0}^{4} c_{3,n}Q_n(t),\\
&T_4(t) = \textstyle\sum_{n=0}^{4} c_{4,n}Q_n(t),\\
&R_4(t) = \textstyle\sum_{n=0}^{4} c_{5,n}Q_n(t).
\end{aligned} \tag{27}$$

Using Lemma 2.2, the expression (27) is expressed as follows:

$$\begin{aligned}
S_4(t) &= \mathbf{F}_4(t)\mathbf{K}_4\mathbf{C}_{1,4},\\
I_4(t) &= \mathbf{F}_4(t)\mathbf{K}_4\mathbf{C}_{2,4},\\
A_4(t) &= \mathbf{F}_4(t)\mathbf{K}_4\mathbf{C}_{3,4},\\
T_4(t) &= \mathbf{F}_4(t)\mathbf{K}_4\mathbf{C}_{4,4},\\
R_4(t) &= \mathbf{F}_4(t)\mathbf{K}_4\mathbf{C}_{5,4}.
\end{aligned} \tag{28}$$

Here, $\mathbf{F}_4(t) = \begin{bmatrix} 1 & t & t^2 & t^3 & t^4 \end{bmatrix}$, $\mathbf{C}_{i,4} = \begin{bmatrix} c_{i,0} & c_{i,1} & c_{i,2} & c_{i,3} & c_{i,4} \end{bmatrix}^T$, $(i = 1, 2, 3, 4, 5)$,

$$\mathbf{K}_4 = \begin{bmatrix} 2 & 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 6 & 0 \\ 0 & 0 & 4 & 0 & 16 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 16 \end{bmatrix}.$$

For range [0, 200], the collocation points become $t_0 = 0, t_1 = 50, t_2 = 100, t_3 = 150, t_4 = 200$ using (12). Thus, we arrive

$$\begin{aligned}
&\mathbf{G}_0\mathbf{C}_{1,4} + \mathbf{D}_{1,0}\mathbf{C}_{2,4} + (0.03 + 0.0196)\mathbf{M}_0\mathbf{C}_{1,4} = 0.55,\\
&\mathbf{G}_0\mathbf{C}_{2,4} + \mathbf{D}_{2,0}\mathbf{C}_{2,4} - 0.08\mathbf{M}_0\mathbf{C}_{4,4} = 0,\\
&\mathbf{G}_0\mathbf{C}_{3,4} - 0.15\mathbf{M}_0\mathbf{C}_{2,4} + (0.0909 + 0.0196)\mathbf{M}_0\mathbf{C}_{3,4} - 0.03\mathbf{M}_0\mathbf{C}_{4,4} = 0,\\
&\mathbf{G}_0\mathbf{C}_{4,4} - 0.35\mathbf{M}_0\mathbf{C}_{2,4} + (0.08 + 0.0196 + 0.0667 + 0.03)\mathbf{M}_0\mathbf{C}_{4,4} = 0,\\
&\mathbf{G}_0\mathbf{C}_{5,4} - 0.03\mathbf{M}_0\mathbf{C}_{1,4} + 0.0196\mathbf{M}_0\mathbf{C}_{5,4} = 0,\\
&\mathbf{G}_1\mathbf{C}_{1,4} + \mathbf{D}_{1,1}\mathbf{C}_{2,4} + (0.03 + 0.0196)\mathbf{M}_1\mathbf{C}_{1,4} = 0.55,\\
&\mathbf{G}_1\mathbf{C}_{2,4} + \mathbf{D}_{2,1}\mathbf{C}_{2,4} - 0.08\mathbf{M}_1\mathbf{C}_{4,4} = 0,\\
&\mathbf{G}_1\mathbf{C}_{3,4} - 0.15\mathbf{M}_1\mathbf{C}_{2,4} + (0.0909 + 0.0196)\mathbf{M}_1\mathbf{C}_{3,4} - 0.03\mathbf{M}_1\mathbf{C}_{4,4} = 0,\\
&\mathbf{G}_1\mathbf{C}_{4,4} - 0.35\mathbf{M}_1\mathbf{C}_{2,4} + (0.08 + 0.0196 + 0.0667 + 0.03)\mathbf{M}_1\mathbf{C}_{4,4} = 0,\\
&\mathbf{G}_1\mathbf{C}_{5,4} - 0.03\mathbf{M}_1\mathbf{C}_{1,4} + 0.0196\mathbf{M}_1\mathbf{C}_{5,4} = 0,\\
&\qquad\qquad\vdots\\
&\mathbf{G}_4\mathbf{C}_{1,4} + \mathbf{D}_{1,4}\mathbf{C}_{2,4} + (0.03 + 0.0196)\mathbf{M}_4\mathbf{C}_{1,4} = 0.55,\\
&\mathbf{G}_N\mathbf{C}_{2,N} + \mathbf{D}_{2,4}\mathbf{C}_{2,4} - 0.08\mathbf{M}_4\mathbf{C}_{4,4} = 0,\\
&\mathbf{G}_4\mathbf{C}_{3,4} - 0.15\mathbf{M}_4\mathbf{C}_{2,4} + (0.0909 + 0.0196)\mathbf{M}_4\mathbf{C}_{3,4} - 0.03\mathbf{M}_4\mathbf{C}_{4,4} = 0,\\
&\mathbf{G}_4\mathbf{C}_{4,4} - 0.35\mathbf{M}_4\mathbf{C}_{2,4} + (0.08 + 0.0196 + 0.0667 + 0.03)\mathbf{M}_4\mathbf{C}_{4,4} = 0,\\
&\mathbf{G}_4\mathbf{C}_{5,4} - 0.03\mathbf{M}_4\mathbf{C}_{1,4} + 0.0196\mathbf{M}_4\mathbf{C}_{5,4} = 0,
\end{aligned} \tag{29}$$

where,

$$\mathbf{G}_i = \mathbf{F}_4^{(p)}(t_i)\mathbf{K}_4,$$

$$\mathbf{D}_{1,i} = 0.03\mathbf{F}_4(t_i)\mathbf{K}_4\mathbf{C}_{1,4}\mathbf{F}_4(t_i)\mathbf{K}_4,$$

$$\mathbf{D}_{2,i} = -0.03\mathbf{F}_4(t_i)\mathbf{K}_4\mathbf{C}_{1,4}\mathbf{F}_4(t_i)\mathbf{K}_4 + (0.0196 + 0.15 + 0.35)\mathbf{F}_4(t_i)\mathbf{K}_4,$$

$$\mathbf{M}_i = \mathbf{F}_4(t_i)\mathbf{K}_4.$$

$$\mathbf{F}_4(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{F}_4(50) = \begin{bmatrix} 1 & 50 & 50^2 & 50^3 & 50^4 \end{bmatrix},$$

$$\mathbf{F}_4(100) = \begin{bmatrix} 1 & 100 & 100^2 & 100^3 & 100^4 \end{bmatrix},$$

$$\mathbf{F}_4(150) = \begin{bmatrix} 1 & 150 & 150^2 & 150^3 & 150^4 \end{bmatrix}, \quad \mathbf{F}_4(200) = \begin{bmatrix} 1 & 200 & 200^2 & 200^3 & 200^4 \end{bmatrix},$$

$$\mathbf{F}_4^{(p)}(t_i) = \begin{bmatrix} 0 & \dfrac{\Gamma(2)}{\Gamma(2-p)}t_i^{1-p} & \dfrac{\Gamma(3)}{\Gamma(3-p)}t_i^{2-p} & \cdots & \dfrac{\Gamma(5)}{\Gamma(5-p)}t_i^{4-p} \end{bmatrix}, \quad \mathbf{F}_4^{(1)}(t_i) = \begin{bmatrix} 0 & 1 & 2t_i & 3t_i^2 & 4t_i^3 \end{bmatrix},$$

$$\mathbf{F}_4^{(0.95)}(t_i) = \begin{bmatrix} 0 & 1.0272e+00t_i^{0.05} & 1.9566e+00t_i^{1.05} & 2.8633e+00t_i^{2.05} & 3.7552e+00t_i^{3.05} \end{bmatrix},$$

$$\mathbf{F}_4^{(0.85)}(t_i) = \begin{bmatrix} 0 & 1.0718e+00t_i^{0.15} & 1.8639e+00t_i^{1.15} & 2.6008e+00t_i^{2.15} & 3.3027e+00t_i^{3.15} \end{bmatrix}.$$

When the relations (10) are used, we have

$$\begin{aligned} \mathbf{L}_4\mathbf{C}_{1,4} &= S_0, \\ \mathbf{L}_4\mathbf{C}_{2,4} &= I_0, \\ \mathbf{L}_4\mathbf{C}_{3,4} &= A_0, \\ \mathbf{L}_4\mathbf{C}_{4,4} &= T_0, \\ \mathbf{L}_4\mathbf{C}_{5,4} &= R_0. \end{aligned} \tag{30}$$

Here, $\mathbf{L}_4 = \mathbf{F}_4(0)\mathbf{K}_4$. If systems (29) and (30) are solved using MATLAB, approximate solutions are calculated according to various p values.

Graphical solutions have a lot of importance. For example, thanks to graphical solutions, we can easily understand the behavior of the function, examine the properties of the function, see how it changes for different values of the function, and predict how it changes for different values of the function. Additionally, since we can examine more than one function on the same graph, we can compare the behavior of the functions. In this study, graphical solutions are used since the comparisons are made for different values of the parameter representing the fractional order derivative and for different values of the selected number N.

Fig. 1 shows the graph of functions $S_3(t)$, $I_3(t)$, $A_3(t)$, $T_3(t)$, $R_3(t)$ for various fractional orders on interval $t \in [0, 200]$. It is concluded from this plot that the curves of various values of p have similar tendencies. Nevertheless, their values are slightly different.

Fig. 2 demonstrates the graph of solution functions for $N = 3$, $N = 4$, $N = 5$ and the fractional order $p = 1$. Figs. 3–5 indicate the graph of solution functions, respectively, for the fractional orders $p = 0.95$, $p = 0.9$ and $p = 0.85$. When the 200-day oncoming alteration is investigated, it is examined that first 4 population, firstly, decrease, then increase, then increase again, that is, they have an oscillatory behavior. As for the behavior of $R_N(t)$, it can be said that it tends to increase. Also, it can

be observed that the number of $I_N(t)$, $A_N(t)$ and $T_N(t)$ going towards zero as $t \to \infty$. Namely, there are no patients needing treatment since HIV-infectious and full-blown AIDS people eventually vanish from population.

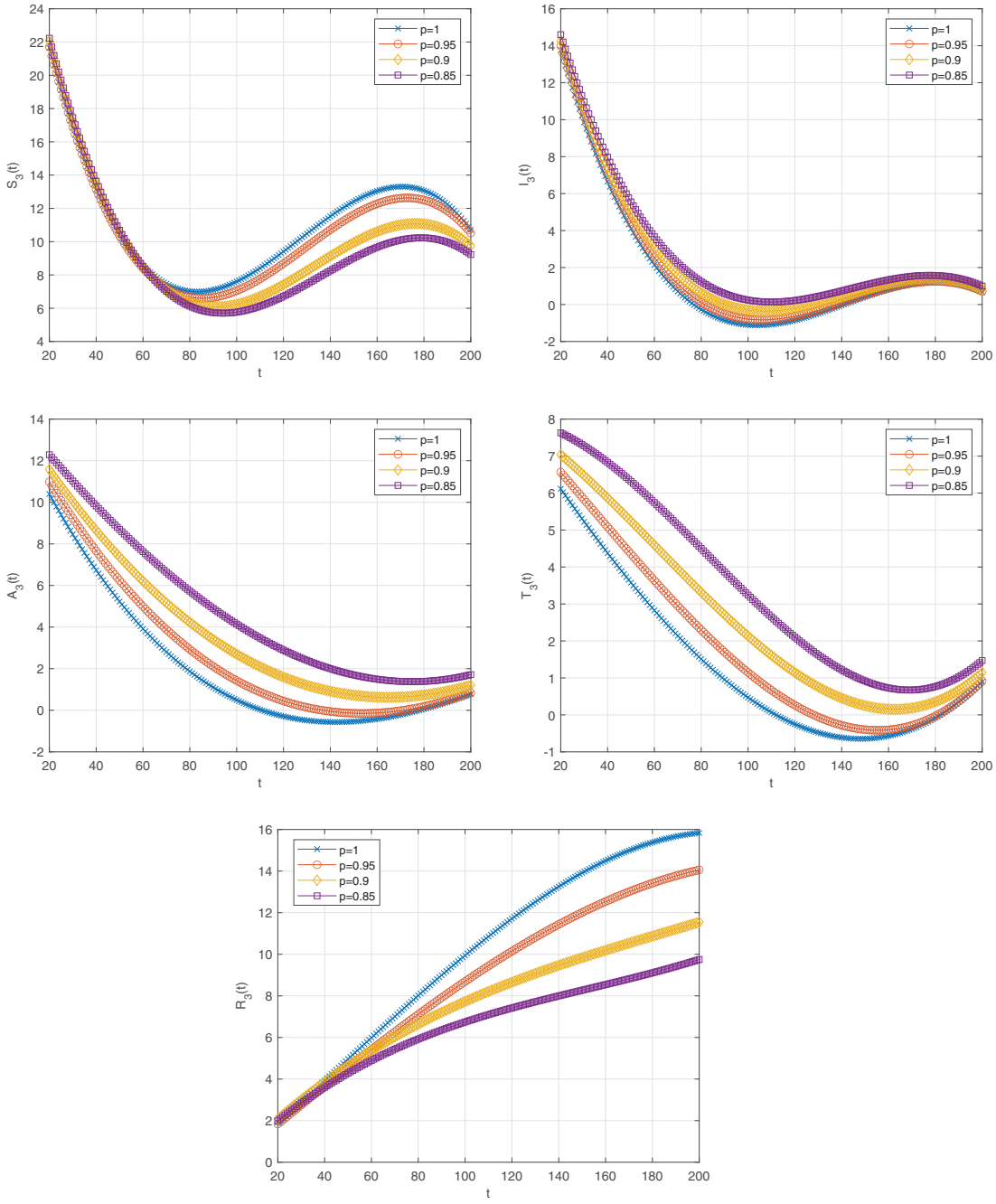

Figure 1: Graph of functions $S_3(t)$, $I_3(t)$, $A_3(t)$, $T_3(t)$, $R_3(t)$ of (26) when $p = 1$, $p = 0.95$, $p = 0.9$, $p = 0.85$

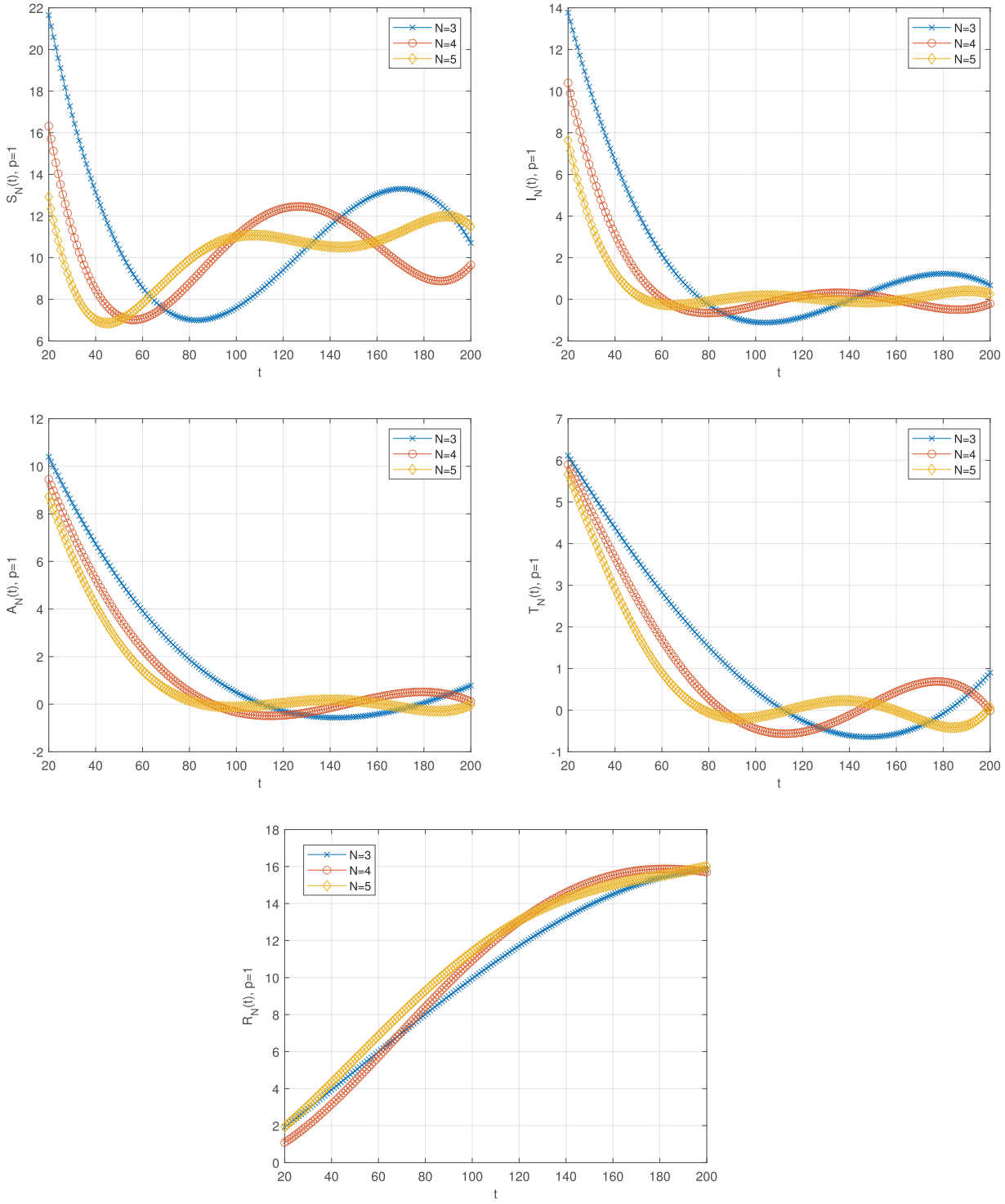

Figure 2: Graph of solution functions of (26) when $p = 1$ for $N = 3, N = 4, N = 5$

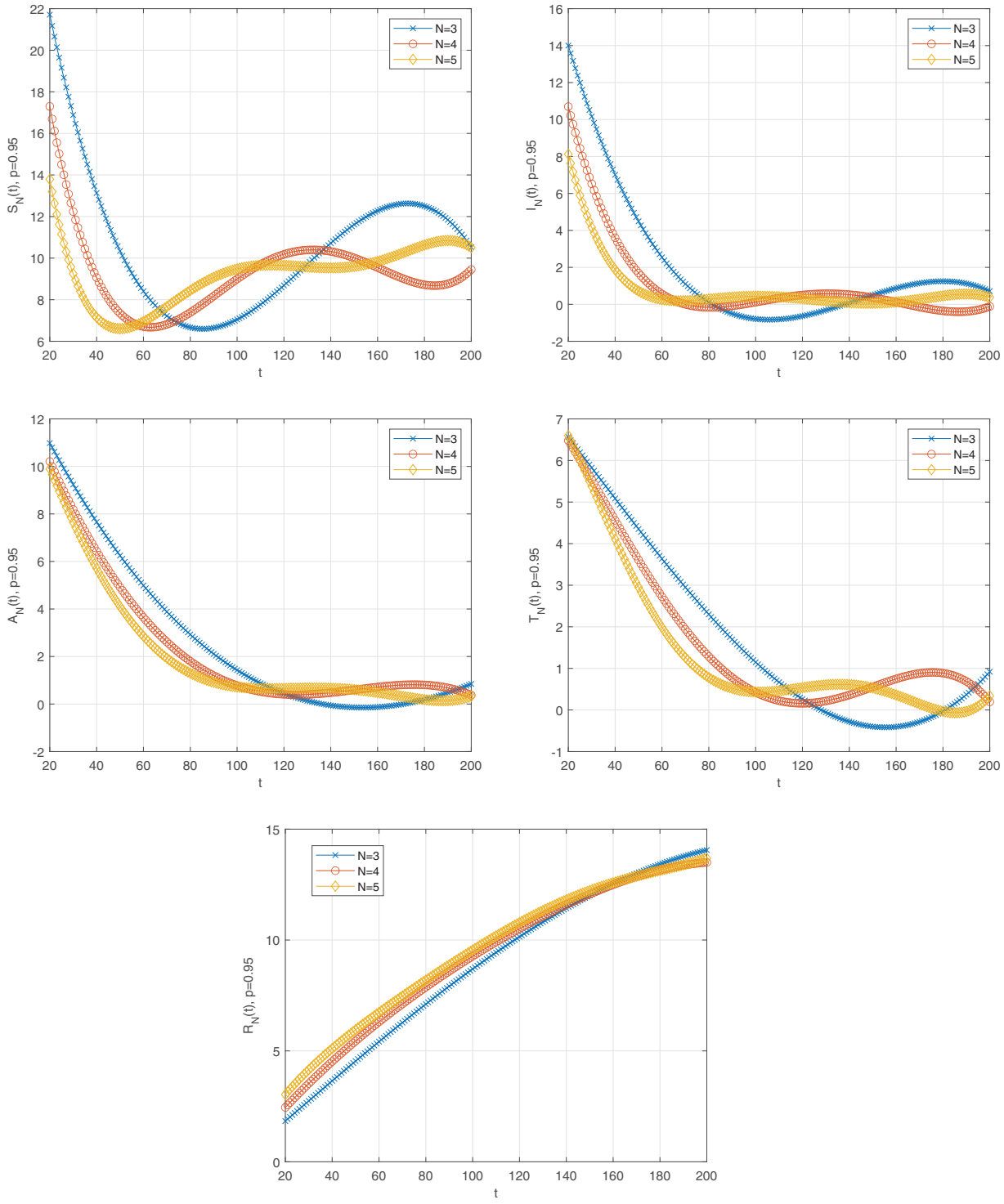

Figure 3: Graph of solution functions of (26) when $p = 0.95$ for $N = 3, N = 4, N = 5$

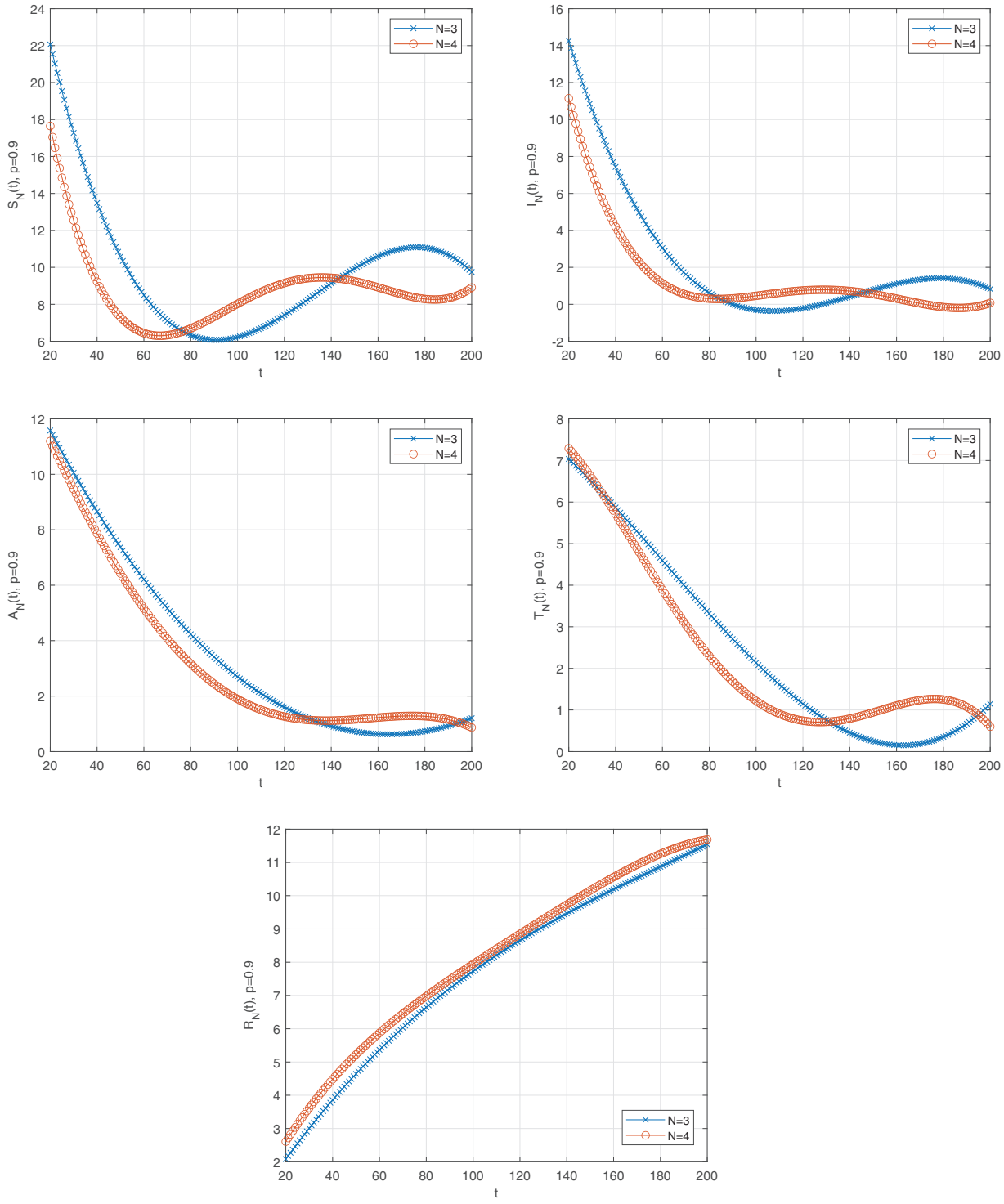

Figure 4: Graph of solution functions of (26) when $p = 0.9$ for $N = 3, N = 4$

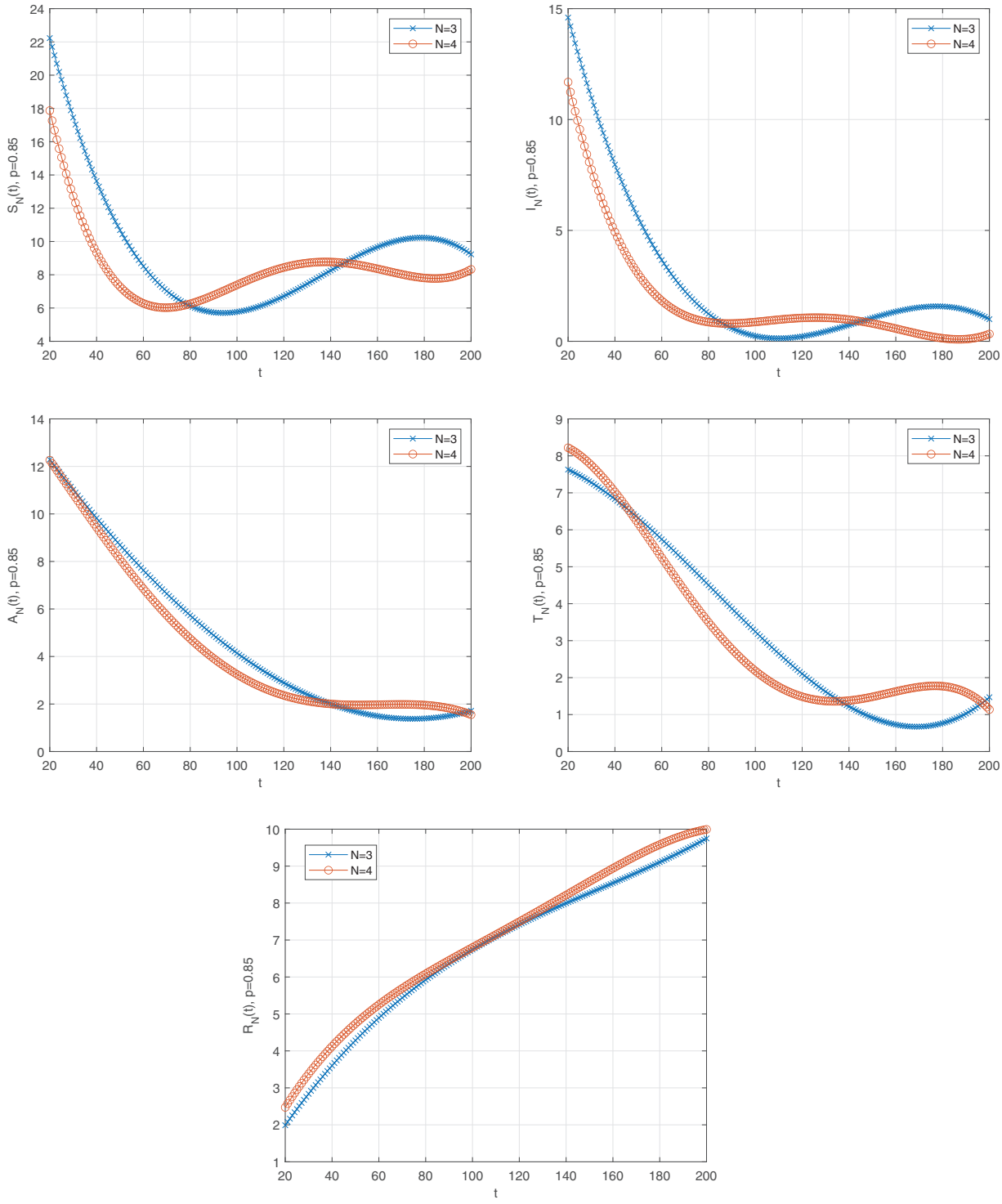

Figure 5: Graph of solution functions of (26) when $p = 0.85$ for $N = 3, N = 4$

Figs. 6–9 show the graph of the estimated error functions, respectively, for various fractional orders ($p = 1$, $p = 0.95$, $p = 0.9$ and $p = 0.85$). According to these figures, we observe that outcomes have quite decent errors. Besides, we understand that errors reduce when values of N rise. Fig. 10 visualizes the estimated error functions of (26) when $(N, M) = (4, 5)$ for $p = 1$, $p = 0.95$, $p = 0.9$, $p = 0.85$. Although very close results are obtained according to this figure, we can say that the best result with a very small difference is obtained when $p = 0.85$. Consequently, the results demonstrate influence of our method in reaching decent accuracy.

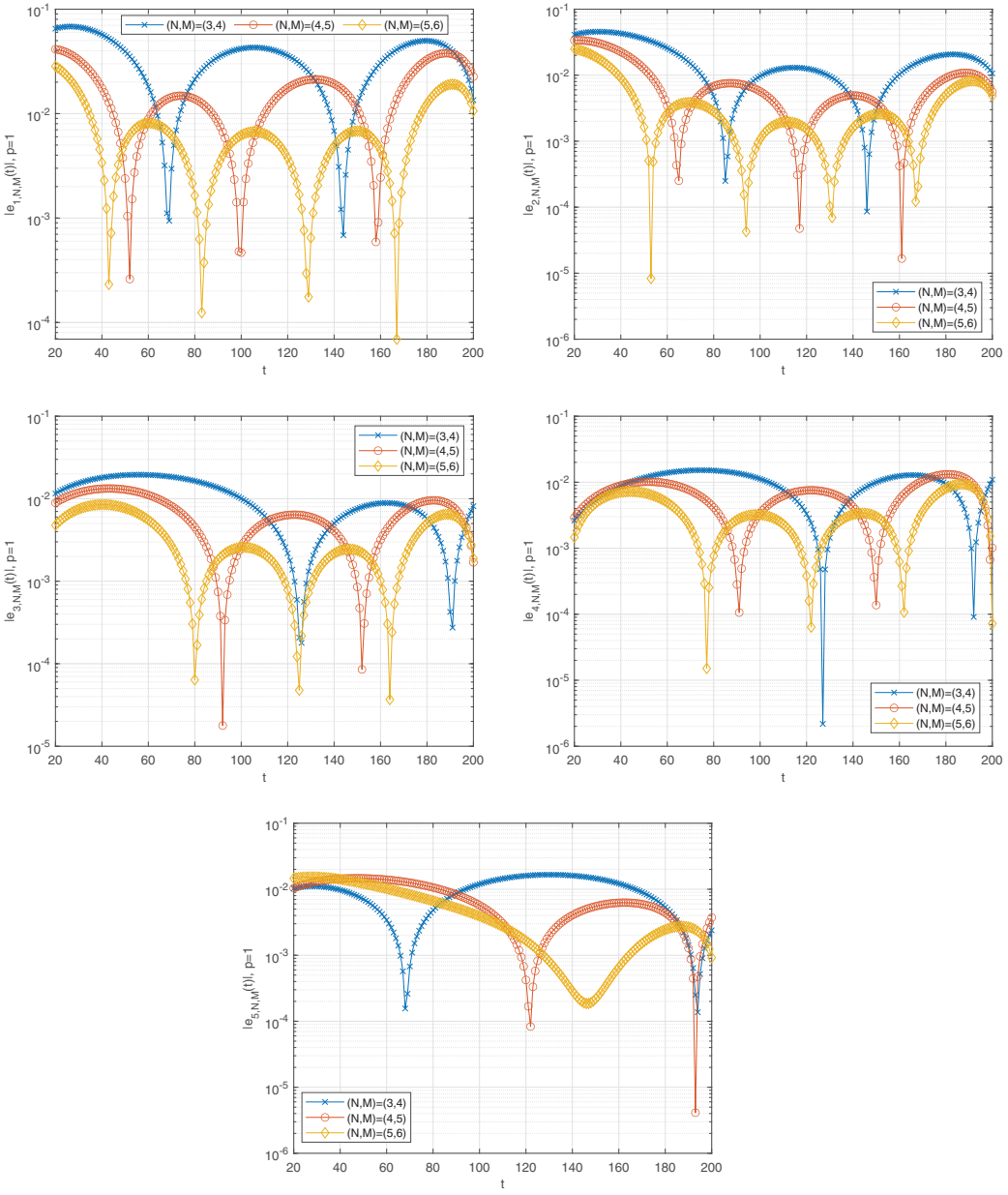

Figure 6: Graph of the estimated errors of (26) when $p = 1$ for $(N, M) = (3, 4), (N, M) = (4, 5), (N, M) = (5, 6)$

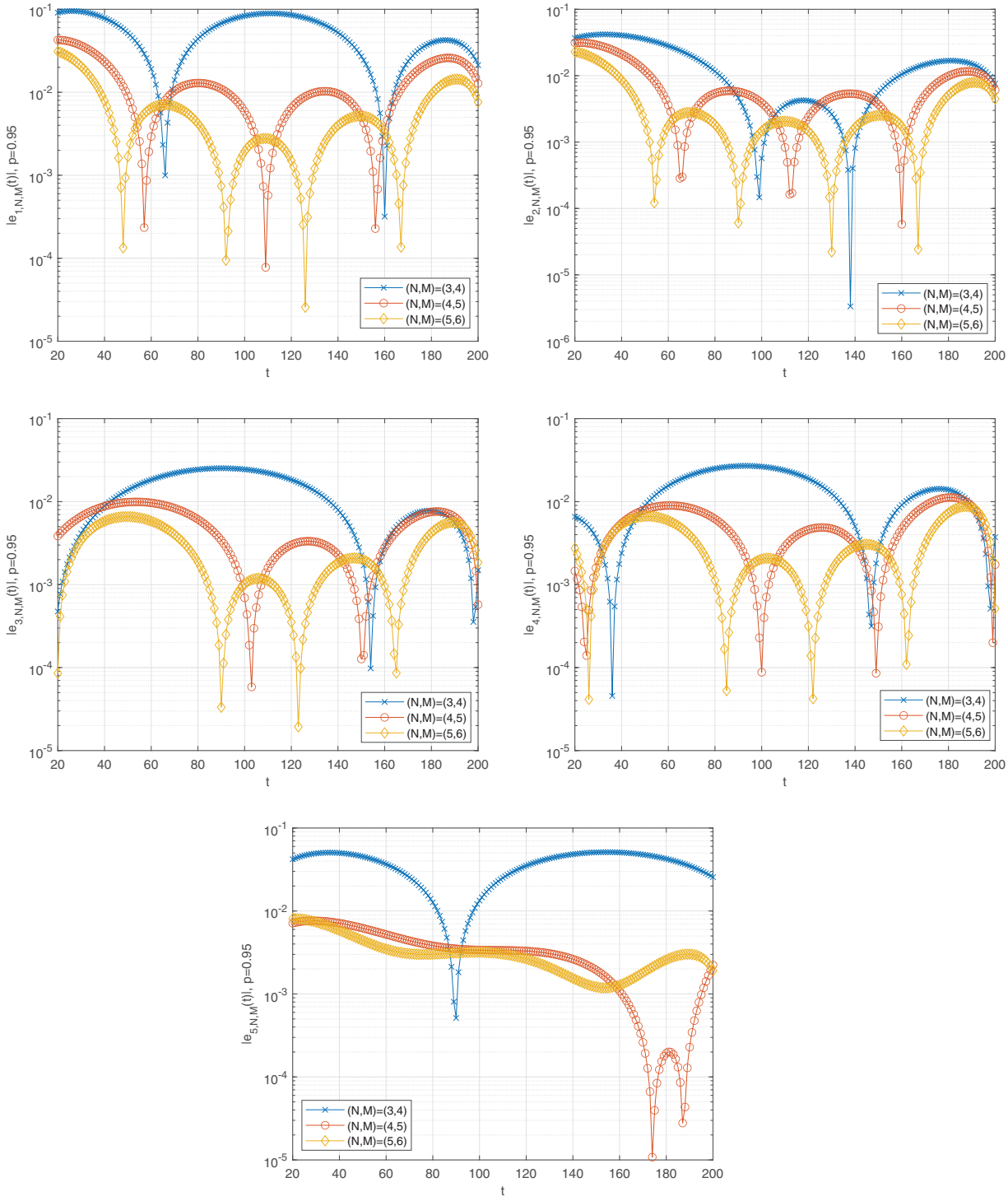

Figure 7: Graph of the estimated errors of (26) when $p = 0.95$ for $(N, M) = (3, 4), (N, M) = (4, 5), (N, M) = (5, 6)$

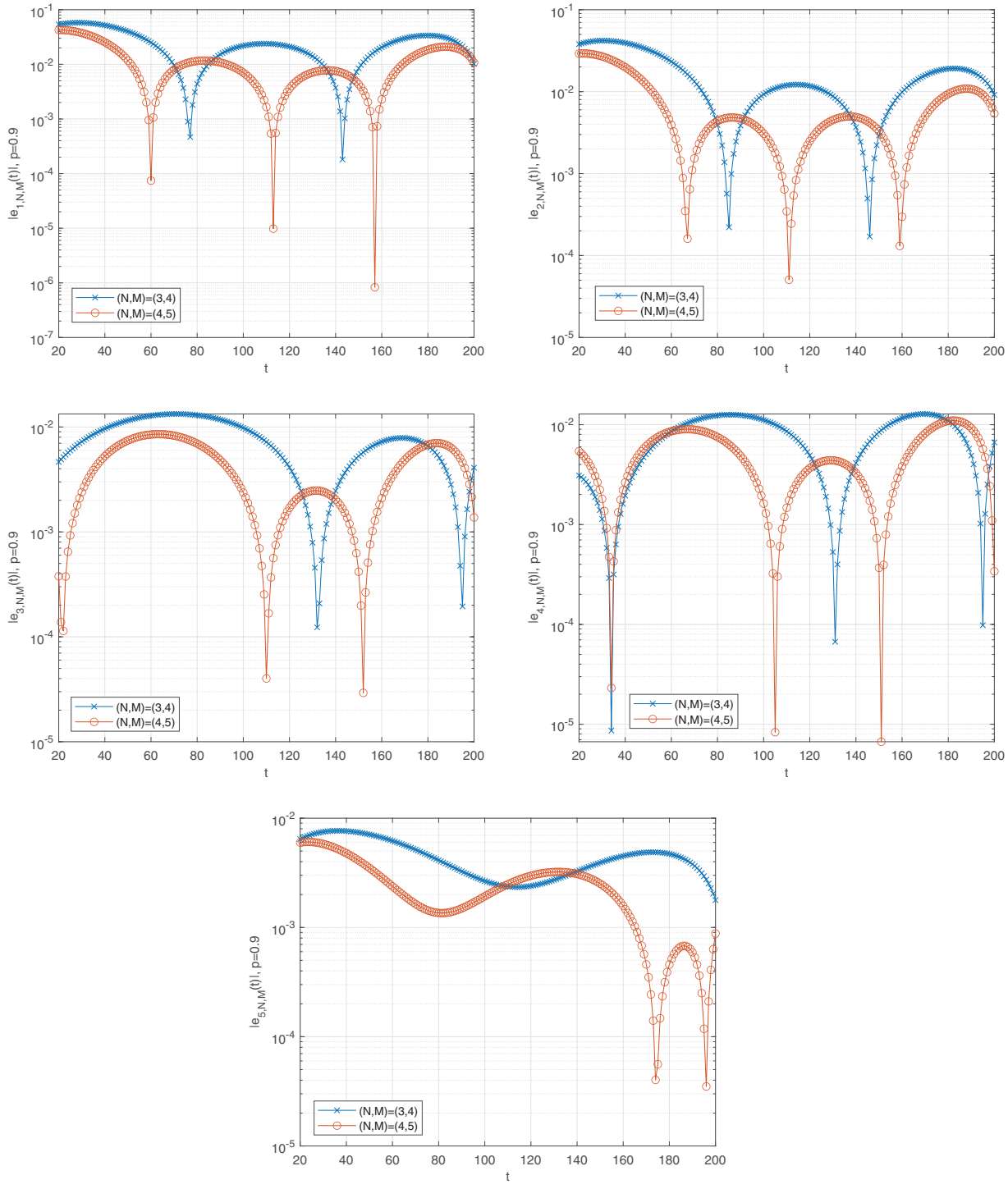

Figure 8: Graph of the estimated errors of (26) when $p = 0.9$ for $(N, M) = (3, 4), (N, M) = (4, 5)$

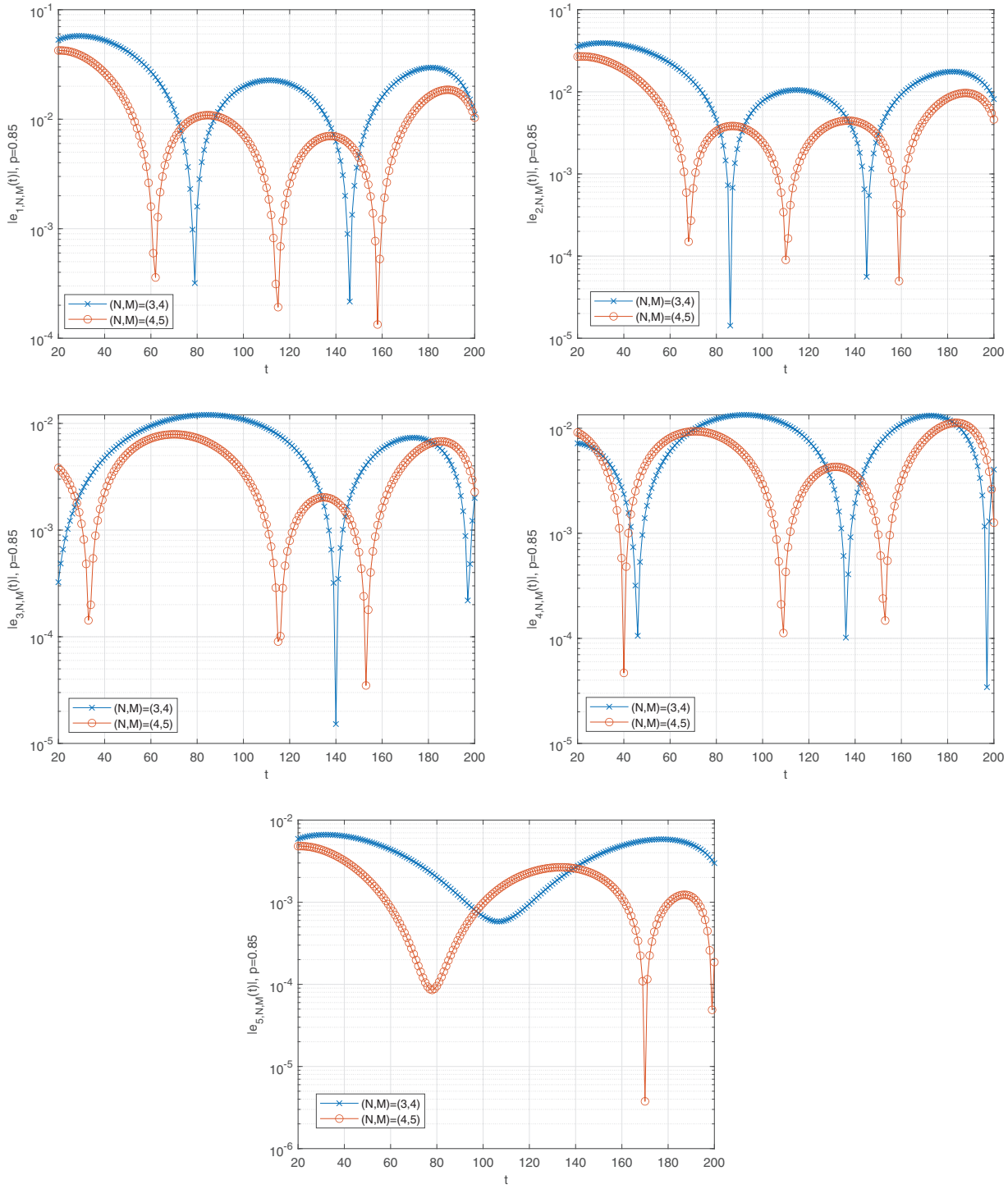

Figure 9: Graph of the estimated errors of (26) when $p = 0.85$ for $(N, M) = (3, 4), (N, M) = (4, 5)$

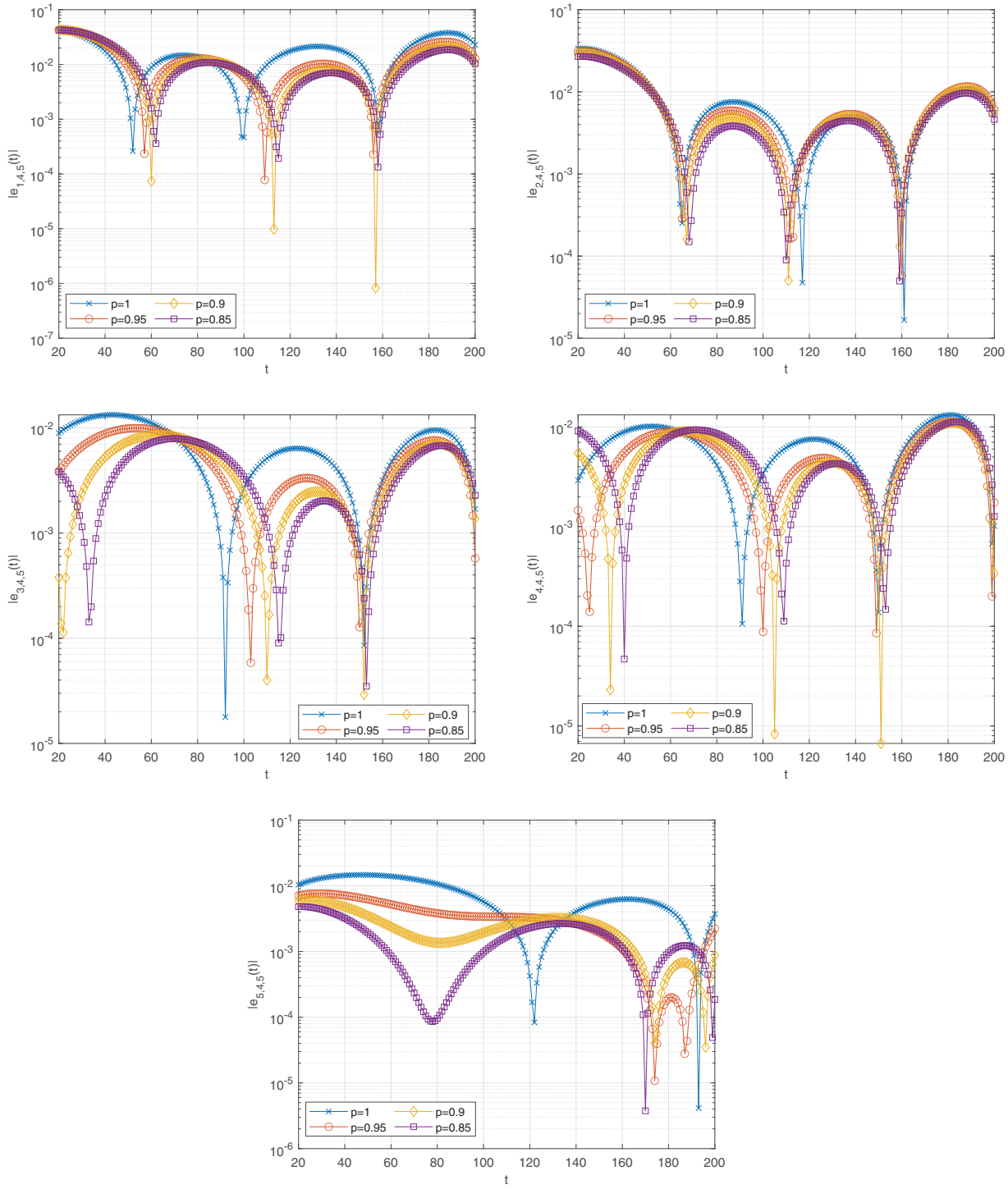

Figure 10: Graph of the estimated errors of (26) when $(N, M) = (4, 5)$ for $p = 1$, $p = 0.95$, $p = 0.9$, $p = 0.85$

Moreover, Table 3 gives the values at some t points of the estimated absolute errors of (26) when $(N, M) = (4, 5)$ for $p = 1$, $p = 0.95$, $p = 0.9$, $p = 0.85$. Accordingly, we observe that similar results are obtained for different values of p.

Table 3: The values at some t points of the estimated absolute errors of (26) when $(N, M) = (4, 5)$ for $p = 1$, $p = 0.95$, $p = 0.9$, $p = 0.85$

	$\lvert e_{1,4,5}(t)\rvert$			
t_i	$p = 1$	$p = 0.95$	$p = 0.9$	$p = 0.85$
50	2.3910e−03	8.9747e−03	1.1728e−02	1.3258e−02
100	4.6649e−04	5.8821e−03	7.0330e−03	7.2184e−03
150	1.1080e−02	4.9688e−03	4.3317e−03	4.4549e−03
200	2.2611e−02	1.2819e−02	1.0739e−02	1.0339e−02
	$\lvert e_{2,4,5}(t)\rvert$			
t_i	$p = 1$	$p = 0.95$	$p = 0.9$	$p = 0.85$
50	1.2554e−02	1.1874e−02	1.1486e−02	1.1163e−02
100	5.6864e−03	3.8995e−03	3.0726e−03	2.4172e−03
150	3.7843e−03	3.6982e−03	3.2702e−03	2.8712e−03
200	5.5861e−03	6.0612e−03	5.3999e−03	4.6093e−03
	$\lvert e_{3,4,5}(t)\rvert$			
t_i	$p = 1$	$p = 0.95$	$p = 0.9$	$p = 0.85$
50	1.2902e−02	9.8565e−03	7.4432e−03	5.2558e−03
100	2.6405e−03	6.9235e−04	2.3717e−03	3.4575e−03
150	8.5115e−04	1.2712e−04	4.1735e−04	6.2026e−04
200	1.6922e−03	5.7311e−04	1.3729e−03	2.2816e−03
	$\lvert e_{4,4,5}(t)\rvert$			
t_i	$p = 1$	$p = 0.95$	$p = 0.9$	$p = 0.85$
50	1.0120e−02	8.0145e−03	6.3914e−03	4.8049e−03
100	3.4014e−03	8.7960e−05	1.6239e−03	2.9369e−03
150	1.3768e−04	3.1038e−04	3.6835e−04	9.7087e−04
200	1.0134e−03	1.7429e−03	3.3990e−04	1.2625e−03
	$\lvert e_{5,4,5}(t)\rvert$			
t_i	$p = 1$	$p = 0.95$	$p = 0.9$	$p = 0.85$
50	1.4633e−02	6.2005e−03	3.5017e−03	1.9163e−03
100	5.7269e−03	3.3905e−03	1.9278e−03	9.7569e−04
150	5.5190e−03	1.9385e−03	2.5161e−03	2.0500e−03
200	3.7243e−03	2.2333e−03	8.7976e−04	1.8643e−04

An advantage of the method is that it is prone to computer programming language. Thus, the desired changes can be made to the method easily at any time. For example, the results can be obtained quickly by changing the p value, which represents the fractional order derivative, on the same code. Another advantage of the method is that the results can be obtained in a short time thanks to the code created in MATLAB. Even if the selected N value is very small (like 3), successful results are obtained from the method. This is an important advantage of the presented method. The biggest advantage of the method is the ease of handling nonlinear terms. The method has another important advantage. Thanks to the error estimation method presented in Section 4, information can be obtained about the error in the presented method. Because the exact solution of the model is unknown and there is no study in the literature that clearly gives the results of the application of this model. For this reason, it can be observed from the estimated error functions that the results of the method are effective. However, the method also has some disadvantages, such as making mistakes when entering data into the code. Another disadvantage of the method is that if the N value is chosen large in the method, the complexity of the operations in the MATLAB algorithm increases and the large size of the matrices is the reason for calculation errors. However, since small N values such as $3, 4$ and 5 are chosen in applications, such a situation is not encountered. Since there are many parameters and variables in the presented model, it is necessary to be very careful when creating code in MATLAB. This can be given as an example of the difficulty of the method. As a result of this article, we observe that ARV treatment is a good treatment method for people with HIV infection and full-blown AIDS. Since the discussed model can be developed for different epidemic models, both existing and that may emerge in the future, this study makes a great contribution to the literature. As a result of the study, it contributes to the quality of life of people, as the behavior of the disease can be understood and a solution can be found by applying treatment. On the other hand, thanks to the results obtained from this article, this study can guide many scientists who continue their research today, and thus the method becomes usable and applicable, making a great contribution to both science and humanity.

6 Conclusions

In this article, the PLCM is presented to solve FHEMTC. This method is based on Pell-Lucas functions and the collocation method. The fractional derivative is defined in the Caputo type, and the fractional differentiation matrices are derived for PLPs. An error analysis is investigated for the presented method. The error estimation technique is constituted by using residual function. This technique is important because it is possible to comment on the error when an exact solution for the system is not known. In addition, to show the method's accuracy and efficiency, we analyze four cases of the fractional order derivative within the range $[0, 200]$. Owing to applications, we figure out that the outcomes have quite decent errors. Also, there are no patients needing treatment since HIV-infectious and full-blown AIDS people eventually vanish from the population. Additionally, we understand that the errors decrease when values of N increase. The results demonstrate the influence of our method in reaching decent righteousness. The results can be quickly obtained by adjusting the p-value, which represents the fractional order derivative, within the same code. The reason for this efficiency is the optimized code developed for MATLAB. Even if the selected N value is very small (like 3), successful results are obtained from the method. These two situations are the important advantages of the presented method. The main advantage of the collocation method compared to other methods is that the structure of the method is simple and the computational cost is low. It also provides an easier and simpler procedure for solving various problems involving differential equations that model real-world phenomena. According to the results, we deduce that the application of this method can be very straightforward for similar epidemic models.

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Ethics Approval: Not applicable.

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