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SHORT COMMUNICATION

New Definitions of the Isometric Latitude and the Mercator Projection

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ABSTRACT

The short communication discusses the interrelationships of loxodromes, isometric latitudes and the normal aspect of Mercator projection. It is shown that by applying the isometric latitude, a very simple equation of the loxodrome on the sphere is reached. The consequence of this is that the isometric latitude can be defined using the generalized longitude, and not only using the latitude, as was common until now. Generalized longitude is the longitude defined for every real number; modulo 2π of generalized and usual longitude are congruent. Since the image of the loxodrome in the plane of the Mercator projection is a straight line, the isometric latitude can also be defined using this projection. Finally, a new definition of the Mercator projection is given, according to which it is a normal aspect cylindrical projection in which the images of the loxodromes on the sphere are straight lines in the plane of the projection that, together with the images of the meridians in the projection, form equal angles, as do the loxodromes with the meridians on the sphere. The short communication provides additional knowledge to all those who are interested in the theory of maps in navigation and have a piece of requisite mathematical knowledge, as well as an interest in map projections. It will be useful for teachers and students studying cartography and GIS, navigation or applied mathematics.

KEYWORDS

Loxodrome; isometric latitude; mercator projection

1 Introduction

We encounter the loxodrome in mathematics, cartography and seafaring. According to Britannica et al. [1], a loxodrome is a line on the globe that intersects all meridians at the same angle. A loxodrome is also the route a ship sails when it keeps the same course. The loxodrome at latitude $\varphi = 0°$, with a course of 90° or 270°, is the great circle of the Earth (equator). The loxodromes at other northern or southern latitudes, with a course of 90° or 270°, are small circles (parallels). Loxodromes with a course of 0° or 180° are great circles (meridians). In all other cases, loxodromes are spirals that twist toward the poles.

There is relatively detailed cartographic literature on loxodromes, isometric latitude and Mercator projection [2–5].

Loxodrome and Mercator projection are closely related to navigation [6–9]. The loxodrome was specially investigated by Alexander [10], Kos et al. [11,12], Elhashash [13], Petrović [14,15], Weintrit et al. [16], Babaarslan et al. [17], Kovalchuk et al. [18] and Lambrinos et al. [19]. Alexander [10] mainly deals with the historical development and connection of the loxodrome with the Mercator projection. Petrović [14] considers the loxodrome on the ellipsoid of revolution, but only gives equations without a more detailed derivation and without concrete applications.

Isometric latitude (see details in Section 3) appears in conformal mappings [20,21]. The Mercator projection is one of the most famous map projections. Even in recent times, it has been researched and written about by many, e.g., Kawase [22], Abee [23], Lapaine et al. [24], Pápay [25] and Viličić et al. [26]. Lapaine et al. [24] investigate a new variant of the Mercator projection, the web-Mercator projection. Viličić et al. [26] deal with the transverse Mercator projection and the problem of secant cylinders.

In this short communication, we start with the derivation of the loxodrome equation on the sphere in the geographic parameterization. Then, instead of geographic latitude, we introduce isometric latitude as a parameter. This shows how to arrive at a very simple equation of the loxodrome. It is a linear relationship between the isometric latitude and geographic longitude, with the fact that the longitude should be taken in a generalized sense, i.e., from the interval $(-\infty, \infty)$. This, in turn, allows us to define the isometric latitude in a new way using the loxodrome and longitude.

After that, we consider the normal aspect Mercator projection of the sphere in the usual way and using the isometric latitude. Then we derive the equation of the loxodrome image in that projection. This gives us the possibility of a new interpretation of isometric latitude using the Mercator projection. Finally, the idea to approach the Mercator projection in a new way is presented. We define it as a normal aspect cylindrical projection in which the images of loxodromes on the sphere are straight lines in the plane of the projection that make the same angles as the images of the meridians in the projection as loxodromes with the meridians on the sphere.

2 The Equation of a Loxodrome on the Sphere

Let us recall that for $R = \text{const.}$

$$
\begin{aligned}
x &= x(\varphi, \lambda) = R\cos\varphi\cos\lambda \\
y &= y(\varphi, \lambda) = R\cos\varphi\sin\lambda \\
z &= z(\varphi, \lambda) = R\sin\varphi \\
(\varphi, \lambda) &\in \Omega = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [-\pi, \pi], (x, y, z) \in \mathrm{R}^3
\end{aligned}
\tag{1}
$$

Defines a sphere with its center at the origin of the coordinate system and the radius R. Curves on the sphere for which $\varphi = \text{const.}$ we call parallels, and those for which $\lambda = \text{const.}$ meridians.

The coefficients of the first differential form of this mapping are

$$E = R^2, F = 0, G = R^2\cos^2\varphi.$$

The differential expressions for any curve on the sphere are

$$ds^2 = R^2 d\varphi^2 + R^2\cos^2\varphi d\lambda^2 \tag{2}$$

$$\cos\ \alpha\ ds = Rd\varphi \tag{3}$$

$$\sin\alpha\ ds = \pm R\cos\varphi\ d\lambda \tag{4}$$

$$\tan\alpha = \pm\cos\varphi\frac{d\lambda}{d\varphi} \tag{5}$$

where α is the angle between the observed curve and the meridian. Let us agree that the angle α will have a value from the interval $(0, 2\pi)$. It is the azimuth that will be measured clockwise so that the relationships shown in Table 1 apply.

Table 1: Basic relationships between the latitude, longitude, and azimuth

Latitude	Longitude	Azimuth
$\varphi_1 < \varphi_2$	$\lambda_1 < \lambda_2$	$\alpha \in \left(0, \frac{\pi}{2}\right)$
$\varphi_1 > \varphi_2$	$\lambda_1 < \lambda_2$	$\alpha \in \left(\frac{\pi}{2}, \pi\right)$
$\varphi_1 > \varphi_2$	$\lambda_1 > \lambda_2$	$\alpha \in \left(\pi, \frac{3\pi}{2}\right)$
$\varphi_1 < \varphi_2$	$\lambda_1 > \lambda_2$	$\alpha \in \left(\frac{3\pi}{2}, 2\pi\right)$

If we accept the relations from Table 1, and the fact that the length of the arc of the curve must be positive, then in Eqs. (4) and (5), it is sufficient to take only the positive sign.

Let it be $\alpha = \text{const}$. The differential equation of the loxodrome or rhumb line on the sphere is then, e.g., (3), and it can be solved in the following simple way:

$\cos\alpha \int ds = R \int d\varphi$,

Which after integration gives

$$s\cos\alpha = R(\varphi - \varphi_1) \tag{6}$$

and it is the equation of the loxodrome connecting the latitude φ and the arc length s. That loxodrome passes through a point with latitude φ_1 and at that point the arc length is 0.

Loxodromes on a sphere are generally spiral curves that wrap around each pole an infinite number of times (Fig. 1) and never reach it, although their length is finite. The length of the loxodrome from pole to pole is equal to the length of the arc of the meridian divided by the cosine of the angle α. Indeed, in Eq. (6) we should put $\varphi_1 = -\frac{\pi}{2}$, $s_1 = 0$, $\varphi = \frac{\pi}{2}$, so we get $s = \frac{R\pi}{\cos\alpha}, \alpha \neq \frac{\pi}{2}, \alpha \neq \frac{3\pi}{2}$.

If we start with the differential Eq. (4) we cannot integrate it immediately. First we should express φ s by means of λ or s. Therefore, we prefer to take Eq. (5) which can be integrated if we write it in the form

$$d\lambda = \tan\alpha\frac{d\varphi}{\cos\varphi}. \tag{7}$$

After integration we get

$$\lambda = \tan\alpha \ln\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) + \beta = \tan\alpha \tanh^{-1}(\sin\varphi) + \beta. \tag{8}$$

We note that according to (8) $\lambda \in (-\infty, \infty)$ for $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore, λ is a generalized geographical longitude [27].

Figure 1: Loxodrome on a sphere

If we want the loxodrome to pass through the point with geographical coordinates (φ_1, λ_1), it is necessary to take the integration constant β in the following way:

$$\beta = \lambda_1 - \tan\alpha \ln\tan\left(\frac{\pi}{4} + \frac{\varphi_1}{2}\right) = \lambda_1 - \tan\alpha \tanh^{-1}(\sin\varphi_1)\,. \tag{9}$$

Finally, if we want the relationship between λ and s, we can write

$$\lambda = \tan\alpha \ln\tan\left(\frac{\pi}{4} + \frac{s\cos\alpha}{2R} + \frac{\varphi_1}{2}\right) + \beta = \tan\alpha \tanh^{-1}\left(\sin\left(\frac{s\cos\alpha}{R} + \varphi_1\right)\right) + \beta. \tag{10}$$

If $s = 0$, then $\lambda = \lambda_1$.

3 Isometric Latitude and Loxodrome

The isometric latitude q on the sphere is defined in the theory of map projections by the geographic latitude φ and the differential equation [2].

$$dq = \frac{d\varphi}{\cos\varphi}. \tag{11}$$

The purpose of the isometric latitude is to give a parametrization, in which the Gaussian fundamental coefficients E and G are equal to each other. The solution of differential Eq. (11) is

$$q = \ln\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) = \frac{1}{2}\ln\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right) = \tanh^{-1}(\sin\varphi)\,. \tag{12}$$

with the assumption that we took for the integration constant the value that gives $q = 0$ for $\varphi = 0$.

Note that $q \in (-\infty, \infty)$ for $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The inverse relation is

$$\varphi = \sin^{-1}(\tanh q) = 2\tan^{-1}(\exp(q)) - \frac{\pi}{2}. \tag{13}$$

These relations are easily derived from the definition of isometric latitude

$$\tanh q = \sin\varphi,\ \sinh q = \tan\varphi,\ \cosh q = \frac{1}{\cos\varphi}, \tag{14}$$

$$\tanh\frac{q}{2} = \tan\frac{\varphi}{2}, \exp\text{ (q)} = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$$

Furthermore, the differential Eq. (7) written by using the isometric latitude q becomes very simple and reads

$$d\lambda = \tan\ \alpha\ dq. \tag{15}$$

After integration, we get the equation of the loxodrome on the sphere in the form

$$\lambda = q\tan\alpha + \beta \tag{16}$$

where β is the integration constant. In that equation, λ is the generalized longitude or longitude in a broader sense. The corresponding value of longitude λ' from the interval $(-\pi, \pi)$ will be obtained as a remainder when dividing by 2π, i.e., by applying the formula.

$$\lambda' = \lambda - 2\pi\,\mathrm{sgn}(\lambda)\left[\frac{|\lambda| + \pi}{2\pi}\right], \tag{17}$$

where $\mathrm{sgn}(\lambda)$ is equal to 1, 0 or –1, according to whether $\mathrm{sgn}(\lambda)$ is greater than, equal to or less than zero, while the square brackets indicate the largest integer function, i.e., $[x]$ is the largest integer that is less than x or equal to x.

If we want the loxodrome to pass through the point with coordinates (q_1, λ_1), it is necessary to take the integration constant β in the following way:

$$\beta = \lambda_1 - q_1\tan\alpha. \tag{18}$$

3.1 Special Cases

Meridians and parallels are special cases of loxodromes. For meridians, $\alpha = k\pi, k = 0, 1, 2$ and for parallels, $\alpha = \frac{\pi}{2} + k\pi, k = 0, 1$.

Indeed, if we take $\alpha = k\pi, k = 0, 1, 2$, then (6) turns into $s = R(\varphi - \varphi_1)$, and (7) into $\lambda = \lambda_1$. For $\alpha = \frac{\pi}{2} + k\pi, k = 0, 1$, (6) becomes $\varphi = \varphi_1$, and the differential equation $ds = R\cos\varphi_1 d\lambda$ gives the solution $s = R\cos\varphi_1(\lambda - \lambda_1)$.

3.2 The Equation of a Loxodrome on a Sphere Expressed Using Isometric Latitude

The equation of the loxodrome on the sphere expressed using geographic coordinates is (8). Considering the relation between geographic and isometric latitudes (12), (8) can be written in the form

$$\lambda = \tan\alpha\ q + \beta. \tag{19}$$

For constant values of α and β, Eq. (18) represents a linear relationship between longitude λ and isometric latitude q.

4 Mercator Projection of the Sphere

The Mercator projection is a conformal cylindrical projection. This means that the basic equations of the normal aspect projection are

$$x = a\lambda, y = y(\varphi), \tag{20}$$

where φ and λ are the latitude and longitude, a is a constant, and $y(\varphi)$ is a function that must be determined so that the projection is conformal. Let us assume that the sphere of radius R should be conformally mapped into the plane according to Eq. (20). The condition for this mapping to be conformal reads [2,3]

$$h = k, \tag{21}$$

where h and k are local linear scale factors along the meridian and parallel, respectively. From the expression [2,3]:

$$h = \frac{dy}{Rd\varphi}, k = \frac{a}{R\cos\varphi} \tag{22}$$

It follows according to (21):

$$\frac{dy}{Rd\varphi} = \frac{a}{R\cos\varphi}, \tag{23}$$

i.e.,

$$dy = a\frac{d\varphi}{\cos\varphi}, \tag{24}$$

and from there

$$y = a\int \frac{d\varphi}{\cos\varphi} + K = a\ln\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) + K, \tag{25}$$

where K is the constant of integration. The constants a and K can be chosen in different ways. For example, if we set the conditions $\varphi = 0$ and $x = 0$, it follows that $K = 0$. Furthermore, if we want $k = 1$ for some $\varphi = \varphi_0$, we will get $a = R\cos\varphi_0$. Finally, the normal aspect Mercator projection is given by equations.

$$x = a\lambda, y = a\ln\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right). \tag{26}$$

Let us note at the end that the equations of the Mercator projection (26) can be written in a very simple form using the isometric latitude q (12).

$$x = a\lambda, y = aq. \tag{27}$$

4.1 Loxodrome in the Normal Aspect Mercator Projection

The equation of the loxodrome on the sphere is (19). If we substitute (19) in (27), we will get the equation of the loxodrome in the Mercator projection.

$$x = a(\tan\alpha\ q + \beta), y = aq. \tag{28}$$

Eq. (28) represents the straight line equation in parametric form. The parameter here is the isometric latitude q. By eliminating that parameter, we can obtain the equation of the straight line in an explicit, implicit or any other form.

From (19) we can get

$$q = \cot\alpha\ (\lambda - \beta) \tag{29}$$

and then from (27)

$$x = a\lambda, y = a\cot\alpha\ (\lambda - \beta). \tag{30}$$

Eq. (30) again represents the straight line equation in parametric form. The parameter is now the generalized longitude λ, $\lambda \in (-\infty, \infty)$. If we need an ordinary longitude, we can get it using Eq. (17). In a similar way, we could write the equation of the loxodrome in the plane of the Mercator projection parameterized by latitude φ or arc length s.

Fig. 2 shows the loxodrome in the normal aspect Mercator projection with the assumptions $\alpha = 75^\circ = \dfrac{5\pi}{12}, \beta = 0$.

Figure 2: Loxodrome in the normal aspect Mercator projection

Although the geometric interpretations of latitude and longitude and geocentric and reduced latitude are well known, a similar interpretation of isometric latitude is not easy to find. For example, Heck [28] says in his famous monograph: "While the latitude φ can be given a geometric meaning of the direction of the normal on the surface, the numerical values of the isometric latitude cannot be clearly interpreted." In Lapaine's article, a connection between loxodrome and isometric latitude was observed, and on this basis a new, very simple definition of isometric latitude was given [29]. This is also proof that the isometric latitude, contrary to Heck's claim, can be clearly interpreted.

Now we will give a new definition of the isometric latitude q on the sphere using the Mercator projection.

Definition. The isometric latitude of any point on the sphere is proportional to the ordinate y of the image of that point in the normal aspect Mercator projection, $q = \frac{y}{a}$. The proportionality factor is $\frac{1}{a} = \frac{1}{R\cos\varphi_0}$, where R is the radius of the sphere to be mapped, and $\varphi_0 = 0$ is the latitude of the standard parallel. If $R = 1$ and $\varphi_0 = 0$, then the isometric latitude of a point on the sphere is equal to the ordinate y of the image of that point in the normal aspect Mercator projection.

5 A New Approach to Normal Aspect Mercator Projection

A common approach to deriving the equations of the normal aspect Mercator projection is to look for a cylindrical projection that satisfies the conformality condition (Section 4 in this short communication). When we have the equations of the normal aspect Mercator projection, then we derive from them the equation of the loxodrome in that projection and show that it is always a straight line. The new approach to the derivation of the equations of this projection does not start with setting the conformality condition. Instead, we set the condition that each loxodrome in the normal aspect cylindrical projection is mapped as a straight line. The equations of the normal aspect Mercator projection will emerge from this condition. Let us remind that when Mercator made his famous map, he had in mind the rectilinearity of loxodromes, not conformality.

Let us start from the equations of any normal aspect cylindrical projection (20) where φ and λ are the latitude and longitude, respectively, a is a constant, and $y(\varphi)$ is a function to be determined assuming that each loxodrome on the sphere is mapped by a normal aspect cylindrical projection as a straight line in the plane of the projection that forms an equal angle α with the positive direction of the y axis as does the loxodrome with all meridians on the sphere. The equation of the loxodrome on the sphere is given by the Eq. (16). If we substitute (16) in (20), we will get the equations of the loxodrome in the plane of the normal aspect cylindrical projection.

$$x = a(q\tan\alpha + \beta), y = y(\varphi) \tag{31}$$

For (31) to be straight line equations in parametric form with the parameter q, which form the angle α with the positive direction of the y axis, the equation for y must be of the form

$$y = aq + b \tag{32}$$

where b is a constant. Considering (12), we have

$$y = a\ln\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) + b. \tag{33}$$

Therefore, the equations of the normal aspect cylindrical projection, which has the property that every loxodrome on the sphere that forms an angle α with the meridians is mapped to a straight line in the projection plane that forms the same angle α with the images of the meridians are

$$x = a\lambda, y = a\ln\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) + b. \tag{34}$$

In addition to the usual condition in map projections that $y = 0$ for $\varphi = 0$, it follows $b = 0$, and we have

$$x = a\lambda, y = a\ln\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) \tag{35}$$

where we recognize the equations of the normal aspect Mercator projection.

6 Conclusion

It is known that instead of geographic latitude, it is convenient to introduce isometric latitude as a parameter when it comes to the issue of preserving angles [2,30–32]. We have shown that in this way, we arrive at a very simple equation of the loxodrome on the sphere. It is a linear relationship between the isometric latitude and geographic longitude, with the fact that longitude should be taken in a generalized sense, i.e., from the interval $(-\infty, \infty)$. This made it possible to define the isometric latitude using the loxodrome and longitude.

The normal aspect of the Mercator projection of the sphere can be defined in the usual way or using isometric latitude. We have shown that the introduction of the isometric latitude is very clever when deriving the equation of the loxodrome image in that projection. Furthermore, it enabled a new definition of isometric latitude using the normal aspect Mercator projection.

When Mercator made his map, he had in mind the rectilinearity of the loxodrome, not conformality. The Mercator projection is usually defined as a cylindrical conformal projection, and the novelty of this paper is that this is a consequence of the new definition. Namely, this projection can also be defined as a normal aspect cylindrical projection in which the images of the loxodromes from the sphere are straight lines in the plane of the projection that form the same angles as the images of the meridians in the projection as the loxodromes with the meridians on the sphere. Thus, the article, in a certain way, connects Mercator's original idea with today's usual approach to his projection, as a conformal cylindrical projection. In this way, we enrich the theory of map projections and expand the horizons of the user's knowledge.

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