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Dielectric Breakdown Model For An Electrically Impermeable Crack In A Piezoelectric Material

Tong-Yi Zhang[1]

Abstract: The present work presents a strip Dielectric Breakdown (DB) model for an electrically impermeable crack in a piezoelectric material. In the DB model, the dielectric breakdown region is assumed to be a strip along the crack's front line. Along the DB strip, the electric field strength is equal to the dielectric breakdown strength. The DB model is exactly in analogy with the mechanical Dugdale model. Two energy release rates emerge from the analysis. An applied energy release rate appears when evaluating *J*-integral along a contour surrounding both the dielectric breakdown strip and the crack tip, whereas a local energy release rate appears when evaluating *J*-integral along an infinitesimal contour surrounding only the crack tip. Under small yielding conditions, the local energy release rate, if used as a failure criterion, gives a linear relationship between the applied stress intensity factor and the applied electric intensity factor.

1 INTRODUCTION

Piezoelectric ceramics have become preferred materials for a wide variety of electronic and mechatronic devices due to their pronounced piezoelectric, dielectric, and pyroelectric properties. However, piezoelectric ceramics are brittle and susceptible to cracking at all scales ranging from electric domains to devices. Various defects, such as domain walls, grain boundaries, flaws and pores, impurities and inclusions, etc, exist in piezoelectric ceramics. The defects cause geometric, electric, thermal, and mechanical discontinuities and thus induce high stress and/or electric field concentrations, which may induce crack initiation, crack growth, partial discharge, and cause dielectric breakdown, fracture and failure. Due to the importance of the reliability of these devices, there has been tremendous interest in studying the fracture and failure behaviors of such materials (Beom and Atluri, 2003; dos Santos e Lucato et al. 2002; Schneider et al. 2003; Shieh et al. 2003; Landis 2003; Shindo et al. 2003).

The theoretical results (Zhang and Tong 1996; Zhang et al. 1998; Zhang et al. 2002) based on linear electroelasticity show that the electric field is extremely high at the crack tip. Especially, if the electrically impermeable boundary conditions are approximately applied to an electrically insulated crack, the theoretical predicated electric field will approach infinity at the impermeable crack tip when applied electric field has a component in the direction perpendicular to the crack line. Even when the electrically permeable boundary conditions are approximately applied to an electrically insulated crack, the theoretical predicated electric field, in terms of the electric field strength, will be about 1000 times higher in magnitude than the applied electric field for most piezoelectric materials because most piezoelectric materials have about a 1000 times higher dielectric constant in magnitude than the crack interior (air or vacuum). Under such a high local electric field, electrical non-linearity may occur near the crack tip. The strip polarization saturation (PS) model (Gao et al. 1997) was developed to explore the effects of the electrical non-linearity on piezoelectric fracture. The PS model takes the advantage that the constitutive relationship between the electric displacement and the electric field strength is similar to that between the stress and the strain. For that reason, the PS model corresponds to a mechanical Dugdale model in which the strain remains a constant value as the stress increases. McMeeking (2001) gave comprehensive and suggestive comments on the PS model. From the energy point of view, the electric displacement behaves like the strain, while the electric field strength functions like the mechanical stress. Therefore, the PS model does not correspond to the classical Dugdale model (Dugdal 1960) in which the stress is equal to the yield strength along the strip in front of a crack tip.

[1] Department of Mechanical Engineering
Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong, China

Zhang and Gao (2004) proposed a strip dielectric breakdown (DB) model, which is exactly analogous to the classical Dugdale model from the energy point of view because, in the DB model, the electric field strength on a strip adjacent to a crack tip is taken as a constant. The physical arguments for the DB model are described as follows. As discussed above, a very high electric field exists near a crack tip when the piezoelectric material is subjected to mechanical and/or electric loads. The local electric field may be much higher than the dielectric breakdown strength. The dielectric breakdown strength is defined as a critical electric field at which dielectric discharge occurs and leads to dielectric breakdown. The characteristics of the breakdown strength are similar to those of the mechanical fracture strength (Dissado and Fothergill 1992), which is very sensitive to solid defects such as flaws, voids and cracks, thereby indicating that partial discharge may occur at the crack tip due to the high electric field. As a result, a local partial discharge zone or electric breakdown zone, like a plastic deformation zone, is formed adjacent to the crack tip, and in the partial discharge zone the electric field cannot exceed the dielectric breakdown strength, E_b, which is analogous to the yield strength in mechanically plastic deformation. In analogy with the mechanical Dugdale model, the dielectric breakdown region is assumed to be a strip along the crack's front line, where the electric field strength is equal to the dielectric breakdown strength, E_b.

In developing the BD model, Zhang and Gao (2004) used the simplified electroelasticity constitutive equations and an electrically impermeable crack because the simplified electroelasticity constitutive equations and an electrically impermeable crack were used in the PS model (Gao et al. 1997). In this way, Zhang and Gao (2004) were able to compare the results from the BD model with those from the PS model. The advantage of using the simplified constitutive equations is that the formulation process and final results are simple and thus give more insights into the physical picture. For simplicity, Zhang and Gao (2004) considered a semi-infinitely long impermeable crack in their study, which means that the partial discharge zone is much shorter than the crack length and thus the small-scale yielding condition is satisfied. In the present work, we shall study an electrically impermeable crack with a finite length.

2 ANALYSIS and RESULTS

2.1 Basic equations

The simplified constitutive equation (Gao et al. 1997) reduces the number of the independent material constants to a minimum and takes the form:

$$
\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}
= M \begin{bmatrix} 1 & * & * & 0 & 0 & 0 \\ * & 1 & * & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}
\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}
- e \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}, \tag{1a}
$$

$$
\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix}
= e \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix}
\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}
+ \kappa \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}, \tag{1b}
$$

when the poling direction is along the x_3-axis, where * means that the corresponding constant will not appear in the model, σ_{ij}, ε_{ij}, D_j and E_i are the stress, the strain, the electric displacement and the electric field, respectively. Only three positive independent material constants, M, e and κ, are used to represent, on a qualitative basis, the elastic, piezoelectric and dielectric properties of the material. The static equilibrium and kinematic equations are given by

$$\sigma_{ij,j} = 0, \quad D_{i,i} = 0, \tag{2}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad E_i = -\varphi_{,i}, \tag{3}$$

respectively, where u_i and φ denote the displacement and the electric potential, respectively.

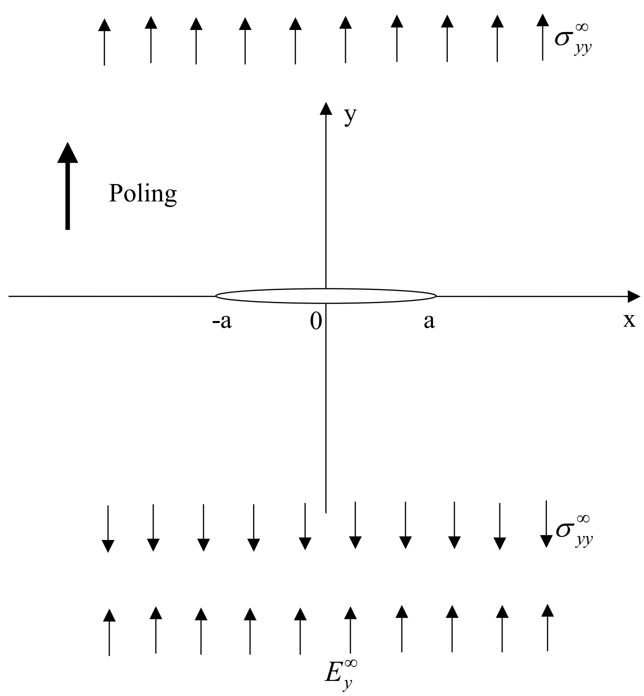

Figure 1 : An electrically impermeable crack of a finite length under remotely electrical and mechanical loading.

Consider an electrically impermeable crack with a finite length perpendicular to the poling direction in a transversely isotropic piezoelectric material. The coordinate system is set up such that the crack center is at the origin and the crack lies on the x-axis from $-a$ to a, as shown in Fig. 1. In the sample approach (Gao et al. 1997), the material is constrained to move only in the y-direction such that

$$u_x = 0, \quad u_y = u\,(x,y), \tag{4a}$$

$$\varphi = \varphi(x,y)\,. \tag{4b}$$

Rearranging the constitutive equations by substituting Eq. (4) into Eq. (3) and then using Eq. (1), we have

$$\begin{aligned} \sigma_{yx} &= Mu_{,x} + e\varphi_{,x}, \\ \sigma_{yy} &= Mu_{,y} + e\varphi_{,y}, \\ D_x &= eu_{,x} - \kappa\varphi_{,x}, \\ D_y &= eu_{,y} - \kappa\varphi_{,y}. \end{aligned} \tag{5}$$

If u and φ are expressed as imaginary parts of two analytic functions, i.e.,

$$u = Im[U(z)], \tag{6a}$$

$$\varphi = Im[\Phi(z)], \tag{6b}$$

the static equilibrium conditions of Eq. (2) are satisfied automatically, where $z = x + iy$. In this case, the kinematic and constitutive equations are given by

$$\begin{aligned} \varepsilon_{yy} + i2\varepsilon_{yx} &= U'(z), \\ E_y + iE_x &= -\Phi'(z), \end{aligned} \tag{7}$$

$$\begin{aligned} \sigma_{yy} + i\sigma_{yx} &= M\,U'(z) + e\Phi'(z), \\ D_y + iD_x &= eU'(z) - \kappa\Phi'(z). \end{aligned} \tag{8}$$

The formulation is formally identical to that for an analysis of a mode III crack (Pak 1990; Zhang and Tong 1996; Zhang et al. 2002), which allows us to modify the solution of a mode III crack for the present study.

For an electrically impermeable crack, the boundary conditions along the crack faces are

$$\sigma_{yy} = 0 \quad \text{and} \quad D_y = 0 \quad \text{for} \quad y = 0 \quad \text{and} \quad |x| < a. \tag{9}$$

Note that the boundary condition $\sigma_{yx} = 0$ cannot be enforced here due to the constraint that material is allowed to move in the y-direction only.

2.2 *Applied energy release rate*

When the crack is loaded by remotely uniform stress, σ_{yy}^{∞}, and electric field, E_y^{∞}, the solution is obtained by modifying Eq. (3.98) in the previous work (Zhang et al. 2002) and using the impermeable boundary conditions, which is given by

$$\varepsilon_{yy} + i2\varepsilon_{yx} = \frac{\sigma_{yy}^{\infty} + e\,E_y^{\infty}}{M}\frac{z}{\sqrt{z^2 - a^2}}, \tag{10a}$$

$$E_y + iE_x = E_y^{\infty}\frac{z}{\sqrt{z^2 - a^2}}, \tag{10b}$$

$$\sigma_{yy} + i\sigma_{yx} = \sigma_{yy}^{\infty}\frac{z}{\sqrt{z^2 - a^2}}, \tag{10c}$$

$$D_y + iD_x = \left[\frac{e\sigma_{yy}^{\infty}}{M} + \left(\kappa + \frac{e^2}{M}\right) E_y^{\infty}\right] \frac{z}{\sqrt{z^2 - a^2}}. \tag{10d}$$

From Eqs. (10a) and (10b), we can calculate the crack opening, Δu_y, and the voltage, $\Delta\varphi$, cross the crack faces, respectively, which take the forms of

$$\Delta u_y = \frac{\sigma_{yy}^{\infty} + e\, E_y^{\infty}}{M} \sqrt{a^2 - x^2}, \tag{11a}$$

$$\Delta\varphi = -E_y^{\infty} \sqrt{a^2 - x^2}. \tag{11b}$$

Eq. (11a) indicates that in order to open the crack, the following inequality must be satisfied

$$\sigma_{yy}^{\infty} + e\, E_y^{\infty} > 0. \tag{12}$$

Usually, applied mechanical loading is always in tension and applied electrical field could be positive or negative. The inequality is automatically satisfied under mechanical and positively electrical loading, whereas the level of a negative electrical field should be lower than a critical value determined by the applied mechanical load, i.e.,

$$E_y^{\infty} > -\sigma_{yy}^{\infty}/e. \tag{13}$$

The crack opening condition must be satisfied when a failure criterion is proposed and applied.

The stress, strain, electric field strength, and electric displacement intensity factors at the right crack tip are defined by

$$\begin{aligned} K_\sigma &= \lim_{x\to a} \sqrt{2\pi(x-a)}\sigma_{yy}, \\ K_\varepsilon &= \lim_{x\to a} \sqrt{2\pi(x-a)}\varepsilon_{yy}, \\ K_E &= \lim_{x\to a} \sqrt{2\pi(x-a)}E_y, \\ K_D &= \lim_{x\to a} \sqrt{2\pi(x-a)}D_y. \end{aligned} \tag{14}$$

Substituting Eq. (10) into Eq. (14) yields the intensity factors produced by the applied loads

$$\begin{aligned} &K_\sigma^{(a)} = \sqrt{\pi a}\sigma_{yy}^{\infty}, \quad K_\varepsilon^{(a)} = \tfrac{1}{M}\left(K_\sigma^{(a)} + eK_E^{(a)}\right), \\ &K_E^{(a)} = \sqrt{\pi a}E_y^{\infty}, \quad K_D^{(a)} = \tfrac{e}{M}K_\sigma^{(a)} + \kappa\left(1 + \tfrac{e^2}{M\kappa}\right)K_E^{(a)}. \end{aligned} \tag{15}$$

where the superscript "(a)" denotes the applied. Equation (15) also satisfies the constitutive relationships given by Eq. (8). From the definitions of the intensity factors, we can express the mechanical and electrical fields near the crack in terms of the four intensity factors, i.e.,

$$\begin{aligned} &\sigma_{yy} = \tfrac{K_\sigma}{\sqrt{2\pi r}}\cos\tfrac{\theta}{2}, \quad \sigma_{yx} = -\tfrac{K_\sigma}{\sqrt{2\pi r}}\sin\tfrac{\theta}{2}, \\ &\varepsilon_{yy} = \tfrac{K_\varepsilon}{\sqrt{2\pi r}}\cos\tfrac{\theta}{2}, \quad 2\varepsilon_{yx} = -\tfrac{K_\varepsilon}{\sqrt{2\pi r}}\sin\tfrac{\theta}{2}, \\ &E_y = \tfrac{K_E}{\sqrt{2\pi r}}\cos\tfrac{\theta}{2}, \quad E_x = -\tfrac{K_E}{\sqrt{2\pi r}}\sin\tfrac{\theta}{2}, \\ &D_y = \tfrac{K_D}{\sqrt{2\pi r}}\cos\tfrac{\theta}{2}, \quad D_x = -\tfrac{K_D}{\sqrt{2\pi r}}\sin\tfrac{\theta}{2}, \end{aligned} \tag{16}$$

where r is the distance from the crack tip and θ is the polar angle. Using Eq. (16) and the J-integral (Cherepanov 1979; Zhang et al. 2002),

$$J = \int_\Gamma (hn_1 - \sigma_{ij}n_ju_{i,1} + D_iE_1n_i)\, d\Gamma, \tag{17}$$

where h is the electric enthalpy per unit volume defined by $h = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} - \frac{1}{2}D_iE_i$, one obtains the relationship between the energy release rate and the field intensity factors,

$$J = \frac{1}{2}(K_\sigma K_\varepsilon - K_D K_E). \tag{18a}$$

The energy release rate can apparently be divided into the mechanical energy release rate, J_M, and the electrical energy release rate, J_E, (Zhang et al. 2002) such that

$$\begin{aligned} &J = J_M + J_E, \\ &J_M = \tfrac{1}{2}K_\sigma K_\varepsilon, \\ &J_E = -\tfrac{1}{2}K_D K_E. \end{aligned} \tag{18b}$$

It should be emphasized that Eq. (18) is valid as long as Eq. (16) is valid along the path of the J-integral. In linear analysis, i.e., without considering the breakdown zone, Eq. (18) gives

$$J^{(a)} = \frac{1}{2}\left(K_\sigma^{(a)}K_\varepsilon^{(a)} - K_D^{(a)}K_E^{(a)}\right). \tag{19}$$

Inserting Eq. (15) into Eq. (19) leads to the expression of the applied energy release rate with two independent loading parameters as

$$2J^{(a)} = \frac{1}{M}\left(K_\sigma^{(a)}\right)^2 - \kappa\left(1 + \frac{e^2}{M\kappa}\right)\left(K_E^{(a)}\right)^2. \tag{20}$$

Equation (20) is consistent with the result of Eq. (3.115) in the previous work (Zhang et al. 2002) for an impermeable crack under mode III loading.

2.3 Interaction of an electric dislocation with the finite impermeable crack

An electric dislocation is studied here prior to discussion of the DB model. In general, a piezoelectric dislocation may have eight characteristics, as discussed in the previous review article (Zhang et al. 2002), which are three jumps in three mechanical displacements, yielding the Burgers vector, a jump in electric potential, three line forces per unit length along three coordinate directions, and a line charge per unit length. The electric dislocation studied here has only one characteristic, the potential jump, $\Delta\varphi_b$. The electric dislocation produces an electric field, but it does not produce any mechanical displacement. The complex potentials for such an electric dislocation located at z_d in an infinite domain without any cracks are given by

$$U = 0, \quad \Phi = b_\varphi \ln(z - z_d), \tag{21}$$

where

$$b_\varphi = \frac{\Delta\varphi_b}{2\pi}. \tag{22}$$

Then, the electric and mechanical fields produced by the electric dislocation are given by

$$\varepsilon_{yy} + i2\varepsilon_{yx} = 0, \tag{23a}$$

$$E_y + iE_x = -\frac{b_\varphi}{z - z_d}, \tag{23b}$$

$$\sigma_{yy} + i\sigma_{yx} = -e(E_y + iE_x), \tag{23c}$$

$$D_y + iD_x = \kappa(E_y + iE_x). \tag{23d}$$

Eq. (23) indicates that the electric dislocation does not produce any strain field, but it induces a stress field through the piezoelectric effect. The solution of the electric dislocation in an infinite medium will be used in the development of the DB model.

When an electric dislocation is located at z_d near the impermeable crack, as shown in Fig. 2, the boundary conditions of Eq. (9) should be satisfied. The complex potentials satisfying the boundary conditions are given in Eq. (4.39) in the previous work (Zhang et al. 2002) and take

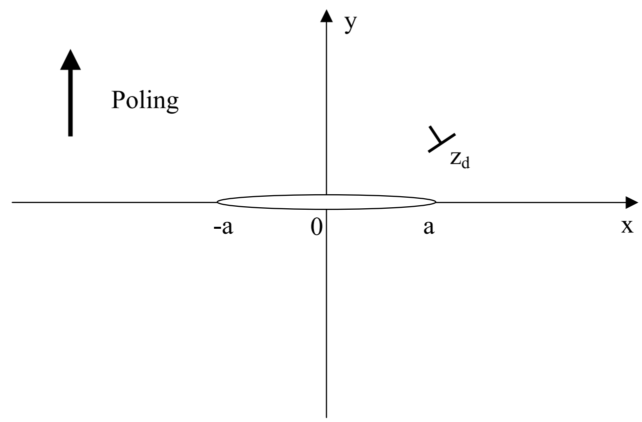

Figure 2 : An electric dislocation near a finite impermeable crack.

the following form:

$$U = 0,$$
$$\Phi = b_\varphi \ln\left(z + \sqrt{z^2 - a^2} - z_d - \sqrt{z_d^2 - a^2}\right) - b_\varphi \ln\left(\frac{a^2 - (z + \sqrt{z^2 - a^2})(\overline{z_d} + \sqrt{\overline{z_d^2} - a^2})}{z + \sqrt{z^2 - a^2}}\right), \tag{24}$$

where the overbar denotes the conjugate of a complex variable. The electric dislocation produces an electric field, which is given by

$$E_y + iE_x = -\frac{b_\varphi}{\sqrt{z^2 - a^2}}\left[\frac{z + \sqrt{z^2 - a^2}}{z + \sqrt{z^2 - a^2} - z_d - \sqrt{z_d^2 - a^2}} + \frac{a^2}{a^2 - (z + \sqrt{z^2 - a^2})(\overline{z_d} + \sqrt{\overline{z_d^2} - a^2})}\right]. \tag{25}$$

Eq. (25) can be obtained by modifying Eq. (4.40) in the previous work (Zhang et al. 2002). Again, the electric dislocation does not produce any strain field when it is near an electrically impermeable crack. The stresses and the electric displacements can be calculated through the constitutive equations, i.e., Eqs. (23c) and (23d). More important is that Eqs. (23c) and (23d) indicate that the boundary conditions along the impermeable crack faces, i.e., Eq. (9), can be specially simplified as

$$E_y = 0 \quad \text{for} \quad y = 0 \quad \text{and} \quad |x| < a. \tag{26}$$

Caution should be used that Eq. (26) is applied only to the electric dislocations, which have only one characteristic, the potential jump, $\Delta\varphi_b$.

When the electric dislocation is located on the $x-$axis, Substituting Eq. (25) into the intensity factor definition of Eq. (14) leads to

$$K_E^{(d)} = -\frac{\Delta\varphi_b}{\sqrt{\pi a}}\left(\frac{a}{x_d+\sqrt{x_d^2-a^2}-a}\right). \tag{27a}$$

where the superscript "(d)" denotes the electric dislocation. The electric dislocation does not produce any strain intensity factor. The stress intensity factor and the electric displacement intensity factor are calculated through the constitutive relationships,

$$K_\sigma^{(d)} = -eK_E^{(d)}, \tag{27b}$$

$$K_D^{(d)} = \kappa K_E^{(d)}. \tag{27c}$$

Eq. (27) is a special case of the intensity factors produced by a general dislocation at an impermeable crack tip, which is given by Eq. (4.41) in the previous work (Zhang et al. 2002).

2.4 Dielectric breakdown model

Following the successful treatment of the problem of plastic yielding at a crack tip by Dugdale (1960), Bilby et al. (1963) developed a dislocation model to formulate the strip plastic yielding. In the dislocation model (Bilby et al. 1963), the crack and the strip plastic zone are simulated by an array of dislocations. We shall adopt the dislocation approach to develop the DB model.

Figure 3 shows the proposed DB model, where ($-c$, $-a$) and (a, c) denote the dielectric breakdown strips. The remote boundary conditions for this problem are

$$\sigma_{yy}+i\sigma_{yx}=\sigma_{yy}^\infty, \quad E_y+iE_x=E_y^\infty, \quad \text{for} \quad z\to\infty. \tag{28}$$

The boundary conditions along the crack faces are the same as these given by Eq. (9). In addition, the boundary conditions along the dielectric breakdown strips are

$$u_y^+=u_y^-, \quad E_y=E_b, \quad \text{for} \quad y=0 \quad \text{and} \quad a<|x|<c. \tag{29}$$

Eqs. (7) and (8) indicate that the total solutions are obtained if we have the solutions of the strain field and the

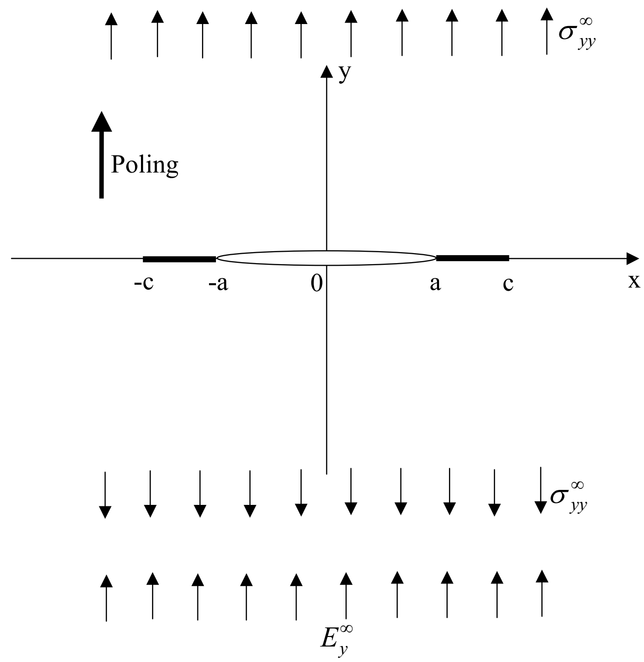

Figure 3 : The schematic of the dielectric breakdown model.

electrical field. Since the electric dislocation does not produce any strains, we can separately derive the strain field and the electrical field. The strain field is completely determined by the applied loads, which is given by Eq. (10a). The electric field will be derived from the dislocation model.

Consider a linear array of electric dislocations distributed from $-c$ to c, as shown in Fig. 4. The equilibrium equation for the array may be written as

$$-b_\varphi\int_{-c}^{c}\frac{f(x')}{x-x'}dx'+E_y^\infty=0 \quad \text{for} \quad y=0 \quad \text{and} \quad |x|<a, \tag{30a}$$

and as

$$-b_\varphi\int_{-c}^{c}\frac{f(x')}{x-x'}dx'+E_y^\infty=E_b \quad \text{for} \quad y=0 \quad \text{and} \quad a<|x|<c, \tag{30b}$$

where $f(x)$ is the distribution function of the dislocations and the Cauchy principal values of the integral are to be taken at $x=x'$ to avoid divergence. The solution to this

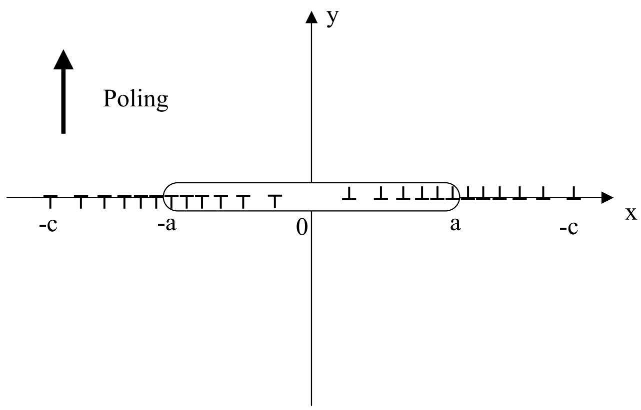

Figure 4 : The schematic distribution of an electric dislocation array.

singular integral equation may be obtained by the method developed by Muskhelishvili (1953) and is given by

$$f(x) = -\frac{2E_b}{\pi\Delta\varphi_b}\ln\left[\left|\frac{x\sqrt{c^2-a^2}+a\sqrt{c^2-x^2}}{x\sqrt{c^2-a^2}-a\sqrt{c^2-x^2}}\right|\right]. \tag{31}$$

To ensure that there is no singularity at $|x| = c$, the following equation must be satisfied

$$\frac{a}{c} = \cos\left(\frac{\pi E_y^\infty}{2E_b}\right). \tag{32}$$

From Eq. (32) we can calculate the size of the dielectric breakdown zone,

$$r_b = c - a = a\,\sec\left(\frac{\pi E_y^\infty}{2E_b}\right) - a. \tag{33}$$

Under the small yielding condition, $r_b << a$, which corresponds to $\pi E_y^\infty/(2E_b) \to 0$, Eq. (33) can be approximately reduced to

$$r_b = \frac{a}{2}\left(\frac{\pi E_y^\infty}{2E_b}\right)^2. \tag{34a}$$

Using Eq. (15), we can further express the size of the dielectric breakdown zone in terms of the electric intensity factor, $K_E^{(a)}$, under the small yielding condition as

$$r_b = \frac{\pi}{2}\left(\frac{K_E^{(a)}}{2E_b}\right)^2. \tag{34b}$$

The mechanical and electric fields are finally calculated from

$$\varepsilon_{yy} + i2\varepsilon_{yx} = \frac{\sigma_{yy}^\infty + e\,E_y^\infty}{M}\frac{z}{\sqrt{z^2-a^2}}, \tag{35a}$$

$$E_y + iE_x = -b_\varphi\int_{-c}^{c}\frac{f(x')}{z-x'}dx', \tag{35b}$$

$$\sigma_{yy} + i\sigma_{yx} = M(\varepsilon_{yy} + i2\varepsilon_{yx}) - e(E_y + iE_x), \tag{35c}$$

$$D_y + iD_x = e(\,\varepsilon_{yy} + i2\varepsilon_{yx}) + \kappa(E_y + iE_x). \tag{35d}$$

When the mechanical and electrical fields are available, we are able to calculate the local intensity factors, which turn out to be

$$K_\varepsilon^{(l)} = \sqrt{\pi a}\frac{\sigma_{yy}^\infty + eE_y^\infty}{M},$$

$$K_E^{(l)} = \lim_{x\to a}\sqrt{2\pi(x-a)}\left[-b_\varphi\int_{-c}^{c}\frac{f(x')}{x-x'}dx'\right] = 0,$$

$$K_\sigma^{(l)} = \sqrt{\pi a}\left(\sigma_{yy}^\infty + eE_y^\infty\right),\quad K_D^{(l)} = e\sqrt{\pi a}\frac{\sigma_{yy}^\infty + eE_y^\infty}{M}, \tag{36a}$$

where the superscript "(l)" denotes the local. The local intensity factors can be expressed in terms of the applied intensity factors as

$$K_\varepsilon^{(l)} = \frac{K_\sigma^{(a)} + eK_E^{(a)}}{M},\quad K_E^{(l)} = 0,$$

$$K_\sigma^{(l)} = K_\sigma^{(a)} + eK_E^{(a)},\quad K_D^{(l)} = e\frac{K_\sigma^{(a)} + eK_E^{(a)}}{M}. \tag{36b}$$

Eq. (36) indicates that the impermeable crack is completely shielded electrically such that the local electric intensity factor, $K_E^{(l)}$, nulls. However, comparing Eq. (36a) with Eq. (15) indicates that the local stress intensity factor depends not only on the remotely applied mechanical load, but also on the remotely applied electrical load, making it differ from the applied stress intensity.

The local energy release rate corresponds to the J-integral along an infinitesimal local contour enclosing just the crack tip. The local fields are dominated by the local intensity factors. Thus, Eq. (18) holds. Substituting Eq. (36) into Eq. (18) gives the local energy release rate as

$$2J_{DB}^{(l)} = \frac{1}{M}\left(K_\sigma^{(a)} + eK_E^{(a)}\right)^2, \tag{37}$$

where $J_{DB}^{(l)}$ indicates the local J-integral based on the strip dielectric breakdown model. As expected, Eq. (37) is identical to Eq. (201) in the previous work (Zhang and Gao 2004) for a semi-infinite impermeable crack.

If we re-express the local energy release rate, $J_{PS}^{(l)}$, derived from the PS model (Gao et al 1997), i.e., Eq. (39) in Gao et al. paper in terms of the applied stress intensity factor and the applied electric intensity factor, $J_{PS}^{(l)}$ takes the form of

$$2J_{PS}^{(l)} = \frac{1}{M}\left(1+\frac{e^2}{M\kappa}\right)\left(K_\sigma^{(a)} + eK_E^{(a)}\right)^2. \tag{38}$$

Comparing Eq. (37) with Eq. (38), one finds

$$\frac{J_{PS}^{(l)}}{J_{DB}^{(l)}} = 1+\frac{e^2}{M\kappa} > 1. \tag{39}$$

This means that the PS model gives a higher value of the local energy release rate than that derived from the DB model. However, Eqs. (37) and (38) both indicate that positive electric field will assist an applied mechanical stress to propagate the impermeable crack if the local J-integral is adopted as a failure criterion, while a negative electric field will retard crack propagation. Note that to ensure the crack open, Eq. (13) must be satisfied if the electrical field is negative. The relationship between the applied stress intensity factor and the applied electric intensity factor is identical in the two models. In this sense, the DB model gives the same result as the PS model.

As described above, the energy release rate can be apparently divided in the electric energy release rate and mechanical energy release rate. Due to the piezoelectric effect, an electric field contributes to the mechanical release rate and a stress field is involved in the electric energy release rate. In the DB model the electric field strength at the crack tip is completely shielded by the dielectric breakdown zone, whereas the electric displacement at the crick tip is completely shielded by the polarization saturation zone in the PS model. Either the complete shielding of the electric field strength or the complete shielding of the electric displacement leads to a zero value of the local electric energy release rate and results in the local energy release rate to be purely local mechanical energy release rate.

When the local energy release rate is purely mechanical, applying the local stress intensity factor as a failure may be equivalent to applying the local energy release rate. In the DB model, the local stress intensity factor is given by $K_{\sigma,DB}^{(l)} = K_\sigma^{(a)} + eK_E^{(a)}$, whereas the local stress intensity factor is given by $K_{\sigma,PS}^{(l)} = [1+e^2/(M\kappa)]\cdot(K_\sigma^{(a)} + eK_E^{(a)})$ in the PS model. Clearly, the ratio of $K_{\sigma,PS}^{(l)}/K_{\sigma,DB}^{(l)}$ is the same as that of $J_{PS}^{(l)}/J_{DB}^{(l)}$, as shown in Eq. (39).

3 CONCLUDING REMARKS

The present work reports the DB model for an electrically impermeable crack with a finite length. Although the approach used in the present work differs from the approach used in the previous work for a semi-infinite permeable crack (Zhang and Gao, 2004), the derived results are the same as those reported in the previous work when small yielding conditions are applied, which is expected. For the sake of explicitness, the analysis of the DB model has been restricted to the case of an impermeable crack. Theoretically, we shall be able to extend the DB model to the cases of permeable cracks and/or semi-permeable cracks. The more challenging task is to experimentally verify the DB model.

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