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A Refined Asymptotic Theory for the Nonlinear Analysis of Laminated Cylindrical Shells

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Abstract: Within the framework of the three-dimensional (3D) nonlinear elasticity, a refined asymptotic theory is developed for the nonlinear analysis of laminated circular cylindrical shells. In the present formulation, the basic equations including the nonlinear relations between the finite strains (Green strains) and displacements, the nonlinear equilibrium equations in terms of the Kirchhoff stress components and the generalized Hooke's law for a monoclinic elastic material are considered. After using proper nondimensionalization, asymptotic expansion, successive integration and then bringing the effects of transverse shear deformation into the leading-order level, we obtain recursive sets of the governing equations for various orders. It is shown that the von Karman-type first-order shear deformation theory (FSDT) is derived as a first-order approximation to the 3D nonlinear theory. The differential operators in the linear terms of governing equations for the leading order problem remain identical to those for the higher-order problems. The nonlinear terms related to the unknowns of the current order appear in a regular pattern and the other nonhomogeneous terms can be calculated by the lower-order solutions. It is also illustrated that the nonlinear analysis of laminated circular cylindrical shells can be made in a hierarchic and consistent way.

keyword: Asymptotic theory, FSDT, nonlinear analysis, cylindrical shells, 3D elasticity, perturbation

1 Introduction

The research topics on the nonlinear analysis of laminated composite shells have received considerable attention by the researchers due to the increasing use of composite materials in structures in the industrial applications. Most of the articles related to the present subject in the literature are based on the von Karman nonlinear theory [Chia (1980)]. With the assumed two-dimensional (2D) displacement models, such as the Kirchhoff's displacement model and several refined displacement models, various 2D nonlinear analyses of laminated plates and shells were presented.

By application of the dynamic virtual work principle, Xu, Xia and Chia (1996) derived a set of 2D nonlinear equations of transverse motion for the nonlinear dynamic analysis of laminated shells. Numerical results in nonlinear vibration of symmetric cross-ply laminated conical shells were presented and compared with the results available in the literature. The nonlinear vibration of unsymmetrically laminated moderately thick shallow shells was studied by Xu and Chia (1994). In their paper, Fourier-Bessel series solution was formulated for the nonlinear analysis and the effects of transverse shear deformation and rotatory inertia were considered. In conjunction with a modified variational principle of 3D nonlinear theory and a higher-order displacement model, Librescu (1987) proposed a refined geometrically nonlinear theory of anisotropic laminated shells of arbitrary shape. By using an improved von Karman's nonlinear deformation-strain relation and accounting for the effects of normal stress and strain, Tan, Tian and Du (2000) proposed a six-variable geometrical nonlinear shear deformation theory. Comprehensive reviews on the static and dynamic nonlinear analyses of laminated plates and shells have been made in the literature [Chia (1988), Sathyamoorthy (1987), Moussaoui and Benamar (2002)].

The 3D nonlinear analysis of circular cylindrical shells under transverse pressure on the lateral surfaces is an important class of structural problems. After a close literature survey, it is found that most of the articles are studied for the 3D linear analysis using the Frobenius method or other classical methods [Bhaskar and Varadan (1993), Yuan and Kim (2000)]. The literature on applying classic methods to the 3D nonlinear analysis of axisymmetric laminated cylindrical shells is scarce.

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In recent papers, Wu, Tarn and Chi (1996a, b) presented the asymptotic theories of doubly curved laminated shells. The linear classical shell theory (CST) was derived as a first-order approximation to the 3D linear theory. The 3D linear elasticity solutions for the static and dynamic problems of laminated shells were presented. To accelerate the convergent speed, Wu, Tarn and Chen (1997) and Wu, Tarn and Tang (1998) further developed a refined asymptotic theory by considering the effect of transverse shear strains at the leading order level. The linear first-order shear deformation theory (FSDT) was derived as a first-order approximation to the 3D linear theory in the refined asymptotic theory. The refined asymptotic theory was applied to the static and dynamic analyses of doubly curved laminated shells and the satisfactory results were obtained.

In the present paper we aim at developing a refined asymptotic theories for the 3D nonlinear analysis of laminated cylindrical shells. It is an extension of the earlier papers [Wu, Tarn and Chen (1997)] by considering the geometrically nonlinear effect in the asymptotic formulation. In the Lagrangian description, the nonlinear relations between finite strains and displacements, the nonlinear equilibrium equations in terms of Kirchhoff stresses, and the generalized Hooke's law for a monoclinic elastic material are regarded as the basic equations of the 3D nonlinear theory. After applying a standard perturbation approach and considering the effect of transverse shear deformations at the leading order level in advance, we obtain a series of nonlinear governing equations leading with the von Karman-type FSDT theory. The von Karman-type FSDT theory is derived as a first-order approximation to the 3D nonlinear theory. The 3D nonlinear solutions of the simply supported cross-ply laminated cylindrical strips under cylindrical bending are presented. Convergence of the present asymptotic solutions is examined. The deviations between the present 3D nonlinear solutions and the accurate 3D linear solutions available in the literature are evaluated.

2 Basic equations of nonlinear elasticity

A laminated circular cylindrical shell of uniform thickness $2h$ subject to the transverse loads $\overline{q}_z^{\pm}$ on the lateral surfaces is considered in the present formulation. A set of the cylindrical coordinates (x, θ, r) is adopted for the derivation. R and L denote the radius and length of the cylindrical shell, respectively.

The Green strains in the Lagrangian description are related to the displacements in the circular cylindrical coordinates as [Saada (1974)]

$$\varepsilon_x = e_x + \left[(u_{x,x})^2 + (u_{\theta,x})^2 + (u_{r,x})^2\right]/2 \tag{1}$$

$$\varepsilon_\theta = e_\theta + \left\{(u_{x,\theta}/r)^2 + [(u_{\theta,\theta}/r) + (u_r/r)]^2 + [(u_{r,\theta}/r) - (u_\theta/r)]^2\right\}/2 \tag{2}$$

$$\varepsilon_r = e_r + \left[(u_{x,r})^2 + (u_{\theta,r})^2 + (u_{r,r})^2\right]/2 \tag{3}$$

$$\gamma_{xr} = e_{xr} + [(u_{x,x})(u_{x,r}) + (u_{\theta,x})(u_{\theta,r}) + (u_{r,x})(u_{r,r})] \tag{4}$$

$$\gamma_{\theta r} = e_{\theta r} + [(u_{x,\theta}/r)(u_{x,r}) + (u_{\theta,\theta}/r + u_r/r)(u_{\theta,r}) + (u_{r,\theta}/r - u_\theta/r)(u_{r,r})] \tag{5}$$

$$\gamma_{x\theta} = e_{x\theta} + [(u_{x,\theta}/r)(u_{x,x}) + (u_{\theta,\theta}/r + u_r/r)(u_{\theta,x}) + (u_{r,\theta}/r - u_\theta/r)(u_{r,x})] \tag{6}$$

where the commas denote differentiation with respect to the suffix variables; u_x, u_θ and u_r are the displacement components; e_x, e_θ, e_r, e_{xr}, $e_{\theta r}$ and $e_{x\theta}$ are the infinitesimal (or linear) strain components used in the linear elasticity and expressed in terms of displacements as

$$e_x = u_{x,x}, \quad e_\theta = (u_{\theta,\theta} + u_r)/r, \quad e_r = u_{r,r},$$
$$e_{xr} = u_{x,r} + u_{r,x}, \quad e_{\theta r} = u_{\theta,r} - (u_\theta/r) + (u_{r,\theta}/r),$$
$$e_{x\theta} = (u_{x,\theta}/r) + u_{\theta,x}$$

The stress-strain relations for a monoclinic material are considered as linear and are given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_r \\ \tau_{\theta r} \\ \tau_{xr} \\ \tau_{x\theta} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \varepsilon_r \\ \gamma_{\theta r} \\ \gamma_{xr} \\ \gamma_{x\theta} \end{Bmatrix} \tag{7}$$

where $\sigma_x, \sigma_\theta, \sigma_r, \tau_{xr}, \tau_{\theta r}, \tau_{x\theta}$ are the Kirchhoff stress components.

Under the consideration of finite deformations, the stress equilibrium equations in the absence of the body forces [Washizu (1974), Novozhilov (1953)] are

$$r\tilde{\sigma}_{x,x} + \tilde{\tau}_{\theta x,\theta} + (r\tilde{\tau}_{rx})_{,r} = 0 \tag{8}$$

$$r\tilde{\tau}_{x\theta,x} + \tilde{\sigma}_{\theta,\theta} + (r\tilde{\tau}_{r\theta})_{,r} + \tilde{\tau}_{\theta r} = 0 \tag{9}$$

$$r\tilde{\tau}_{xr,x} + \tilde{\tau}_{\theta r,\theta} + (r\tilde{\sigma}_r)_{,r} - \tilde{\sigma}_\theta = 0 \tag{10}$$

where $\tilde{\sigma}_i$ $(i = x, \theta, r)$ and $\tilde{\tau}_{ij}$ $(i, j = x, \theta, r)$ are the Piola stress components.

It is noted that the Piola stress tensors are unsymmetric and Kirchhoff stress tensors are symmetric. The relations between them are in the form of [Washizu (1974), Novozhilov (1953)] are

$$\begin{bmatrix} \tilde{\sigma}_x & \tilde{\tau}_{x\theta} & \tilde{\tau}_{xr} \\ \tilde{\tau}_{\theta x} & \tilde{\sigma}_\theta & \tilde{\tau}_{\theta r} \\ \tilde{\tau}_{rx} & \tilde{\tau}_{r\theta} & \tilde{\sigma}_r \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{x\theta} & \tau_{xr} \\ \tau_{x\theta} & \sigma_\theta & \tau_{\theta r} \\ \tau_{xr} & \tau_{\theta r} & \sigma_r \end{bmatrix}$$
$$\begin{bmatrix} (1+e_x) & (e_{x\theta}/2+\omega_r) & (e_{xr}/2-\omega_\theta) \\ (e_{x\theta}/2-\omega_r) & (1+e_\theta) & (e_{\theta r}/2+\omega_x) \\ (e_{xr}/2+\omega_\theta) & (e_{\theta r}/2-\omega_x) & (1+e_r) \end{bmatrix} \tag{11}$$

where ω_i are rotations of an element about the i axis $(i = x, \theta, r)$ and are related to the displacements as

$$\omega_x = [-u_{\theta,r} + (u_{r,\theta}/r) - (u_\theta/r)]/2$$

$$\omega_\theta = (u_{x,r} - u_{r,x})/2$$

$$\omega_r = [-(u_{x,\theta}/r) + u_{\theta,x}]/2$$

In the present formulation the displacements and transverse stresses are taken as the primary field variables. After a straightforward derivation where the in-surface stresses and strains are eliminated directly using Eqs.(1)-(10), the basic nonlinear equations can be rearranged in the form of

$$u_{r,r} = -[(\tilde{c}_{13}\partial_x + \tilde{c}_{36}\partial_\theta/r) \quad (\tilde{c}_{36}\partial_x + \tilde{c}_{23}\partial_\theta/r)]$$
$$\left\{ \begin{matrix} u_x \\ u_\theta \end{matrix} \right\} - (\tilde{c}_{23}/r)u_r + (\sigma_r/c_{33}) - [\ \tilde{c}_{13} \quad \tilde{c}_{23} \quad \tilde{c}_{36}\ ]$$
$$\left\{ \begin{matrix} \left[(u_{x,x})^2 + (u_{\theta,x})^2 + (u_{r,x})^2\right]/2 \\ \left[(u_{x,\theta}/r)^2 + (u_{\theta,\theta}/r + u_r/r)^2 + (u_{r,\theta}/r - u_\theta/r)^2\right]/2 \\ (u_{x,x})(u_{x,\theta}/r) + (u_{\theta,x})(u_{\theta,\theta}/r + u_r/r) + (u_{r,x})(u_{r,\theta}/r - u_\theta/r) \end{matrix} \right\}$$
$$-\left[(u_{x,r})^2 + (u_{\theta,r})^2 + (u_{r,r})^2\right]/2 \tag{12}$$

$$\left\{ \begin{matrix} u_{x,r} \\ u_{\theta,r} \end{matrix} \right\} = \begin{bmatrix} 0 & 0 \\ 0 & (1/r) \end{bmatrix} \left\{ \begin{matrix} u_x \\ u_\theta \end{matrix} \right\} - \begin{bmatrix} \partial_x \\ (1/r)\partial_\theta \end{bmatrix} u_r$$
$$+ \begin{bmatrix} c_{55} & c_{45} \\ c_{45} & c_{44} \end{bmatrix}^{-1} \left\{ \begin{matrix} \tau_{xr} \\ \tau_{\theta r} \end{matrix} \right\}$$
$$- \begin{bmatrix} (u_{x,x})(u_{x,r}) + (u_{\theta,x})(u_{\theta,r}) + (u_{r,x})(u_{r,r}) \\ (u_{x,\theta}/r)(u_{x,r}) + (u_{\theta,\theta}/r + u_r/r)(u_{\theta,r}) + (u_{r,\theta}/r - u_\theta/r)(u_{r,r}) \end{bmatrix} \tag{13}$$

$$(1/r)\left\{ \begin{matrix} (r\tau_{xr})_{,r} \\ (1/r)\left(r^2\tau_{\theta r}\right)_{,r} \end{matrix} \right\}$$
$$= -\begin{bmatrix} \partial_x & 0 & (1/r)\partial_\theta \\ 0 & (1/r)\partial_\theta & \partial_x \end{bmatrix} \left\{ \begin{matrix} \sigma_x \\ \sigma_\theta \\ \tau_{x\theta} \end{matrix} \right\}$$
$$- \begin{bmatrix} u_{x,xx} & u_{x,\theta\theta}/r^2 & 2u_{x,x\theta}/r \\ u_{\theta,xx} & u_{\theta,\theta\theta}/r^2 & 2u_{\theta,x\theta}/r \end{bmatrix} \left\{ \begin{matrix} \sigma_x \\ \sigma_\theta \\ \tau_{x\theta} \end{matrix} \right\}$$
$$- \begin{bmatrix} u_{x,x} & u_{x,\theta}/r \\ u_{\theta,x} & u_{\theta,\theta}/r \end{bmatrix}$$

$$
\begin{aligned}
&\begin{bmatrix} \partial_x & 0 & (1/r)\partial_\theta \\ 0 & (1/r)\partial_\theta & \partial_x \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \tau_{x\theta} \end{Bmatrix} \\
&-\left(\begin{bmatrix} u_{x,x} & u_{x,\theta}/r \\ u_{\theta,x} & u_{\theta,\theta}/r \end{bmatrix} \begin{Bmatrix} \tau_{xr} \\ \tau_{\theta r} \end{Bmatrix} \right)_{,r} \\
&-\left(\begin{bmatrix} \partial_x & (1/r)\partial_\theta \end{bmatrix} \begin{bmatrix} \tau_{xr}u_{x,r} & \tau_{xr}u_{\theta,r} \\ \tau_{\theta r}u_{x,r} & \tau_{\theta r}u_{\theta,r} \end{bmatrix} \right)^{\mathrm{T}} \\
&-\left(\begin{bmatrix} \tau_{xr}/r & \tau_{\theta r}/r \end{bmatrix} \begin{bmatrix} u_{x,x} & u_{\theta,x} \\ u_{x,\theta}/r & u_{\theta,\theta}/r \end{bmatrix} \right)^{\mathrm{T}} \\
&-(1/r) \begin{bmatrix} 0 \\ \tau_{x\theta,x}u_r + 2\tau_{x\theta}u_{r,x} + \sigma_{\theta,\theta}u_r/r \\ \quad +2\sigma_\theta u_{r,\theta}/r \\ +\tau_{\theta r}u_{r,r} + \tau_{\theta r}u_r/r - \sigma_\theta u_\theta/r \end{bmatrix} \\
&-(1/r) \begin{bmatrix} u_{x,r} \\ u_{\theta,r} \end{bmatrix} \sigma_r \\
&-\left(\begin{bmatrix} u_{x,r} \\ u_{\theta,r} \end{bmatrix} \sigma_r \right)_{,r} - \begin{bmatrix} 0 \\ \frac{\tau_{\theta r}u_r}{r} \end{bmatrix}_{,r}
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&(1/r)(r\sigma_r)_{,r} \\
&= (1/r)\sigma_\theta - \begin{bmatrix} \partial_x & (1/r)\partial_\theta \end{bmatrix} \begin{Bmatrix} \tau_{xr} \\ \tau_{\theta r} \end{Bmatrix} \\
&\quad - \begin{bmatrix} u_{r,xx} & u_{r,\theta\theta}/r^2 & 2u_{r,x\theta}/r \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \tau_{x\theta} \end{Bmatrix} \\
&\quad - \begin{bmatrix} u_{r,x} & u_{r,\theta}/r \end{bmatrix} \\
&\qquad \begin{bmatrix} \partial_x & 0 & (1/r)\partial_\theta \\ 0 & (1/r)\partial_\theta & \partial_x \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \tau_{x\theta} \end{Bmatrix} \\
&\quad - \begin{bmatrix} \partial_x & (1/r)\partial_\theta \end{bmatrix} \begin{bmatrix} u_{r,r}\tau_{xr} \\ u_{r,r}\tau_{\theta r} \end{bmatrix} \\
&\quad +(1/r) \begin{bmatrix} \partial_x & (1/r)\partial_\theta \end{bmatrix} \begin{bmatrix} \tau_{x\theta}u_\theta \\ \sigma_\theta u_\theta \end{bmatrix} \\
&\quad +(1/r) \begin{bmatrix} \tau_{x\theta} & \sigma_\theta \end{bmatrix} \begin{bmatrix} u_{\theta,x} \\ u_{\theta,\theta}/r \end{bmatrix} \\
&\quad + (r\tau_{\theta r,r}u_\theta + 2r\tau_{\theta r}u_{\theta,r} + \sigma_\theta u_r)/r^2 \\
&\quad -(1/r)\left(r \begin{bmatrix} \tau_{xr} & \tau_{\theta r} \end{bmatrix} \begin{bmatrix} u_{r,x} \\ u_{r,\theta}/r \end{bmatrix} \right)_{,r} \\
&\quad -(1/r)(r\sigma_r u_{r,r})_{,r}
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&\begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \tau_{x\theta} \end{Bmatrix} \\
&= \begin{bmatrix} Q_{11}\partial_x + (Q_{16}/r)\partial_\theta & Q_{16}\partial_x + (Q_{12}/r)\partial_\theta \\ Q_{12}\partial_x + (Q_{26}/r)\partial_\theta & Q_{26}\partial_x + (Q_{22}/r)\partial_\theta \\ Q_{16}\partial_x + (Q_{66}/r)\partial_\theta & Q_{66}\partial_x + (Q_{26}/r)\partial_\theta \end{bmatrix} \\
&\quad \begin{Bmatrix} u_x \\ u_\theta \end{Bmatrix} + \begin{bmatrix} Q_{12}/r \\ Q_{22}/r \\ Q_{26}/r \end{bmatrix} u_r + \begin{bmatrix} \tilde{c}_{13} \\ \tilde{c}_{23} \\ \tilde{c}_{36} \end{bmatrix} \sigma_r \\
&\quad + \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \\
&\quad \begin{Bmatrix} \left[(u_{x,x})^2 + (u_{\theta,x})^2 + (u_{r,x})^2 \right]/2 \\ \left[(u_{x,\theta}/r)^2 + (u_{\theta,\theta}/r + u_r/r)^2 \right. \\ \left. \quad + (u_{r,\theta}/r - u_\theta/r)^2 \right]/2 \\ (u_{x,\theta}/r)\,u_{x,x} + (u_{\theta,\theta}/r + u_r/r)\,u_{\theta,x} \\ + (u_{r,\theta}/r - u_\theta/r)\,u_{r,x} \end{Bmatrix}
\end{aligned}
\tag{16}
$$

where $\partial_x = \partial/\partial_x$, $\partial_\theta = \partial/\partial\theta$ and $Q_{ij} = c_{ij} - (c_{i3}c_{j3}/c_{33})$
The associated boundary conditions on the lateral surfaces are given by

$$
\begin{aligned}
&\begin{Bmatrix} \tau_{xr} \\ \tau_{\theta r} \end{Bmatrix} + \begin{bmatrix} u_{x,x} & u_{x,\theta}/r \\ u_{\theta,x} & u_{\theta,\theta}/r + u_r/r \end{bmatrix} \begin{Bmatrix} \tau_{xr} \\ \tau_{\theta r} \end{Bmatrix} \\
&+ \begin{bmatrix} u_{x,r} \\ u_{\theta,r} \end{bmatrix} \sigma_r = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}
\end{aligned}
\tag{17a}
$$

$$\sigma_r+\left[\begin{array}{cc} u_{r,x} & u_{r,\theta}/r-u_\theta/r \end{array}\right]\left\{\begin{array}{c}\tau_{xr}\\ \tau_{\theta r}\end{array}\right\}+u_{r,r}\sigma_r=\left\{\begin{array}{ll}\overline{q}_z^+ & \text{on} \quad r=R+h\\ -\overline{q}_z^- & \text{on} \quad r=R-h\end{array}\right. \tag{17b}$$

The admissible boundary conditions on the edge boundary surfaces are given as follows:

Along the edges x=constants,

$$\sigma_x+\sigma_x u_{x,x}+\tau_{x\theta}u_{x,\theta}/r+\tau_{xr}u_{x,r}=\overline{p}_{xx} \quad \text{or} \quad u_x=\overline{u}_x \tag{18a}$$

$$\tau_{x\theta}+\sigma_x u_{\theta,x}+\tau_{x\theta}u_{\theta,\theta}/r+\tau_{x\theta}u_r/r+\tau_{xr}u_{\theta,r}=\overline{p}_{x\theta} \quad \text{or} \quad u_\theta=\overline{u}_\theta \tag{18b}$$

$$\tau_{xr}+\sigma_x u_{r,x}+\tau_{x\theta}u_{r,\theta}/r-\tau_{x\theta}u_\theta/r+\tau_{xr}u_{r,r}=\overline{p}_{xr} \quad \text{or} \quad u_r=\overline{u}_r \tag{18c}$$

Along the edges θ = constants,

$$\tau_{x\theta}+\tau_{x\theta}u_{x,x}+\sigma_\theta u_{x,\theta}/r+\tau_{\theta r}u_{x,r}=\overline{p}_{\theta x} \quad \text{or} \quad u_x=\overline{u}_x \tag{19a}$$

$$\sigma_\theta+\tau_{x\theta}u_{\theta,x}+\sigma_\theta u_{\theta,\theta}/r+\sigma_\theta u_r/r+\tau_{\theta r}u_{\theta,r}=\overline{p}_{\theta\theta} \quad \text{or} \quad u_\theta=\overline{u}_\theta \tag{19b}$$

$$\tau_{\theta r}+\tau_{x\theta}u_{r,\theta}/r+\sigma_\theta u_{r,\theta}/r-\sigma_\theta u_\theta/r+\tau_{\theta r}u_{r,r}=\overline{p}_{\theta r} \quad \text{or} \quad u_r=\overline{u}_r \tag{19c}$$

where $\overline{p}_{xx}$, $\overline{p}_{x\theta}$, $\overline{p}_{xr}$ are the traction components prescribed along the edges x=constants, and $\overline{p}_{xx}$, $\overline{p}_{x\theta}$, $\overline{p}_{xr}$ along the edges at θ=constants; $\overline{u}_x$, $\overline{u}_\theta$, $\overline{u}_r$ are the displacement components prescribed along the edges.

3 Nondimensionalization

A set of dimensionless field variables is used in the present work and defined as follows:

$$x_1=x/R\varepsilon, \quad x_2=\theta,\varepsilon \quad x_3=(r-R)/R\varepsilon^2; \tag{20a}$$

$$u_1=u_x/R\varepsilon^3, \quad u_2=u_\theta/R\varepsilon^3, \quad u_3=u_r/R\varepsilon^2; \tag{20b}$$

$$\sigma_1=\sigma_x/Q\varepsilon^2, \quad \sigma_2=\sigma_\theta/Q\varepsilon^2, \quad \tau_{12}=\tau_{x\theta}/Q\varepsilon^2; \tag{20c}$$

$$\tau_{13}=\tau_{xr}/Q\varepsilon^3, \quad \tau_{23}=\tau_{\theta r}/Q\varepsilon^3, \quad \sigma_3=\sigma_r/Q\varepsilon^4 \tag{20d}$$

where $\varepsilon^2=h/R$ is a perturbation parameter, usually much less than 1. Q stands for a reference elastic modulus.

After introducing the set of dimensionless variables given in Eq.(20) into the formulation, we rewrite the basic equations of 3D nonlinear elasticity Eqs.(12)-(16) in the dimensionless form of

$$\begin{aligned}u_{3,3}= & -\tfrac{1}{2}(u_{3,3})^2-\varepsilon^2\mathbf{L}_1\mathbf{u}-\varepsilon^2(\tilde{c}_{23}/\tilde{r})u_3\\ & -\varepsilon^2\mathbf{L}_2\left[\begin{array}{l}(u_{3,1})^2/2\\ (u_{3,2})^2/2\\ (u_{3,1})(u_{3,2})\end{array}\right]\\ & -\varepsilon^2\left[(u_{1,3})^2+(u_{2,3})^2\right]/2+\varepsilon^4(Q/c_{33})\sigma_3\\ & -\varepsilon^4\mathbf{L}_2\left\{\begin{array}{l}\left[(u_{1,1})^2+(u_{2,1})^2\right]/2\\ (u_{1,2})^2/2+(u_{2,2}+u_3)^2/2\\ -(u_{3,2})u_2+h(u_2)^2/2R\\ (u_{1,1})(u_{1,2})+(u_{2,1})\\ (u_{2,2}+u_3)-(u_{3,1})u_2\end{array}\right\}\end{aligned} \tag{21}$$

$$\begin{aligned}\mathbf{u}_{,3}= & -\mathbf{D}u_3-(\mathbf{D}u_3)u_{3,3}-\varepsilon^2\mathbf{L}_3\mathbf{u}\\ & -\varepsilon^2\mathbf{L}_4u_3+\varepsilon^2\mathbf{S}\sigma_s\\ & -\varepsilon^2\left[\begin{array}{l}\tilde{r}(u_{1,1})(u_{1,3})+\tilde{r}(u_{2,1})(u_{2,3})\\ +x_3(u_{3,1})(u_{3,3})\\ (u_{2,3})(u_{2,2}+u_3)-u_2(u_{3,3})\\ +(u_{1,2})(u_{1,3})\end{array}\right]\\ & +\varepsilon^4x_3\mathbf{S}\sigma_s\end{aligned} \tag{22}$$

$$
\begin{aligned}
\boldsymbol{\sigma}_{s,3} = & -\mathbf{L}_5\,\boldsymbol{\sigma}_m - \varepsilon^2 \begin{bmatrix} \tilde{r}u_{1,11} & u_{1,22}/\tilde{r} & 2u_{1,12} \\ \tilde{r}u_{2,11} & u_{2,22}/\tilde{r} & 2u_{2,12} \end{bmatrix} \boldsymbol{\sigma}_m \\
& - \varepsilon^2\,\mathbf{L}_6 \boldsymbol{\sigma}_s - \varepsilon^2 \begin{bmatrix} u_{1,1} & u_{1,2}/\tilde{r} \\ u_{2,1} & u_{2,2}/\tilde{r} \end{bmatrix} \mathbf{L}_5\,\boldsymbol{\sigma}_m \\
& - \varepsilon^2 \left(\begin{bmatrix} u_{1,1} & u_{1,2}/\tilde{r} \\ u_{2,1} & u_{2,2}/\tilde{r} \end{bmatrix} \boldsymbol{\sigma}_s \right)_{,3} \\
& - \varepsilon^2 \left(\mathbf{D}^{\mathrm{T}} \boldsymbol{\sigma}_s \begin{bmatrix} u_{1,3} \\ u_{2,3} \end{bmatrix}^{\mathrm{T}} \right)^{\mathrm{T}} \\
& - \varepsilon^2 \left(\begin{bmatrix} u_{1,3} \\ u_{2,3} \end{bmatrix} \sigma_3 \right)_{,3} - \varepsilon^2 \begin{bmatrix} 0 \\ (\tau_{23}\,u_3/\tilde{r})_{,3} \end{bmatrix} \\
& - \varepsilon^2 \begin{bmatrix} 0 \\ (\tau_{12,1})u_3 + 2\tau_{12}(u_{3,1}) + (\sigma_{2,2})u_3/\tilde{r} \\ +2\sigma_2(u_{3,2})/\tilde{r} + \tau_{23}(u_{3,3}) \end{bmatrix} \\
& - \varepsilon^4 (1 + x_3\partial_3) \begin{bmatrix} u_{1,1} & u_{1,2}/\tilde{r} \\ u_{2,1} & u_{2,2}/\tilde{r} \end{bmatrix} \boldsymbol{\sigma}_s \\
& - \varepsilon^4 (1 + x_3\,\partial_3) \begin{bmatrix} u_{1,3} \\ u_{2,3} \end{bmatrix} \sigma_3 \\
& - \varepsilon^4 \begin{bmatrix} x_3(\tau_{13}\,u_{1,3})_{,1} \\ x_3(\tau_{13}\,u_{2,3})_{,1} + (1 + x_3\,\partial_3)(\tau_{23}\,u_3/\tilde{r}) \\ -\sigma_2\,u_2/\tilde{r} \end{bmatrix}
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\sigma_{3,3} = & \; \sigma_2 - \mathbf{D}^{\mathrm{T}} \boldsymbol{\sigma}_s - (\mathbf{D}\,u_3)^{\mathrm{T}} (\mathbf{L}_5\,\boldsymbol{\sigma}_m) \\
& - (\mathbf{L}_7\,u_3)\boldsymbol{\sigma}_m - \mathbf{D}^{\mathrm{T}} (u_{3,3}\,\boldsymbol{\sigma}_s) \\
& - ([u_{3,1} \quad u_{3,2}/\tilde{r}]\boldsymbol{\sigma}_s)_{,3} \\
& - (u_{3,3}\,\sigma_3)_{,3} + \varepsilon^2 (u_2\,\tau_{12})_{,1} + \varepsilon^2 (u_2\,\sigma_2)_{,2}/\tilde{r} \\
& + \varepsilon^2 (u_2\,\tau_{23}/\tilde{r})_{,3} - \varepsilon^2 (1 + x_3\,\partial_3) \\
& \quad ([u_{3,1} \quad u_{3,2}/\tilde{r}]\boldsymbol{\sigma}_s + u_{3,3}\,\sigma_3 + \sigma_3) \\
& - \varepsilon^2 x_3 [(u_{3,3}\,\tau_{13})_{,1} + \tau_{13,1}] \\
& + \varepsilon^2 [u_{2,1}\,\tau_{12} + u_{2,2}\,\sigma_2/\tilde{r} + u_3\,\sigma_2/\tilde{r} \\
& \qquad + u_{2,3}\,\tau_{23} + x_3\,u_{3,2}\,\sigma_{2,2}/\tilde{r} \\
& \qquad + x_3\,u_{3,2}\,\tau_{12,1}] \\
& + \varepsilon^4 (1 + x_3\,\partial_3)(u_2\,\tau_{23}/\tilde{r})
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\boldsymbol{\sigma}_m = & \; \mathbf{L}_8\,\mathbf{u} + \mathbf{L}_9\,u_3 + \mathbf{L}_{11} \begin{bmatrix} (u_{3,1})^2/2 \\ (u_{3,2})^2/2 \\ (u_{3,1})(u_{3,2}) \end{bmatrix} \\
& + \varepsilon^2 \mathbf{L}_{10}\,\sigma_3 \\
& + \varepsilon^2\,\mathbf{L}_{11} \left\{ \begin{array}{l} \left[(u_{1,1})^2 + (u_{2,1})^2\right]/2 \\ (u_{2,2} + u_3)^2/2 - (u_{3,2})u_2 \\ +(u_2)^2(h/2R) + (u_{1,2})^2/2 \\ (u_{1,1})(u_{1,2}) \\ +(u_{2,1})(u_{2,2} + u_3) - (u_{3,1})u_2 \end{array} \right\}
\end{aligned}
\tag{25}
$$

where

$$
\mathbf{u} = \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\}, \quad \boldsymbol{\sigma}_s = \left\{ \begin{array}{c} \tau_{13} \\ \tau_{23} \end{array} \right\},
$$

$$
\boldsymbol{\sigma}_m = \left\{ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{array} \right\}, \quad \mathbf{D} = \begin{bmatrix} \partial_1 \\ \partial_2 \end{bmatrix},
$$

$$
\mathbf{S} = Q \begin{bmatrix} c_{55} & c_{45} \\ c_{45} & c_{44} \end{bmatrix}^{-1},
$$

$$
\mathbf{L}_1 = [\tilde{c}_{13}\,\partial_1 + (\tilde{c}_{36}/\tilde{r})\partial_2 \qquad \tilde{c}_{36}\,\partial_1 + (\tilde{c}_{23}/\tilde{r})\partial_2],
$$

$$
\mathbf{L}_2 = [\; \tilde{c}_{13} \quad \tilde{c}_{23}/\tilde{r}^2 \quad \tilde{c}_{36}/\tilde{r} \;],
$$

$$
\mathbf{L}_3 = \begin{bmatrix} x_3\,\partial_3 & 0 \\ 0 & (x_3\,\partial_3 - 1) \end{bmatrix}, \quad \mathbf{L}_4 = \begin{bmatrix} x_3\,\partial_1 \\ 0 \end{bmatrix},
$$

$$
\mathbf{L}_5 = \begin{bmatrix} \tilde{r}\partial_1 & 0 & \partial_2 \\ 0 & \partial_2 & \tilde{r}\,\partial_1 \end{bmatrix},
$$

$$
\mathbf{L}_6 = \begin{bmatrix} 1 + x_3\,\partial_3 & 0 \\ 0 & 2 + x_3\,\partial_3 \end{bmatrix},
$$

$$
\mathbf{L}_7 = \begin{bmatrix} \tilde{r}\,\partial_{11} \\ (1/\tilde{r})\,\partial_{22} \\ 2\partial_{12} \end{bmatrix}^{\mathrm{T}},
$$

$$
\mathbf{L}_8 = \begin{bmatrix} \tilde{Q}_{11}\partial_1 + (\tilde{Q}_{16}/\tilde{r})\partial_2 & \tilde{Q}_{16}\partial_1 + (\tilde{Q}_{12}/\tilde{r})\partial_2 \\ \tilde{Q}_{12}\partial_1 + (\tilde{Q}_{26}/\tilde{r})\partial_2 & \tilde{Q}_{26}\partial_1 + (\tilde{Q}_{22}/\tilde{r})\partial_2 \\ \tilde{Q}_{16}\partial_1 + (\tilde{Q}_{66}/\tilde{r})\partial_2 & \tilde{Q}_{66}\partial_1 + (\tilde{Q}_{26}/\tilde{r})\partial_2 \end{bmatrix},
$$

$$
\mathbf{L}_9 = \begin{bmatrix} \tilde{Q}_{12}/\tilde{r} \\ \tilde{Q}_{22}/\tilde{r} \\ \tilde{Q}_{26}/\tilde{r} \end{bmatrix}, \quad \mathbf{L}_{10} = \begin{bmatrix} \tilde{c}_{13} \\ \tilde{c}_{23} \\ \tilde{c}_{36} \end{bmatrix},
$$

$$
\mathbf{L}_{11} = \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12}/\tilde{r}^2 & \tilde{Q}_{16}/\tilde{r} \\ \tilde{Q}_{12} & \tilde{Q}_{22}/\tilde{r}^2 & \tilde{Q}_{26}/\tilde{r} \\ \tilde{Q}_{16} & \tilde{Q}_{26}/\tilde{r}^2 & \tilde{Q}_{66}/\tilde{r} \end{bmatrix},
$$

$\tilde{Q}_{ij} = Q_{ij}/Q, \quad \tilde{r} = r/R$

After introduction of Eq.(20) into (17), the associated boundary conditions on the lateral surfaces are rewritten in the dimensionless form of

$$\tau_{13} + \varepsilon^2 \left[u_{1,1}\tau_{13} + u_{1,2}\tau_{23}/\tilde{r} + u_{1,3}\sigma_3\right] = 0 \tag{26a}$$

$$\begin{aligned}\tau_{23} + \varepsilon^2 [u_{2,1}\tau_{13} + u_{2,2}\tau_{23}/\tilde{r} \\ + u_3\tau_{23}/\tilde{r} + u_{2,3}\sigma_3] = 0\end{aligned} \tag{26b}$$

$$\begin{aligned}\sigma_3 + u_{3,1}\tau_{13} + u_{3,2}\tau_{23}/\tilde{r} + u_{3,3}\sigma_3 \\ -\varepsilon^2 (u_2\tau_{23}/\tilde{r}) = \begin{cases} \overline{q}_3^+ & \text{on} \quad x_3 = 1 \\ -\overline{q}_3^- & \text{on} \quad x_3 = -1 \end{cases}\end{aligned} \tag{26c}$$

where $\overline{q}_3^\pm = \overline{q}_z^\pm / Q\varepsilon^4$.

Similarly, the dimensionless boundary conditions on the edge surfaces are obtained and are rewritten as follows:

Along the edges at x_1=constants,

$$\begin{aligned}\sigma_1 + \varepsilon^2 [\sigma_1 u_{1,1} + \tau_{12} u_{1,2}/\tilde{r} + \tau_{13} u_{1,3}] = \overline{p}_{11} \\ \text{or} \quad u_1 = \overline{u}_1\end{aligned} \tag{27a}$$

$$\begin{aligned}\tau_{12} + \varepsilon^2 [\sigma_1 u_{2,1} + \tau_{12} u_{2,2}/\tilde{r} + \tau_{12} u_3/\tilde{r} \\ + \tau_{13} u_{2,3}] = \overline{p}_{12} \quad \text{or} \quad u_2 = \overline{u}_2\end{aligned} \tag{27b}$$

$$\begin{aligned}\tau_{13} + \sigma_1 u_{3,1} + \tau_{12} u_{3,2}/\tilde{r} + \tau_{13} u_{3,3} \\ -\varepsilon^2 [\tau_{12} u_2/\tilde{r}] = \overline{p}_{13} \quad \text{or} \quad u_3 = \overline{u}_3\end{aligned} \tag{27c}$$

Along the edges at x_2=constants,

$$\begin{aligned}\tau_{12} + \varepsilon^2 [\tau_{12} u_{1,1} + \sigma_2 u_{1,2}/\tilde{r} + \tau_{23} u_{1,3}] = \overline{p}_{21} \\ \text{or} \quad u_1 = \overline{u}_1\end{aligned} \tag{28a}$$

$$\begin{aligned}\sigma_2 + \varepsilon^2 [\tau_{12} u_{2,1} + \sigma_2 u_{2,2}/\tilde{r} + \sigma_2 u_3/\tilde{r} \\ + \tau_{23} u_{2,3}] = \overline{p}_{22} \quad \text{or} \quad u_2 = \overline{u}_2\end{aligned} \tag{28b}$$

$$\begin{aligned}\tau_{23} + \tau_{12} u_{3,1} + \sigma_2 u_{3,2}/\tilde{r} + \tau_{23} u_{3,3} \\ -\varepsilon^2 [\sigma_2 u_2/\tilde{r}] = \overline{p}_{23} \quad \text{or} \quad u_3 = \overline{u}_3\end{aligned} \tag{28c}$$

where $\overline{p}_{11} = \overline{p}_{xx}/Q\varepsilon^2$, $\overline{p}_{12} = \overline{P}_{x\theta}/Q\varepsilon^2$, $\overline{p}_{13} = \overline{p}_{xr}/Q\varepsilon^3$; $\overline{p}_{21} = \overline{p}_{\theta x}/Q\varepsilon^2$, $\overline{p}_{22} = \overline{p}_{\theta\theta}/Q\varepsilon^2$, $\overline{p}_{23} = \overline{p}_{\theta r}/Q\varepsilon^3$; and $\overline{u}_1 = \overline{u}_x/R\varepsilon^3$, $\overline{u}_2 = \overline{u}_\theta/R\varepsilon^3$, $\overline{u}_3 = \overline{u}_r/R\varepsilon^2$.

4 Asymptotic expansion

Following the similar derivation process in an early paper [Wu, Tarn and Chen (1997)], we bring the effect of transverse shear deformations in Eq.(22) to the stage at the leading order and introduce two auxiliary variables (ψ_1 and ψ_2) associated with the transverse shear deformations. The transverse shear strains are therefore given as

$$\varepsilon^2 \mathbf{S}\sigma_s = \psi + \varepsilon^2 \mathbf{S}\hat{\sigma}_s \tag{29}$$

where $\psi = \{\ \psi_1(x_1, x_2) \quad \psi_2(x_1, x_2)\ \}^T$, $\hat{\sigma}_s = \{\ \hat{\tau}_{13}(x_1, x_2) \quad \hat{\tau}_{23}(x_1, x_2)\ \}^T$; ψ_1 and ψ_2 denote the shear rotations at middle surface. $\hat{\tau}_{13}$ and $\hat{\tau}_{23}$ are the differences between the actual stresses and assumed stresses.

In the formulation, the dimensionless displacements and stresses are expanded as a series of even order powers [Nayfeh (1981)] and given as

$$\begin{aligned}f(x_1, x_2, x_3, \varepsilon) = f^{(0)}(x_1, x_2, x_3) + \varepsilon^2 f^{(1)}(x_1, x_2, x_3) \\ + \varepsilon^4 f^{(2)}(x_1, x_2, x_3) + \ldots,\end{aligned} \tag{30}$$

Substituting Eq.(29) into Eq.(22), applying Eq.(30) to Eqs.(21)-(28) and then collecting coefficients of equal powers of ε, we obtain the following sets of equations at various levels.

Order ε^0 :

$$u_3^{(0)}{}_{,3} = -\frac{1}{2}\left(u_3^{(0)}{}_{,3}\right)^2 \tag{31}$$

$$\mathbf{u}^{(0)}{}_{,3} = -\mathbf{D}\,u_3^{(0)} - \left(\mathbf{D}\,u_3^{(0)}\right) u_3^{(0)}{}_{,3} + \psi_0 \tag{32}$$

$$\sigma_s^{(0)}{}_{,3} = -\,\mathbf{L}_5\,\sigma_m^{(0)} \tag{33}$$

$$\begin{aligned}\sigma_3^{(0)}{}_{,3} = \sigma_2^{(0)} - \mathbf{D}^{\mathrm{T}}\boldsymbol{\sigma}_s^{(0)} - \left(\mathbf{D}u_3^{(0)}\right)^{\mathrm{T}}\left(\mathbf{L}_5\boldsymbol{\sigma}_m^{(0)}\right) \\ - \left(\mathbf{L}_7\,u_3^{(0)}\right)\boldsymbol{\sigma}_m^{(0)} - \mathbf{D}^{\mathrm{T}}\left(u_3^{(0)}{}_{,3}\,\boldsymbol{\sigma}_s^{(0)}\right) \\ - \left(\left[u_3^{(0)}{}_{,1} \quad u_3^{(0)}{}_{,2}/\tilde{r}\right]\boldsymbol{\sigma}_s^{(0)}\right)_{,3} - \left(u_3^{(0)}{}_{,3}\,\sigma_3^{(0)}\right)_{,3}\end{aligned} \tag{34}$$

$$\boldsymbol{\sigma}_m^{(0)} = \mathbf{L}_8\,\mathbf{u}^{(0)} + \mathbf{L}_9\,u_3^{(0)} + \mathbf{L}_{11}\begin{bmatrix} \left(u_3^{(0)}{}_{,1}\right)^2/2 \\ \left(u_3^{(0)}{}_{,2}\right)^2/2 \\ \left(u_3^{(0)}{}_{,1}\right)\left(u_3^{(0)}{}_{,2}\right) \end{bmatrix} \tag{35}$$

With shear rotations as auxiliary variables, two additional equations related to the moment equilibrium across the thickness are needed together with Eqs.(33) in formulating the refined theory. The equations are obtained by multiplying Eqs.(33) by x_3, integrating over the thickness, using the integration by parts and lateral boundary conditions. These two additional equations at ε^0-order level are derived as

$$\int_{-1}^{1} \mathbf{C}_s \boldsymbol{\psi}_0 \, dx_3 - \int_{-1}^{1} x_3 \, \mathbf{L}_5 \boldsymbol{\sigma}_m^{(0)} \, dx_3 = 0 \tag{36}$$

where $\mathbf{C}_s = (R/h)\,\mathbf{S}^{-1}$.

Proceeding to the higher-order problems, we obtain the differential equations as follows.

Order ε^{2k} (k=1, 2, 3,...):

$$\begin{aligned}
u_3^{(k)},_3 =& -\frac{1}{2}\sum_{i=0}^{k}\left(u_3^{(k-i)},_3\right)\left(u_3^{(i)},_3\right) - \mathbf{L}_1\,\mathbf{u}^{(k-1)} \\
& - (\tilde{c}_{23}/\tilde{r})\,u_3^{(k-1)} - \sum_{i=0}^{k-1}\mathbf{L}_2\begin{bmatrix}\left(u_3^{(k-1-i)},_1\right)\left(u_3^{(i)},_1\right)/2\\ \left(u_3^{(k-1-i)},_2\right)\left(u_3^{(i)},_2\right)/2\\ \left(u_3^{(k-1-i)},_1\right)\left(u_3^{(i)},_2\right)\end{bmatrix} \\
& - \sum_{i=0}^{k-1}\left[\left(u_1^{(k-1-i)},_3\right)\left(u_1^{(i)},_3\right) + \left(u_2^{(k-1-i)},_3\right)\left(u_2^{(i)},_3\right)\right]/2 \\
& + (Q/c_{33})\,\sigma_3^{(k-2)} \\
& - \sum_{i=0}^{k-2}\mathbf{L}_2\left\{\begin{array}{l}\left[\left(u_1^{(k-2-i)},_1\right)\left(u_1^{(i)},_1\right) + \left(u_2^{(k-2-i)},_1\right)\left(u_2^{(i)},_1\right)\right]/2\\[2ex] \left(u_1^{(k-2-i)},_2\right)\left(u_1^{(i)},_2\right)/2 + \left(u_2^{(k-2-i)},_2 + u_3^{(k-2-i)}\right)\\ \quad\left(u_2^{(i)},_2 + u_3^{(i)}\right)/2 - (u_3^{(k-2-i)},_2)\,u_2^{(i)}\\ +h\left(u_2^{(k-2-i)}\right)u_2^{(i)}/2R\\[2ex] (u_1^{(k-2-i)},_1)(u_1^{(i)},_2)\\ +(u_2^{(k-2-i)},_1)(u_2^{(i)},_2 + u_3^{(i)})\\ -(u_3^{(k-1-i)},_1)\,u_2^{(i)}\end{array}\right\}
\end{aligned} \tag{37}$$

$$\begin{aligned}
\mathbf{u}^{(k)},_3 =& -\mathbf{D}\,u_3^{(k)} + \psi_k - \sum_{i=0}^{k}(\mathbf{D}\,u_3^{(k-i)})\,u_3^{(i)},_3 \\
& - \mathbf{L}_3\ \mathbf{u}^{(k-1)} - \mathbf{L}_4\,u_3^{(k-1)} + \mathbf{S}\ \hat{\sigma}_s^{(k-1)} + x_3\ \mathbf{S}\sigma_s^{(k-2)} \\
& - \sum_{i=0}^{k-1}\begin{bmatrix}\tilde{r}\left(u_1^{(k-1-i)},_1\right)\left(u_1^{(i)},_3\right) + \tilde{r}\left(u_2^{(k-1-i)},_1\right)\\ \left(u_2^{(i)},_3\right) + x_3\left(u_3^{(k-1-i)},_1\right)\left(u_3^{(i)},_3\right)\\[2ex] \left(u_2^{(k-1-i)},_3\right)\left(u_2^{(i)},_2 + u_3^{(i)}\right) - u_2^{(k-1-i)}\\ \left(u_3^{(i)},_3\right) + (u_1^{(k-1-i)},_2)(u_1^{(i)},_3)\end{bmatrix}
\end{aligned} \tag{38}$$

$$\begin{aligned}
\sigma_s^{(k)},_3 =& -\mathbf{L}_5\,\sigma_m^{(k)} - \mathbf{L}_6\,\sigma_s^{(k-1)} \\
& - \sum_{i=0}^{k-1}\begin{bmatrix}\tilde{r}u_1^{(k-1-i)},_{11} & u_1^{(k-1-i)},_{22}/\tilde{r} & 2u_1^{(k-1-i)},_{12}\\ \tilde{r}u_2^{(k-1-i)},_{11} & u_2^{(k-1-i)},_{22}/\tilde{r} & 2u_2^{(k-1-i)},_{12}\end{bmatrix}\sigma_m^{(i)} \\
& - \sum_{i=0}^{k-1}\begin{bmatrix}u_1^{(k-1-i)},_1 & u_1^{(k-1-i)},_2/\tilde{r}\\ u_2^{(k-1-i)},_1 & u_2^{(k-1-i)},_2/\tilde{r}\end{bmatrix}\mathbf{L}_5\sigma_m^{(i)} \\
& - \sum_{i=0}^{k-1}\left(\begin{bmatrix}u_1^{(k-1-i)},_1 & u_1^{(k-1-i)},_2/\tilde{r}\\ u_2^{(k-1-i)},_1 & u_2^{(k-1-i)},_2/\tilde{r}\end{bmatrix}\sigma_s^{(i)}\right)_{,3} \\
& - \sum_{i=0}^{k-1}\left(\mathbf{D}^{\mathrm{T}}\sigma_s^{(k-1-i)}\begin{bmatrix}u_1^{(i)},_3\\ u_2^{(i)},_3\end{bmatrix}^{\mathrm{T}}\right)^{\mathrm{T}} \\
& - \sum_{i=0}^{k-1}\left(\begin{bmatrix}u_1^{(k-1-i)},_3\\ u_2^{(k-1-i)},_3\end{bmatrix}\sigma_3^{(i)}\right)_{,3} \\
& - \sum_{i=0}^{k-1}\begin{bmatrix}0\\ (\tau_{23}^{(k-1-i)}u_3^{(i)}/\tilde{r}),_3\end{bmatrix} \\
& - \sum_{i=0}^{k-1}\begin{bmatrix}0\\[2ex] \left(\tau_{12}^{(k-1-i)},_1\right)u_3^{(i)} + 2\tau_{12}^{(k-1-i)}\left(u_3^{(i)},_1\right)\\ + \left(\sigma_2^{(k-1-i)},_2\right)u_3^{(i)}/\tilde{r}\\ +2\sigma_2^{(k-1-i)}\left(u_3^{(i)},_2\right)/\tilde{r} + \tau_{23}^{(k-1-i)}\left(u_3^{(i)},_3\right)\end{bmatrix}
\end{aligned}$$

$$-\sum_{i=0}^{k-2}(1+x_3\partial_3)\begin{bmatrix} u_1^{(k-2-i)},_1 & u_1^{(k-2-i)},_2/\tilde{r} \\ u_2^{(k-2-i)},_1 & u_2^{(k-2-i)},_2/\tilde{r}\end{bmatrix}\sigma_s^{(i)}$$
$$-\sum_{i=0}^{k-2}(1+x_3\partial_3)\begin{bmatrix} u_1^{(k-2-i)},_3 \\ u_2^{(k-2-i)},_3\end{bmatrix}\sigma_3^{(i)}$$
$$-\sum_{i=0}^{k-2}\begin{bmatrix} x_3\left(\tau_{13}^{(k-2-i)}u_1^{(i)},_3\right),_1 \\ \\ x_3\left(\tau_{13}^{(k-2-i)}u_2^{(i)},_3\right),_1 \\ +(1+x_3\partial_3)\left(\tau_{23}^{(k-2-i)}u_3^{(i)}/\tilde{r}\right) \\ -\sigma_2^{(k-2-i)}u_2^{(i)}/\tilde{r}\end{bmatrix} \tag{39}$$

$$\begin{aligned}\sigma_3^{(k)},_3 = {} & \sigma_2^{(k)} - \mathbf{D}^{\mathrm{T}}\boldsymbol{\sigma}_s^{(k)} - \sum_{i=0}^{k}(\mathbf{D}u_3^{(k-i)})^{\mathrm{T}}(\mathbf{L}_5\boldsymbol{\sigma}_m^{(i)}) \\ & -\sum_{i=0}^{k}(\mathbf{L}_7u_3^{(k-i)})\boldsymbol{\sigma}_m^{(i)} - \sum_{i=0}^{k}\mathbf{D}^{\mathrm{T}}(u_3^{(k-i)},_3\boldsymbol{\sigma}_s^{(i)}) \\ & -\sum_{i=0}^{k}\left(\begin{bmatrix} u_3^{(k-i)},_1 & u_3^{(k-i)},_2/\tilde{r}\end{bmatrix}\boldsymbol{\sigma}_s^{(i)}\right),_3 \\ & -\sum_{i=0}^{k}\left(u_3^{(k-i)},_3\sigma_3^{(i)}\right),_3 + \sum_{i=0}^{k-1}\left(u_2^{(k-1-i)}\tau_{12}^{(i)}\right),_1 \\ & +\sum_{i=0}^{k-1}\left(u_2^{(k-1-i)}\sigma_2^{(i)}\right),_2/\tilde{r} + \sum_{i=0}^{k-1}\left(u_2^{(k-1-i)}\tau_{23}^{(i)}/\tilde{r}\right),_3 \\ & -\sum_{i=0}^{k-1}\left[x_3\left(u_3^{(k-1-i)},_3\tau_{13}^{(i)}\right),_1\right] - x_3\tau_{13}^{(k-1)},_1 \\ & -\sum_{i=0}^{k-1}(1+x_3\partial_3)\left(\begin{bmatrix} u_3^{(k-1-i)},_1 & u_3^{(k-1-i)},_2/\tilde{r}\end{bmatrix}\boldsymbol{\sigma}_s^{(i)}\right. \\ & \left.+u_3^{(k-1-i)},_3\sigma_3^{(i)}\right) - (1+x_3\partial_3)\sigma_3^{(k-1)} \\ & +\sum_{i=0}^{k-1}\left[u_2^{(k-1-i)},_1\tau_{12}^{(i)} + u_2^{(k-1-i)},_2\sigma_2^{(i)}/\tilde{r}\right. \\ & \qquad \left.+u_3^{(k-1-i)}\sigma_2^{(i)}/\tilde{r} + u_2^{(k-1-i)},_3\tau_{23}^{(i)}\right] \\ & +\sum_{i=0}^{k-1}\left[x_3u_3^{(k-1-i)},_2\sigma_2^{(i)},_2/\tilde{r} + x_3u_3^{(k-1-i)},_2\tau_{12}^{(i)},_1\right] \\ & +\sum_{i=0}^{k-2}(1+x_3\partial_3)\left(u_2^{(k-2-i)}\tau_{23}^{(i)}/\tilde{r}\right)\end{aligned} \tag{40}$$

$$\begin{aligned}\sigma_m^{(k)} = {} & \mathbf{L}_8\,\mathbf{u}^{(k)} + \mathbf{L}_9\,u_3^{(k)} + \mathbf{L}_{10}\,\sigma_3^{(k-1)} \\ & +\sum_{i=0}^{k}\mathbf{L}_{11}\begin{bmatrix}\left(u_3^{(k-i)},_1\right)\left(u_3^{(i)},_1\right)/2 \\ \left(u_3^{(k-i)},_2\right)\left(u_3^{(i)},_2\right)/2 \\ \left(u_3^{(k-i)},_1\right)\left(u_3^{(i)},_2\right)\end{bmatrix} \\ & +\sum_{i=0}^{k-1}\mathbf{L}_{11}\left\{\begin{matrix}\left[\left(u_1^{(k-1-i)},_1\right)\left(u_1^{(i)},_1\right) + \left(u_2^{(k-1-i)},_1\right)\left(u_2^{(i)},_1\right)\right]/2 \\ \\ \left(u_2^{(k-1-i)},_2+u_3^{(k-1-i)}\right)\left(u_2^{(i)},_2+u_3^{(i)}\right)/2 \\ -\left(u_3^{(k-1-i)},_2\right)u_2^{(i)} \\ +(h/2R)\left(u_2^{(k-1-i)}\right)u_2^{(i)} \\ +\left(u_1^{(k-1-i)},_2\right)\left(u_1^{(i)},_2\right)/2 \\ \\ \left[\left(u_1^{(k-1-i)},_1\right)\left(u_1^{(i)},_2\right) + \left(u_2^{(k-1-i)},_1\right)\right. \\ \left.\left(u_2^{(i)},_2+u_3^{(i)}\right) - \left(u_3^{(k-1-i)},_1\right)u_2^{(i)}\right]\end{matrix}\right\}\end{aligned} \tag{41}$$

$$\int_{-1}^{1}\mathbf{C}_s\boldsymbol{\psi}_k\,dx_3 = \int_{-1}^{1}\boldsymbol{\sigma}_s^{(k)}\,dx_3 \tag{42}$$

where the superscript j of the displacement and stress components $f^{(j)}$ must be a zero or positive integer; otherwise the terms will vanish.

The associated dimensionless boundary conditions for various orders are described as follows.

On the inner and outer surfaces the following traction conditions must be satisfied,

At the ε^0-order level:

$$\begin{aligned}&\left[\tau_{13}^{(0)},\quad \tau_{23}^{(0)},\right. \\ &\qquad \left.\left(\sigma_3^{(0)} + \tau_{13}^{(0)}u_3^{(0)},_1 + \tau_{23}^{(0)}u_3^{(0)},_2/\tilde{r} + \sigma_3^{(0)}u_3^{(0)},_3\right)\right] \\ &= \left[0,\quad 0,\quad \pm\overline{q}_3^{\pm}\right] \quad \text{on} \quad x_3 = \pm 1,\end{aligned} \tag{43}$$

At the ε^{2k}-order (k=1, 2, ...) level:

$$\begin{aligned}\tau_{13}^{(k)} + \sum_{i=0}^{k-1}&\left[\tau_{13}^{(k-1-i)}u_1^{(i)},_1 + (1/\tilde{r})\tau_{23}^{(k-1-i)}u_1^{(i)},_2\right. \\ &\left.+\sigma_3^{(k-1-i)}u_1^{(i)},_3\right] = 0 \qquad \text{on} \quad x_3 = \pm 1\end{aligned} \tag{44a}$$

$$\tau_{23}^{(k)}+\sum_{i=0}^{k-1}\Big[\tau_{13}^{(k-1-i)}u_2^{(i)},_1+(1/\tilde{r})\tau_{23}^{(k-1-i)}u_2^{(i)},_2 +(1/\tilde{r})\tau_{23}^{(k-1-i)}u_3^{(i)}+\sigma_3^{(k-1-i)}u_2^{(i)},_3\Big]=0 \quad \text{on } x_3=\pm1 \tag{44b}$$

$$\sigma_3^{(k)}+\sum_{i=0}^{k}\Big[\tau_{13}^{(k-i)}u_3^{(i)},_1+\tau_{23}^{(k-i)}u_3^{(i)},_2/\tilde{r}+\sigma_3^{(k-i)}\quad u_3^{(i)},_3\Big] -\sum_{i=0}^{k-1}\Big(\tau_{23}^{(k-1-i)}u_2^{(i)}/\tilde{r}\Big)=0 \quad \text{on } x_3=\pm1 \tag{44c}$$

Along the edges one member of each pair of the following quantities must be satisfied.

At the ε^0-order level:

Along x_1= constants,

$$\sigma_1^{(0)}=\overline{p}_{11} \quad \text{or} \quad u_1^{(0)}=\overline{u}_1 \tag{45a}$$

$$\tau_{12}^{(0)}=\overline{p}_{12} \quad \text{or} \quad u_2^{(0)}=\overline{u}_2, \tag{45b}$$

$$\tau_{13}^{(0)}+\sigma_1^{(0)}u_3^{(0)},_1+\tau_{12}^{(0)}u_3^{(0)},_2/\tilde{r} +\tau_{13}^{(0)}u_3^{(0)},_3=\overline{p}_{13} \quad \text{or} \quad u_3^{(0)}=\overline{u}_3 \tag{45c}$$

Along x_2=constants,

$$\tau_{21}^{(0)}=\overline{p}_{21} \quad \text{or} \quad u_1^{(0)}=\overline{u}_1, \tag{46a}$$

$$\sigma_2^{(0)}=\overline{p}_{22} \quad \text{or} \quad u_2^{(0)}=\overline{u}_2, \tag{46b}$$

$$\tau_{23}^{(0)}+\tau_{12}^{(0)}u_3^{(0)},_1+\sigma_2^{(0)}u_3^{(0)},_2 +\tau_{13}^{(0)}u_3^{(0)},_3=\overline{p}_{23} \quad \text{or} \quad u_3^{(0)}=\overline{u}_3 \tag{46c}$$

At theε^{2k}-order (k=1, 2, ...) level:

Along x_1= constants,

$$\sigma_1^{(k)}+\sum_{i=0}^{k-1}\Big[\sigma_1^{(k-1-i)}u_1^{(i)},_1+\tau_{12}^{(k-1-i)}u_1^{(i)},_2/\tilde{r} +\tau_{13}^{(k-1-i)}u_1^{(i)},_3\Big]=0 \quad \text{or} \quad u_1^{(k)}=0 \tag{47a}$$

$$\tau_{12}^{(k)}+\sum_{i=0}^{k-1}\Big[\sigma_1^{(k-1-i)}u_2^{(i)},_1+\tau_{12}^{(k-1-i)}u_2^{(i)},_2/\tilde{r} +\tau_{12}^{(k-1-i)}u_3^{(i)}/\tilde{r}+\tau_{13}^{(k-1-i)}u_2^{(i)},_3\Big]=0 \quad \text{or} \quad u_2^{(k)}=0 \tag{47b}$$

$$\tau_{13}^{(k)}+\sum_{i=0}^{k}\Big(\sigma_1^{(k-i)}u_3^{(i)},_1+\tau_{12}^{(k-i)}u_3^{(i)},_2/\tilde{r}+\tau_{13}^{(k-i)}u_3^{(i)},_3\Big) -\sum_{i=0}^{k-1}(\tau_{12}^{(k-1-i)}u_2^{(i)}/\tilde{r})=0 \quad \text{or} \quad u_3^{(k)}=0 \tag{47c}$$

Along x_2=constants,

$$\tau_{12}^{(k)}+\sum_{i=0}^{k-1}\Big[\tau_{12}^{(k-1-i)}u_1^{(i)},_1+\sigma_2^{(k-1-i)}u_1^{(i)},_2/\tilde{r} +\tau_{23}^{(k-1-i)}u_1^{(i)},_3\Big]=0 \quad \text{or} \quad u_1^{(k)}=0 \tag{48a}$$

$$\sigma_2^{(k)}+\sum_{i=0}^{k-1}\Big[\tau_{12}^{(k-1-i)}u_2^{(i)},_1+\sigma_2^{(k-1-i)}u_2^{(i)},_2/\tilde{r} +\sigma_2^{(k-1-i)}u_3^{(i)}/\tilde{r}+\tau_{23}^{(k-1-i)}u_2^{(i)},_3\Big]=0 \quad \text{or} \quad u_2^{(k)}=0 \tag{48b}$$

$$\tau_{23}^{(k)}+\sum_{i=0}^{k}\Big(\tau_{12}^{(k-i)}u_3^{(i)},_1+\sigma_2^{(k-i)}u_3^{(i)},_2/\tilde{r}+\tau_{23}^{(k-i)}u_3^{(i)},_3\Big) -\sum_{i=0}^{k-1}\Big(\sigma_2^{(k-1-i)}u_2^{(i)}/\tilde{r}\Big)=0 \quad \text{or} \quad u_3^{(k)}=0 \tag{48c}$$

5 Successive integration and the von Karman theory

The asymptotic equations Eq.(31)-(35) can be integrated with respect to x_3 in succession. The associated lateral boundary conditions (43) at the inner surfaces (x_3=-1) will be satisfied in process of the integration. As a result, we obtain at the leading order

$$u_3^{(0)}=u_3^0\,(x_1,\,x_2) \tag{49}$$

$$\mathbf{u}^{(0)}=\,\mathbf{u}^0\,(x_1,\,x_2)\,-\,x_3\boldsymbol{\phi}_0 \tag{50}$$

$$\boldsymbol{\sigma}_m^{(0)}=\mathbf{L}_8\left(\mathbf{u}^0+x_3\boldsymbol{\phi}_0\right)+\mathbf{L}_9u_3^0 +\mathbf{L}_{11}\begin{bmatrix}\left(u_{3,1}^0\right)^2/2\\ \left(u_{3,2}^0\right)^2/2\\ \left(u_{3,1}^0\right)\left(u_{3,2}^0\right)\end{bmatrix} \tag{51}$$

$$\boldsymbol{\sigma}_s^{(0)}=-\int_{-1}^{x_3}\left[\mathbf{L}_5\mathbf{L}_8\left(\mathbf{u}^0-\eta\boldsymbol{\phi}_0\right)+\mathbf{L}_5\mathbf{L}_9\,u_3^0\right]d\eta -\int_{-1}^{x_3}\left\{\mathbf{L}_5\mathbf{L}_{11}\begin{bmatrix}\left(u_{3,1}^0\right)^2/2\\ \left(u_{3,2}^0\right)^2/2\\ \left(u_{3,1}^0\right)\left(u_{3,2}^0\right)\end{bmatrix}\right\}d\eta \tag{52}$$

$$\sigma_3^{(0)} = -\left[\; u^0_{3,1} \quad u^0_{3,2}/\tilde{r} \;\right] \sigma_s^{(0)} + \overline{q}_3^{-} + \int_{-1}^{x_3} \left[\sigma_2^{(0)} - \left(\mathbf{D}u_3^0\right)^{\mathrm{T}} \left(\mathbf{L}_5 \sigma_m^{(0)}\right) - \left(\mathbf{L}_7 u_3^0\right) \sigma_m^{(0)}\right] d\eta + \int_{-1}^{x_3} \left\{ (x_3-\eta)\, \mathbf{D}^{\mathrm{T}} \left[\mathbf{L}_5\mathbf{L}_8 \left(\mathbf{u}^0 + \eta\, \phi_0\right) + \mathbf{L}_5\mathbf{L}_9 u_3^0\right] \right\} d\eta + \int_{-1}^{x_3} \left\{ (x_3-\eta)\, \mathbf{L}_5\mathbf{L}_{11} \begin{bmatrix} \left(u^0_{3,1}\right)^2/2 \\ \left(u^0_{3,2}\right)^2/2 \\ \left(u^0_{3,1}\right)\left(u^0_{3,2}\right) \end{bmatrix} \right\} d\eta \quad (53)$$

where $u_3^0(x_1,\ x_2)$, $\mathbf{u}^0 = \left\{u_1^0(x_1, x_2) \quad u_2^0(x_1, x_2)\right\}^{\mathrm{T}}$ represent the displacements on the middle surface; $\boldsymbol{\phi}_0 = -\mathbf{D}\, u_3^0 + \boldsymbol{\psi}_0$.

After imposing lateral boundary conditions Eq.(43) at outer surface (x_3=1) on Eqs.(52)-(53) and then making a simple manipulation, we obtain the governing equations at the leading order

$$K_{11}\, u_1^0 + K_{12}\, u_2^0 + K_{13}\, u_3^0 + K_{14}\, \phi_1^0 + K_{15}\, \phi_2^0 = -u^0_{3,1}\left(K_{11}\, u_3^0\right) - u^0_{3,2}\left(\tilde{K}_{12}\, u_3^0\right) \quad (54)$$

$$K_{12}\, u_1^0 + K_{22}\, u_2^0 + K_{23}\, u_3^0 + K_{24}\, \phi_1^0 + K_{25}\, \phi_2^0 = -u^0_{3,1}\left(K_{12}\, u_3^0\right) - u^0_{3,2}\left(\tilde{K}_{22}\, u_3^0\right) \quad (55)$$

$$K_{13}\, u_1^0 + K_{23}\, u_2^0 + K_{33}\, u_3^0 + K_{34}\, \phi_1^0 + K_{35}\, \phi_2^0 = -(\overline{q}_3^{+} - \overline{q}_3^{-}) + \left[u^0_{3,1} \quad u^0_{3,1}\right] \mathbf{M}_1\, u_3^0 - \left[u^0_{3,11} \quad u^0_{3,22} \quad 2u^0_{3,12}\right] \left\{ \mathbf{M}_2 \begin{bmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \\ \phi_1^0 \\ \phi_2^0 \end{bmatrix} + \mathbf{M}_3 \begin{bmatrix} \left(u^0_{3,1}\right)^2/2 \\ \left(u^0_{3,2}\right)^2/2 \\ \left(u^0_{3,1}\right)\left(u^0_{3,2}\right) \end{bmatrix} \right\} \quad (56)$$

$$K_{14}u_1^0 + K_{24}u_2^0 + K_{34}u_3^0 + K_{44}\phi_1^0 + K_{45}\phi_2^0 = -u^0_{3,1}\left(K_{14}u_3^0\right) - u^0_{3,2}\left(\tilde{K}_{24}u_3^0\right) \quad (57)$$

$$K_{15}u_1^0 + K_{25}u_2^0 + K_{35}u_3^0 + K_{45}\phi_1^0 + K_{55}\phi_2^0 = -u^0_{3,1}\left(K_{15}u_3^0\right) - u^0_{3,2}\left(\tilde{K}_{25}u_3^0\right) \quad (58)$$

where

$$K_{11} = -(\hat{A}_{11}\partial_{11} + 2\tilde{A}_{16}\partial_{12} + \overline{A}_{66}\partial_{22}),$$

$$K_{12} = -\left[\hat{A}_{16}\partial_{11} + \left(\tilde{A}_{12} + \tilde{A}_{66}\right)\partial_{12} + \overset{\frown}{A}_{26}\partial_{22}\right],$$

$$K_{13} = -\left(\tilde{A}_{12}\partial_1 + \overline{A}_{26}\partial_2\right),$$

$$K_{14} = -\left(\hat{B}_{11}\partial_{11} + 2\tilde{B}_{16}\partial_{12} + \overline{B}_{66}\partial_{22}\right),$$

$$K_{15} = -\left[\hat{B}_{16}\partial_{11} + \left(\tilde{B}_{12} + \tilde{B}_{66}\right)\partial_{12} + \overline{B}_{26}\partial_{22}\right],$$

$$\tilde{K}_{12} = -\left[\tilde{A}_{16}\partial_{11} + \left(\overline{A}_{12} + \overline{A}_{66}\right)\partial_{12} + \hat{A}_{26}\partial_{22}\right],$$

$$K_{22} = -\left(\hat{A}_{66}\partial_{11} + 2\tilde{A}_{26}\partial_{12} + \overline{A}_{22}\partial_{22}\right),$$

$$K_{23} = -\left(\tilde{A}_{26}\partial_1 + \tilde{A}_{22}\partial_2\right),$$

$$K_{24} = -\left[\hat{B}_{16}\partial_{11} + (\tilde{B}_{12} + \tilde{B}_{66})\partial_{12} + \overline{B}_{26}\partial_{22}\right],$$

$$K_{25} = -\left[\hat{B}_{66}\partial_{11} + 2\tilde{B}_{26}\partial_{12} + \overline{B}_{22}\partial_{22}\right],$$

$$\tilde{K}_{22} = -\left(\tilde{A}_{66}\partial_{11} + 2\overline{A}_{26}\partial_{12} + \overset{\frown}{A}_{22}\partial_{22}\right),$$

$$K_{33} = \tilde{A}_{55}\partial_{11} + 2\tilde{A}_{45}\partial_{12} + \tilde{A}_{44}\partial_{22} - \overline{A}_{22},$$

$$K_{34} = -\left[\left(\tilde{B}_{12} - \tilde{A}_{55}\right)\partial_1 + \left(\overline{B}_{26} - \tilde{A}_{45}\right)\partial_2\right]$$

$$K_{35} = -\left[\left(\tilde{B}_{26} - \tilde{A}_{45}\right)\partial_1 + (\overline{B}_{22} - \tilde{A}_{44})\partial_2\right],$$

$$K_{44} = -\left(\hat{D}_{11}\partial_{11} + 2\tilde{D}_{16}\partial_{12} + \overline{D}_{66}\partial_{22} - \tilde{A}_{55}\right),$$

$$K_{45} = -\left[\hat{D}_{16}\partial_{11} + (\tilde{D}_{12} + \tilde{D}_{66})\partial_{12} + \overline{D}_{26}\partial_{22} - \tilde{A}_{45}\right],$$

$$\tilde{K}_{24} = -\left[\tilde{B}_{16}\partial_{11} + (\overline{B}_{12} + \overline{B}_{66})\partial_{12} + \hat{B}_{26}\partial_{22}\right],$$

$$K_{55} = -\left(\hat{D}_{66}\partial_{11} + 2\tilde{D}_{26}\partial_{12} + \overline{D}_{22}\partial_{22} - \tilde{A}_{44}\right),$$

$$\tilde{K}_{25} = -\left(\tilde{B}_{66}\partial_{11} + 2\overline{B}_{26}\partial_{12} + \overset{\frown}{B}_{22}\partial_{22}\right),$$

$$\mathbf{M}_1 = \begin{bmatrix} \left(\tilde{A}_{12}\partial_1 + \overline{A}_{26}\partial_2\right)/2 \\ \left(\overline{A}_{26}\partial_1 + A_{22}\partial_2\right)/2 \end{bmatrix},$$

$$\mathbf{M}_2 = \begin{bmatrix} \left(\hat{A}_{11}\partial_1 + \tilde{A}_{16}\partial_2\right) & \left(\overline{A}_{12}\partial_1 + \overset{\frown}{A}_{26}\partial_2\right) & \cdots & \left(\tilde{A}_{16}\partial_1 + \overline{A}_{66}\partial_2\right) \\ \left(\hat{A}_{16}\partial_1 + \tilde{A}_{12}\partial_2\right) & \left(\overline{A}_{26}\partial_1 + \overset{\frown}{A}_{22}\partial_2\right) & \cdots & \left(\tilde{A}_{66}\partial_1 + \overline{A}_{26}\partial_2\right) \\ \tilde{A}_{12} & \overset{\frown}{A}_{22} & \cdots & \overline{A}_{26} \\ \left(\hat{B}_{11}\partial_1 + \tilde{B}_{16}\partial_2\right) & \left(\overline{B}_{12}\partial_1 + \overset{\frown}{B}_{26}\partial_2\right) & \cdots & \left(\tilde{B}_{16}\partial_1 + \overline{B}_{66}\partial_2\right) \\ \left(\hat{B}_{16}\partial_1 + \tilde{B}_{12}\partial_2\right) & \left(\overline{B}_{26}\partial_1 + \overset{\frown}{B}_{22}\partial_2\right) & \cdots & \left(\tilde{B}_{66}\partial_1 + \overline{B}_{26}\partial_2\right) \end{bmatrix}^T$$

$$\mathbf{M}_3 = \begin{bmatrix} \hat{A}_{11} & \overline{A}_{12} & \tilde{A}_{16} \\ \overline{A}_{12} & \breve{A}_{22} & \overset{\frown}{A}_{26} \\ \tilde{A}_{16} & \overset{\frown}{A}_{26} & \overline{A}_{66} \end{bmatrix},$$

$$\begin{bmatrix}\hat{A}_{ij} & \tilde{A}_{ij} & \overline{A}_{ij} & \widehat{A}_{ij} & \breve{A}_{ij}\end{bmatrix} = \int_{-1}^{1} \tilde{Q}_{ij} \begin{bmatrix}\tilde{r} & 1 & 1/\tilde{r} & 1/\tilde{r}^2 & 1/\tilde{r}^3\end{bmatrix} dx_3,$$

$$\begin{bmatrix}\hat{B}_{ij} & \tilde{B}_{ij} & \overline{B}_{ij} & \widehat{B}_{ij}\end{bmatrix} = \int_{-1}^{1} \tilde{Q}_{ij} x_3 \begin{bmatrix}\tilde{r} & 1 & 1/\tilde{r} & 1/\tilde{r}^2\end{bmatrix} dx_3,$$

$$\begin{bmatrix}\hat{D}_{ij} & \tilde{D}_{ij} & \overline{D}_{ij}\end{bmatrix} = \int_{-1}^{1} \tilde{Q}_{ij} (x_3)^2 \begin{bmatrix}\tilde{r} & 1 & 1/\tilde{r}\end{bmatrix} dx_3.$$

After a close examination of the governing equations of the von Karman-type FSDT in terms of displacements, we find that those equations are reproduced from Eqs.(54)-(58) by imposing a geometry assumption of the circular cylindrical shell: $\zeta/R << 1$. In the present notations, this implies $\tilde{r} \cong 1$, $\hat{A}_{ij} = \tilde{A}_{ij} = \overline{A}_{ij} = \widehat{A}_{ij} = \breve{A}_{ij} = A_{ij}/Qh$, $\hat{B}_{ij} = \tilde{B}_{ij} = \overline{B}_{ij} = \widehat{B}_{ij} = B_{ij}/Q_{ij}h^2$, $\hat{D}_{ij} = \tilde{D}_{ij} = \overline{D}_{ij} = D_{ij}/Qh^3$, $\tilde{K}_{12} = K_{12}$, $\tilde{K}_{22} = K_{22}$, $\tilde{K}_{24} = K_{24}$, $\tilde{K}_{25} = K_{25}$, where A_{ij}, B_{ij}, D_{ij} denote the extension, extension-bending and bending stiffness, respectively. Thus, it can be observed that the von Karman-type FSDT becomes a first-order approximation to the 3D nonlinear elasticity theory.

The governing equations of ε^0-order Eqs.(54)-(58) and the edge boundary conditions Eq.(43) compose a well-posed boundary-valued problem. The modified Newton-Raphson method [Zienkiewicz (1976)] has been widely used for solving this type of problems. After the nonlinear problem at the ε^0-order level is solved, the associated ε^0-order solution can be obtained by using Eqs.(49)-(53) and Eq.(35).

Carrying on the analysis to order ε^2 by integrating Eqs.(37)-(42) in succession with k=1, we obtain

$$u_3^{(1)} = u_3^1(x_1,x_2) + \varphi_{31}(x_1,x_2,x_3), \tag{59}$$

$$\mathbf{u}^{(1)} = \mathbf{u}^1(x_1, x_2) + x_3\,\boldsymbol{\phi}_1 + \boldsymbol{\varphi}_1(x_1, x_2, x_3) \tag{60}$$

$$\boldsymbol{\sigma}_m^{(1)} = \mathbf{L}_8(\mathbf{u}^1 + x_3\boldsymbol{\phi}_1) + \mathbf{L}_9 u_3^1 + \mathbf{L}_{11}\begin{bmatrix}(u_{3,1}^0)(u_{3,1}^1)\\(u_{3,2}^0)(u_{3,2}^1)\\(u_{3,1}^0)(u_{3,2}^1)+(u_{3,2}^0)(u_{3,1}^1)\end{bmatrix} + \mathbf{p}_1 \tag{61}$$

$$\begin{aligned}\boldsymbol{\sigma}_s^{(1)} = &-\begin{bmatrix}u_1^{(0)},_1 & u_1^{(0)},_2/\tilde{r}\\ u_2^{(0)},_1 & u_2^{(0)},_2/\tilde{r}\end{bmatrix}\boldsymbol{\sigma}_s^{(0)} - \begin{bmatrix}\sigma_3^{(0)}u_1^{(0)},_3\\ \sigma_3^{(0)}u_2^{(0)},_3 + \tau_{23(0)}u_{3(0)}/\tilde{r}\end{bmatrix}\\ &-\int_{-1}^{x_3}\left[\mathbf{L}_5\mathbf{L}_8(\mathbf{u}^1+\eta\phi_1)+\mathbf{L}_5\mathbf{L}_9u_3^1\right]d\eta\\ &-\int_{-1}^{x_3}\left\{\mathbf{L}_5\mathbf{L}_{11}\begin{bmatrix}(u_{3,1}^0)(u_{3,1}^1)\\(u_{3,2}^0)(u_{3,2}^1)\\(u_{3,1}^0)(u_{3,2}^1)+(u_{3,2}^0)(u_{3,1}^1)\end{bmatrix}\right\}d\eta\\ &-\mathbf{f}_1(x_1,x_2,x_3)\end{aligned} \tag{62}$$

$$\begin{aligned}\sigma_3^{(1)} = &-\sum_{i=0}^{1}\left[u_3^{(1-i)},_1 \quad u_3^{(1-i)},_2/\tilde{r}\right]\boldsymbol{\sigma}_s^{(i)}\\ &-\sum_{i=0}^{1}\left(u_3^{(1-i)},_3\right)\sigma_3^{(i)} + \left(u_2^{(0)}\tau_{23}^{(0)}/\tilde{r}\right)\\ &+\int_{-1}^{x_3}\left[\sigma_2^{(1)} - \sum_{i=0}^{1}(\mathbf{D}u_3^{1-i})^{\mathrm{T}}(\mathbf{L}_5\boldsymbol{\sigma}_m^{(i)}) - \sum_{i=0}^{1}(\mathbf{L}_7u_3^{1-i})\boldsymbol{\sigma}_m^{(i)}\right]d\eta\\ &+\int_{-1}^{x_3}\left\{(x_3-\eta)\mathbf{D}^{\mathrm{T}}\left[\mathbf{L}_5\mathbf{L}_8(\mathbf{u}^1+\eta\boldsymbol{\phi}_1)+\mathbf{L}_5\mathbf{L}_9u_3^1\right]\right\}d\eta\\ &+\int_{-1}^{x_3}\left\{(x_3-\eta)\mathbf{D}^{\mathrm{T}}\mathbf{L}_5\mathbf{L}_{11}\begin{bmatrix}(u_{3,1}^0)(u_{3,1}^1)\\(u_{3,2}^0)(u_{3,2}^1)\\(u_{3,1}^0)(u_{3,2}^1)+(u_{3,2}^0)(u_{3,1}^1)\end{bmatrix}\right\}d\eta\\ &-f_{31}(x_1,x_2,x_3)\end{aligned} \tag{63}$$

where u_3^1 and $\mathbf{u}^1 = \left\{u_1^1(x_1, x_2) \quad u_2^1(x_1, x_2)\right\}^{\mathrm{T}}$ represent the ε^2-order modifications to the displacements on the middle surface, and the relevant functions $\mathbf{g}_1$, f_{31}, $\mathbf{f}_1$, $\boldsymbol{\phi}_{31}$, ϕ_1 are given in Appendix I.

Upon imposing the associated lateral boundary conditions (44) on Eqs.(62) and (63), we obtain the governing equations for ε^2-order as follows:

$$\begin{aligned}&K_{11}u_1^1 + K_{12}u_2^1 + K_{13}u_3^1 + K_{14}\phi_1^1 + K_{15}\phi_2^1\\ &= -u_{3,1}^0(K_{11}u_3^1) - u_{3,1}^1(K_{11}u_3^0)\\ &\quad - u_{3,2}^0(\tilde{K}_{12}u_3^1) - u_{3,2}^1(\tilde{K}_{12}u_3^0)\\ &\quad + f_{11}(x_3=1)\end{aligned} \tag{64}$$

$$K_{12}u_1^1+K_{22}u_2^1+K_{23}u_3^1+K_{24}\phi_1^1+K_{25}\phi_2^1$$
$$=-u_{3,1}^0(K_{12}u_3^1)-u_{3,1}^1(K_{12}u_3^0)$$
$$-u_{3,2}^0(\tilde{K}_{22}u_3^1)-u_{3,2}^1(\tilde{K}_{22}u_3^0)$$
$$+f_{21}(x_3=1) \quad (65)$$

$$K_{13}u_1^1+K_{23}u_2^1+K_{33}u_3^1+K_{34}\phi_1^1+K_{35}\phi_2^1$$
$$=\begin{bmatrix}u_{3,1}^0 & u_{3,2}^0\end{bmatrix}\mathbf{M}_1 u_3^1+\begin{bmatrix}u_{3,1}^1 & u_{3,2}^1\end{bmatrix}\mathbf{M}_1 u_3^0$$
$$-\begin{bmatrix}u_{3,11}^0 & u_{3,22}^0 & 2u_{3,12}^0\end{bmatrix}$$
$$\left\{\mathbf{M}_2\begin{bmatrix}u_1^1\\ u_1^2\\ u_1^3\\ \phi_1^1\\ \phi_1^2\end{bmatrix}+\mathbf{M}_3\begin{bmatrix}(u_{3,1}^0)(u_{3,1}^1)\\ (u_{3,2}^0)(u_{3,2}^1)\\ (u_{3,1}^0)(u_{3,2}^1)+(u_{3,1}^1)(u_{3,2}^0)\end{bmatrix}\right\}$$
$$-\begin{bmatrix}u_{3,11}^1 & u_{3,22}^1 & 2u_{3,12}^1\end{bmatrix}$$
$$\left\{\mathbf{M}_2\begin{bmatrix}u_1^0\\ u_2^0\\ u_3^0\\ \phi_1^0\\ \phi_2^0\end{bmatrix}+\mathbf{M}_3\begin{bmatrix}(u_{3,1}^0)^2/2\\ (u_{3,2}^0)^2/2\\ (u_{3,1}^0)(u_{3,2}^0)\end{bmatrix}\right\}+f_{31}(x_3=1) \quad (66)$$

$$K_{14}u_1^1+K_{24}u_2^1+K_{34}u_3^1+K_{44}\phi_1^1+K_{45}\phi_2^1$$
$$=-u_{3,1}^0\left(K_{14}u_3^1\right)-u_{3,1}^1\left(K_{14}u_3^0\right)$$
$$-u_{3,2}^0\left(\tilde{K}_{24}u_3^1\right)-u_{3,2}^1\left(\tilde{K}_{24}u_3^0\right)$$
$$+f_{11}(x_3=1)-g_{11}(x_3=1) \quad (67)$$

$$K_{15}u_1^1+K_{25}u_2^1+K_{35}u_3^1+K_{45}\phi_1^1+K_{55}\phi_2^1$$
$$=-u_{3,1}^0\left(K_{15}u_3^1\right)-u_{3,1}^1\left(K_{15}u_3^0\right)$$
$$-u_{3,2}^0\left(\tilde{K}_{25}u_3^1\right)-u_{3,2}^1\left(\tilde{K}_{25}u_3^0\right)$$
$$+f_{21}(x_3=1)-g_{21}(x_3=1) \quad (68)$$

After a close examination between Eqs.(54)-(58) and Eqs.(64)-(68), it can be found that the differential operators of the linear terms (K_{ij}, i, j=1-5) remain identical, the nonlinear terms related to the unknowns of the current order appear in a regular pattern, and the other nonhomogeneous terms can be calculated from the lower-order solutions.

The solution procedure used in the ε^0-order problem can be repeatedly applied to the ε^1-order problem. Again, after the ε^1-order problem is solved the associated ε^1-order modifications can be obtained by using Eqs.(59)-(63). In view of the recurrence among the problems of various orders, the 3D nonlinear solutions can be determined in a hierarchic and consistent way.

6 Illustrative examples

To make the relevant computation simple, we demonstrate the present 3D nonlinear analysis of the simply supported cross-ply laminated cylindrical strips subjected to cylindrical bending (Figure 1). In these cases, the field variables are naturally dependent on the circumferential and thickness coordinates only and independent on the axial coordinate. The present formulation is then reduced by letting the derivatives of the field variables with respect to the axial coordinate zero. With the conjunction of the differential quadrature and modified Newton-Raphson methods, the present asymptotic solutions for various orders are computed. Since the DQ method is well developed and has been used to yield satisfactory results for 3D elasticity analysis in the early papers [Wu and Hung (1999); Wu and Wu (2000)], it is omitted here for brevity. The ratio of radius-to-thickness of the cylindrical strip is taken as $R/2h$=10, 20. The material properties are E_L/E_T=40, G_{LT}/E_T=0.5, G_{TT}/E_T=0.2, $\nu_{LT}=\nu_{TT}$=0.25.

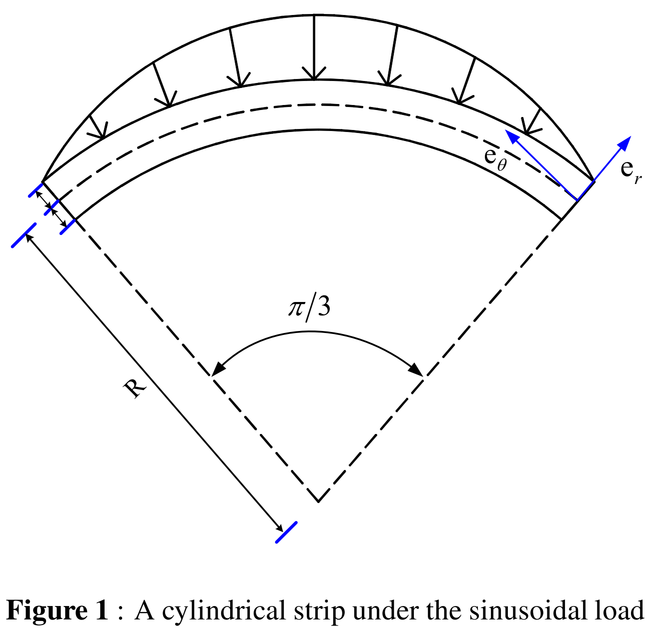

Figure 1 : A cylindrical strip under the sinusoidal load

The lateral load applied on the upper surface is given as $\overline{q}_z^+=q_0\sin(\pi\theta/\Phi)$ where the maximum angle Φ is taken as $\pi/3$. The normalized parameters are defined as $\overline{u}_r = u_r/2h$, $(\overline{\sigma}_x \quad \overline{\sigma}_\theta) = (\sigma_x \quad \sigma_\theta)/S^2 q_0$ and $S = R/2h$. The dimensionless forms of load magnitude and its increment are defined as $(\ \overline{q} \quad \Delta\overline{q}\ ) = (\ q_0 \quad \Delta q_0\ )\ S^4/E_T$. Table 1 shows the convergence study of the dimensionless central deflection of [90/0] cylindrical strips where $R/2h$=20. It is observed from Table 1 that the present asymptotic solutions yield convergence at the ε^4−order level in the cases of moderately load magnitude ($\overline{q}$=10 and $\overline{u}_r$=0.3044) and at the ε^6−order level in the cases of large load magnitude ($\overline{q}$=40 and $\overline{u}_r$=1.3555) with $\Delta\overline{q}$=1. The 3D nonlinear solutions are shown to be about 6% larger than 3D linear solutions in the cases of $\overline{q}$=20 and about 13% in the cases of $\overline{q}$=40. The results of in-surface stresses are presented in Tables 2-3 and compared with the accurate linear solutions available in the literature [Jingt and Tzeng (1995)]. It is shown that the present convergent solutions are in excellent agreement with the accurate linear solutions in the cases of small load magnitude ($\overline{q}$=0.1). The deviations between the present convergent solutions and the accurate linear solutions become larger as the load magnitude is large. The present convergent results of in-surface stresses are larger than those of accurate linear solutions about 13% in the cases of moderately thick shells ($R/2h$=10) and about 5% in the cases of thin shells ($R/2h$=20) with $\overline{q}$=20.

Table 1 : Convergence study of dimensionless central deflection of [90/0] cylindrical strips ($R/2h$=20)

$\overline{q}$	Present 3D linear sols.	$\Delta\overline{q}$	Present 3D nonlinear sols. ε^0	ε^2	ε^4	ε^6
10	0.2961	5	0.23380	0.29123	0.30240	0.30439
		2	0.23380	0.29123	0.30240	0.30439
		1	0.23380	0.29123	0.30240	0.30439
20	0.5922	5	0.47362	0.59602	0.62259	0.62815
		2	0.47362	0.59602	0.62259	0.62815
		1	0.47362	0.59602	0.62259	0.62814
30	0.8883	5	0.71991	0.91649	0.96439	0.97620
		2	0.71991	0.91649	0.96439	0.97619
		1	0.71990	0.91647	0.96438	0.97617
40	1.1844	5	0.97316	1.25519	1.33286	1.35551
		2	0.97316	1.25519	1.33286	1.35551
		1	0.97315	1.25518	1.33284	1.35546

Table 2 : Dimensionless in-surface stresses of [90/0] cylindrical strips at the mid-span ($R/2h$=10)

$\overline{q}$		Accurate linear sol. [Jingt and Tzeng (1995)]	Present 3D nonlinear sols. ε^0	ε^2	ε^4	ε^6
0.1	$\overline{\sigma}_x(R+h)$	0.0466	0.0352	0.0456	0.0465	0.0466
	$\overline{\sigma}_x(R-h)$	-0.0162	-0.0136	-0.0159	-0.0162	-0.0162
	$\overline{\sigma}_\theta(R+h)$	0.1763	0.1408	0.1723	0.1760	0.1763
	$\overline{\sigma}_\theta(R-h)$	-2.5947	-2.1808	-2.5433	-2.5898	-2.5954
1	$\overline{\sigma}_x(R+h)$	0.0466	0.0353	0.0457	0.0467	0.0468
	$\overline{\sigma}_x(R-h)$	-0.0162	-0.0137	-0.0159	-0.0163	-0.0163
	$\overline{\sigma}_\theta(R+h)$	0.1763	0.1411	0.1729	0.1767	0.1773
	$\overline{\sigma}_\theta(R-h)$	-2.5947	-2.1853	-2.5519	-2.6011	-2.6069
20	$\overline{\sigma}_x(R+h)$	0.0466	0.0370	0.0497	0.0524	0.0534
	$\overline{\sigma}_x(R-h)$	-0.0162	-0.0143	-0.0174	-0.0184	-0.0187
	$\overline{\sigma}_\theta(R+h)$	0.1763	0.1479	0.1889	0.1994	0.2033
	$\overline{\sigma}_\theta(R-h)$	-2.5947	-2.2908	-2.7876	-2.9407	-2.9979

7 Concluding remarks

By means of the method of perturbation, a refined asymptotic theory is presented for the 3D nonlinear analysis of laminated composite cylindrical shells. Without making any static or kinematic assumptions in advance, we decompose the 3D nonlinear theory into a series of 2D nonlinear theories for various orders. The von Karman-type FSDT theory is derived as a first-order approximation to the 3D nonlinear theory. The present asymptotic formulation reveals that the 3D nonlinear solutions can be obtained by solving the von Karman-type FSDT equations in a hierarchic and consistent manner. Application of the present asymptotic theory to the nonlinear analysis of the laminated cylindrical strips under cylindrical bending is made. It is shown that convergence of the present asymptotic formulation is fast. In the cases of thin shells ($R/2h$=20), the deviations between the 3D nonlinear solutions and 3D linear solutions are up to about 13% as the maximum transverse deflection is in the same order as the thickness. These deviations become larger as the shell is thick.

Table 3 : Dimensionless in-surface stresses of [90/0] cylindrical strips at the mid-span ($R/2h$=20)

$\bar{q}$		Accurate linear sol. [Jingt and Tzeng (1995)]	Present 3D nonlinear sols. ε^0	ε^2	ε^4	ε^6
0.1	$\bar{\sigma}_x(R+h)$	0.0427	0.0362	0.0421	0.0427	0.0427
	$\bar{\sigma}_x(R-h)$	-0.0155	-0.0134	-0.0153	-0.0155	-0.0155
	$\bar{\sigma}_\theta(R+h)$	0.1684	0.1450	0.1657	0.1681	0.1684
	$\bar{\sigma}_\theta(R-h)$	-2.4774	-2.1512	-2.4407	-2.4738	-2.4775
1	$\bar{\sigma}_x(R+h)$	0.0427	0.0363	0.0421	0.0427	0.0428
	$\bar{\sigma}_x(R-h)$	-0.0155	-0.0135	-0.0153	-0.0155	-0.0155
	$\bar{\sigma}_\theta(R+h)$	0.1684	0.1451	0.1660	0.1685	0.1687
	$\bar{\sigma}_\theta(R-h)$	-2.4774	-2.1532	-2.4448	-2.4789	-2.4829
20	$\bar{\sigma}_x(R+h)$	0.0427	0.0370	0.0438	0.0449	0.0451
	$\bar{\sigma}_x(R-h)$	-0.0155	-0.0137	-0.0159	-0.0163	-0.0164
	$\bar{\sigma}_\theta(R+h)$	0.1684	0.1480	0.1727	0.1771	0.1779
	$\bar{\sigma}_\theta(R-h)$	-2.4774	-2..1969	-2.5423	-2.6045	-2.6180

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Appendix: The Expressions of $\tilde{\mathbf{f}}_1$, f_{31} and the Relevant Terms

The expressions of $\tilde{\mathbf{f}}_1$, f_{31} and the relevant terms for the ε^2-order corrections are

$$\mathbf{p}_1 = \left\{ \begin{array}{c} p_{11}(x_1,\, x_2,\, x_3) \\ p_{21}(x_1,\, x_2,\, x_3) \\ p_{31}(x_1,\, x_2,\, x_3) \end{array} \right\} = \mathbf{L}_8\boldsymbol{\varphi}_1 + \mathbf{L}_9\varphi_{31} + \mathbf{L}_{10}\sigma_3^{(0)}$$
$$+\mathbf{L}_{11}\left[\begin{array}{c} (u^0_{3,1})(\varphi_{31,1}) \\ (u^0_{3,2})(\varphi_{31,2}) \\ (u^0_{3,1})(\varphi_{31,2}) + (u^0_{3,2})(\varphi_{31,1}) \end{array} \right]$$
$$+\mathbf{L}_{11}\left\{ \begin{array}{c} \left[(u_1^{(0)},_1)^2 + (u_2^{(0)},_1)^2\right]/2 \\ \\ (u_2^{(0)},_2 + u_3^0)^2/2 - (u^0_{3,2})u_2^{(0)} \\ +(u_2^{(0)})^2(h/2R) + (u_1^{(0)},_2)^2/2 \\ \\ (u_1^{(0)},_1)(u_1^{(0)},_2) + (u_2^{(0)},_1) \\ (u_2^{(0)},_2 + u_3^0) - (u^0_{3,1})u_2^{(0)} \end{array} \right\} \tag{69}$$

$$f_{31}(x_1,x_2,x_3)$$
$$= -\int_{-1}^{x_3} \Big\{ \mathbf{D}^{\mathrm{T}}\mathbf{f} + \mathbf{D}^{\mathrm{T}} \left[\begin{array}{c} u_1^{(0)},_3\,\sigma_3^{(0)} \\ u_2^{(0)},_3\,\sigma_3^{(0)} + \tau_{23(0)}u_{3(0)}/\tilde{r} \end{array} \right]$$
$$+\mathbf{D}^{\mathrm{T}} \left[\begin{array}{c} u_1^{(0)},_1\,\tau_{13}^{(0)} + u_1^{(0)},_2\,\tau_{23}^{(0)}/\tilde{r} \\ u_2^{(0)},_1\,\tau_{13}^{(0)} + u_2^{(0)},_2\,\tau_{23}^{(0)}/\tilde{r} \end{array} \right]$$
$$-\left[\ \tilde{r}u^0_{3,11} \quad (u^0_{3,22}/\tilde{r} - 1) \quad 2u^0_{3,12}\ \right]\mathbf{g}_1$$
$$-\left[\ \tilde{r}\varphi_{31,11} \quad (\phi_{31,22}/\tilde{r}) \quad 2\varphi_{31,12}\ \right]\boldsymbol{\sigma}_m^{(0)}$$
$$-(\mathbf{D}^{\mathrm{T}}\varphi_{31})(\mathbf{L}_5\boldsymbol{\sigma}_m^{(0)}) - \mathbf{D}^T(\varphi_{31,3}\boldsymbol{\sigma}_s^{(0)})$$
$$-(1 + x_3\partial_3)(u_3^{(0)},_1\,\tau_{13}^{(0)} + u_3^{(0)},_2\,\tau_{23}^{(0)}/\tilde{r} + \sigma_3^{(0)})$$
$$-x_3\left[\tau_{13}^{(0)},_1 - u^0_{3,2}\,\sigma_{22}^{(0)},_2/\tilde{r} - u^0_{3,2}\,\tau_{12}^{(0)},_1\right]$$
$$+\left[\sigma_2^{(0)}u_3^0/\tilde{r} + \tau_{23}^{(0)}\phi_2^0\right]$$
$$+\Big[\tau_{12}^{(0)},_1\,u_2^{(0)} + 2\tau_{12}^{(0)}u_2^{(0)},_1$$
$$+\sigma_2^{(0)},_2\,u_2^{(0)}/\tilde{r} + 2\sigma_2^{(0)}u_2^{(0)},_2\Big]\Big\}\, d\eta \tag{70}$$

$$\mathbf{f_1} = \left\{ \begin{array}{c} f_{1k}(x_1,x_2,x_3) \\ f_{2k}(x_1,x_2,x_3) \end{array} \right\}$$
$$= \int_{-1}^{x_3} \Bigg\{ \mathbf{L}_5\mathbf{p}_1 + \left[\begin{array}{ccc} \tilde{r}u_1^{(0)},_{11} & (u_1^{(0)},_{22}/\tilde{r}) & 2u_1^{(0)},_{12} \\ \tilde{r}u_2^{(0)},_{11} & (u_2^{(0)},_{22}/\tilde{r}) & 2u_2^{(0)},_{12} \end{array} \right] \boldsymbol{\sigma}_m^{(0)}$$
$$+\left[\begin{array}{cc} u_1^{(0)},_1 & u_1^{(0)},_2/\tilde{r} \\ u_2^{(0)},_1 & u_2^{(0)},_2/\tilde{r} \end{array} \right] \mathbf{L}_5\boldsymbol{\sigma}_m^{(0)}$$
$$+\left[\begin{array}{c} \left(\tau_{13}^{(0)}\phi_1^0\right),_1 + \left(\tau_{23}^{(0)}\phi_1^0\right),_2 \\ \left(\tau_{13}^{(0)}\phi_2^0\right),_1 + \left(\tau_{23}^{(0)}\phi_2^0\right),_2 \end{array} \right]$$
$$+\left[\begin{array}{c} (1 + x_3\partial_3)\tau_{13}^{(0)} \\ (2 + x_3\partial_3)\tau_{23}^{(0)} \end{array} \right]$$
$$+\left[\begin{array}{c} 0 \\ u_3^0\tau_{12}^{(0)},_1 + 2u^0_{3,1}\,\tau_{12}^{(0)} \end{array} \right]$$
$$+\left[\begin{array}{c} 0 \\ u_3^0\sigma_2^{(0)},_2/\tilde{r} + 2u^0_{3,2}\,\sigma_2^{(0)}/\tilde{r} \end{array} \right] \Bigg\} d\eta \tag{71}$$

$$\varphi_{31}(x_1,x_2,x_3) = -\int_0^{x_3} \left[\mathbf{L}_1\mathbf{u}^{(0)} + (\tilde{c}_{23}/\tilde{r})u_3^0\right] d\eta$$
$$-\int_0^{x_3} \Bigg\{ \mathbf{L}_2 \left[\begin{array}{c} (u^0_{3,1})^2/2 \\ (u^0_{3,2})^2/2 \\ (u^0_{3,1})(u^0_{3,2}) \end{array} \right]$$
$$+\left[(\phi_1^0)^2 + \phi_2^0)^2\right]/2 \Bigg\} d\eta \tag{72}$$

$$\boldsymbol{\varphi}_1 = \left\{ \begin{array}{c} \varphi_{11}(x_1,x_2,x_3) \\ \varphi_{21}(x_1,x_2,x_3) \end{array} \right\}$$
$$= \int_0^{x_3} \left[\mathbf{S}\hat{\boldsymbol{\sigma}}_s^{(0)} - \mathbf{D}\varphi_{31}\right] d\eta + \int_0^{x_3} \Big\{ -\varphi_{31,3}\,\mathbf{D}u_3^0$$
$$-\left[\begin{array}{c} -\tilde{r}\phi_1^0 u_1^{(0)},_1 + \tilde{r}\phi_2^{(0)}u_2^{(0)},_1 + \eta\left(\phi_1^0 + u^0_{3,1}\right) \\ \phi_1^0 u_1^{(0)},_2 + \phi_2^0\left(\eta + u_3^0 + u_2^{(0)},_2\right) - u_2^{(0)} \end{array} \right] \Big\} d\eta \tag{73}$$

$$\mathbf{g}_1 = \left\{ \begin{array}{c} g_{11}(x_1,x_2,x_3) \\ g_{21}(x_1,x_2,x_3) \end{array} \right\}$$
$$= \int_0^{x_3} \Bigg\{ \mathbf{f} + \left[\begin{array}{c} u_1^{(0)},_1 \\ u_2^{(0)},_1 \end{array} \right] \tau_{13}^{(0)} + \left[\begin{array}{c} u_1^{(0)},_2/\tilde{r} \\ u_2^{(0)},_2/\tilde{r} \end{array} \right] \tau_{23}^{(0)}$$
$$+\boldsymbol{\phi}_0\,\sigma_3^{(0)} + \left[\begin{array}{c} 0 \\ \tau_{23}^{(0)}\,u_3^{(0)}/\tilde{r} \end{array} \right] \Bigg\} dx_3 \tag{74}$$