 CMC, vol.2, no.4, pp.267-276, 2005

Solution of Maxwell's Equations Using the MQ Method

D.L. Young [1,3], C.S. Chen [2], T.K. Wong [3]

Abstract: A meshless time domain numerical method based on the radial basis functions using multiquadrics (MQ) is employed to simulate electromagnetic field problems by directly solving the time-varying Maxwell's equations without transforming to simplified versions of the wave or Helmholtz equations. In contrast to the conventional numerical schemes used in the computational electromagnetism such as FDTD, FETD or BEM, the MQ method is a truly meshless method such that no mesh generation is required. It is also easy to deal with the appropriate partial derivatives, divergences, curls, gradients, or integrals like semi-analytic solutions. For illustration purposes, the MQ method is employed to calculate the propagations of the electric and magnetic waves in the homogeneous, isotropic, and non-lossy 2D rectangular waveguide as well as 3D cavity resonator. Good agreements are obtained as compared to analytical solutions. By directly solving the Maxwell's equations, the MQ scheme provides a very simple and effective numerical scheme for the computational electromagnetism.

keyword: Multiquadrics (MQ) method, Maxwell's equations, Meshless time-domain method, 2D waveguide, 3D cavity resonator.

1 Introduction

Regarding direct solutions of the time-dependent Maxwell's equations, the finite difference time domain (FDTD) and finite element time domain (FETD) methods are most commonly used in the realm of computational electromagnetism (CE). The FDTD scheme was first developed by Yee (1966) in the mid-1960s. This method was well developed and widely applied in the computation of different fields of electric and magnetic wave propagations [Taflove and Hagness (2000)]. With the advents of finite element technique in 1970's and 1980's, using the FETD method to solve the Maxwell's equations also became very popular in the CE community, especially for complicate geometry [Volakis *et al.* (1984), Silvester and Ferrari (1996)].

In many fields, such as microstrip, radio science, and scattering problems, etc. [Mei *et al.* (1984), Zhang and Mei (1988), Madsen and Siolkowski (1988), Fusco (1988)] in electromagnetic engineering, the FDTD method was applied to solve the Maxwell's equations. Similar to the FDTD method, the FETD scheme was also employed to solve the Maxwell's equations for the wave propagations of microwave cavity, electromagnetic wave scattering, and radiation [Thng and Booton (1994), Lee (1994), Kolbehdari and Sadiku (1998), Dibben and Metaxas (1996), Cangellaris (1991), Snaks and Lee (1995), Morgan *et al.* (1996), Morgan *et al.* (1998)] among others.

In the development of above conventional mesh oriented numerical methods, the mesh generation is required. These mesh-dependent schemes are very tedious in the process of mesh generation, especially for the three-dimensional problems with complicate geometry. In recent years, various meshless methods have been developed to alleviate the difficulty of mesh generation. Most of these meshless methods are evolved from the finite element method (FEM) or boundary element method (BEM) such as the meshless local Petrov-Galerkin (MLPG) method [Han and Atluri (2004)]. Meshless methods have also been applied to solve electromagnetic problems and radar scattering problems [Young and Ruan (2004), Qian et al (2004), Hassan et al (2004), Mittra and Prakash (2004)]. Meshless schemes using radial basis functions (RBFs) have also attracted great attention in science and engineering, due to their special features of simplicity and effectiveness. RBFs are very effective in modeling multivariate scattered problems with irregular domain, since they only depend upon distances between pairs of points in the computational field. In 1990, Kansa developed a domain-type mesh-

[1] Corresponding author, Fax and Tel: +886-2-23626114, E-mail: dlyoung@ntu.edu.tw
[2] Department of Mathematics, University of Southern Mississippi, Hattiesburg, MS 39406, USA
[3] Department of Civil Engineering & Hydrotech Research Institute, National Taiwan University, Taipei, Taiwan

less method using multiquadrics (MQ), which is widely circulated in the science and engineering communities. Similar to other meshless methods, the MQ method requires neither domain nor surface meshes. In RBF literature, the MQ method has been ranked as the best performance in multivariate interpolation. After introduced by Kansa (1990), the MQ method was found equally effective to solve various types of partial differential equations.

In this study, the MQ scheme is employed directly to solve the Maxwell's equations. Hardy (1971) first introduced MQ to approximate topographic surfaces from scattered data of surveying and mapping problems. The MQ scheme is a truly scattered, grid free scheme for approximating surfaces and bodies in any dimensions. Later applications to the computational fluid mechanics can be found in recent literature, which include the Burger's equation, shallow water wave equations, Stokes equations, and Navier-Stokes equations, etc. [Hon and Mao (1998), Hon et al. (1999), Young et al. (2004), Mai-Duy and Tran-Cong (2001)]. In comparison to other numerical methods of linear or quadratic convergence, both Maydych and Nelson (1992) and Cheng *et al.* (2003) have shown that if free shape parameter is chosen properly, the MQ scheme produces exponential convergence for surface interpolation. Therefore, with relatively small amount of collocation points, the MQ method can provide highly accurate numerical solutions. Furthermore, MQ scheme is not only an excellent method for accurate approximation of function values, but also for the approximation of their partial derivatives. As a result, the partial derivatives, divergences, curls, gradients, and integrals can be easily handled. Using MQ method, neither the fundamental solution nor singular integration is required, which are essential when using boundary element method (BEM). Note that the BEM has been widely applied in the CE community.

In this paper we demonstrate that the MQ method is a powerful tool for solving the Maxwell's equations. The propagations of the electric and magnetic waves in the homogeneous, isotropic and non-lossy 2D rectangular waveguide and 3D cavity resonator are considered to show the feasibility of the MQ method. We note that the analytical solutions of these two examples are available. Other numerical methods such as FETD scheme [Morgan *et al.* (1996)] and spectral-domain method [Omar and Schunemann (1991)] were also employed to study these two problems. The successful applications of these two benchmark problems have provided the opportunity for solving more general and complicated CE problems of the Maxwell's equations in the future studies.

2 Maxwell's Equations

2.1 Governing Equations

The electromagnetic fields are governed by the Maxwell's equations and are described as follows:

$$\nabla \cdot \overline{B} = 0 \tag{1}$$

$$\nabla \cdot \overline{D} = \rho_v \tag{2}$$

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \tag{3}$$

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} \tag{4}$$

where $\overline{E}$ is the electric field intensity, $\overline{H}$ is the magnetic field intensity, $\overline{D}$ is the electric flux density, $\overline{B}$ is the magnetic flux density, $\overline{J}$ is the electric current density, ρ_v is the electric charge density.

In order to simplify the programming problem of electromagnetic fields, we assume: (i) The waveguide or resonator is electrically and magnetically filled with linear, homogenous, isotropic, and source free dielectric material. (ii) The medium obeys the Ohm's law. (iii) All materials in the vacuum are assumed to be non-lossy.

Under these assumptions, the general Maxwell's equations (1) to (4) can be simplified as follows:

$$\nabla \cdot \overline{E} = 0 \tag{5}$$

$$\nabla \cdot \overline{H} = 0 \tag{6}$$

$$\nabla \times \overline{E} = -\mu \frac{\partial \overline{H}}{\partial t} \tag{7}$$

$$\nabla \times \overline{H} = \varepsilon \frac{\partial \overline{E}}{\partial t} \tag{8}$$

These simplified equations are the governing equations for the electromagnetic fields adopted in this study. As a first attempt, we assume the permittivity ε and permeability μ to be constants. As far as the MQ scheme is concerned, extension for solving more general Maxwell's equations (1) to (4) will cause no further difficulty.

2.2 Boundary Conditions

In this study, we assume the boundary material is a perfect electric conductor. The boundary conditions for the perfect conductor are shown as following:

$$\overline{n}\cdot\vec{H}=0 \tag{9}$$

$$\overline{n}\times\overline{E}=0 \tag{10}$$

3 Numerical Analysis

In this paper, the MQ method is used to deal with the Maxwell's equations. For three-dimensional time dependent Maxwell's equations, the time derivative term can be discretized by the finite difference method with the time integrating coefficient θ from time steps n to $n+1$ for Equations (7) and (8). Through the rest of this paper, we will only discretize the x component of Equation (8). The other remaining equations will not be further elaborated, since they can be obtained in a similar way. From Equation (8), it follows that

$$\varepsilon(E_x^{(n+1)}-E_x^{(n)})=\Delta t\theta\left(\frac{\partial H_z^{(n+1)}}{\partial y}-\frac{\partial H_y^{(n+1)}}{\partial z}\right)+\Delta t(1-\theta)\left(\frac{\partial H_z^{(n)}}{\partial y}-\frac{\partial H_y^{(n)}}{\partial z}\right) \tag{11}$$

According to the MQ scheme [Kansa (1990)], the electric and magnetic fields can be assumed as follows:

$$E_{x_i}=\sum_{j=1}^{m}a_{E_{x_j}}\sqrt{r_{ij}^2+c^2} \tag{12}$$

$$H_{y_i}=\sum_{j=1}^{m}a_{H_{y_j}}\sqrt{r_{ij}^2+c^2} \tag{13}$$

$$H_{z_i}=\sum_{j=1}^{m}a_{H_{z_j}}\sqrt{r_{ij}^2+c^2} \tag{14}$$

$$\left(\frac{\partial H_z}{\partial y}\right)_i=\sum_{j=1}^{m}a_{H_{z_j}}\left(\frac{y_i-y_j}{\sqrt{r_{ij}^2+c^2}}\right) \tag{15}$$

etc, where $r_{ij}^2=(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2$ for *i, j = 1,..., m*, and m is the number of the total node points contained in the domain and boundary, and c is called the shape parameter and is pre-assigned in the simulation. Therefore, the Equation (11) then becomes:

$$\begin{aligned}&\varepsilon\sum_{j=1}^{m}a_{E_{x_j}}^{(n+1)}\sqrt{r_{ij}^2+c^2}\\&-\theta\Delta t\left(\sum_{j=1}^{m}a_{H_{z_j}}^{(n+1)}\frac{y_i-y_j}{\sqrt{r_{ij}^2+c^2}}-\sum_{j=1}^{m}a_{H_{y_j}}^{(n+1)}\frac{z_i-z_j}{\sqrt{r_{ij}^2+c^2}}\right)\\&=\varepsilon\sum_{j=1}^{m}a_{E_{x_j}}^{(n)}\sqrt{r_{ij}^2+c^2}\\&+\Delta t(1-\theta)\left(\sum_{j=1}^{m}a_{H_{z_j}}^{(n)}\frac{y_i-y_j}{\sqrt{r_{ij}^2+c^2}}-\sum_{j=1}^{l}a_{H_{y_j}}^{(n)}\frac{z_i-z_j}{\sqrt{r_{ij}^2+c^2}}\right),\\&\qquad i=1,\dots,l\end{aligned} \tag{16}$$

where l is number of the total node points in the domain. The boundary conditions from Equation (9) are satisfied by collocation:

$$\sum_{j=1}^{m}a_{H_{z_j}}^{(n+1)}\sqrt{r_{ij}^2+c^2}=B_{H_{z_i}boundary\_condition},\ i=l+1,\dots,m. \tag{17}$$

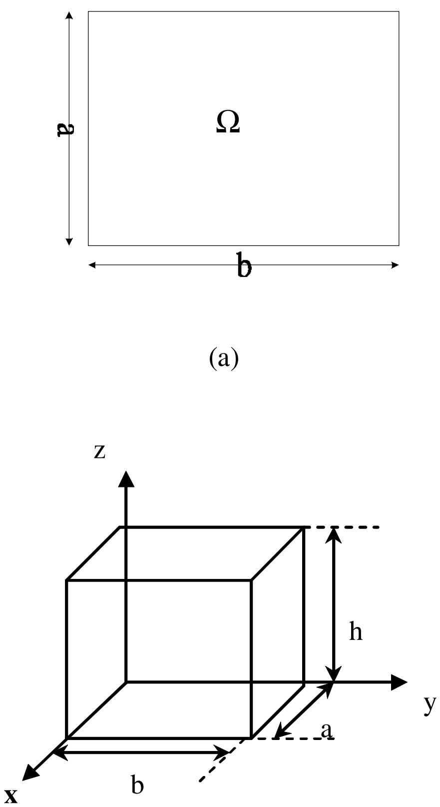
Equations (7) and (8) as well as the imposition of boundary conditions of Equations (9) and (10) can be undertaken in a similar way. Once these equations are obtained, a matrix system is formed and unknown coefficients can be obtained. The electric and magnetic fields for the whole domain can then be calculated from the known coefficients of equations. In the time evolution process, we need only to invert the field distance matrix in the Equations (16) and (17) once. In this way, high efficiency in numerical computation can be achieved.

4 Numerical Results and Discussions

A two-dimensional rectangular waveguide and a three-dimensional cavity resonator are chosen to demonstrate the capability of using the meshless MQ method to solve the simplified Maxwell's equations.

4.1 Two-Dimensional Rectangular Waveguide

The profile of the 2D rectangular waveguide is shown in Figure 1(a). We consider $a=1$ and $b=1$ to model the waveguide. To further simplify the modeling, the parameters ε and μ are all set equal to 1. The time step is



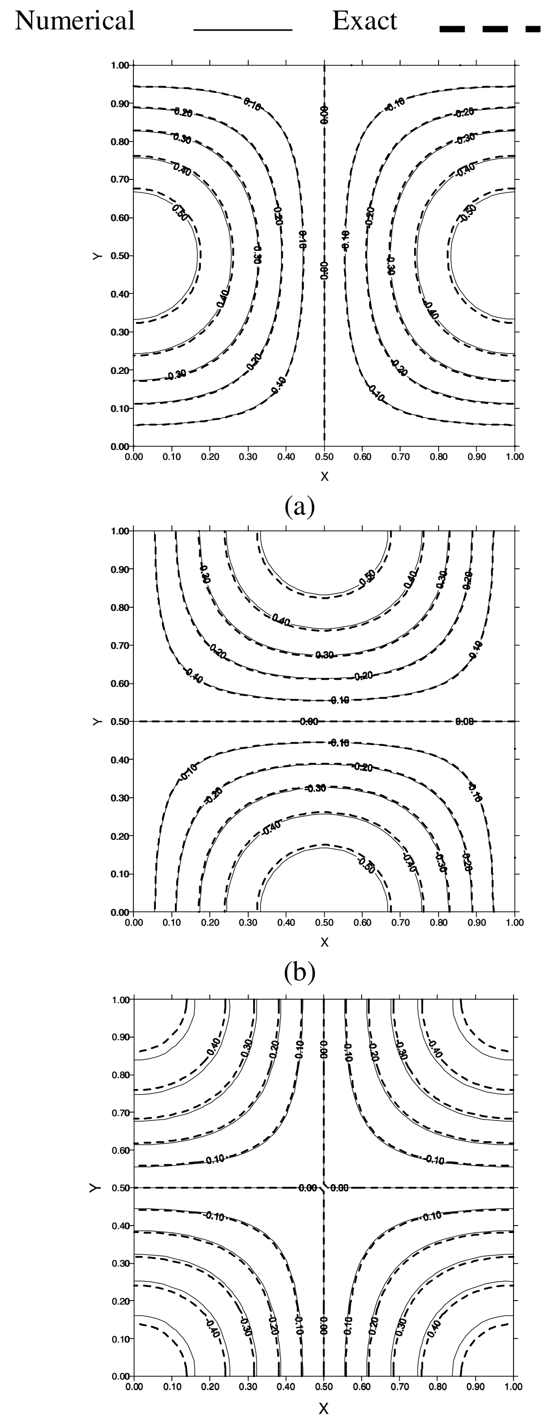
(a)

(b)

Figure 1 : (a) Geometry of the 2D rectangular waveguide (b) Geometry of the 3D cavity resonator

chosen to satisfy the stability constraint of the Courant-Friedrich-Levy (CFL) condition such that $\Delta t = 0.001$. The time integrating coefficient $\theta = 0.5$ is selected. The initial condition can be chosen freely. However, we take the initial conditions by setting time equal to zero from the analytic solutions, to make a direct comparison with the analytic solutions in this study. The analytical solutions of TM_z mode as well as TE_z mode can be found from the standard textbooks [Cheng (1989), Bowman et al. (1987), Balanis (1989)]. Only the TE_z mode is selected for illustration in the present numerical computations.

The MQ's shape parameter $c = 0.246$, and the node points 14*14 are taken. Figures 2 and 3 show the corresponding plane and solid contours of E_x, E_y and H_z for TE_z mode at time $t = 8$ respectively. Figure 4 shows the

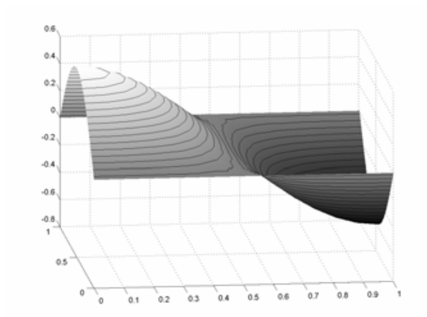


(a)

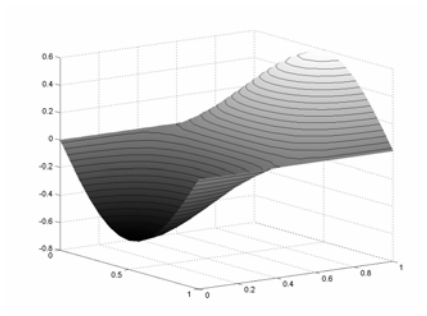
(b)

(c)

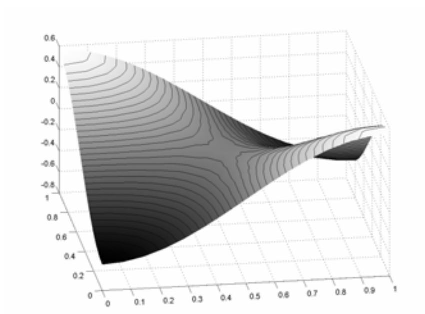
Figure 2 : Plane contours (a) E_x (b) E_y (c) H_z(node points 14*14, c=0.246, θ=0.5) for TE_z mode at t=8 of 2D waveguide



(a)

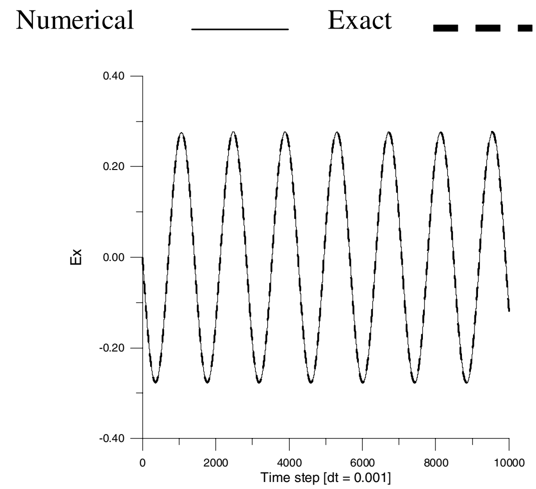


(b)

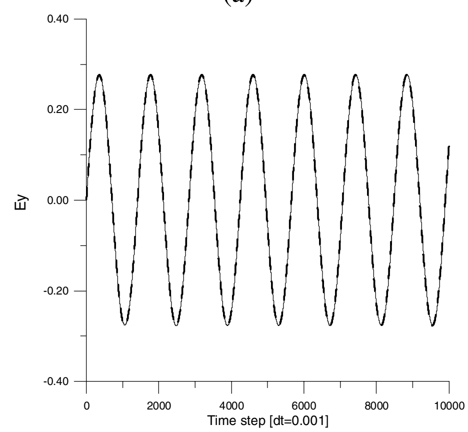


(c)

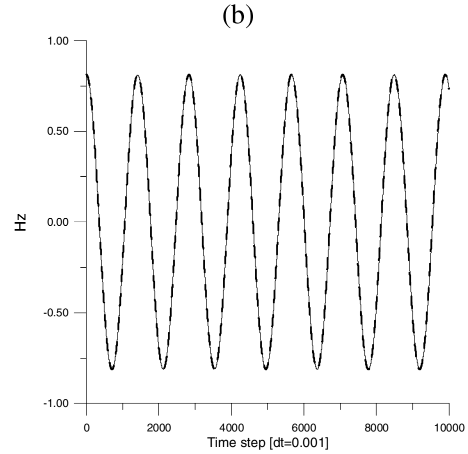
Figure 3 : Solid contours (a) E_x (b) E_y (c) H_z(node points 14*14, c=0.246, θ=0.5) for TE_z mode at t=8 of 2D waveguide



(a)



(b)



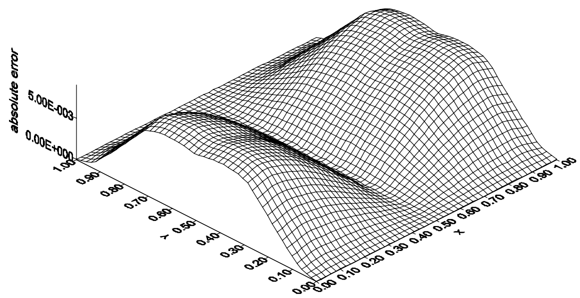
(c)

Figure 4 : The time history (a) E_x (b) E_y (c) H_z(at point (x,y)=(0.1428,0.1428), node points 14*14, c=0.246, θ=0.5) for TE_z mode of 2D waveguide

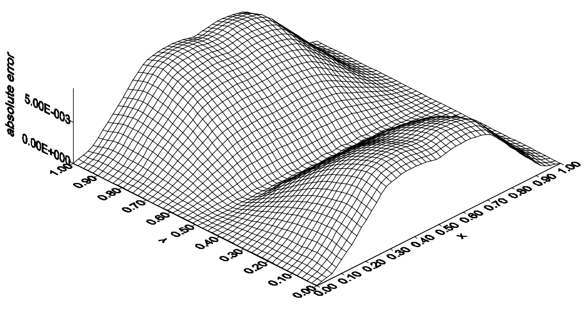
time history of E_x, E_y and H_z for TE_z mode at the fixed point $(x, y) = (0.1428, 0.1428)$. The profiles of the overall error distribution in the above cases are shown in Figure 5. These results show good agreement between the analytical solutions and the MQ method.

4.2 Three-Dimensional Cavity Resonator

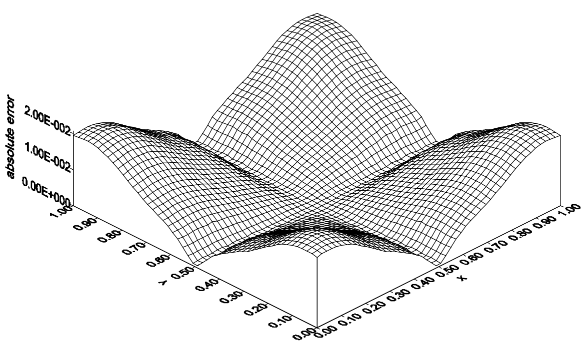
The profile of the 3D cavity resonator is shown in Figure 1(b). The dimensions of the resonator are $a = b = h =$



(a)



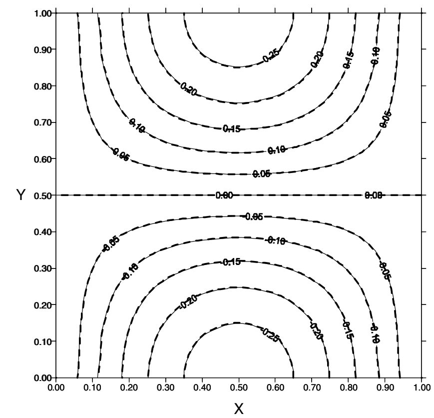
(b)



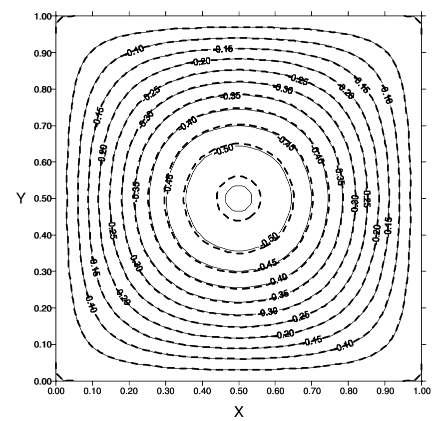
(c)

Figure 5 : The error distributions of (a) E_x (b) E_y (c) H_z(node points 14*14, c=0.246, θ=0.5) for TE_z mode at t=8 of 2D waveguide

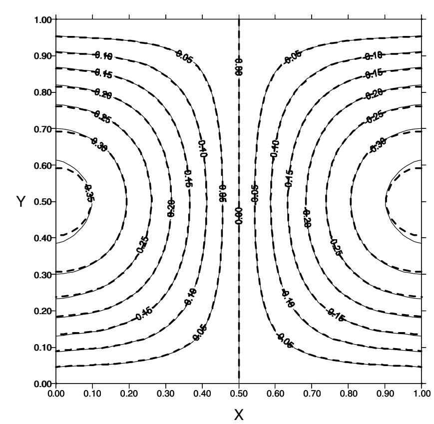
Numerical ——— Exact – – –



(a)



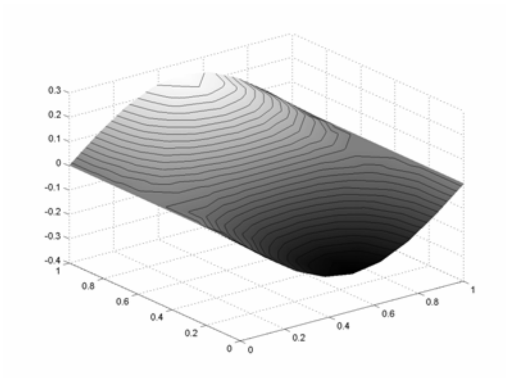
(b)



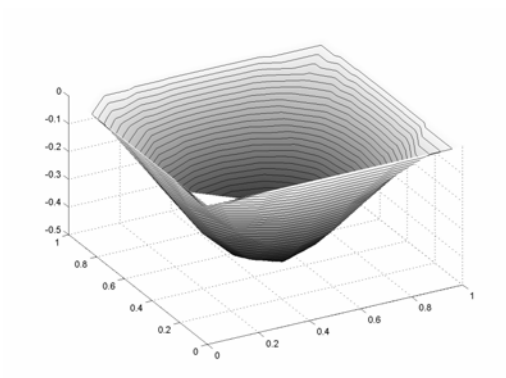
(c)

Figure 6 : Plane contours (a) E_y (b) E_z (c) H_y at plane z=0.75(node points 8*8*8, c=0.5) for TM mode at t=4.5 of 3D resonator

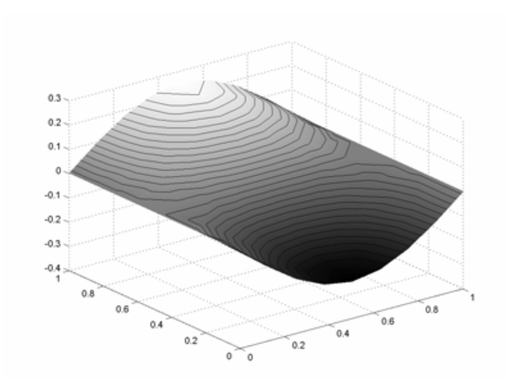
1. To further simplify the calculation, the parameters of μ, ε are all set to 1. Only the TM mode is selected for comparison. The node points 8*8*8, $\Delta t = 0.005$, and $\theta = 0.5$ are considered. We take shape parameter $c = 0.5$ in this case. The analytical solutions of TM mode and TE mode are also found from references [Cheng (1989),



(a)

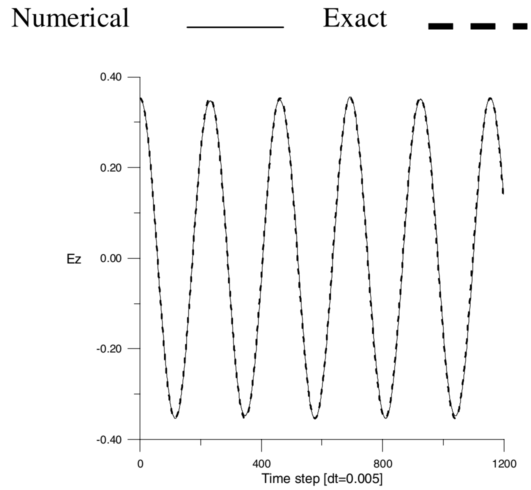


(b)

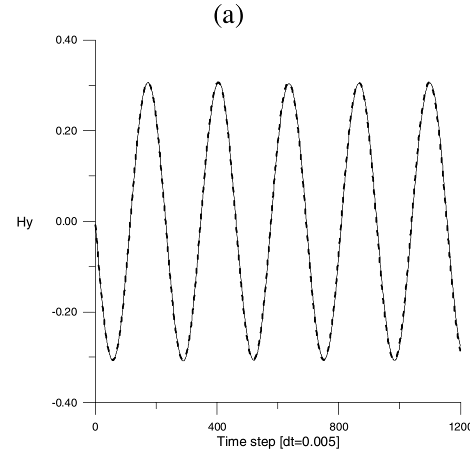


(c)

Figure 7 : Solid contours (a) E_y (b) E_z (c) H_y at plane z=0.75(node points 8*8*8, c=0.5) for TM mode at t=4.5 of 3D resonator



(a)



(b)

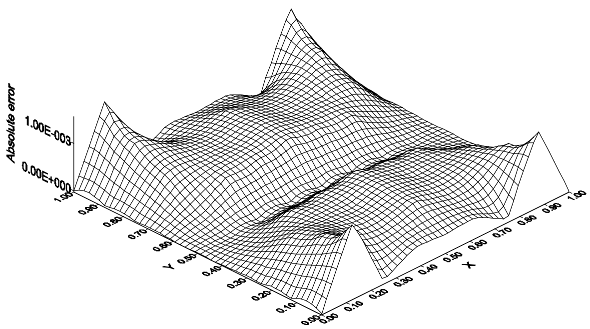
Figure 8 : The time history (a) E_z (b) H_y (at point (x,y,z)=(0.75,0.25,0.25), TM mode of 3D resonator

Bowman et al. (1987), Balanis (1989)].

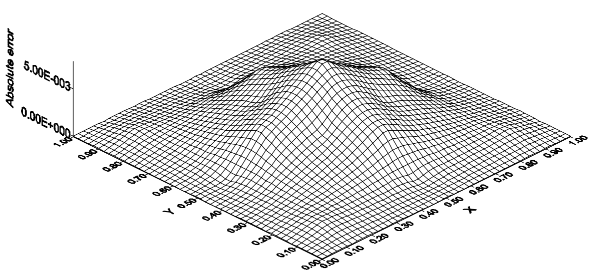
Figures 6 and 7 illustrate the plane and solid contours of E_y,E_z,H_y respectively for the TM mode along (x,y,z) = $(x,y,\ 0.75)$ plane at $t = 4.5$ (900-th time step). Figure 8 shows the harmonic evolutions of time histories of E_z, and H_y for the TM mode at fixed point (x,y,z) = (0.75, 0.25, 0.25). The profiles of corresponding error distribution are shown in Figure 9**.** The results also show very good agreement between the analytical solution and the MQ method.

5 Conclusions

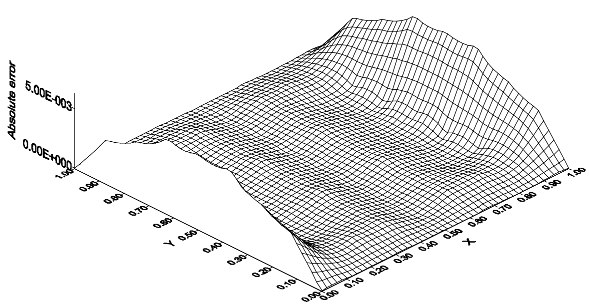
Without transforming the Maxwell's equations into wave or Helmholtz equations, the time-varying Maxwell's equations are simulated directly in the time-space do-



(a)



(b)



(c)

Figure 9 : The error distributions of (a) E_y (b) E_z (c) H_y at plane z=0.75(node points 8*8*8, c=0.5) for TM mode at t=4.5 of 3D resonator

main using MQ method. In this paper, the MQ method is employed to model the propagation of the electromagnetic waves in the simple 2D rectangular waveguide and 3D cavity resonator. In both cases, good agreements are obtained comparing to the analytical solutions. Since the MQ time domain method does not require a mesh generation, the method is considered free from boundary and domain integrations, the integral of singularity, and frequency searching. The MQ scheme is an excellent method not only for very accurate interpolation of the field variables, but also for the approximation of appropriate partial derivatives, divergences, curls, gradients or integrals. Numerical results show that the MQ method approximates the electric and magnetic wave propagations very well by even using very coarse node points. The good performance of the MQ scheme has demonstrated that it is a powerful and potential tool for the numerical solutions of electromagnetic field problems. Even though we only consider standard cases in this study, we believe that the MQ method is capable to simulate the multimode, high frequency, and very complex electronic and electric devices. We will continue to investigate these cases in our future studies.

Acknowledgement: The first and third authors acknowledge the financial support from the National Science Council of Taiwan to carry out this work under the Grant No. NSC 93-2611-E-002-001. The second author also acknowledges the support of NATO Collaborative Linkage Grant under the reference PST.CLG.980398.

References

Balanis, C. A. (1989): *Advanced engineering electromagnetics.* John Wiley & Sons, New York.

Bowman, J. J.; Senior, T. B. A.; Uslenghi, P. L. E. (1987): *Electromagnetic and acoustic scattering by simple shapes.* Hemisphere Publishing Corporation, New York.

Cangellaris, A. C. (1991): Time-domain finite element for electromagnetic wave propagation and scattering. *IEEE Trans Magn*, vol.27, pp.3780-3785.

Cheng, A H. D.; Golberg, M. A.; Kansa, E. J.; Zammito, G. (2003): Exponential convergence and H-c multiquadric collocation method for partial differential equations. *Numer Meth Part Differ Eq*, vol.19, pp.571-594.

Cheng, D. K. (1989): *Field and wave electromagnetics.* Addison-Wesley.

Dibben, D.C.; Metaxas, R. (1996): Time domain finite element analysis of multimode microwave applications. *IEEE Trans Magn*, vol.32, pp.942-945.

Fusco, M. (1988): FDTD algorithm in curvilinear coordinates. *IEEE Trans Antennas Propagat*, vol. 38, pp.76-89.

Han Z.D.; Atluri, S.N. (2004): A meshless local Petrov-Galerkin (MLPG) approach for 3-dimensional elasto-dynamics. *CMC : Computers, Materials, & Continua*, vol.1,no.2,pp.129-140.

Hassan O.; Morgan K.; Jones J.; Larwood B.; Weatherill N. P. (2004): Parallel 3D time domain electromagnetic scattering simulations on unstructured meshes, *CMES: Computer Modeling in Engineering & Sciences,* vol. 5, no. 5, pp. 383-394.

Hardy, R. L. (1971): Multiquadric equations of topography and other irregular surfaces. *Geophys Res*, vol.176, pp. 905-915.

Hon, Y. C.; Cheung, K. F.; Mao, X. Z.; Kansa, E. J. (1999): Multiquadric solution for shallow water equations. *J Hydraul Engng*, vol.125, no.5, pp. 524-533.

Hon, Y. C; Mao, X. Z. (1998): An efficient numerical scheme for Burger's elliptical equation. *Appl Math Comput*, vol. 95, pp. 37-50.

Kansa, E. J. (1990): Multiquadrics - a scattered data approximation scheme with applications to computational fluid-dynamics-II: Solutions to parabolic, hyperbolic and elliptic partial differential equations. *Comput Math Applic*, vol.19,no.8/9, pp.147-161.

Kolbehdari, M. A.; Sadiku, M. S. (1998): Explicit and implicit time domain finite element methods using whiteny forms. *IEEE Southeastcon' 98 Proceedings*, pp.73-76.

Lee, J. F. (1994): WETD-A finite element time-domain approach for solving Maxwell's equations. *IEEE Microwave Guided Wave Lett*, vol.4, pp.11-13.

Madsen, N. K.; Siolkowski, R. W. (1988): Numerical solution of Maxwell's equations in the time domain using irregular nonorthogonal grids. *Wave Motions*, vol.10, pp. 583-596.

Madych, W. R.; Nelson, S. A. (1992): Bounds on multivariate polynomials and exponential estimates for multivariate interpolation. *J Approx Theory*, vol.70, pp.94-114.

Mai-Duy, N.; Tran-Cong, T. (2001): Numerical solution of Navier-Stokes equations using multiquadrics radial basis function networks. *Int J Numer Meth Fluids,* vol.37, pp. 65-86.

Mei, K. K.; Changellaris, A.; Angelakos, D. J. (1984): Conformal time domain finite difference method. *Radio Science*, vol.9, no.5, pp.1145-1147.

Mittra, Raj; Prakash, V.V.S. (2004): The characteristic basis function method: A new technique for fast solution of radar scattering problems, *CMES: Computer Modeling in Engineering & Sciences,*vol. 5, no. 5, pp. 435-442.

Morgan, K.; Brookes, P. J.; Hassan, O.; Weatherill, N. P. (1998): Parallel processing for the simulation of problems involving scattering of electromagnetic waves. *Comput Meth App Mech Engng*, vol.152, pp.157-174.

Morgan, K.; Hassan, O.; Perarie, J. (1996): A time domain unstructured grid approach to the simulation of electromagnetic scattering in piecewise homogeneous media. *Comput Meth App Mech Engng*, vol.134, pp.17-36.

Omar, A. S.; Schunemann, K. F. (1991): Application of the generalized spectral-domain technique to the analysis of rectangular waveguides with rectangular and circular metal inserts. *IEEE Tran Microwave Theory Tech*, vol.39, no.6, pp. 944-952.

Qian, Z.Y.; Han, Z.D.; Atluri, S.N. (2004): Directly derived non-hyper-singular boundary integral equations for acoustic problems, and their solution through Petrov-Galerkin schemes, *CMES: Computer Modeling in Engineering & Sciences,* vol. 5, no. 6, pp. 541-562.

Snaks, Z. S; Lee, J. F. (1995): A finite-element time-domain method using prism elements for microwave cavities. *IEEE Trans Electromagn Compat*, vol.37, pp.519-527.

Silvester, P. P.; Ferrari, R. L. (1996): *Finite elements for electrical engineers*. Cambridge University Press, Cambridge, UK.

Taflove, A.; Hagness, S. C. (2000): *Computational electrodynamics: the finite difference time-domain method*. Artech House, Boston.

Thng, C. H.; Booton, R. C. (1994): Edge-element time-domain method for solving Maxwell's equations. *IEEE MTT-S International*, vol.2, pp.693-696.

Volakis, J. L.; Chatterjee, A.; Kempel, L. C. (1998): *Finite element method for electromagnetics*. IEEE Antennas & Propagation Society, Sponsor, IEEE Press.

Yee, K. S. (1966): Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media. *IEEE Trans Antennas Propagat*, vol.AP-14, no.3, pp.302-307.

Young, D. L.; Jane, S. C.; Lin, C. Y.; Chiu, C. L.; Chen, K. C. (2004): Solutions of 2D and 3D Stokes laws using multiquadrics method. *Engng Anal Bound Ele*, vol. 28, pp. 1233-1243.

Young, D.L.; Ruan J.W. (2004): Method of fundamental solutions for scattering problems of electromagnetic waves, *CMES: Computer Modeling in Engineering & Sciences,* vol. 7, no. 2, pp. 223-232.

Zhang, X. L.; Mei, K. K. (1988): Time-domain finite difference approach to the calculation of the frequency-dependent characteristics of microstrip discontinuities. *IEEE Trans. Microwave Theory Tech*, vol.36, no.12, pp.1775-1787.