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Asymptotic Solutions for Multilayered Piezoelectric Cylinders under Electromechanical Loads

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Abstract: Based on the three-dimensional (3D) piezoelectricity, we presented asymptotic solutions for multilayered piezoelectric hollow cylinders using the method of perturbation. The material properties in the general formulation are firstly regarded to be heterogeneous through the thickness, and then specified as the layerwise step functions in the cases of multilayered cylinders. The transverse normal load and normal electric displacement are respectively applied on the lateral surfaces of the cylinders. The boundary conditions of cylinders are considered to be simply supported at the two edges. In the formulation the twenty-two basic equations of piezoelectricity are reduced to eight differential equations in terms of eight primary variables of elastic and electric fields. After performing nondimensionalization, asymptotic expansion and successive integration, we finally decompose the 3D problem into a series of 2D problems with the same governing equations for various orders except for the nonhomogeneous terms. In view of the recurrent property, it is illustrated that the present asymptotic solutions can be obtained in a hierarchic manner and asymptotically approach 3D piezoelectricity solutions.

keyword: Piezoelectric material, Cylinders, Exact solutions, Piezoelectricity, Asymptotic expansion, Perturbation

1 Introduction

In recent years, laminated composite structures bonded with piezoelectric actuators and sensors on the outer surfaces of the structures were widely used as intelligent or smart structures in the engineering applications. Determination of exact solutions of piezoelectric laminates naturally becomes an attractive research subject. It is well known that there are twenty-two basic equations of three-dimensional (3D) piezoelectricity governing the electro-elastic behavior of piezoelectric laminated structures. In view of the considerable number of equations, the exact solution of piezoelectricity is hard to be obtained and is scarce in the literature.

Heyliger (1997) presented exact solutions for the static behavior of simply-supported laminated piezoelectric cylinders. In his analysis, the laminated piezoelectric cylinders are subjected to either transverse normal loads or electric potential on the outer surfaces. The primary field variables are expanded as Fourier series in the in-surface directions where the edge boundary conditions in axial direction and periodical continuity conditions in the circumferential direction are satisfied. By using the set of double Fourier series functions, Heyliger reduced basic equations of 3D piezoelectricity as a set of ordinary differential equations. The Frobenious method is used to evaluate the coupling effects of elastic and electric fields on the structural behavior of multilayered piezoelectric cylinders. The similar methodology is used to obtain exact solutions of orthotropic cylindrical shells with piezoelectric layers under the cylindrical bending type of lateral loads or electric potential (Chen, Shen and Wang, 1996). By expanding the field variables as a product of an exponential function and a power series in the thickness coordinate, Kapuria, Sengupta and Dumir (1997) also used the aforementioned methodology to present exact solutions for simply-supported piezoelectric cylindrical shells under axisymmetric electromechanical loads.

Tarn (2002) presented a state space formalism for electro-thermo-elastic analysis of a linear piezoelectric body. Exact solutions both for a piezoelectric half-space under a line of electromechanical loading and for an infinite piezoelectric plate with an elliptical notch subjected to in-plane loads are determined. Lee and Jiang (1996) presented an analytical approach for the electromechanical analysis of laminated piezoelectric structures. In their analysis, the state space equations for a 3D piezoelectric lamina are derived. By using the method of transfer matrix, Lee and Jiang studied the coupled electro-elastic

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behavior of piezoelectric laminates. Comprehensive reviews of theoretical analysis and numerical modeling for piezoelectric laminates were presented by Saravanos and Heyliger (1999) and Gopinathan, Varadan and Varadan (2000).

An alternative analytical approach to obtain exact solutions of doubly curved laminated piezoelectric shells was proposed by Wu and his colleagues (2005, 2006). By means of the perturbation method, the 3D asymptotic formulations for the static and dynamic analyses of laminated piezoelectric shells were developed. Several benchmark problems were studied by applying the asymptotic approach.

After a close examination on the exact analysis of piezoelectric laminates in the literature, we found that most of the aforementioned articles dealt with the structural behavior of piezoelectric laminates under applied electric potential rather than normal electric displacement on the lateral surfaces. Based on the generalized Hamilton's principle, Tiersten (1969) indicated that there are two possibilities for electric loading conditions on the lateral surfaces (i.e., either electric potential or normal electric displacement is prescribed). To replenish the benchmark solutions for 3D piezoelectricity, we presented exact solutions for multilayered piezoelectric hollow cylinders under transverse normal loads and normal electric displacement. By using the different dimensionless forms of electric field variables from the previous work (Wu et al., 2005), it turns the variable of electric potential to be one of the primary variables in the governing equations for various orders problems. Since the successive integration is performed through the thickness coordinate to the differential equation of normal electric displacement, unlike to that of electric potential in the previous work (Wu et al., 2005), the present analysis for applied normal electric displacement cases can then be considered. The coupled electro-elastic effect on the structural behavior of piezoelectric cylinders subjected to two different types of electric loads was evaluated using the present and previous asymptotic formulations, respectively. The illustrative examples show that both two formulations yield satisfactory results and the present asymptotic solutions converge rapidly.

2 Basic equations of piezoelectricity

Consider a multilayered piezoelectric hollow cylinder as shown in Fig. 1 in the present formulation. The cylindrical coordinates system with variables x, θ, r is used and located on the middle surface of the cylinder. $2h$, L and R stand for the total thickness, the length and the curvature radii to the middle surface of the cylinder, respectively.

The linear constitutive equations of piezoelectric material in the cylindrical coordinates system are given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_r \\ \tau_{\theta r} \\ \tau_{xr} \\ \tau_{x\theta} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \varepsilon_r \\ \gamma_{\theta r} \\ \gamma_{xr} \\ \gamma_{x\theta} \end{Bmatrix} - \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{32} \\ 0 & 0 & e_{33} \\ e_{14} & e_{24} & 0 \\ e_{15} & e_{25} & 0 \\ 0 & 0 & e_{36} \end{bmatrix} \begin{Bmatrix} E_x \\ E_\theta \\ E_r \end{Bmatrix} \tag{1}$$

$$\begin{Bmatrix} D_x \\ D_\theta \\ D_r \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & e_{14} & e_{15} & 0 \\ 0 & 0 & 0 & e_{24} & e_{25} & 0 \\ e_{31} & e_{32} & e_{33} & 0 & 0 & e_{36} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \varepsilon_r \\ \gamma_{\theta r} \\ \gamma_{xr} \\ \gamma_{x\theta} \end{Bmatrix} + \begin{bmatrix} \eta_{11} & \eta_{12} & 0 \\ \eta_{12} & \eta_{22} & 0 \\ 0 & 0 & \eta_{33} \end{bmatrix} \begin{Bmatrix} E_x \\ E_\theta \\ E_r \end{Bmatrix}, \tag{2}$$

where $\sigma_x, \sigma_\theta, \sigma_r, \tau_{xr}, \tau_{\theta r}, \tau_{x\theta} and \varepsilon_x, \varepsilon_\theta, \varepsilon_r, \gamma_{xr}, \gamma_{\theta r}, \gamma_{x\theta}$ denote the stress and strain components, respectively. D_x, D_θ, D_r and E_x, E_θ, E_r denote the components of electric displacement and electric fields, respectively. c_{ij}, e_{ij} and η_{ij} are the elastic coefficients, piezoelectric coefficients and dielectric coefficients, respectively, relative to the geometrical axes of the hollow cylinder. The material is regarded to be heterogeneous through the thickness (i.e., $c_{ij}(\zeta)$, $e_{ij}(\zeta)$ and $\eta_{ij}(\zeta)$). For a typical multilayered piezoelectric cylinder considered in the paper, its material properties are the layerwise step functions through the thickness coordinate.

The kinematic equations in terms of the cylindrical coordinates $x, \theta and r$ are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \varepsilon_r \\ \gamma_{\theta r} \\ \gamma_{xr} \\ \gamma_{x\theta} \end{Bmatrix} = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \frac{1}{r}\partial_\theta & \frac{1}{r} \\ 0 & 0 & \partial_r \\ 0 & -\frac{1}{r}+\partial_r & \frac{1}{r}\partial_\theta \\ \partial_r & 0 & \partial_x \\ \frac{1}{r}\partial_\theta & \partial_x & 0 \end{bmatrix} \begin{Bmatrix} u_x \\ u_\theta \\ u_r \end{Bmatrix}, \quad (3)$$

in which $u_x, u_\theta and u_r$ are the displacement components; $\partial_x = \partial/\partial_x$, $\partial_\theta = \partial/\partial_\theta$, $\partial_r = \partial/\partial_r$.

The stress equilibrium equations without body forces are given by

$$r\sigma_{x,x} + \tau_{x\theta,\theta} + \tau_{xr} + r\tau_{xr,r} = 0, \quad (4)$$

$$\gamma\tau_{x\theta,x} + \sigma_{\theta,\theta} + r\tau_{\theta r,r} + 2\tau_{\theta r} = 0, \quad (5)$$

$$r\tau_{xr,x} + \tau_{\theta r,\theta} + r\sigma_{r,r} + \sigma_r - \sigma_\theta = 0. \quad (6)$$

The charge equation of the piezoelectric material without electric charge density is

$$D_{x,x} + \frac{1}{r}D_{\theta,\theta} + D_{r,r} + \frac{1}{r}D_r = 0. \quad (7)$$

The relations between the electric field and electric potential are

$$E_x = -\Phi_{,x}, \quad (8a)$$

$$E_\theta = -\frac{1}{r}\Phi_{,\theta}, \quad (8b)$$

$$E_r = -\Phi_{,r}, \quad (8c)$$

where Φ denotes the electric potential.

The boundary conditions of the problem are specified as follows:

On the lateral surface the transverse load $\overline{q}_r^{\pm}(x,\theta)$ and normal electric displacement $\overline{D}_r^{\pm}(x,\theta)$ are prescribed,

$$[\tau_{xr} \quad \tau_{\theta r}] = [0 \quad 0] \qquad \text{on} \quad r = R \pm h, \quad (9a)$$

$$\sigma_r = \overline{q}_r^{\pm}(x,\theta) \qquad \text{on} \quad r = R \pm h, \quad (9b)$$

$$D_r = \overline{D}_r^{\pm}(x,\theta) \qquad \text{on} \quad r = R \pm h. \quad (10)$$

The edge boundary conditions require one member of each pair of the following quantities be satisfied,

$$n_1\sigma_x + n_2\tau_{x\theta} = \overline{p}_1^* \qquad \text{or} \qquad u_x = \overline{u}_x^*, \quad (11a)$$

$$n_1\tau_{x\theta} + n_2\sigma_\theta = \overline{p}_2^* \qquad \text{or} \qquad u_\theta = \overline{u}_\theta^*, \quad (11b)$$

$$n_1\tau_{xr} + n_2\tau_{\theta r} = \overline{p}_3^* \qquad \text{or} \qquad u_r = \overline{u}_r^*, \quad (11c)$$

where $\overline{p}_1^*, \overline{p}_2^*$ and $\overline{p}_3^*$ are the applied edge loads; $\overline{u}_x^*, \overline{u}_\theta^*$ and $\overline{u}_r^*$ are the prescribed edge displacements; $n_1 and n_2$denote the outward unit normal at a point along the edge.

In addition, the edges are suitably grounded so that the electric potential Φ at the edges are zero and given by

$$\Phi = 0. \quad (12)$$

According to Eqs.(1)-(8), it is listed that there are twenty-two basic equations for the present electroelastic analysis of piezoelectric hollow cylinders. For a 3D analysis, we must determine the aforementioned twenty-two unknown variables satisfying the basic equations (Eqs.(1)-(8)) in the shell domain, the boundary conditions at outer surfaces (Eqs. (9)-(10)) and the edges (Eqs. (11)-(12)). In the following derivation, we present an asymptotic formulation for the 3D analysis of single-layer and multilayered hybrid piezoelectric hollow cylinders.

3 Nondimensionalization

A set of dimensionless coordinates and variables are defined as

$$x_1 = x/R\varepsilon, \quad x_2 = \theta/\varepsilon, \quad x_3 = \zeta/h \quad \text{and} \quad r = R + \zeta;$$

$$u_1 = u_x/R\varepsilon, \quad u_2 = u_\theta/R\varepsilon, \quad u_3 = u_r/R;$$

$$\sigma_1 = \sigma_x/Q, \quad \sigma_2 = \sigma_\theta/Q, \quad \tau_{12} = \tau_{x\theta}/Q;$$

$$\tau_{13} = \tau_{xr}/Q\varepsilon, \quad \tau_{23} = \tau_{\theta r}/Q\varepsilon, \quad \sigma_3 = \sigma_r/Q\varepsilon^2;$$

$$D_1 = D_x/\varepsilon^{(j-1)}e, \quad D_2 = D_\theta/\varepsilon^{(j-1)}e, \quad D_3 = D_r/e;$$

$$\phi = \Phi e/\varepsilon^j RQ; \quad (13)$$

where $\varepsilon^2 = h/R$; Q and e denote a reference elastic moduli and a reference piezoelectric moduli, respectively. In the present formulation, the superscript j is taken as zero that corresponds to the analysis where the normal electric displacement is prescribed on the lateral surfaces; whereas j=2 is used in the previous work (Wu et al., 2005) where the electric potential is prescribed on the lateral surfaces.

By eliminating the secondary field variables (σ_x, σ_θ, $\tau_{x\theta}$, D_x, D_θ, ε_x, ε_θ, ε_r, γ_{xr}, $\gamma_{\theta r}$, $\gamma_{x\theta}$, E_x, E_θ, E_r) from

Eqs.(1)-(8), and then introducing the set of dimensionless coordinates and variables (Eq. (13)) in the resulting equations, we can rewrite the basic differential equations in the form of

$$u_{3,3} = -\varepsilon^2 \mathbf{L}_1 \mathbf{u} - \varepsilon^2 \tilde{l}_{33} u_3 + \varepsilon^4 \tilde{l}_{34} \sigma_3 + \varepsilon^2 \tilde{l}_{35} D_3, \quad (14)$$

$$\mathbf{u}_{,3} = -\mathbf{D} u_3 + \varepsilon^2 \mathbf{L}_2 \mathbf{u} + \varepsilon^2 \mathbf{S} \sigma_s + \varepsilon^4 \mathbf{L}_3 \sigma_s + \mathbf{L}_4 \phi, \quad (15)$$

$$D_{3,3} = -\mathbf{D}^T \mathbf{d} - \varepsilon^2 x_3 D_{1,1} - \varepsilon^2 \tilde{l}_{44} D_3, \quad (16)$$

$$\sigma_{s,3} = -\mathbf{L}_5 \mathbf{u} - \mathbf{L}_6 u_3 - \varepsilon^2 \mathbf{L}_7 \sigma_s - \varepsilon^2 \gamma_\theta \mathbf{L}_1^T \sigma_3 - \mathbf{L}_8 D_3, \quad (17)$$

$$\sigma_{3,3} = \mathbf{L}_9 \mathbf{u} + \tilde{l}_{63} u_3 - \mathbf{D}^T \sigma_s - \varepsilon^2 \mathbf{L}_{10} \sigma_s - \varepsilon^2 \tilde{l}_{64} \sigma_3 + \tilde{l}_{65} D_3, \quad (18)$$

$$\phi_{,3} = -\varepsilon^2 (1/\gamma_\theta) \mathbf{L}_8^T \mathbf{u} - \varepsilon^2 (1/\gamma_\theta) \tilde{l}_{65} u_3 + \varepsilon^4 \tilde{l}_{35} \sigma_3 - \varepsilon^2 \tilde{l}_{81} D_3, \quad (19)$$

where

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}, \quad \mathbf{D} = \begin{Bmatrix} \partial_1 \\ \partial_2 \end{Bmatrix}, \quad \mathbf{S} = \begin{bmatrix} \tilde{l}_{14} & \tilde{l}_{15} \\ \tilde{l}_{15} & \tilde{l}_{25} \end{bmatrix},$$

$$\sigma_s = \begin{Bmatrix} \tau_{13} \\ \tau_{23} \end{Bmatrix}, \quad \mathbf{d} = \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix},$$

$$\mathbf{L}_1 = \begin{bmatrix} \tilde{l}_{31} & \tilde{l}_{32} \end{bmatrix}, \quad \mathbf{L}_2 = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{l}_{22} \end{bmatrix},$$

$$\mathbf{L}_3 = \begin{bmatrix} 0 & 0 \\ \tilde{l}_{26} & \tilde{l}_{27} \end{bmatrix}, \quad \mathbf{L}_4 = \begin{bmatrix} \tilde{l}_{18} \\ \tilde{l}_{28} \end{bmatrix},$$

$$\mathbf{L}_5 = \begin{bmatrix} \tilde{l}_{41} & \tilde{l}_{42} \\ \tilde{l}_{51} & \tilde{l}_{52} \end{bmatrix}, \quad \mathbf{L}_6 = \begin{bmatrix} \tilde{l}_{43} \\ \tilde{l}_{53} \end{bmatrix},$$

$$\mathbf{L}_7 = \begin{bmatrix} \tilde{l}_{44} & 0 \\ 0 & \tilde{l}_{55} \end{bmatrix}, \quad \mathbf{L}_8 = \begin{bmatrix} \tilde{l}_{46} \\ \tilde{l}_{56} \end{bmatrix},$$

$$\mathbf{L}_9 = \begin{bmatrix} \tilde{l}_{61} & \tilde{l}_{62} \end{bmatrix}, \quad \mathbf{L}_{10} = \begin{bmatrix} x_3 \partial_1 & 0 \end{bmatrix},$$

$\gamma_\theta = 1 + \varepsilon^2 x_3$,and $\tilde{l}_{ij}$ are given in Appendix A.

The in-surface stresses and electric displacements can be expressed in terms of the primary variables as follows:

$$\sigma_p = \mathbf{B}_1 \mathbf{u} + \mathbf{B}_2 u_3 + \varepsilon^2 \mathbf{B}_3 \sigma_3 + \mathbf{B}_4 D_3 \quad (20)$$

$$\mathbf{d} = \varepsilon^2 \mathbf{B}_5 \sigma_s + \mathbf{B}_6 \phi, \quad (21)$$

where

$$\sigma_p = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{12} \\ \tilde{b}_{21} & \tilde{b}_{22} \\ \tilde{b}_{31} & \tilde{b}_{32} \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} \tilde{b}_{13} \\ \tilde{b}_{23} \\ \tilde{b}_{33} \end{bmatrix},$$

$$\mathbf{B}_3 = \begin{bmatrix} \tilde{b}_{14} \\ \tilde{b}_{24} \\ \tilde{b}_{34} \end{bmatrix}, \quad \mathbf{B}_4 = \begin{bmatrix} \tilde{b}_{15} \\ \tilde{b}_{25} \\ \tilde{b}_{35} \end{bmatrix}, \quad \mathbf{B}_5 = \begin{bmatrix} \tilde{b}_{41} & \tilde{b}_{42} \\ \tilde{b}_{51} & \tilde{b}_{52} \end{bmatrix},$$

$$\mathbf{B}_6 = \begin{bmatrix} \tilde{b}_{43} \\ \tilde{b}_{53} \end{bmatrix},$$

and $\tilde{b}_{ij}$ are given in Appendix A.

The dimensionless form of boundary conditions of the problem are specified as follows:

On the lateral surface the transverse load and electric potential are prescribed,

$$[\tau_{13} \quad \tau_{23}] = [0 \quad 0] \qquad \text{on} \quad x_3 = \pm 1, \quad (22a)$$

$$\sigma_3 = \overline{q}_3^{\pm}(x_1, x_2) \qquad \text{on} \quad x_3 = \pm 1, \quad (22b)$$

$$D_3 = \overline{D}_3^{\pm}(x_1, x_2) \qquad \text{on} \quad x_3 = \pm 1. \quad (23)$$

At the edges one member of each pair of the following quantities is satisfied,

$$n_1 \sigma_1 + n_2 \tau_{12} = \overline{p}_{n1} \qquad \text{or} \quad u_1 = \overline{u}_1, \quad (24a)$$

$$n_1 \tau_{12} + n_2 \sigma_2 = \overline{p}_{n2} \qquad \text{or} \quad u_2 = \overline{u}_2, \quad (24b)$$

$$n_1 \tau_{13} + n_2 \tau_{23} = \overline{p}_{n3} \qquad \text{or} \quad u_3 = \overline{u}_3. \quad (24c)$$

In addition,

$$\phi = 0, \quad (25)$$

where $\overline{q}_3^{\pm} = \overline{q}_r^{\pm}/Q\varepsilon^2$; $\overline{D}_3^{\pm} = \overline{D}_r^{\pm}/e$; $(\overline{p}_{n1}, \overline{p}_{n2}, \overline{p}_{n3}) = (\overline{p}_1^*/Q, \overline{p}_2^*/Q, \overline{p}_3^*/Q\varepsilon)$; $(\overline{u}_1, \overline{u}_2, \overline{u}_3) = (\overline{u}_1^*/R\varepsilon, \overline{u}_\theta^*/R\varepsilon, \overline{u}_r^*/R)$.

4 Asymptotic expansions

Since Eqs.(14)-(19) contain terms involving only even powers of ε, we therefore asymptotically expand the primary variables in the powers ε^2 as given by

$$f(x_1, x_2, x_3, \varepsilon) = f^{(0)}(x_1, x_2, x_3) + \varepsilon^2 f^{(1)}(x_1, x_2, x_3) + \varepsilon^4 f^{(2)}(x_1, x_2, x_3) + \ldots \quad (26)$$

Substituting Eq.(26) into Eqs.(14)-(19) and collecting coefficients of equal powers of ε, we obtain the following sets of recurrence equations.

Order ε^0 :

$$u_3^{(0)},_3 = 0, \tag{27}$$

$$\mathbf{u}^{(0)},_3 = -\mathbf{D}u_3^{(0)} + \mathbf{L}_4\phi^{(0)}, \tag{28}$$

$$\phi^{(0)},_3 = 0, \tag{29}$$

$$\sigma_s^{(0)},_3 = -\mathbf{L}_5\mathbf{u}^{(0)} - \mathbf{L}_6 u_3^{(0)} - \mathbf{L}_8 D_3^{(0)}, \tag{30}$$

$$\sigma_3^{(0)},_3 = \mathbf{L}_9\mathbf{u}^{(0)} + \tilde{l}_{63}u_3^{(0)} - \mathbf{D}^T\sigma_s^{(0)} + \tilde{l}_{65}D_3^{(0)}, \tag{31}$$

$$D_3^{(0)},_3 = -\mathbf{D}^T\mathbf{d}^{(0)}, \tag{32}$$

$$\sigma_p^{(0)} = \mathbf{B}_1\mathbf{u}^{(0)} + \mathbf{B}_2 u_3^{(0)} + \mathbf{B}_4 D_3^{(0)}, \tag{33}$$

$$\mathbf{d}^{(0)} = \mathbf{B}_6\phi^{(0)}; \tag{34}$$

Order ε^{2k} (k=1, 2, 3,...):

$$u_3^{(k)},_3 = -\mathbf{L}_1\mathbf{u}^{(k-1)} - \tilde{l}_{33}u_3^{(k-1)} + \tilde{l}_{34}\sigma_3^{(k-2)} + \tilde{l}_{35}D_3^{(k-1)}, \tag{35}$$

$$\mathbf{u}^{(k)},_3 = -\mathbf{D}u_3^{(k)} + \mathbf{L}_4\phi^{(k)} + \mathbf{L}_2\mathbf{u}^{(k-1)} + \mathbf{S}\sigma_s^{(k-1)} + \mathbf{L}_3\sigma_s^{(k-2)}, \tag{36}$$

$$\phi^{(k)},_3 = -(1/\gamma_\theta)\,\mathbf{L}_8^T\mathbf{u}^{(k-1)} - (1/\gamma_\theta)\,\tilde{l}_{65}u_3^{(k-1)} + \tilde{l}_{35}\sigma_3^{(k-2)} - \tilde{l}_{81}D_3^{(k-1)} \tag{37}$$

$$\sigma_s^{(k)},_3 = -\mathbf{L}_5\mathbf{u}^{(k)} - \mathbf{L}_6 u_3^{(k)} - \mathbf{L}_7\sigma_s^{(k-1)} - \gamma_\theta\mathbf{L}_1^T\sigma_3^{(k-1)} - \mathbf{L}_8 D_3^{(k)}, \tag{38}$$

$$\sigma_3^{(k)},_3 = \mathbf{L}_9\mathbf{u}^{(k)} + \tilde{l}_{63}u_3^{(k)} - \mathbf{D}^T\sigma_s^{(k)} - \mathbf{L}_{10}\sigma_s^{(k-1)} - \tilde{l}_{64}\sigma_3^{(k-1)} + \tilde{l}_{65}D_3^{(k)}, \tag{39}$$

$$D_3^{(k)},_3 = -\mathbf{D}^T\mathbf{d}^{(k)} - x_3 D_{1,1}^{(k-1)} - \tilde{l}_{44}D_3^{(k-1)}, \tag{40}$$

$$\sigma_p^{(k)} = \mathbf{B}_1\mathbf{u}^{(k)} + \mathbf{B}_2 u_3^{(k)} + \mathbf{B}_3\sigma_3^{(k-1)} + \mathbf{B}_4 D_3^{(k)}, \tag{41}$$

$$\mathbf{d}^{(k)} = \mathbf{B}_5\sigma_s^{(k-1)} + \mathbf{B}_6\phi^{(k)}. \tag{42}$$

The transverse loads and electric potential at the lateral surfaces are given as

Order ε^0 :

$$\begin{bmatrix}\tau_{13}^{(0)} & \tau_{23}^{(0)}\end{bmatrix} = \begin{bmatrix}0 & 0\end{bmatrix} \quad \text{on} \quad x_3 = \pm 1, \tag{43a}$$

$$\sigma_3^{(0)} = \overline{q}_3^{\pm}(x_1, x_2) \quad \text{on} \quad x_3 = \pm 1, \tag{43b}$$

$$D_3^{(0)} = \overline{D}_3^{\pm}(x_1, x_2) \quad \text{on} \quad x_3 = \pm 1. \tag{44}$$

Order ε^{2k} (k = 1, 2, 3,...):

$$\begin{bmatrix}\tau_{13}^{(k)} & \tau_{23}^{(k)} & \sigma_3^{(k)}\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\end{bmatrix} \quad \text{on} \quad x_3 = \pm 1, \tag{45}$$

$$D_3^{(k)} = 0 \quad \text{on} \quad x_3 = \pm 1. \tag{46}$$

Along the edges one member of each pair of the following quantities must be satisfied,

Order ε^0 :

$$n_1\sigma_1^{(0)} + n_2\tau_{12}^{(0)} = \overline{p}_{n1} \quad \text{or} \quad u_1^{(0)} = \overline{u}_1, \tag{47a}$$

$$n_1\tau_{12}^{(0)} + n_2\sigma_2^{(0)} = \overline{p}_{n2} \quad \text{or} \quad u_2^{(0)} = \overline{u}_2, \tag{47b}$$

$$n_1\tau_{13}^{(0)} + n_2\tau_{23}^{(0)} = \overline{p}_{n3} \quad \text{or} \quad u_3^{(0)} = \overline{u}_3, \tag{47c}$$

$$\phi^{(0)} = 0. \tag{48}$$

Order ε^{2k} (k = 1, 2, 3,...):

$$n_1\sigma_1^{(k)} + n_2\tau_{12}^{(k)} = 0 \quad \text{or} \quad u_1^{(k)} = 0, \tag{49a}$$

$$n_1\tau_{12}^{(k)} + n_2\sigma_2^{(k)} = 0 \quad \text{or} \quad u_2^{(k)} = 0, \tag{49b}$$

$$n_1\tau_{13}^{(k)} + n_2\tau_{23}^{(k)} = 0 \quad \text{or} \quad u_3^{(k)} = 0, \tag{49c}$$

$$\phi^{(k)} = 0. \tag{50}$$

5 Successive integration

Examination of the sets of asymptotic equations, it is found that the analysis can be carried on by integrating those equations through the thickness direction. For brevity of the derivation, the material properties of each layer are assumed to be orthotropic. We therefore integrate Eqs.(27)-(29) to obtain

$$u_3^{(0)} = u_3^0(x_1, x_2), \tag{51}$$

$$\phi^{(0)} = \phi^0(x_1, x_2), \tag{52}$$

$$\mathbf{u}^{(0)} = \mathbf{u}^0 - x_3 \mathbf{D} u_3^0 - \left[\begin{array}{c} \tilde{E}_{00}^{15}(x_3)\,\partial_1 \\ \tilde{E}_{00}^{24}(x_3)\,\partial_2 \end{array} \right] \phi^0, \tag{53}$$

where $u_3^0(x_1,x_2)$, $\mathbf{u}^0 = \left[u_1^0(x_1,x_2) \quad u_2^0(x_1,x_2)\right]^T$ and $\phi_0(x_1,x_2)$ represent the displacements and electric potential on the middle surface; $\tilde{E}_{00}^{kl}(x_3) = \int_0^{x_3} (Qe_{kl}/ec_{ll})d\eta$.

By observation from Eq.(53), it is noted that the in-surface displacements at the leading-order level are dependent on the electric potential. Based on the previous study, we may consider Eqs.(51)-(53) as the basic kinematics assumptions for the analysis of thin piezoelectric shells.

Proceeding to derive the governing equations for the leading order, we successively integrate Eqs.(30)-(32) through the thickness with using the lateral boundary conditions on x_3=-1 (i.e., Eqs.(43)-(44)), we obtain

$$D_3^{(0)} = \overline{D}_3^- + \tilde{F}_{00}^{15}(x_3)\,\phi^0{}_{,11} + \overline{F}_{00}^{24}(x_3)\,\phi^0{}_{,22}, \tag{54}$$

$$\sigma_s^{(0)} = -\int_{-1}^{x_3} \left[\mathbf{L}_5 \left(\mathbf{u}^0 - \eta \mathbf{D} u_3^0 - \left[\begin{array}{c} \tilde{E}_{00}^{15}(\eta)\,\partial_1 \\ \tilde{E}_{00}^{24}(\eta)\,\partial_2 \end{array} \right] \phi^0 \right) + \mathbf{L}_6 u_3^0 + \mathbf{L}_8 D_3^{(0)} \right] d\eta, \tag{55}$$

$$\begin{aligned} \sigma_3^{(0)} = \overline{q}_3^- &+ \int_{-1}^{x_3} \left[\mathbf{L}_9 \left(\mathbf{u}^0 - \eta \mathbf{D} u_3^0 - \left[\begin{array}{c} \tilde{E}_{00}^{15}(\eta)\,\partial_1 \\ \tilde{E}_{00}^{24}(\eta)\,\partial_2 \end{array} \right] \phi^0 \right) + \tilde{l}_{63} u_3^0 + \tilde{l}_{65} D_3^{(0)} \right] d\eta \\ &+ \int_{-1}^{x_3} (x_3-\eta)\mathbf{D}^T \left[\mathbf{L}_5 \left(\mathbf{u}^0 - \eta \mathbf{D} u_3^0 - \left[\begin{array}{c} \tilde{E}_{00}^{15}(\eta)\,\partial_1 \\ \tilde{E}_{00}^{24}(\eta)\,\partial_2 \end{array} \right] \phi^0 \right) + \mathbf{L}_6 u_3^0 + \mathbf{L}_8 D_3^{(0)} \right] d\eta, \end{aligned} \tag{56}$$

where

$$\left[\begin{array}{c} \hat{F}_{00}^{kl}(x_3) \\ \tilde{F}_{00}^{kl}(x_3) \\ \overline{F}_{00}^{kl}(x_3) \end{array} \right] = \int_{-1}^{x_3} \left(\frac{Q}{e^2} \right) \left(\frac{e_{kl}^2}{c_{ll}} + \eta_{kk} \right) \left[\begin{array}{c} \gamma_\theta \\ 1 \\ 1/\gamma_\theta \end{array} \right] d\eta.$$

Imposition of the remaining lateral boundary conditions on x_3=1 (i.e., Eqs.(43)-(44)) in Eqs.(54)-(56) yields

$$K_{11}u_1^0 + K_{12}u_2^0 + K_{13}u_3^0 + K_{14}\phi^0 = \hat{F}_{31}\overline{D}_3^-{}_{,1}, \tag{57}$$

$$K_{21}u_1^0 + K_{22}u_2^0 + K_{23}u_3^0 + K_{24}\phi^0 = \tilde{F}_{32}\overline{D}_3^-{}_{,2}, \tag{58}$$

$$\begin{aligned} &K_{31}u_1^0 + K_{32}u_2^0 + K_{33}u_3^0 + K_{34}\phi^0 \\ &\quad = \overline{q}_3^+ - \overline{q}_3^- - \tilde{F}_{32}\overline{D}_3^- + \hat{H}_{31}\overline{D}_3^-{}_{,11} + \tilde{H}_{32}\overline{D}_3^-{}_{,22}, \end{aligned} \tag{59}$$

$$K_{41}u_1^0 + K_{42}u_2^0 + K_{43}u_3^0 + K_{44}\phi^0 = \overline{D}_3^+ - \overline{D}_3^-, \tag{60}$$

in which

$$K_{11} = -\hat{A}_{11}\partial_{11} - \overline{A}_{66}\partial_{22},$$

$$K_{12} = -\left(\tilde{A}_{12} + \tilde{A}_{66}\right)\partial_{12},$$

$$K_{13} = \hat{B}_{11}\partial_{111} + \left(\tilde{B}_{12} + \tilde{B}_{66} + \overline{B}_{66}\right)\partial_{122} \quad -\tilde{A}_{12}\partial_1,$$

$$K_{14} = \left(\hat{E}_{11}^{15} - \hat{F}_{31}^{15}\right)\partial_{111} + \left(\overline{E}_{66}^{15} + \tilde{E}_{12}^{24} + \tilde{E}_{66}^{24} - \hat{F}_{31}^{24}\right)\partial_{122},$$

$$K_{21} = -\left(\tilde{A}_{21} + \tilde{A}_{66}\right)\partial_{12},$$

$$K_{22} = -\hat{A}_{66}\partial_{11} - \overline{A}_{22}\partial_{22},$$

$$K_{23} = \left(\tilde{B}_{21} + \tilde{B}_{66} + \hat{B}_{66}\right)\partial_{112} + \overline{B}_{22}\partial_{222} - \overline{A}_{22}\partial_2,$$

$$K_{24} = \left(\tilde{E}_{21}^{15} + \tilde{E}_{66}^{15} + \hat{E}_{66}^{24} - \tilde{F}_{32}^{15}\right)\partial_{112} + \left(\overline{E}_{22}^{24} - \tilde{F}_{32}^{24}\right)\partial_{222},$$

$$K_{31} = -\hat{B}_{11}\partial_{111} - \left(\tilde{B}_{21} + \tilde{B}_{66} + \overline{B}_{66}\right)\partial_{122} \quad +\tilde{A}_{21}\partial_1,$$

$$K_{32} = -\left(\tilde{B}_{12} + \tilde{B}_{66} + \hat{B}_{66}\right)\partial_{112} - \overline{B}_{22}\partial_{222} + \overline{A}_{22}\partial_2,$$

$$\begin{aligned} K_{33} = &\hat{D}_{11}\partial_{1111} + \left(\tilde{D}_{12} + \tilde{D}_{21} + \overline{D}_{66} + 2\tilde{D}_{66} + \hat{D}_{66}\right)\partial_{1122} \\ &+ \overline{D}_{22}\partial_{2222} - \left(\tilde{B}_{12} + \tilde{B}_{21}\right)\partial_{11} - 2\overline{B}_{22}\partial_{22} + \overline{A}_{22}, \end{aligned}$$

$$\begin{aligned} K_{34} = &\left(\hat{G}_{11}^{15} - \hat{H}_{31}^{15}\right)\partial_{1111} + \left(\overline{G}_{66}^{15} + \tilde{G}_{12}^{24} + \tilde{G}_{66}^{24} + \tilde{G}_{21}^{15}\right. \\ &\left. + \tilde{G}_{66}^{15} + \hat{G}_{66}^{24} - \hat{H}_{31}^{24} - \tilde{H}_{32}^{15}\right)\partial_{1122} + \left(\overline{G}_{22}^{24} - \tilde{H}_{32}^{24}\right)\partial_{2222} \\ &- \left(\tilde{E}_{21}^{15} - \tilde{F}_{32}^{15}\right)\partial_{11} - \left(\overline{E}_{22}^{24} - \tilde{F}_{32}^{24}\right)\partial_{22}, \end{aligned}$$

$$K_{41} = K_{42} = K_{43} = 0,$$

$$K_{44} = \tilde{F}^{15}\partial_{11} + \overline{F}^{24}\partial_{22},$$

$$\left[\begin{array}{c} \hat{A}_{ij} \\ \tilde{A}_{ij} \\ \overline{A}_{ij} \end{array} \right] = \int_{-1}^{1} \tilde{Q}_{ij} \left[\begin{array}{c} \gamma_\theta \\ 1 \\ 1/\gamma_\theta \end{array} \right] dx_3,$$

$$\left[\begin{array}{c} \hat{B}_{ij} \\ \tilde{B}_{ij} \\ \overline{B}_{ij} \end{array} \right] = \int_{-1}^{1} \tilde{Q}_{ij} x_3 \left[\begin{array}{c} \gamma_\theta \\ 1 \\ 1/\gamma_\theta \end{array} \right] dx_3,$$

$$\left[\begin{array}{c} \hat{D}_{ij} \\ \tilde{D}_{ij} \\ \overline{D}_{ij} \end{array} \right] = \int_{-1}^{1} \tilde{Q}_{ij} x_3^2 \left[\begin{array}{c} \gamma_\theta \\ 1 \\ 1/\gamma_\theta \end{array} \right] dx_3,$$

$$\begin{bmatrix} \hat{E}_{ij}^{kl} \\ \tilde{E}_{ij}^{kl} \\ \overline{E}_{ij}^{kl} \end{bmatrix} = \int_{-1}^{1} \tilde{Q}_{ij} \begin{bmatrix} \gamma_\theta \\ 1 \\ 1/\gamma_\theta \end{bmatrix} \int_{0}^{x_3} \left(\frac{Qe_{kl}}{ec_{ll}} \right) d\eta dx_3,$$

$$\begin{bmatrix} \hat{F}_{3i}^{15} \\ \tilde{F}_{3i}^{15} \\ \overline{F}_{3i}^{15} \end{bmatrix} = \int_{-1}^{1} \left(\frac{e}{Q} \right) \left(\frac{c_{3i}e_{33} - e_{3i}c_{33}}{e_{33}^2 + c_{33}\eta_{33}} \right) \begin{bmatrix} \gamma_\theta \\ 1 \\ 1/\gamma_\theta \end{bmatrix} \int_{-1}^{x_3} \left(\frac{Q}{e^2} \right) \left(\frac{e_{15}^2}{c_{55}} + \eta_{11} \right) d\eta dx_3,$$

$$\begin{bmatrix} \hat{F}_{3i}^{24} \\ \tilde{F}_{3i}^{24} \\ \overline{F}_{3i}^{24} \end{bmatrix} = \int_{-1}^{1} \left(\frac{e}{Q} \right) \left(\frac{c_{3i}e_{33} - e_{3i}c_{33}}{e_{33}^2 + c_{33}\eta_{33}} \right) \begin{bmatrix} \gamma_\theta \\ 1 \\ 1/\gamma_\theta \end{bmatrix} \int_{-1}^{x_3} \left(\frac{Q}{e^2\gamma_\theta} \right) \left(\frac{e_{24}^2}{c_{44}} + \eta_{22} \right) d\eta dx_3,$$

$$\begin{bmatrix} \hat{F}_{3i} \\ \tilde{F}_{3i} \\ \overline{F}_{3i} \end{bmatrix} = \int_{-1}^{1} \left(\frac{c_{3i}e_{33} - e_{3i}c_{33}}{e_{33}^2 + c_{33}\eta_{33}} \right) \left(\frac{e}{Q} \right) \begin{bmatrix} \gamma_\theta \\ 1 \\ 1/\gamma_\theta \end{bmatrix} dx_3,$$

$$\begin{bmatrix} \hat{F}^{kl} \\ \tilde{F}^{kl} \\ \overline{F}^{kl} \end{bmatrix} = \int_{-1}^{1} \left(\frac{Q}{e^2} \right) \left(\frac{e_{kl}^2}{c_{ll}} + \eta_{kk} \right) \begin{bmatrix} \gamma_\theta \\ 1 \\ 1/\gamma_\theta \end{bmatrix} dx_3,$$

$$\begin{bmatrix} \hat{G}_{ij}^{kl} \\ \tilde{G}_{ij}^{kl} \\ \overline{G}_{ij}^{kl} \end{bmatrix} = \int_{-1}^{1} x_3 \tilde{Q}_{ij} \begin{bmatrix} \gamma_\theta \\ 1 \\ 1/\gamma_\theta \end{bmatrix} \int_{0}^{x_3} \left(\frac{Qe_{kl}}{ec_{ll}} \right) d\eta dx_3,$$

$$\begin{bmatrix} \hat{H}_{3i}^{15} \\ \tilde{H}_{3i}^{15} \\ \overline{H}_{3i}^{15} \end{bmatrix} = \int_{-1}^{1} x_3 \left(\frac{e}{Q} \right) \left(\frac{c_{3i}e_{33} - e_{3i}c_{33}}{e_{33}^2 + c_{33}\eta_{33}} \right) \begin{bmatrix} \gamma_\theta \\ 1 \\ 1/\gamma_\theta \end{bmatrix} \int_{-1}^{x_3} \left(\frac{Q}{e^2} \right) \left(\frac{e_{15}^2}{c_{55}} + \eta_{11} \right) d\eta dx_3,$$

$$\begin{bmatrix} \hat{H}_{3i}^{24} \\ \tilde{H}_{3i}^{24} \\ \overline{H}_{3i}^{24} \end{bmatrix} = \int_{-1}^{1} x_3 \left(\frac{e}{Q} \right) \left(\frac{c_{3i}e_{33} - e_{3i}c_{33}}{e_{33}^2 + c_{33}\eta_{33}} \right) \begin{bmatrix} \gamma_\theta \\ 1 \\ 1/\gamma_\theta \end{bmatrix} \int_{-1}^{x_3} \left(\frac{Q}{e^2\gamma_\theta} \right) \left(\frac{e_{24}^2}{c_{44}} + \eta_{22} \right) d\eta dx_3,$$

$$\begin{bmatrix} \hat{H}_{3i} \\ \tilde{H}_{3i} \\ \overline{H}_{3i} \end{bmatrix} = \int_{-1}^{1} x_3 \left(\frac{c_{3i}e_{33} - e_{3i}c_{33}}{e_{33}^2 + c_{33}\eta_{33}} \right) \left(\frac{e}{Q} \right) \begin{bmatrix} \gamma_\theta \\ 1 \\ 1/\gamma_\theta \end{bmatrix} dx_3.$$

Solution of Eqs.(57)-(60) must be supplemented with the edge boundary conditions Eqs.(47)-(48) to constitute a well-posed boundary value problem. Once $u_1^0, u_2^0 and u_3^0$ and ϕ^0 are determined, the leading-order solutions of displacements, transverse shear and normal stresses, in-surface stresses, in-surface and normal electric displacements can be obtained by Eqs.(51)-(53), Eqs.(55)-(56), Eq.(33), Eq.(34) and Eq.(54), respectively.

Proceed to order ε^{2k} (k=1, 2, 3, etc) following the same line as was done before, we readily obtain

$$u_3^{(k)} = u_3^k(x_1,x_2) + \varphi_{3k}(x_1,x_2,x_3), \tag{61}$$

$$\phi^{(k)} = \phi^k(x_1,x_2) + \varphi_{4k}(x_1,x_2,x_3), \tag{62}$$

$$\mathbf{u}^{(k)} = \mathbf{u}^k - x_3\mathbf{D}u_3^k - \begin{bmatrix} \tilde{E}_{00}^{15}(x_3)\partial_1 \\ \tilde{E}_{00}^{24}(x_3)\partial_2 \end{bmatrix} \phi^k + \varphi^k, \tag{63}$$

$$D_3^{(k)} = \tilde{F}_{00}^{15}(x_3)\phi^k{}_{,11} + \overline{F}_{00}^{24}(x_3)\phi^k{}_{,22} - f_{4k}(x_1,x_2,x_3), \tag{64}$$

$$\sigma_s^{(k)} = -\int_{-1}^{x_3} \left[\mathbf{L}_5 \left(\mathbf{u}^k - \eta\mathbf{D}u_3^k - \begin{bmatrix} \tilde{E}_{00}^{15}(x_3)\partial_1 \\ \tilde{E}_{00}^{24}(x_3)\partial_2 \end{bmatrix} \phi^k \right) + \mathbf{L}_6 u_3^k + \mathbf{L}_8 D_3^k \right] d\eta - \mathbf{f}^k(x_1,x_2,x_3), \tag{65}$$

$$\begin{aligned}\sigma_3^{(k)} = &\int_{-1}^{x_3} \left[\mathbf{L}_9 \left(\mathbf{u}^k - \eta\mathbf{D}u_3^k - \begin{bmatrix} \tilde{E}_{00}^{15}(x_3)\partial_1 \\ \tilde{E}_{00}^{24}(x_3)\partial_2 \end{bmatrix} \phi^k \right) + \tilde{l}_{63}u_3^k + \tilde{l}_{65}D_3^k \right] d\eta \\ &+ \int_{-1}^{x_3} (x_3-\eta)\mathbf{D}^T \left[\mathbf{L}_5 \left(\mathbf{u}^k - \eta\mathbf{D}u_3^k - \begin{bmatrix} \tilde{E}_{00}^{15}(x_3)\partial_1 \\ \tilde{E}_{00}^{24}(x_3)\partial_2 \end{bmatrix} \phi^k \right) + \mathbf{L}_6 u_3^k + \mathbf{L}_8 D_3^k \right] d\eta - f_{3k}(x_1,x_2,x_3),\end{aligned} \tag{66}$$

where

$$\varphi_{3k}(x_1,x_2,x_3) = -\int_{0}^{x_3} \left[\mathbf{L}_1\mathbf{u}^{(k-1)} + \tilde{l}_{33}u_3^{(k-1)} - \tilde{l}_{35}D_3^{(k-1)} - \tilde{l}_{34}\sigma_3^{(k-2)} \right] d\eta,$$

$$\mathbf{u}^k = \left[u_1^k(x_1,x_2) \quad u_2^k(x_1,x_2) \right]^T,$$

$$D_3^k = \tilde{F}_{00}^{15}(x_3)\phi^k{}_{,11} + \overline{F}_{00}^{24}(x_3)\phi^k{}_{,22},$$

$$\varphi^k = \left\{ \begin{array}{c} \varphi_{1k}(x_1,x_2,x_3) \\ \varphi_{2k}(x_1,x_2,x_3) \end{array} \right\} = -\int_0^{x_3} (\mathbf{D}\varphi_{3k} - \mathbf{L}_4\varphi_{4k} - \mathbf{L}_2\mathbf{u}^{(k-1)} - \mathbf{S}\sigma_s^{(k-1)} - \mathbf{L}_3\sigma_s^{(k-2)})d\eta$$

$$\varphi_{4k} = -\int_0^{x_3} \left[(1/\gamma_\theta)\mathbf{L}_8^T\mathbf{u}^{(k-1)} + (1/\gamma_\theta)\tilde{l}_{65}u_3^{(k-1)} - \tilde{l}_{35}\sigma_3^{(k-2)} + \tilde{l}_{81}D_3^{(k-1)}\right]d\eta,$$

$$\mathbf{f}^k = \int_{-1}^{x_3} \left[\mathbf{L}_5\varphi^k + \mathbf{L}_6\varphi_{3k} + \mathbf{L}_7\sigma_s^{(k-1)} + \gamma_\theta\mathbf{L}_1^T\sigma_3^{(k-1)} - \mathbf{L}_8 f_{4k}\right]d\eta,$$

$$f_{3k} = -\int_{-1}^{x_3} \left[\mathbf{L}_9\varphi^k + \tilde{l}_{63}\varphi_{3k} - \mathbf{L}_{10}\sigma_s^{(k-1)} - \tilde{l}_{64}\sigma_3^{(k-1)} - \tilde{l}_{65}f_{4k} + \mathbf{D}^T\mathbf{f}^k\right]d\eta,$$

$$f_{4k} = -\int_{-1}^{x_3} \left[\tilde{F}_{00}^{15}(\eta)\,\varphi_{4k,11} + \overline{F}_{00}^{24}(\eta)\,\varphi_{4k,22} - \mathbf{D}^T\mathbf{B}_5\sigma_s^{(k-1)} - \eta D_{1,1}^{(k-1)} - \tilde{l}_{44}D_3^{(k-1)}\right]d\eta.$$

u_3^k, $\mathbf{u}^k$ and ϕ^k represent the modifications to the elastic displacements and electric potential on the middle surface. Upon imposing the associated lateral boundary conditions Eqs.(45)-(46) on Eqs.(64)-(66), we arrive again at the same governing equations at the leading order level, except for nonhomogeneous terms that can be calculated from the lower-order solution. The governing equations for ε^{2k}-order are given by

$$K_{11}u_1^k + K_{12}u_2^k + K_{13}u_3^k + K_{14}\phi^k = f_{1k}(x_1,x_2,1) \tag{67}$$

$$K_{21}u_1^k + K_{22}u_2^k + K_{23}u_3^k + K_{24}\phi^k = f_{2k}(x_1,x_2,1), \tag{68}$$

$$K_{31}u_1^k + K_{32}u_2^k + K_{33}u_3^k + K_{34}\phi^k = f_{3k}(x_1,x_2,1) + \frac{\partial f_{1k}(x_1,x_2,1)}{\partial x_1} + \frac{\partial f_{2k}(x_1,x_2,1)}{\partial x_2}, \tag{69}$$

$$K_{41}u_1^k + K_{42}u_2^k + K_{43}u_3^k + K_{44}\phi^k = f_{4k}(x_1,x_2,1). \tag{70}$$

6 Applications to benchmark problems

The benchmark problems of simply supported, single-layer and multilayered piezoelectric hollow cylinders under lateral mechanical loads or lateral electric displacement on outer surfaces are studied using the present asymptotic formulation. The material of orthotropic or transverse isotropic piezoelectric cylinders is considered so that the following elastic moduli in Eqs.(1)-(2) are identical to zero.

$$(c_{16})_i = (c_{26})_i = (c_{36})_i = (c_{45})_i = 0, \tag{71a}$$

$$(e_{14})_i = (e_{25})_i = (e_{36})_i = 0, \tag{71b}$$

$$(\eta_{12})_i = 0, \tag{71c}$$

where the subscript i denotes the ith layer.

The boundary conditions at two edges are of a shear diaphragm type and specified by

$$\sigma_1 = u_2 = u_3 = 0 \quad \text{on} \quad x_1 = 0 \quad \text{and} \quad x_1 = L/R\varepsilon \tag{72a}$$

The boundary conditions at two edges are suitably grounded and specified by

$$\phi = 0 \quad \text{on} \quad x_1 = 0 \quad \text{and} \quad x_1 = L/R\varepsilon \tag{72b}$$

The mechanical load or normal electric displacement acting on lateral surface of the shell $(\zeta = h)$ is considered in the following illustrative examples and are expressed by the double Fourier series in the dimensionless form of

$$\overline{q}_r^+(x_1,x_2) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \overline{q}_{mn}\sin\tilde{m}x_1\cos\tilde{n}x_2 \tag{73}$$

$$\overline{D}_r^+(x_1,x_2) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \overline{D}_{mn}\sin\tilde{m}x_1\cos\tilde{n}x_2, \tag{74}$$

where $\tilde{m} = m\pi\sqrt{Rh}/L and \tilde{n} = n\sqrt{h/R}$.

The governing equations of the leading-order problem (i.e., Eqs.(57)-(60)) can be easily solved by letting

$$u_1^0 = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} u_{1mn}^0\cos\tilde{m}x_1\cos\tilde{n}x_2, \tag{75}$$

$$u_2^0 = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} u_{2mn}^0\sin\tilde{m}x_1\sin\tilde{n}x_2, \tag{76}$$

$$u_3^0 = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} u_{3mn}^0 \sin\tilde{m}x_1 \cos\tilde{n}x_2 \quad (77)$$

$$\phi^0 = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \phi_{mn}^0 \sin\tilde{m}x_1 \cos\tilde{n}x_2 \quad (78)$$

Substituting Eqs.(75)-(78) into Eqs.(57)-(60) gives

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ 0 & 0 & 0 & k_{44} \end{bmatrix} \begin{Bmatrix} u_{1mn}^0 \\ u_{2mn}^0 \\ u_{3mn}^0 \\ \phi_{mn}^0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \overline{q}_{mn} \\ \overline{D}_{mn} \end{Bmatrix} \quad (79)$$

where

$$k_{11} = \tilde{m}^2\hat{A}_{11} + \tilde{n}^2\overline{A}_{66},$$

$$k_{12} = -\tilde{m}\tilde{n}\left(\tilde{A}_{12} + \tilde{A}_{66}\right),$$

$$k_{13} = -\tilde{m}^3\hat{B}_{11} - \tilde{m}\tilde{n}^2\left(\tilde{B}_{12} + \tilde{B}_{66} + \overline{B}_{66}\right) - \tilde{m}\tilde{A}_{12}$$

$$k_{14} = -\tilde{m}^3\left(\hat{E}_{11}^{15} - \hat{F}_{31}^{15}\right) - \tilde{m}\tilde{n}^2\left(\overline{E}_{66}^{15} + \tilde{E}_{12}^{24} + \tilde{E}_{66}^{24} - \tilde{F}_{31}^{24}\right),$$

$$k_{21} = -\tilde{m}\tilde{n}\left(\tilde{A}_{21} + \tilde{A}_{66}\right),$$

$$K_{22} = \tilde{m}^2\hat{A}_{66} + \tilde{n}^2\overline{A}_{22},$$

$$k_{23} = \tilde{m}^2\tilde{n}\left(\tilde{B}_{21} + \tilde{B}_{66} + \hat{B}_{66}\right) + \tilde{n}^3\overline{B}_{22} + \tilde{n}\overline{A}_{22}$$

$$k_{24} = \tilde{m}^2\tilde{n}\left(\tilde{E}_{21}^{15} + \tilde{E}_{66}^{15} + \hat{E}_{66}^{24} - \tilde{F}_{32}^{15}\right) + \tilde{n}^3\left(\overline{E}_{22}^{24} - \tilde{F}_{32}^{24}\right),$$

$$k_{31} = -\tilde{m}^3\hat{B}_{11} - \tilde{m}\tilde{n}^2\left(\tilde{B}_{21} + \tilde{B}_{66} + \overline{B}_{66}\right) - \tilde{m}\tilde{A}_{21},$$

$$k_{32} = \tilde{m}^2\tilde{n}\left(\tilde{B}_{12} + \tilde{B}_{66} + \hat{B}_{66}\right) + \tilde{n}^3\overline{B}_{22} + \tilde{n}\overline{A}_{22},$$

$$k_{33} = \tilde{m}^4\hat{D}_{11} + \tilde{m}^2\tilde{n}^2\left(\tilde{D}_{12} + \tilde{D}_{21} + \overline{D}_{66} + 2\tilde{D}_{66} + \hat{D}_{66}\right) + \tilde{n}^4\overline{D}_{22} + \tilde{m}^2\left(\tilde{B}_{12} + \tilde{B}_{21}\right) + 2\tilde{n}^2\overline{B}_{22} + \overline{A}_{22},$$

$$k_{34} = \tilde{m}^4\left(\hat{G}_{11}^{15} - \hat{H}_{31}^{15}\right) + \tilde{m}^2\tilde{n}^2\left(\overline{G}_{66}^{15} + \tilde{G}_{12}^{24} + \tilde{G}_{66}^{24} + \tilde{G}_{21}^{15} + \tilde{G}_{66}^{15} + \hat{G}_{66}^{24} - \tilde{H}_{31}^{24} - \tilde{H}_{32}^{15}\right) + \tilde{n}^4\left(\overline{G}_{22}^{24} - \tilde{H}_{32}^{24}\right) + \tilde{m}^2\left(\tilde{E}_{21}^{15} - \tilde{F}_{32}^{15}\right) + \tilde{n}^2\left(\overline{E}_{22}^{24} - \tilde{F}_{32}^{24}\right),$$

$$k_{44} = -\tilde{m}^2\tilde{F}^{15} - \tilde{n}^2\overline{F}^{24}.$$

The electric potential variable ϕ_{mn}^0 can be determined from the last equation in Eq. (79) and be given by

$$\phi_{mn}^0 = \frac{\overline{D}_{mn}}{k_{44}}. \quad (80)$$

According to Eq. (80), we can rewrite Eq. (79) in the form of

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} u_{1mn}^0 \\ u_{2mn}^0 \\ u_{3mn}^0 \end{Bmatrix} = \begin{Bmatrix} -k_{14}\phi_{mn}^0 \\ -k_{24}\phi_{mn}^0 \\ \overline{q}_{mn} - k_{34}\phi_{mn}^0 \end{Bmatrix}. \quad (81)$$

u_{1mn}^0, u_{2mn}^0 and u_{3mn}^0 can be obtained by solving the simultaneously algebraic equations (81). Once u_{1mn}^0, u_{2mn}^0 and u_{3mn}^0 are determined, the ε^0-order solution can be obtained as aforementioned. The summation signs would be dropped for brevity in the following derivation.

Carrying on the solution to higher-order (ε^2-order, k=1, 2, 3, etc), we find that the nonhomogeneous terms for fixed values of m and n in the ε^2-order equations are

$$f_{1k}(x_1,x_2,1) = \tilde{f}_{1k}(1)\cos\tilde{m}x_1\cos\tilde{n}x_2, \quad (82)$$

$$f_{2k}(x_1,x_2,1) = \tilde{f}_{2k}(1)\sin\tilde{m}x_1\sin\tilde{n}x_2, \quad (83)$$

$$f_{3k}(x_1,x_2,1) = \tilde{f}_{3k}(1)\sin\tilde{m}x_1\cos\tilde{n}x_2, \quad (84)$$

$$f_{4k}(x_1,x_2,1) = \tilde{f}_{4k}(1)\sin\tilde{m}x_1\cos\tilde{n}x_2. \quad (85)$$

In view of the recurrence of the equations, the ε^{2k}-order solution can be obtained by letting

$$u_1^k = u_{1mn}^k\cos\tilde{m}x_1\cos\tilde{n}x_2, \quad (86)$$

$$u_2^k = u_{2mn}^k\sin\tilde{m}x_1\sin\tilde{n}x_2, \quad (87)$$

$$u_3^k = u_{3mn}^k\sin\tilde{m}x_1\cos\tilde{n}x_2 \quad (88)$$

$$\phi^k = \phi_{mn}^k\sin\tilde{m}x_1\cos\tilde{n}x_2. \quad (89)$$

Substituting Eqs.(86)-(89) into Eqs.(67)-(70) gives

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ 0 & 0 & 0 & k_{44} \end{bmatrix} \begin{Bmatrix} u_{1mn}^k \\ u_{2mn}^k \\ u_{3mn}^k \\ \phi_{mn}^k \end{Bmatrix} = \begin{Bmatrix} \tilde{f}_{1k}(1) \\ \tilde{f}_{2k}(1) \\ \tilde{f}_{3k}(1) - \tilde{m}\tilde{f}_{1k}(1) + \tilde{n}\tilde{f}_{2k}(1) \\ \tilde{f}_{4k}(1) \end{Bmatrix}, \quad (90)$$

Following the similar solution process of the leading-order level, we obtain

$$\phi_{mn}^k = \frac{\tilde{f}_{4k}(1)}{k_{44}}. \quad (91)$$

Table 1 : Elastic, piezoelectric and dielectric properties of composite and piezoelectric materials

Moduli	PVDT	Graphite/Epoxy	PZT-4
$c_{11}(Gpa)$	0.3e+01	183.433	138.499
$c_{22}(Gpa)$	0.3e+01	11.662	138.499
$c_{33}(Gpa)$	0.3e+01	11.662	114.745
$c_{12}(Gpa)$	0.15e+01	4.363	77.371
$c_{13}(Gpa)$	0.15e+01	4.363	73.643
$c_{23}(Gpa)$	0.15e+01	3.918	73.643
$c_{44}(Gpa)$	0.75	2.870	25.6
$c_{55}(Gpa)$	0.75	7.170	25.6
$c_{66}(Gpa)$	0.75	7.170	30.6
$e_{24}\left(C/m^2\right)$	0.0	0.000	-5.2
$e_{15}\left(C/m^2\right)$	0.0	0.000	-5.2
$e_{31}\left(C/m^2\right)$	-0.15e-02	0.000	15.08
$e_{32}\left(C/m^2\right)$	0.285e-01	0.000	12.72
$e_{33}\left(C/m^2\right)$	-0.51e-01	0.000	12.72
$\eta_{11}\left(F/m\right)$	0.1062e-09	1.53e-08	1.306e-08
$\eta_{22}\left(F/m\right)$	0.1062e-09	1.53e-08	1.306e-08
$\eta_{33}\left(F/m\right)$	0.10838e-09	1.53e-08	1.151e-08

Eq.(90) can then be rewritten as

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} u^k_{1mn} \\ u^k_{2mn} \\ u^k_{3mn} \end{Bmatrix} = \begin{Bmatrix} \tilde{f}_{1k}(1) - k_{14}\phi^k_{mn} \\ \tilde{f}_{2k}(1) - k_{24}\phi^k_{mn} \\ \tilde{f}_{3k}(1) - \tilde{m}\tilde{f}_{1k}(1) + \tilde{n}\tilde{f}_{2k}(1) - k_{34}\phi^k_{mn} \end{Bmatrix}. \quad (92)$$

By solving the system of algebraic equations (92) and using Eq.(91), we may obtain the primary geometric variables $u^k_{1mn}, u^k_{2mn}, u^k_{3mn}$ and ϕ^k_{mn} for the higher-order problem. Afterwards, the ε^2-order corrections are determined from Eqs.(61)-(66) and Eqs.(41)-(42). It is shown that the solution process can be repeatedly applied for various order problems and the asymptotic solutions can be obtained in a hierarchic manner.

6.1 Mechanical loads

6.1.1 single-layer piezoelectric hollow cylinders

For comparison purpose, the present asymptotic formulation is applied to the static analysis of simply-supported, single-layer piezoelectric hollow cylinders. The cylinders are considered to be composed of polyvinyledence fluoride (PVDF) polarized along the radial direction. The elastic, piezoelectric and dielectric properties of PVDF

Table 2 : Mechanical and electric components at the crucial positions in a single-layer piezoelectric cylinder (PVDF) under loading condition of case 1 (*L*/*R*=4).

$R/2h$	ε^{2k}	-$\bar{u}_x(0,+h)$	-$\bar{u}_x(0,-h)$	-$\bar{u}_r(\frac{L}{2},+h)$	-$\bar{u}_r(\frac{L}{2},-h)$	-$\bar{\sigma}_x(\frac{L}{40},+h)$	-$\bar{\sigma}_x(\frac{L}{40},-h)$	-$\bar{\sigma}_\theta(\frac{L}{2},+h)$	-$\bar{\sigma}_\theta(\frac{L}{2},-h)$	-$\bar{\sigma}_r(\frac{L}{2},0)$
6	ε^0	0.3639	0.8818	0.9837	0.9837	0.3743	-0.3945	0.906	1.097	0.5082
	ε^2	0.4716	0.9988	1.0129	1.0662	0.4012	-0.4307	0.994	1.178	0.5475
	ε^4	0.4735	0.9993	1.0182	1.0622	0.4149	-0.4208	0.998	1.172	0.5481
	ε^6	0.4735	0.9992	1.0185	1.0618	0.4140	-0.4216	0.999	1.172	0.5475
	ε^8	0.4735	0.9992	1.0185	1.0618	0.4140	-0.4218	0.999	1.172	0.5475
	ε^{10}	0.4735	0.9992	1.0185	1.0618	0.4140	-0.4218	0.999	1.172	0.5475
	3D solutions (Kapuria et al. ,1997)	0.4708	1.0008	1.0191	1.0620	0.4407	-0.4485	1.003	1.170	0.5623
10	ε^0	0.4301	0.8324	0.9897	0.9897	0.4758	-0.4915	0.946	1.058	0.5069
	ε^2	0.4952	0.9017	1.0069	1.0390	0.4981	-0.5200	1.000	1.105	0.5305
	ε^4	0.4956	0.9018	1.0088	1.0375	0.5072	-0.5137	1.002	1.103	0.5327
	ε^6	0.4957	0.9018	1.0088	1.0374	0.5065	-0.5143	1.002	1.103	0.5312
	ε^8	0.4957	0.9018	1.0088	1.0374	0.5064	-0.5143	1.002	1.103	0.5313
	ε^{10}	0.4957	0.9018	1.0088	1.0374	0.5064	-0.5143	1.002	1.103	0.5313
	3D solutions (Kapuria et al. ,1997)	0.4951	0.9020	1.0087	1.0372	0.4915	-0.5088	1.003	1.102	0.5374
20	ε^0	0.4993	0.7832	0.9902	0.9902	0.5519	-0.5611	0.972	1.029	0.5118
	ε^2	0.5319	0.8173	0.9989	1.0149	0.5603	-0.5726	1.000	1.051	0.5237
	ε^4	0.5321	0.8174	0.9994	1.0146	0.5590	-0.5735	1.001	1.051	0.5231
	ε^6	0.5321	0.8174	0.9994	1.0146	0.5593	-0.5732	1.001	1.051	0.5232
	ε^8	0.5321	0.8174	0.9994	1.0146	0.5593	-0.5732	1.001	1.051	0.5232
	ε^{10}	0.5321	0.8174	0.9994	1.0146	0.5593	-0.5732	1.001	1.051	0.5232
	3D solutions (Kapuria et al. ,1997)	0.5319	0.8175	0.9992	1.0144	0.5620	-0.5730	1.001	1.051	0.5191

material are given in Table 1. The loading conditions on lateral surfaces are considered as case 1 and given by

Case 1. $\overline{q}_r^+ = q_0\text{N / m}^2$, $\overline{q}_r^- = 0\text{N / m}^2$; $\overline{D}_r^+ = 0\text{C}/m^2$, $\overline{D}_r^- = 0\text{C}/m^2$.

The loading condition of case 1 can be regarded as an axisymmetric load. It is reasonable to expect all the field variables are independent upon the circumferential coordinate. In that case, the asymptotic formulation of these axisymmetric piezoelectric cylinders can be further obtained by letting the circumferential wave number (n) be zero in the present one.

The dimensionless variables are denoted as the same forms of those in the Reference (Kapuria, Sengupta and Dumir, 1997) and given as follows:

$$(\overline{u}_x, \overline{u}_r) = (u_x/2h, u_r/2h) \,/\, \left(|q_0|\, S^2/E_T\right),$$

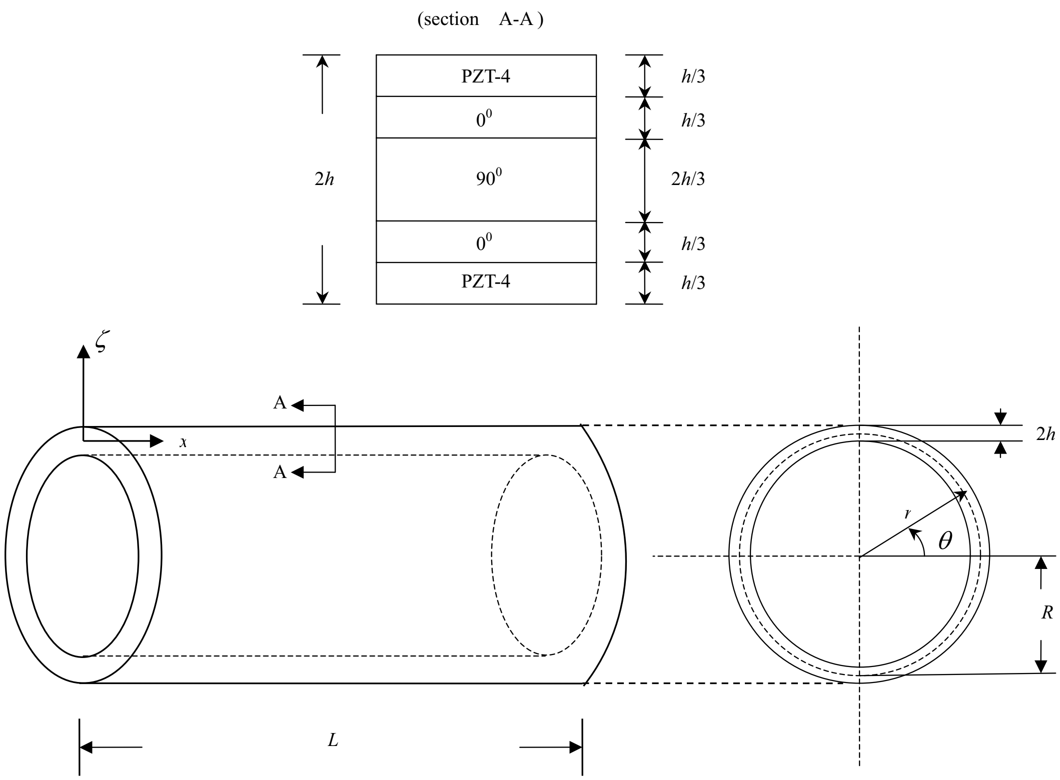

Figure 1 : The geometry and cylindrical coordinates system for a multilayered piezoelectric hollow cylinder.

$(\overline{D}_x, \overline{D}_r) = (D_x/|q_0||d_T|, D_r/|q_0|S|d_T|)$,

$(\overline{\sigma}_x, \overline{\sigma}_\theta, \overline{\sigma}_r, \overline{\tau}_{xr}) = (\sigma_x/S, \sigma_\theta/S, \sigma_r, \tau_{xr})/|q_0|$,

$\overline{\phi} = 1000\Phi E_T|d_T|/2h|q_0|S$,

and $S = R/2h$, $E_T = 2.0GPa$, $|d_T| = 30\text{x}10^{-12}CN^{-1}$.

The Fourier sine series of uniform load q_0 is given as

$$q_0 = \sum_{m=1,3,\cdots}^{M} \frac{4q_0}{m\pi} \sin\frac{m\pi}{L}x.$$

Table 2 shows the present asymptotic solutions of elastic and electric field variables for various orders at crucial positions in the cylinders under a uniform load q_0. Since the convergence of Fourier series form of a uniform load was evaluated by Kapuria, Sengupta and Dumir (1997), it is not repeated in the present analysis. The solution, based on the present asymptotic formulation is computed up to the ε^{10}-order level with L/R=4 and $R/2h$= 6, 10, 20. It is shown that the convergent speed in the cases of thin shells is more rapid than that in the cases of thick shells. The convergent solutions are also compared with the 3D piezoelectricity solutions available in the literature (Kapuria, Sengupta and Dumir, 1997). It is shown that the convergent solutions of the present asymptotic theory are in good agreement with the 3D piezoelectricity solutions.

6.1.2 *Multilayered hybrid piezoelectric hollow cylinders*

The direct piezoelectric effect of [p/0/90/90/0/p] multilayered hybrid piezoelectric hollow cylinders, as shown in Fig. 1, is studied where the character p in square brackets stands for a piezoelectric layer of PZT-4 material. Each layer of the cylinder is of equal thickness. The composite material is made of Graphite/Epoxy. The elastic, piezoelectric and dielectric properties of PZT-4 and Graphite/Epoxy materials are given in Table 1.

Two cases of mechanical loads are given by

Case 2. $\overline{q}_r^+ = q_0 \sin\frac{\pi}{L}x\cos 4\theta \text{N}/\text{m}^2$, $\overline{q}_r^- = 0\text{N}/\text{m}^2$; $\overline{\Phi}_r^+ = 0\text{V}$, $\overline{\Phi}_r^- = 0\text{V}$.

Case 3. $\overline{q}_r^+ = q_0 \sin\frac{\pi}{L}x\cos 4\theta \text{N}/\text{m}^2$, $\overline{q}_r^- = 0\text{N}/\text{m}^2$; $\overline{D}_r^+ = 0\text{C}/\text{m}^2$, $\overline{D}_r^- = 0\text{C}/\text{m}^2$.

For the applied transverse load cases, the dimensionless variables are denoted as

$$\overline{u}_i = u_i c^*/q_0(2h), \quad \overline{\tau}_{ij} = \tau_{ij}/q_0,$$

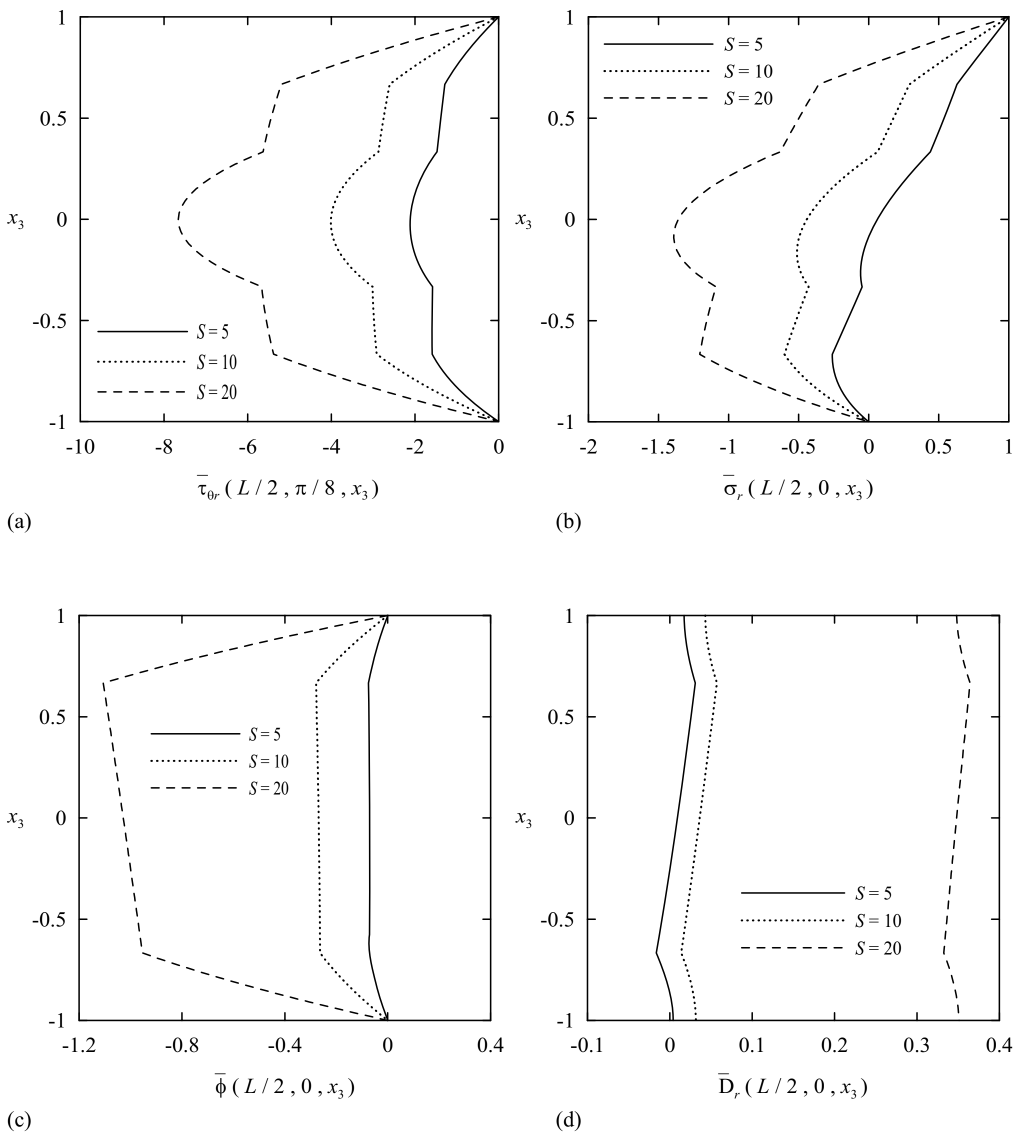

Figure 2 : Variations of mechanical and electric components along the thickness of a [p/0/90/90/0/p] multilayered hybrid piezoelectric cylinder under $\overline{q_r^+} = q_0 \sin(\pi x/L) \cos 4\theta$ N/m^2, $\overline{q_r^-} = 0$ N/m^2, $\overline{\Phi_r^+} = 0$ V and $\overline{\Phi_r^-} = 0$ V on the lateral surfaces.

$\overline{\Phi} = \Phi e^*/q_0(2h), \quad \overline{D_i} = D_i c^*/q_0 e^*;$

where $c^* = 10\text{x}10^9\text{N / m}^2$, $e^* = 10\text{C / m}^2$.

Figures 2-3 illustrate the variations of transverse shear and normal stresses, electric potential and normal electric displacement through the thickness of the cylinder under loading conditions of cases 2 and 3, respectively. The geometric parameters of the cylinder are $L/R = 4$ and S =5, 10, 20 ($S = R/2h$).

By observation of Figs. 2-3, we can find that the prescribed lateral surface conditions are exactly satisfied. It is noted from Figs 2(a), 2(b), 3(a) and 3(b) that the distributions of transverse shear and normal stresses across the thickness are approximately layerwise parabolic and the maximum value occurs in the vicinity of the middle surface of the cylinder. As the cylinder is thin enough (say R/2h=20), the cylinder produces a value of transverse normal stress that exceeds the magnitude of applied load. It is also shown that there is no much difference on the distributions of the transverse stresses through the thickness between loading conditions of case 2 and case 3.

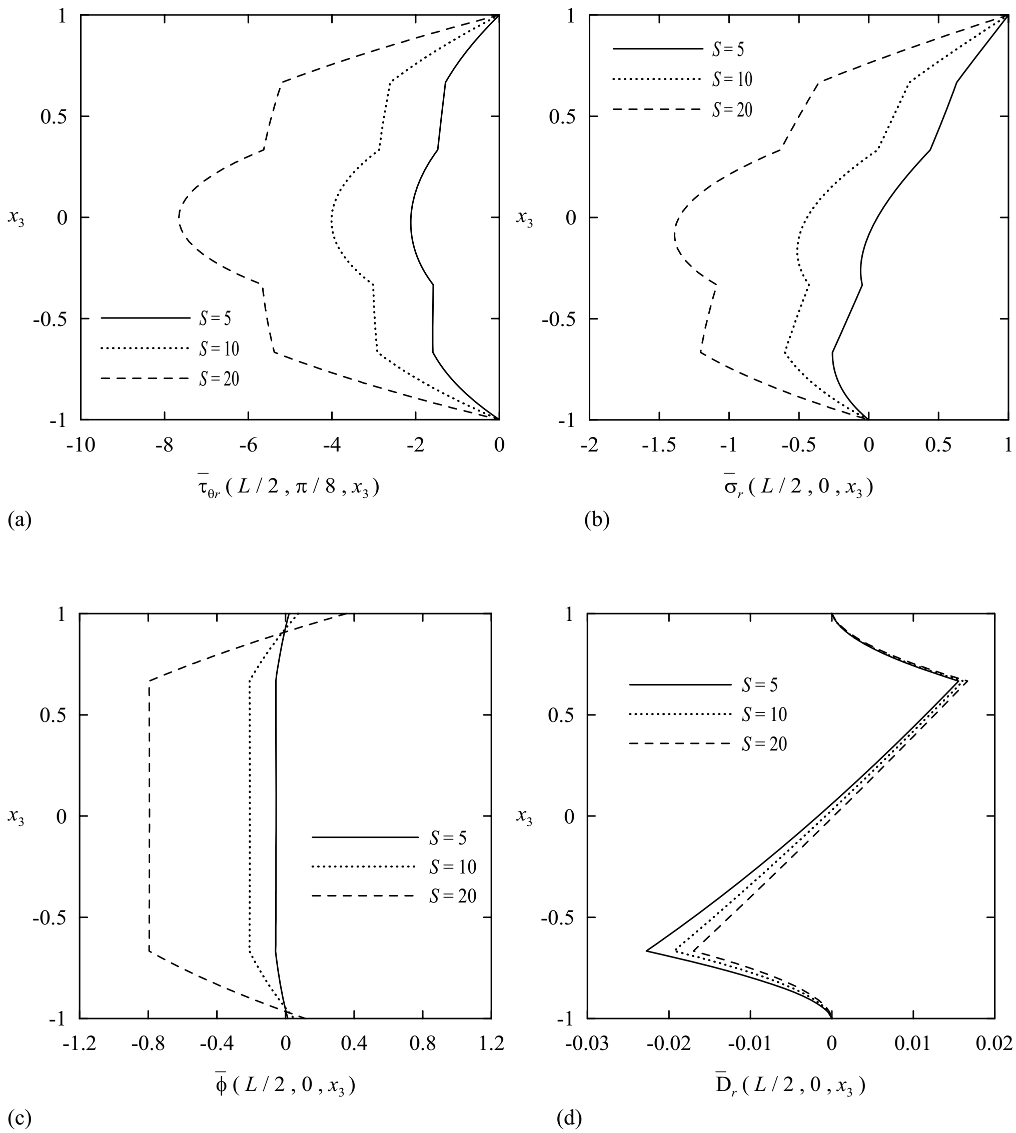

Figure 3 : Variations of mechanical and electric components along the thickness of a [p/0/90/90/0/p] multilayered hybrid piezoelectric cylinder under $\overline{q}_r^+ = q_0 \sin(\pi x/L) \cos 4\theta$ N/m^2, $\overline{q}_r^- = 0$N/m^2, $\overline{D}_r^+ = 0$C/m^2 and $\overline{D}_r^- = 0$ C/m^2 on the lateral surfaces.

It is observed from Figs. 2(c), 2(d), 3(c) and 3(d) that the deviations of normal electric displacement through the thickness coordinate between case 2 and case 3 is more remarkable than that of electric potential. The distributions of electric potential across the thickness are approximately layerwise linear and those of normal electric displacement are approximately layerwise parabolic. The maximum values of electric potential and normal electric displacement occur at the interfaces between the elastic and piezoelectric layers.

6.2 Electric loads

6.2.1 Single-layer piezoelectric hollow cylinders

The single-layer piezoelectric hollow cylinders under two types of electric loads are considered in Tables 3-4. The piezoelectric material of PZT-4 is used for the present analysis and its material properties are given in Table 1.

Two cases of electric loads are given by

Case 4. $\overline{q}_r^+ = 0$N / m^2, $\overline{q}_r^- = 0$N / m^2; $\overline{\Phi}_r^+ = \phi_0 \sin \frac{\pi}{L} x$ V, $\overline{\Phi}_r^- = 0$V.

Table 3 : Mechanical and electric components at the crucial positions in a single-layer piezoelectric cylinder (PZT-4) under loading conditions of case 4 (*L*/*R*=4).

$R/2h$	ε^{2k}	$\bar{u}_x(0,+h)$	$\bar{u}_x(0,-h)$	$\bar{u}_r(\frac{L}{2},+h)$	$\bar{u}_r(\frac{L}{2},-h)$	$\bar{D}_r(\frac{L}{2},+h)$	$\bar{D}_r(\frac{L}{2},-h)$	$\bar{\sigma}_x(\frac{L}{2},+h)$	$\bar{\sigma}_x(\frac{L}{2},-h)$	$\bar{\sigma}_\theta(\frac{L}{2},+h)$	$\bar{\sigma}_\theta(\frac{L}{2},-h)$	$\bar{\tau}_{xr}(0,0)$	$\bar{\sigma}_r(\frac{L}{2},0)$	$\bar{\phi}(\frac{L}{2},0)$
4	ε^0	0.34981	0.32797	-0.11124	-0.11124	-1.3508	-1.3508	-0.9453e-2	0.8276e-2	0.02330	-0.02832	-0.4387e-3	-0.1595e-2	0.49998
	ε^2	0.36105	0.31935	-0.13661	-0.08641	-1.2059	-1.5182	-0.1263	0.1370	-0.08731	0.10210	-0.6412e-2	0.5912e-2	0.53092
	ε^4	0.36213	0.32096	-0.13481	-0.08401	-1.2119	-1.5248	-0.1238	0.1398	-0.08217	0.10978	-0.6381e-2	0.5919e-2	0.53080
	ε^6	0.36215	0.32097	-0.13441	-0.08439	-1.2119	-1.5247	-0.1235	0.1394	-0.08134	0.10875	-0.6375e-2	0.5859e-2	0.53076
	ε^8	0.36214	0.32097	-0.13443	-0.08441	-1.2119	-1.5247	-0.1235	0.1394	-0.08138	0.10869	-0.6375e-2	0.5859e-2	0.53076
	ε^{10}	0.36214	0.32097	-0.13443	-0.08441	-1.2119	-1.5247	-0.1235	0.1394	-0.08138	0.10869	-0.6375e-2	0.5859e-2	0.53076
10	ε^0	0.85859	0.83665	-0.27945	-0.27945	-1.3508	-1.3508	-0.3715e-2	0.3532e-2	0.9803e-2	-0.1059e-1	-0.7124e-4	-0.2545e-3	0.49999
	ε^2	0.86316	0.83325	-0.30465	-0.25448	-1.2872	-1.4180	-0.04899	0.05060	-0.03609	0.03838	-0.9765e-3	0.9307e-3	0.51291
	ε^4	0.86366	0.83381	-0.30386	-0.25359	-1.2882	-1.4191	-0.04856	0.05106	-0.03515	0.03948	-0.9758e-3	0.9309e-3	0.51290
	ε^6	0.86366	0.83381	-0.30380	-0.25365	-1.2882	-1.4191	-0.04854	0.05104	-0.03510	0.03942	-0.9757e-3	0.9293e-3	0.51290
	ε^8	0.86366	0.83381	-0.30380	-0.25365	-1.2882	-1.4191	-0.04854	0.05104	-0.03510	0.03942	-0.9757e-3	0.9293e-3	0.51290
	ε^{10}	0.86366	0.83381	-0.30380	-0.25365	-1.2882	-1.4191	-0.04854	0.05104	-0.03510	0.03942	-0.9757e-3	0.9293e-3	0.51290
100	ε^0	8.4879	8.4660	-2.7970	-2.7970	-1.3508	-1.3508	-0.3647e-3	0.3629e-3	0.1014e-2	-0.1022e-2	-0.7144e-6	-0.2544e-5	0.50000
	ε^2	8.4884	8.4656	-2.8221	-2.7719	-1.3441	-1.3575	-0.4806e-2	0.4822e-2	-0.3685e-2	0.3707e-2	-0.9452e-5	0.9240e-5	0.50132
	ε^4	8.4884	8.4657	-2.8220	-2.7718	-1.3441	-1.3576	-0.4802e-2	0.4826e-2	-0.3674e-2	0.3717e-2	-0.9452e-5	0.9240e-5	0.50132
	ε^6	8.4884	8.4657	-2.8220	-2.7718	-1.3441	-1.3576	-0.4802e-2	0.4826e-2	-0.3674e-2	0.3717e-2	-0.9452e-5	0.9240e-5	0.50132
	ε^8	8.4884	8.4657	-2.8220	-2.7718	-1.3441	-1.3576	-0.4802e-2	0.4826e-2	-0.3674e-2	0.3717e-2	-0.9452e-5	0.9240e-5	0.50132
	ε^{10}	8.4884	8.4657	-2.8220	-2.7718	-1.3441	-1.3576	-0.4802e-2	0.4826e-2	-0.3674e-2	0.3717e-2	-0.9452e-5	0.9240e-5	0.50132

Table 4 : Mechanical and electric components at the crucial positions in a single-layer piezoelectric cylinder (PZT-4) under loading conditions of case 5 (L/R=4).

$R/2h$	ε^{2k}	$\bar{u}_x(0,+h)$	$\bar{u}_x(0,-h)$	$\bar{u}_r(\frac{L}{2},+h)$	$\bar{u}_r(\frac{L}{2},-h)$	$\bar{\phi}(\frac{L}{2},+h)$	$\bar{\phi}(\frac{L}{2},-h)$	$\bar{\sigma}_x(\frac{L}{2},+h)$	$\bar{\sigma}_x(\frac{L}{2},-h)$	$\bar{\sigma}_\theta(\frac{L}{2},+h)$	$\bar{\sigma}_\theta(\frac{L}{2},-h)$	$\bar{\tau}_{xr}(0,0)$	$\bar{\sigma}_r(\frac{L}{2},0)$	$\bar{D}_r(\frac{L}{2},0)$
4	ε^0	-0.48741	0.25308	0.03890	0.03890	-0.1837e+2	-0.1837e+2	0.40139	-0.43559	0.03907	-0.05855	0.02044	-0.3099e-2	0.50000
	ε^2	-0.53609	0.29146	0.04970	0.02958	-0.2093e+2	-0.2055e+2	0.49995	-0.51580	0.09246	-0.10778	0.02475	-0.6290e-2	0.52873
	ε^4	-0.53549	0.29077	0.04905	0.02822	-0.2093e+2	-0.2054e+2	0.49835	-0.51579	0.09074	-0.11097	0.02476	-0.6384e-2	0.52467
	ε^6	-0.53551	0.29072	0.04883	0.02842	-0.2092e+2	-0.2054e+2	0.49825	-0.51551	0.09030	-0.11040	0.02475	-0.6348e-2	0.52469
	ε^8	-0.53551	0.29073	0.04883	0.02843	-0.2092e+2	-0.2054e+2	0.49824	-0.51551	0.09031	-0.11036	0.02475	-0.6348e-2	0.52469
	ε^{10}	-0.53551	0.29073	0.04884	0.02843	-0.2092e+2	-0.2054e+2	0.49824	-0.51551	0.09031	-0.11037	0.02475	-0.6348e-2	0.52469
10	ε^0	-1.22551	0.61456	0.10112	0.10112	-0.1148e+3	-0.1148e+3	0.40981	-0.42359	0.05002	-0.05827	0.8175e-2	-0.1356e-2	0.50000
	ε^2	-1.27599	0.65387	0.11159	0.09240	-0.1208e+3	-0.1205e+3	0.45070	-0.45589	0.07218	-0.07684	0.8890e-2	-0.1865e-2	0.51210
	ε^4	-1.27588	0.65377	0.11128	0.09188	-0.1208e+3	-0.1205e+3	0.45052	-0.45600	0.07187	-0.07733	0.8893e-2	-0.1871e-2	0.51146
	ε^6	-1.27588	0.65377	0.11124	0.09191	-0.1208e+3	-0.1205e+3	0.45051	-0.45599	0.07184	-0.07729	0.8893e-2	-0.1870e-2	0.51146
	ε^8	-1.27588	0.65377	0.11124	0.09191	-0.1208e+3	-0.1205e+3	0.45051	-0.45599	0.07184	-0.07729	0.8893e-2	-0.1870e-2	0.51146
	ε^{10}	-1.27588	0.65377	0.11124	0.09191	-0.1208e+3	-0.1205e+3	0.45051	-0.45599	0.07184	-0.07729	0.8893e-2	-0.1870e-2	0.51146
100	ε^0	-0.1229e+2	6.03518	1.03299	1.03299	-0.1148e+5	-0.1148e+5	0.41483	-0.41622	0.057032	-0.057883	0.8159e-3	-0.1436e-3	0.50000
	ε^2	-0.1235e+2	6.07502	1.04329	1.02465	-0.1154e+5	-0.1154e+5	0.41901	-0.41946	0.059301	-0.059679	0.8232e-3	-0.1487e-3	0.50125
	ε^4	-0.1235e+2	6.07502	1.04325	1.02460	-0.1154e+5	-0.1154e+5	0.41901	-0.41946	0.059298	-0.059684	0.8232e-3	-0.1487e-3	0.50124
	ε^6	-0.1235e+2	6.07502	1.04325	1.02460	-0.1154e+5	-0.1154e+5	0.41901	-0.41946	0.059298	-0.059684	0.8232e-3	-0.1487e-3	0.50124
	ε^8	-0.1235e+2	6.07502	1.04325	1.02460	-0.1154e+5	-0.1154e+5	0.41901	-0.41946	0.059298	-0.059684	0.8232e-3	-0.1487e-3	0.50124
	ε^{10}	-0.1235e+2	6.07502	1.04325	1.02460	-0.1154e+5	-0.1154e+5	0.41901	-0.41946	0.059298	-0.059684	0.8232e-3	-0.1487e-3	0.50124

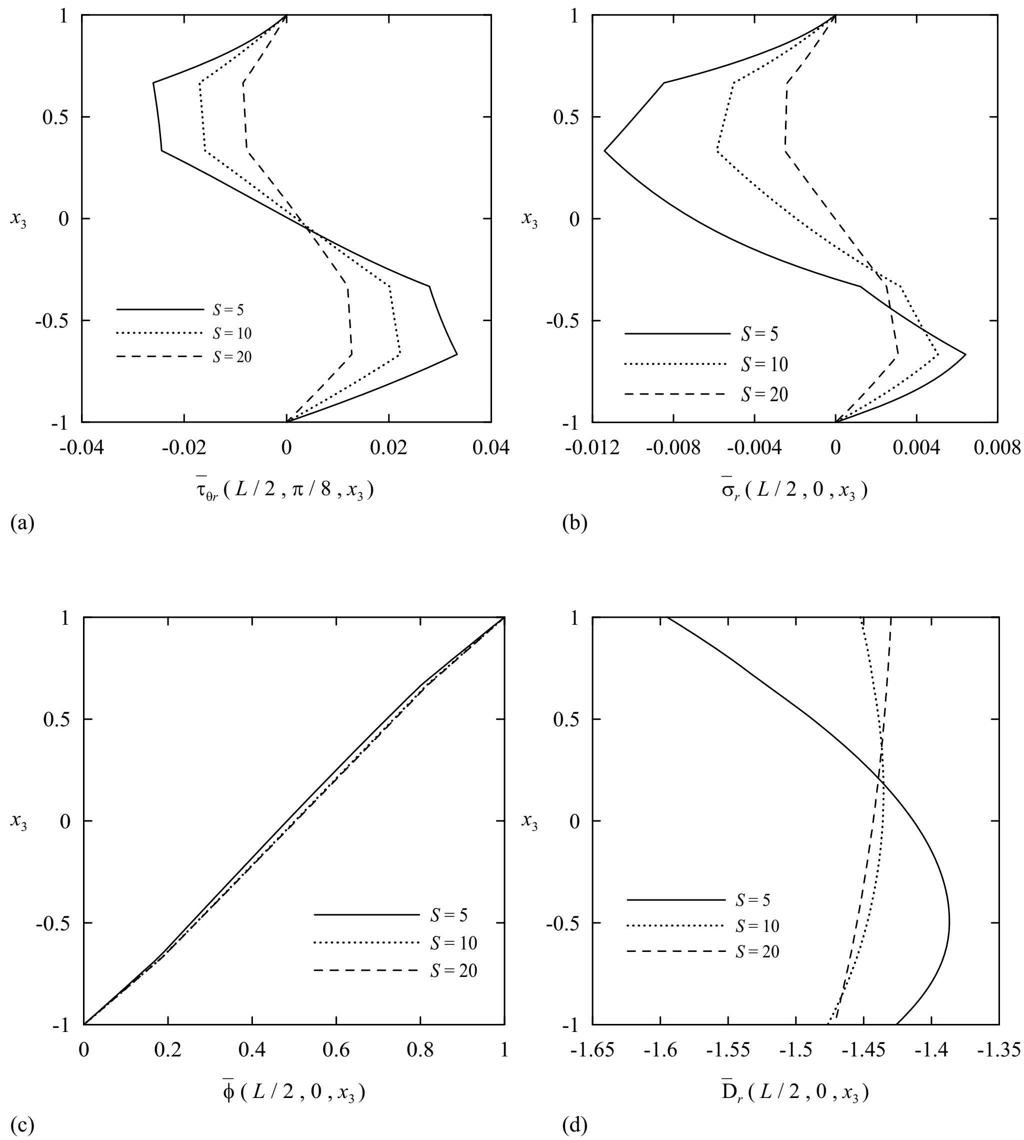

Figure 4 : Variations of mechanical and electric components along the thickness of a [*p*/0/90/90/0/*p*] multilayered hybrid piezoelectric cylinder under $\overline{q}_r^+ = 0$ N/m^2, $\overline{q}_r^- = 0$N/m^2, $\overline{\Phi}_r^+ = \phi_0 \sin(\pi x/L)\cos 4\theta$ V and $\overline{\Phi}_r^- = 0$ V on the lateral surfaces.

Case 5. $\overline{q}_r^+ = 0\text{N}/\text{m}^2$, $\overline{q}_r^- = 0\text{N}/\text{m}^2$; $\overline{D}_r^+ = D_0 \sin\frac{\pi}{L}x\text{C}/\text{m}^2, \overline{D}_r^- = 0\text{C}/\text{m}^2$.

The mechanical and electric field variables are nondimensionalized as follows:

For the applied electric potential case (case 4),

$$\overline{u}_i = u_i c^*/\phi_0 e^*, \quad \overline{\tau}_{ij} = \tau_{ij}(2h)/\phi_0 e^*,$$

$$\overline{\Phi} = \Phi/\phi_0, \quad \overline{D}_i = D_i c^*(2h)/\phi_0 (e^*)^2;$$

For the applied normal electric displacement case (case 5),

$$\overline{u}_i = u_i e^*/D_0(2h), \quad \overline{\tau}_{ij} = \tau_{ij} e^*/D_0 c^*,$$

$$\overline{\Phi} = \Phi(e^*)^2/D_0 c^*(2h), \quad \overline{D}_i = D_i/D_0.$$

The geometric parameters are *L*/*R*=4; *R*/2*h*=4, 10 and 100.

Tables 3-4 show the present asymptotic solutions of elastic and electric field variables for various orders at crucial positions in the cylinders under loading conditions of cases 4 and 5, respectively. Again, it is shown that the

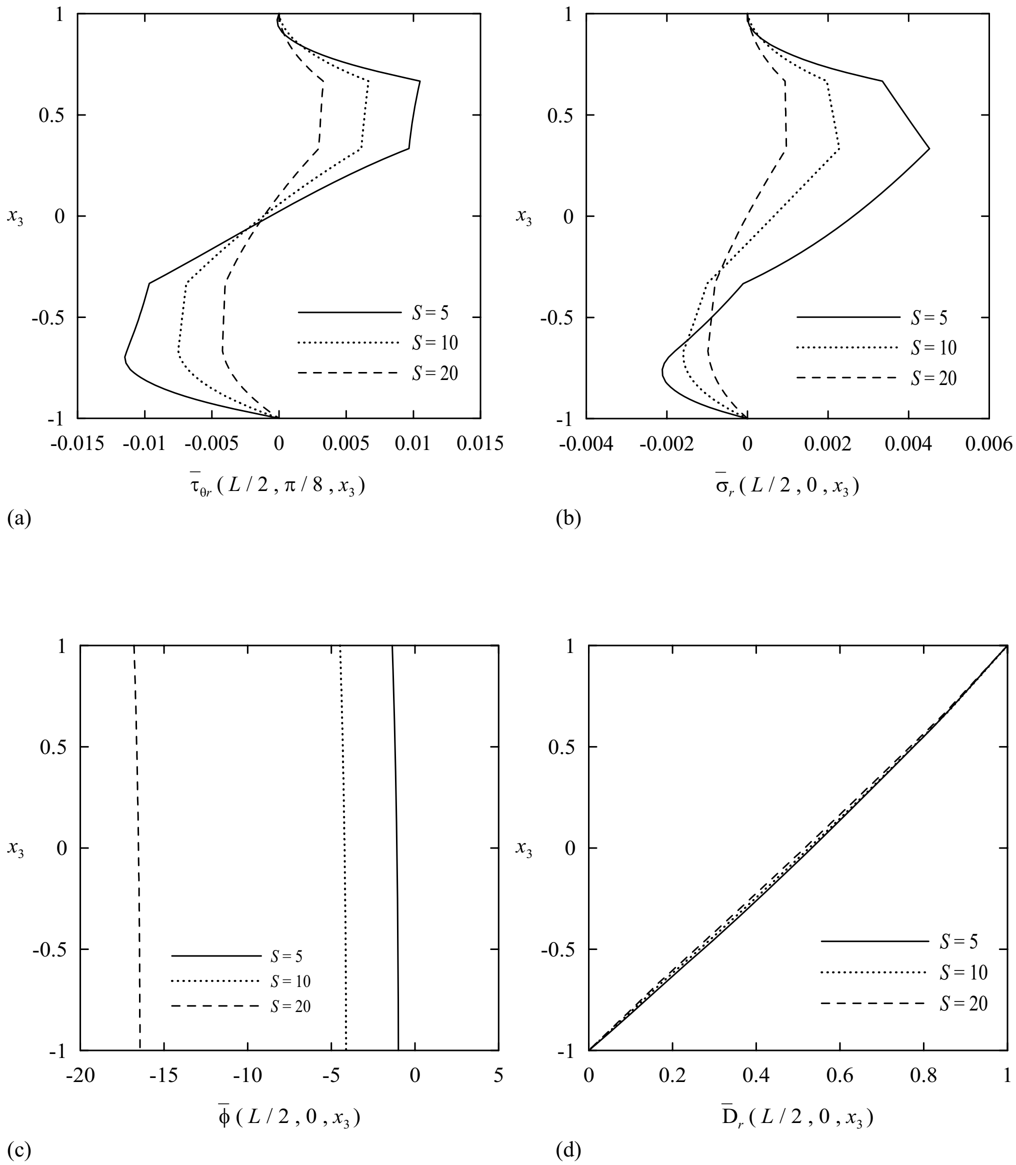

Figure 5 : Variations of mechanical and electric components along the thickness of a [p/0/90/90/0/p] multilayered hybrid piezoelectric cylinder under $\overline{q}_r^+ = 0$ N/m^2, $\overline{q}_r^- = 0$N/m^2, $\overline{D}_r^+ = D_0 \sin(\pi x/L)\cos 4\theta$ C/m^2 and $\overline{D}_r^- = 0$ C/m^2 on the lateral surfaces.

present asymptotic solutions converge rapidly. In general, the convergent solutions are obtained at ε^6-order level in the cases of thick shells (R_α/a_α=4), at ε^4-order level in the cases of moderately thick (R_α/a_α=10) and at ε^2-order level in the cases of thin shells (R_α/a_α=100).

6.2.2 Multilayered hybrid piezoelectric hollow cylinders

The converse piezoelectric effect of [p/0/90/90/0/p] multilayered hybrid piezoelectric hollow cylinders is studied. The configuration, lamination and layer material are the same as used in Example 6.1.2.

Two cases of electric loads are given by

Case 6. $\overline{q}_r^+ = 0\text{N / m}^2$, $\overline{q}_r^- = 0\text{N / m}^2$; $\overline{\Phi}_r^+ = \phi_0 \sin\frac{\pi}{L}x\cos 4\theta$ V, $\overline{\Phi}_r^- = 0$ V.

Case 7. $\overline{q}_r^+ = 0\text{N / m}^2$, $\overline{q}_r^- = 0\text{N / m}^2$; $\overline{D}_r^+ = D_0 \sin\frac{\pi}{L}x\cos 4\theta\text{C/m}^2$, $\overline{D}_r^- = 0\text{C/m}^2$.

The dimensionless variables are the same as used in Example 6.2.1.

Figures 4-5 illustrate the variations of transverse shear and normal stresses, electric potential and normal electric displacement through the thickness coordinate under loading conditions of cases 6 and 7, respectively. It is shown that the distribution of the transverse shear stress across the thickness is approximately layerwise parabolic and the maximum value occurs at the interfaces either between two elastic layers or between the elastic and piezoelectric layers. It is noted from Fig.4(d) that the distributions of normal electric displacement through the thickness coordinate dramatically change in the case of thick shells.

7 Conclusions

Based on the method of perturbation, we develop an asymptotic formulation for the static analysis of multilayered hybrid piezoelectric cylinders under either transverse normal loads or normal electric displacement on the lateral surfaces. The present solutions are illustrated to converge rapidly and be in good agreement with the 3D piezoelectricty solutions available in the literature. A parametric study on the analysis of laminated piezoelectric cylinders under asymmetric electromechanical loads is presented. It is noted that the maximum values of transverse stresses occur at the interfaces either between two elastic layers or between elastic and piezoelectric layers. The distributions of normal electric displacement through the thickness of the shell dramatically change, especially in thick shells. According to the present analysis, it is suggested that an advanced shell theory used for the piezoelectric analysis is more necessary than that used for the elastic analysis.

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Appendix A:

The relevant functions $\tilde{l}_{ij}$ in Eqs.(14)-(19) are given as follows:

$$\tilde{l}_{31} = \left(\frac{e_{31}e_{33}+c_{13}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{\partial}{\partial x_1} + \left(\frac{e_{36}e_{33}+c_{36}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{1}{\gamma_\theta}\frac{\partial}{\partial x_2}, \tag{A.1}$$

$$\tilde{l}_{32} = \left(\frac{e_{36}e_{33}+c_{36}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{\partial}{\partial x_1} + \left(\frac{e_{32}e_{33}+c_{23}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{1}{\gamma_\theta}\frac{\partial}{\partial x_2}, \tag{A.2}$$

$$\tilde{l}_{33} = \left(\frac{e_{32}e_{33}+c_{23}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{1}{\gamma_\theta}, \tag{A.3}$$

$$\tilde{l}_{34} = \frac{\eta_{33}Q}{e_{33}^2+c_{33}\eta_{33}}, \tag{A.4}$$

$$\tilde{l}_{35}=\frac{e_{33}e}{e_{33}^2+c_{33}\eta_{33}}, \tag{A.5}$$

$$\tilde{l}_{22}=(1-x_3\partial_3), \tag{A.6}$$

$$\tilde{l}_{26}=-x_3Qc_{45}/\left(c_{44}c_{55}-c_{45}^2\right), \tag{A.7}$$

$$\tilde{l}_{27}=x_3Qc_{55}/\left(c_{44}c_{55}-c_{45}^2\right), \tag{A.8}$$

$$\tilde{l}_{14}=c_{44}Q/\left(c_{44}c_{55}-c_{45}^2\right), \tag{A.9}$$

$$\tilde{l}_{15}=-c_{45}Q/\left(c_{44}c_{55}-c_{45}^2\right), \tag{A.10}$$

$$\tilde{l}_{25}=c_{55}Q/\left(c_{44}c_{55}-c_{45}^2\right), \tag{A.11}$$

$$\tilde{l}_{18}=\left(\frac{c_{45}e_{14}-c_{44}e_{15}}{c_{44}c_{55}-c_{45}^2}\right)\frac{Q}{e}\frac{\partial}{\partial x_1}+\left(\frac{c_{45}e_{24}-c_{44}e_{25}}{c_{44}c_{55}-c_{45}^2}\right)\frac{Q}{e\gamma_\theta}\frac{\partial}{\partial x_2}, \tag{A.12}$$

$$\tilde{l}_{28}=\left(\frac{c_{45}e_{15}-c_{55}e_{14}}{c_{44}c_{55}-c_{45}^2}\right)\frac{Q\gamma_\theta}{e}\frac{\partial}{\partial x_1}+\left(\frac{c_{45}e_{25}-c_{55}e_{24}}{c_{44}c_{55}-c_{45}^2}\right)\frac{Q}{e}\frac{\partial}{\partial x_2}, \tag{A.13}$$

$$\tilde{l}_{41}=\tilde{Q}_{11}\gamma_\theta\frac{\partial^2}{\partial x^2}+\left(\tilde{Q}_{16}+\tilde{Q}_{61}\right)\frac{\partial^2}{\partial x_1\partial x_2}+\frac{\tilde{Q}_{66}}{\gamma_\theta}\frac{\partial^2}{\partial x_2^2}, \tag{A.14}$$

$$\tilde{l}_{42}=\tilde{Q}_{16}\gamma_\theta\frac{\partial^2}{\partial x_1^2}+(\tilde{Q}_{12}+\tilde{Q}_{66})\frac{\partial^2}{\partial x_1\partial x_2}+\frac{\tilde{Q}_{62}}{\gamma_\theta}\frac{\partial^2}{\partial x_2^2}, \tag{A.15}$$

$$\tilde{l}_{51}=\tilde{Q}_{61}\gamma_\theta\frac{\partial^2}{\partial x_1^2}+(\tilde{Q}_{21}+\tilde{Q}_{66})\frac{\partial^2}{\partial x_1\partial x_2}+\frac{\tilde{Q}_{26}}{\gamma_\theta}\frac{\partial^2}{\partial x_2^2}, \tag{A.16}$$

$$\tilde{l}_{52}=\tilde{Q}_{66}\gamma_\theta\frac{\partial^2}{\partial x_1^2}+\left(\tilde{Q}_{26}+\tilde{Q}_{62}\right)\frac{\partial^2}{\partial x_1\partial x_2}+\frac{\tilde{Q}_{22}}{\gamma_\theta}\frac{\partial^2}{\partial x_2^2}, \tag{A.17}$$

$$\tilde{l}_{43}=\tilde{Q}_{12}\frac{\partial}{\partial x_1}+\frac{\tilde{Q}_{62}}{\gamma_\theta}\frac{\partial}{\partial x_2}, \tag{A.18}$$

$$\tilde{l}_{53}=\tilde{Q}_{62}\frac{\partial}{\partial x_1}+\frac{\tilde{Q}_{22}}{\gamma_\theta}\frac{\partial}{\partial x_2}, \tag{A.19}$$

$$\tilde{l}_{44}=1+x_3\frac{\partial}{\partial x_3}, \tag{A.20}$$

$$\tilde{l}_{55}=2+x_3\frac{\partial}{\partial x_3}, \tag{A.21}$$

$$\tilde{l}_{46}=\left(\frac{c_{13}e_{33}-e_{31}c_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{\gamma_\theta e}{Q}\frac{\partial}{\partial x_1}+\left(\frac{c_{36}e_{33}-e_{36}c_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{e}{Q}\frac{\partial}{\partial x_2}, \tag{A.22}$$

$$\tilde{l}_{56}=\left(\frac{c_{36}e_{33}-e_{36}c_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{\gamma_\theta e}{Q}\frac{\partial}{\partial x_1}+\left(\frac{c_{23}e_{33}-e_{32}c_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{e}{Q}\frac{\partial}{\partial x_2}, \tag{A.23}$$

$$\tilde{l}_{61}=\tilde{Q}_{21}\frac{\partial}{\partial x_1}+\frac{\tilde{Q}_{26}}{\gamma_\theta}\frac{\partial}{\partial x_2}, \tag{A.24}$$

$$\tilde{l}_{62}=\tilde{Q}_{26}\frac{\partial}{\partial x_1}+\frac{\tilde{Q}_{22}}{\gamma_\theta}\frac{\partial}{\partial x_2}, \tag{A.25}$$

$$\tilde{l}_{63}=\frac{\tilde{Q}_{22}}{\gamma_\theta}, \tag{A.26}$$

$$\tilde{l}_{64}=1+x_3\frac{\partial}{\partial x_3}-\left(\frac{e_{32}e_{33}+c_{23}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}}\right), \tag{A.27}$$

$$\tilde{l}_{65}=\frac{e}{Q}\left(\frac{c_{23}e_{33}-e_{32}c_{33}}{e_{33}^2+c_{33}\eta_{33}}\right), \tag{A.28}$$

$$\tilde{l}_{81}=\left(\frac{c_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{e^2}{Q}, \tag{A.29}$$

$$\tilde{Q}_{ij}=\frac{Q_{ij}}{Q}$$

$$\text{and } Q_{ij}=c_{ij}-\left(\frac{e_{33}e_{3j}+c_{j3}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)c_{i3}-\left(\frac{e_{33}c_{j3}-c_{33}e_{3j}}{e_{33}^2+c_{33}\eta_{33}}\right)e_{3i}\quad (i,j=1,2,6) \tag{A.30}$$

The relevant functions $\tilde{b}_{ij}$ in Eqs.(20)-(21) are given by

$$\tilde{b}_{11}=\tilde{Q}_{11}\frac{\partial}{\partial x_1}+\frac{\tilde{Q}_{16}}{\gamma_\theta}\frac{\partial}{\partial x_2},\quad \tilde{b}_{12}=\tilde{Q}_{16}\frac{\partial}{\partial x_1}+\frac{\tilde{Q}_{12}}{\gamma_\theta}\frac{\partial}{\partial x_2}, \tag{A.31}$$

$$\tilde{b}_{21}=\tilde{Q}_{21}\frac{\partial}{\partial x_1}+\frac{\tilde{Q}_{26}}{\gamma_\theta}\frac{\partial}{\partial x_2},\quad \tilde{b}_{22}=\tilde{Q}_{26}\frac{\partial}{\partial x_1}+\frac{\tilde{Q}_{22}}{\gamma_\theta}\frac{\partial}{\partial x_2}, \tag{A.32}$$

$$\tilde{b}_{31}=\tilde{Q}_{61}\frac{\partial}{\partial x_1}+\frac{\tilde{Q}_{66}}{\gamma_\theta}\frac{\partial}{\partial x_2},\quad \tilde{b}_{32}=\tilde{Q}_{66}\frac{\partial}{\partial x_1}+\frac{\tilde{Q}_{62}}{\gamma_\theta}\frac{\partial}{\partial x_2},$$

(A.33)

$$\tilde{b}_{13} = \frac{\tilde{Q}_{12}}{\gamma_\theta}, \qquad \tilde{b}_{23} = \frac{\tilde{Q}_{22}}{\gamma_\theta}, \qquad \tilde{b}_{33} = \frac{\tilde{Q}_{62}}{\gamma_\theta}, \tag{A.34}$$

$$\tilde{b}_{14} = \frac{e_{31}e_{33} + c_{13}\eta_{33}}{e_{33}^2 + c_{33}\eta_{33}}, \quad \tilde{b}_{24} = \frac{e_{32}e_{33} + c_{23}\eta_{33}}{e_{33}^2 + c_{33}\eta_{33}}, \tag{A.35}$$

$$\tilde{b}_{34} = \frac{e_{36}e_{33} + c_{36}\eta_{33}}{e_{33}^2 + c_{33}\eta_{33}}, \tag{A.36}$$

$$\tilde{b}_{15} = \left(\frac{e}{Q}\right)\left(\frac{c_{13}e_{33} - e_{31}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right), \tag{A.37}$$

$$\tilde{b}_{25} = \left(\frac{e}{Q}\right)\left(\frac{c_{23}e_{33} - e_{32}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right), \tag{A.38}$$

$$\tilde{b}_{35} = \left(\frac{e}{Q}\right)\left(\frac{c_{36}e_{33} - e_{36}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right), \tag{A.39}$$

$$\tilde{b}_{41} = \left(\frac{Q}{e}\right)\left(\frac{c_{44}e_{15} - c_{45}e_{14}}{c_{44}c_{55} - c_{45}^2}\right), \tag{A.40}$$

$$\tilde{b}_{42} = \left(\frac{Q}{e}\right)\left(\frac{c_{55}e_{14} - c_{45}e_{15}}{c_{44}c_{55} - c_{45}^2}\right), \tag{A.41}$$

$$\tilde{b}_{51} = \left(\frac{Q}{e}\right)\left(\frac{c_{44}e_{25} - c_{45}e_{24}}{c_{44}c_{55} - c_{45}^2}\right), \tag{A.42}$$

$$b_{52} = \left(\frac{Q}{e}\right)\left(\frac{c_{55}e_{24} - c_{45}e_{25}}{c_{44}c_{55} - c_{45}^2}\right), \tag{A.43}$$

$$\begin{aligned}
\tilde{b}_{43} = &\left[\left(\frac{c_{45}e_{15} - c_{55}e_{14}}{c_{44}c_{55} - c_{45}^2}\right)e_{14}\right. \\
&\left.+\left(\frac{c_{45}e_{14} - c_{44}e_{15}}{c_{44}c_{55} - c_{45}^2}\right)e_{15} - \eta_{11}\right]\frac{Q}{e^2}\frac{\partial}{\partial x_1} \\
&+\left[\left(\frac{c_{45}e_{25} - c_{55}e_{24}}{c_{44}c_{55} - c_{45}^2}\right)e_{14}\right. \\
&\left.+\left(\frac{c_{45}e_{24} - c_{44}e_{25}}{c_{44}c_{55} - c_{45}^2}\right)e_{15} - \eta_{12}\right]\frac{Q}{e^2\gamma_\theta}\frac{\partial}{\partial x_2},
\end{aligned} \tag{A.44}$$

$$\begin{aligned}
\tilde{b}_{53} = &\left[\left(\frac{c_{45}e_{15} - c_{55}e_{14}}{c_{44}c_{55} - c_{45}^2}\right)e_{24}\right. \\
&\left.+\left(\frac{c_{45}e_{14} - c_{44}e_{15}}{c_{44}c_{55} - c_{45}^2}\right)e_{25} - \eta_{12}\right]\frac{Q}{e^2}\frac{\partial}{\partial x_1} \\
&+\left[\left(\frac{c_{45}e_{25} - c_{55}e_{24}}{c_{44}c_{55} - c_{45}^2}\right)e_{24}\right. \\
&\left.+\left(\frac{c_{45}e_{24} - c_{44}e_{25}}{c_{44}c_{55} - c_{45}^2}\right)e_{25} - \eta_{22}\right]\frac{Q}{e^2\gamma_\theta}\frac{\partial}{\partial x_2}
\end{aligned} \tag{A.45}$$