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Simultaneously Estimating the Time-Dependent Damping and Stiffness Coefficients with the Aid of Vibrational Data

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Abstract: For the inverse vibration problem a mathematical method is required to determine unknown parameters from the measurement of vibration data. When both damping and stiffness functions are identified, it is a rather difficult problem. In this paper we will propose a feasible method to simultaneously estimate both the time-dependent damping and stiffness coefficients through three mathematical transformations. First, the second-order equation of motion is transformed into a self-adjoint first-order system by using the concept of integrating factor. Then, we transform these two ODEs into two hyperbolic type PDEs. Finally, we apply a one-step group preserving scheme for the semi-discretizations of PDEs to obtain two uncoupled algebraic equations, of which the first one is used to estimate the damping coefficient while the second one is used to estimate the stiffness coefficient. The estimated results are acceptable for that used in vibrational engineering. We also discuss the use of velocity and acceleration data as inputs in the estimation. However, it leads to a bad result, and is not suggested for the use in estimation.

Keyword: Inverse vibration problem, Time-dependent damping and stiffness coefficients, Integrating factor, Lie-group estimation method

1 Introduction

Many research and effort have been involved in the science of vibrations. The solution of direct problem of forced vibration is concerned with the determination of system's displacement, velocity and acceleration evolving in time when initial conditions, external forces and system parameters are specified exactly. Sometimes we may encounter the problem that some parameters in the system we consider are unknown, and then the resulting problem is an inverse vibration problem. It is concerned with the estimations of damping coefficient [Adhikari and Woodhouse (2001a); Adhikari and Woodhouse (2001b); Ingman and Suzdalnitsky (2001); Liang and Feeny (2006); Liu (2008a)], stiffness [Huang (2001); Shiguemori, Chiwiacowsky and de Campos Velho (2005); Liu (2008a)], as well as external force [Huang (2005); Feldman (2007)] with the aid of measurable vibration data, such as frequency, mode shape, displacement or velocity at different time.

The parameters' identification problem is known to be highly ill-posed in the sense that a small disturbance of measured data may result in a tremendous error in the parameters' estimation. In order to overcome this problem, there have appeared many studies in this field. Although the system we consider is linear, we may require to treat a nonlinear inverse vibration problem.

Let us consider a second-order ordinary differential equation (ODE) describing the forced vibration of a linear structure with time-dependent parameters:

$$\ddot{\phi}+c(t)\dot{\phi}+k(t)\phi=F(t),\ \ 0\leq t\leq t_f, \tag{1}$$

$$\phi(0)=A_0, \tag{2}$$

$$\dot{\phi}(0)=B_0. \tag{3}$$

The direct problem is for the given conditions in Eqs. (2) and (3) and the given functions $c(t)$, $k(t)$ and $F(t)$ in Eq. (1) to find the response of $\phi(t)$ in a time interval of $t\in[0,t_f]$. However, our present inverse vibration problem is to estimate $c(t)$ and

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$k(t)$ with $t \in [0, t_f]$ by using some measured data of $\phi(t)$ and $\dot{\phi}(t)$ in a time interval of $t \in [0, t_f]$.

The present approach is based on three different transformations, and is novel. The readers may appreciate that the present approach is very interesting, which resulting to a closed-form estimating equation without needing of any iteration and initial guess of coefficient functions. More importantly, the novel method does not require to assume a priori the functional forms of unknown coefficients.

Recently, Liu (2006a, 2006b, 2006c) has extended the group preserving scheme (GPS) developed previously by Liu (2001) for ODEs to solve the boundary value problems (BVPs), and the numerical results revealed that the GPS is a rather promising method to effectively solve the two-point BVPs. In the construction of Lie-group method for the calculations of BVPs, Liu (2006a) has introduced the idea of one-step GPS by utilizing the closure property of Lie groups, and hence, the new shooting method has been named the Lie-group shooting method (LGSM). Remarkably, Liu (2008b,2008c) has explored its superiority by using the LGSM to estimate parameters in parabolic type PDEs.

On the other hand, in order to effectively solve the backward in time problems of parabolic type PDEs, a past cone structure and a backward group preserving scheme have been successfully developed, such that the one-step Lie-group numerical methods have been used to solve the backward in time Burgers equation by Liu (2006d), and the backward in time heat conduction equation by Liu, Chang and Chang (2006a).

The Lie-group method is originally used for the BVPs as designed by Liu (2006a, 2006b, 2006c) for direct problems. In a series of papers by the first author and his coworkers, the Lie-group method reveals its excellent behavior on the numerical solutions of different problems, for example, Chang, Liu and Chang (2005) to calculate the sideways heat conduction problem, Chang, Chang and Liu (2006) to treat the boundary layer equation in fluid mechanics, and Liu (2004), Liu, Chang and Chang (2006a), and Chang, Liu and Chang (2007a, 2007b) to treat the backward heat conduction equation, Liu, Chang and Chang (2006b) to treat the Burgers equation, and Liu (2008d) to treat an inverse Sturm-Liouville problem.

It should be stressed that the one-step Lie-group property is usually not shared by other numerical methods, because those methods do not belong to the Lie-group types. This important property as first pointed out by Liu (2006d) was employed to solve the backward in time Burgers equation. After that, Liu (2006e) has used this concept to establish a one-step estimation method to estimate the temperature-dependent heat conductivity, and then extended to estimate heat conductivity and heat capacity by Liu (2006f, 2007) and Liu, Liu and Hong (2007). The Lie-group method possesses a great advantage than other numerical methods due to its group structure, and is a powerful technique to solve the inverse problems of parameters' identification.

This paper will extend those parameters' identification techniques to the inverse vibration problems, which is arranged as follows. We introduce a novel approach of inverse vibration problem in Section 2 by transforming it into a self-adjoint first-order ODEs system, then an identification problem of hyperbolic type PDEs, and then the discretizations of PDEs into a system of ODEs at the discretized times. In Section 3 we give a brief sketch of the GPS for ODEs for a self-content reason. Due to the good property of Lie-group, we will propose an integration technique, such that the one-step GPS can be used to identify the parameters appeared in the introduced PDEs. The resulting algebraic equations are derived in Section 4 when we apply the one-step GPS to identify $c(t)$ and $k(t)$. We demonstrate that how the Lie-group theory can help us to solve these parameters' estimation equations in closed-form. In Section 5 numerical examples are examined to test the Lie-group estimation method (LGEM). Finally, we give some conclusions in Section 6.

2 A novel approach

2.1 Transformation into a self-adjoint system of ODEs

For the second-order ODE in Eq. (1) many teachers may teach the students to write it as a 2D system of ODEs by

$$\frac{d}{dt}\begin{bmatrix} \phi(t) \\ \dot\phi(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k(t) & -c(t) \end{bmatrix}\begin{bmatrix} \phi(t) \\ \dot\phi(t) \end{bmatrix} + \begin{bmatrix} 0 \\ F(t) \end{bmatrix}. \quad (4)$$

In the calculation of direct problem this transformation may be helpful and workable; however, this transformation gives no any help in the calculation of inverse vibration problem.

By letting

$$p(t) := \exp\left[-\int_0^t c(\xi)d\xi\right], \quad (5)$$

from Eq. (1) we may have a self-adjoint system:

$$\frac{d}{dt}\begin{bmatrix} \phi(t) \\ \frac{\dot\phi(t)}{p(t)} \end{bmatrix} = \begin{bmatrix} 0 & p(t) \\ -\frac{k(t)}{p(t)} & 0 \end{bmatrix}\begin{bmatrix} \phi(t) \\ \frac{\dot\phi(t)}{p(t)} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F(t)}{p(t)} \end{bmatrix}. \quad (6)$$

Upon defining

$$y_1(t) := \phi(t), \quad (7)$$

$$y_2(t) := \frac{\dot\phi(t)}{p(t)}, \quad (8)$$

from Eq. (6) we have

$$\dot y_1(t) = p(t)y_2(t), \quad (9)$$

$$\dot y_2(t) = -\frac{k(t)}{p(t)}y_1(t) + \frac{F(t)}{p(t)}. \quad (10)$$

The above two equations are our starting point. Eq. (6) is superior than Eq. (4), because it has a Lie-group transformation of $SL(2,\mathbb{R})$.

2.2 Transformation into a system of PDEs

In the solutions of linear PDEs, a common technique is the seperation of variables, from which the PDEs are transformed into some ODEs. We can reverse this process by considering

$$u(x,t) := (1+x)y_1(t), \quad (11)$$

$$v(x,t) := (1+x)y_2(t), \quad (12)$$

such that Eqs. (9) and (10) are changed to a hyperbolic system of PDEs:

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial u(x,t)}{\partial t} - p(t)v(x,t) + y_1(t), \quad (13)$$

$$\frac{\partial v(x,t)}{\partial x} = \frac{\partial v(x,t)}{\partial t} + \frac{k(t)}{p(t)}u(x,t) - \frac{(1+x)F(t)}{p(t)} + y_2(t), \quad (14)$$

$$u(0,t) = \phi(t), \quad (15)$$

$$v(0,t) = \frac{\dot\phi(t)}{p(t)}, \quad (16)$$

$$u(x,0) = A_0(1+x), \quad (17)$$

$$u(x,t_f) = \phi(t_f)(1+x), \quad (18)$$

$$v(x,0) = B_0(1+x), \quad (19)$$

$$v(x,t_f) = \frac{\dot\phi(t_f)}{p(t_f)}(1+x), \quad (20)$$

where $\phi(t_f)$ and $\dot\phi(t_f)$ are respectively the measured displacement and velocity at time t_f. In Eqs. (13) and (14), $k(t)$ and $p(t)$ are time-dependent functions to be identified, where the domain we consider is $0 \le t \le t_f,\ \ 0 < x \le x_f$. In order to estimate $k(t)$ and $c(t)$ we suppose that $\phi(t)$ and $\dot\phi(t)$ are measurable in a time interval of $0 \le t \le t_f$. The coordinate x is a fictitious one; however, from it together with t we can work in a two-dimensional domain and we can find the variations of $c(t)$ and $k(t)$ more easily in that domain.

2.3 Semi-discretizations

Applying a semi-discrete procedure on the above PDEs yields a coupled system of ODEs. For Eqs. (13) and (14), we adopt the numerical method of line to discretize the time coordinate t by

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=i\Delta t} = \frac{u_{i+1}(x) - u_i(x)}{\Delta t}, \quad (21)$$

$$\left.\frac{\partial v(x,t)}{\partial t}\right|_{t=i\Delta t} = \frac{v_{i+1}(x) - v_i(x)}{\Delta t}, \quad (22)$$

where $\Delta t = t_f/(n+1)$ is a uniform time increment, and $u_i(x) = u(x, i\Delta t)$ and $v_i(x) = v(x, i\Delta t)$ are used for simple notations. Consequently, Eqs. (13) and (14) can be approximated by

$$u_i'(x) = \frac{1}{\Delta t}[u_{i+1}(x) - u_i(x)] - p_i v_i(x) + y_1^i, \quad (23)$$

$$v_i'(x) = \frac{1}{\Delta t}[v_{i+1}(x) - v_i(x)] + \frac{k_i}{p_i} u_i(x) - \frac{(1+x)F_i}{p_i} + y_2^i, \quad (24)$$

where $i = 1, \ldots, n$, $k_i = k(t_i)$, $p_i = p(t_i)$, $F_i = F(t_i)$, $y_1^i = y_1(t_i)$ and $y_2^i = y_2(t_i)$.

In this section we have transformed the inverse vibration problem of the second-order ODE in Eq. (1) into an inverse problem for the PDEs in Eqs. (13) and (14) by estimating $k(t)$ and $p(t)$, and finally an estimation of $2n$ coefficients k_i and p_i in the $2n$-dimensional ODEs system in Eqs. (23) and (24). However, Eq. (23) is a consequence of the definition $\dot{\phi} = p \cdot \dot{\phi}/p$ coming from the first equation in Eq. (6), which will be no useful in the following estimation. Therefore, basing on Eq. (1) we will derive another PDE directly as a supplemented equation in Section 4.

3 GPS for differential equations system

3.1 Group-preserving scheme

Upon letting $\mathbf{w} = (u_1, v_1, \ldots, u_n, v_n)^{\mathrm{T}}$ and $\mathbf{f}$ denoting the right-hand sides of Eqs. (23) and (24) we can write them as a vector form:

$$\mathbf{w}' = \mathbf{f}(\mathbf{w}, x), \ \ \mathbf{w} \in \mathbb{R}^{2n}, \ \ x \in \mathbb{R}. \quad (25)$$

Liu (2001) has embedded Eq. (25) into an augmented differential equations system as follows:

$$\frac{d}{dx}\begin{bmatrix} \mathbf{w} \\ \|\mathbf{w}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{2n\times 2n} & \frac{\mathbf{f}(\mathbf{w},x)}{\|\mathbf{w}\|} \\ \frac{\mathbf{f}^{\mathrm{T}}(\mathbf{w},x)}{\|\mathbf{w}\|} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{w} \\ \|\mathbf{w}\| \end{bmatrix}. \quad (26)$$

It is obvious that the first row in Eq. (26) is the same as the original equation (25), but the inclusion of the second row in Eq. (26) gives us a Minkowskian structure of the augmented state variables of $\mathbf{X} := (\mathbf{w}^{\mathrm{T}}, \|\mathbf{w}\|)^{\mathrm{T}}$, which satisfies the cone condition:

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = 0, \quad (27)$$

where

$$\mathbf{g} := \begin{bmatrix} \mathbf{I}_{2n} & \mathbf{0}_{2n\times 1} \\ \mathbf{0}_{1\times 2n} & -1 \end{bmatrix} \quad (28)$$

is a Minkowski metric, $\mathbf{I}_{2n}$ is the identity matrix of order $2n$, and the superscript T stands for the transpose. In terms of $(\mathbf{w}^{\mathrm{T}}, \|\mathbf{w}\|)$, Eq. (27) becomes

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = \mathbf{w}\cdot\mathbf{w} - \|\mathbf{w}\|^2 = \|\mathbf{w}\|^2 - \|\mathbf{w}\|^2 = 0, \quad (29)$$

where the dot between two vectors denotes their inner product.

Consequently, we have a $2n+1$-dimensional augmented system:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} \quad (30)$$

with a constraint (27), where

$$\mathbf{A} := \begin{bmatrix} \mathbf{0}_{2n\times 2n} & \frac{\mathbf{f}(\mathbf{w},x)}{\|\mathbf{w}\|} \\ \frac{\mathbf{f}^{\mathrm{T}}(\mathbf{w},x)}{\|\mathbf{w}\|} & 0 \end{bmatrix}, \quad (31)$$

satisfying

$$\mathbf{A}^{\mathrm{T}}\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}, \quad (32)$$

is a Lie algebra $so(2n,1)$ of the proper orthochronous Lorentz group $SO_o(2n,1)$. This fact prompts us to devise the group-preserving scheme (GPS), whose discretized mapping $\mathbf{G}$ must exactly preserve the following properties:

$$\mathbf{G}^{\mathrm{T}}\mathbf{g}\mathbf{G} = \mathbf{g}, \quad (33)$$

$$\det \mathbf{G} = 1, \quad (34)$$

$$G_0^0 > 0, \quad (35)$$

where G_0^0 is the 00th component of $\mathbf{G}$.

Although the dimension of the new system is raising one more, it has been shown that the new system permits a GPS given as follows [Liu (2001)]:

$$\mathbf{X}_{\ell+1} = \mathbf{G}(\ell)\mathbf{X}_\ell, \quad (36)$$

where $\mathbf{X}_\ell$ denotes the numerical value of $\mathbf{X}$ at x_ℓ, and $\mathbf{G}(\ell) \in SO_o(2n,1)$ is the group value of $\mathbf{G}$ at x_ℓ. If $\mathbf{G}(\ell)$ satisfies the properties in Eqs. (33)-(35), then $\mathbf{X}_\ell$ satisfies the cone condition in Eq. (27).

The Lie group can be generated from $\mathbf{A} \in so(2n,1)$ by an exponential mapping,

$$\mathbf{G}(\ell) = \exp[\Delta x \mathbf{A}(\ell)] = \begin{bmatrix} \mathbf{I}_{2n} + \frac{(a_\ell - 1)}{\|\mathbf{f}_\ell\|^2}\mathbf{f}_\ell \mathbf{f}_\ell^{\mathrm{T}} & \frac{b_\ell \mathbf{f}_\ell}{\|\mathbf{f}_\ell\|} \\ \frac{b_\ell \mathbf{f}_\ell^{\mathrm{T}}}{\|\mathbf{f}_\ell\|} & a_\ell \end{bmatrix}, \tag{37}$$

where

$$a_\ell := \cosh\left(\frac{\Delta x \|\mathbf{f}_\ell\|}{\|\mathbf{w}_\ell\|}\right), \tag{38}$$

$$b_\ell := \sinh\left(\frac{\Delta x \|\mathbf{f}_\ell\|}{\|\mathbf{w}_\ell\|}\right). \tag{39}$$

Substituting Eq. (37) for $\mathbf{G}(\ell)$ into Eq. (36), we obtain

$$\mathbf{w}_{\ell+1} = \mathbf{w}_\ell + \eta_\ell \mathbf{f}_\ell, \tag{40}$$

$$\|\mathbf{w}_{\ell+1}\| = a_\ell \|\mathbf{w}_\ell\| + \frac{b_\ell}{\|\mathbf{f}_\ell\|}\mathbf{f}_\ell \cdot \mathbf{w}_\ell, \tag{41}$$

where

$$\eta_\ell := \frac{b_\ell \|\mathbf{w}_\ell\| \|\mathbf{f}_\ell\| + (a_\ell - 1)\mathbf{f}_\ell \cdot \mathbf{w}_\ell}{\|\mathbf{f}_\ell\|^2} \tag{42}$$

is an adaptive factor. From $\mathbf{f}_\ell \cdot \mathbf{w}_\ell \geq -\|\mathbf{f}_\ell\|\|\mathbf{w}_\ell\|$ we can prove that

$$\eta_\ell \geq \left[1 - \exp\left(-\frac{\Delta x \|\mathbf{f}_\ell\|}{\|\mathbf{w}_\ell\|}\right)\right] \frac{\|\mathbf{w}_\ell\|}{\|\mathbf{f}_\ell\|} > 0, \ \forall \Delta x > 0. \tag{43}$$

This scheme is group properties preserved for all $\Delta x > 0$, and is called the group-preserving scheme.

3.2 One-step GPS

Applying scheme (40) to Eq. (25) we can compute $\mathbf{w}^f$ by GPS. Throughout this paper the superscript f denotes the value at $x = x_f$, while the superscript 0 denotes the value at $x = 0$. Assume that the total length x_f is divided by K steps, that is, the stepsize we use in the GPS is $\Delta x = x_f / K$.

Starting from $\mathbf{X}^0 = \mathbf{X}(0)$ we want to calculate the value $\mathbf{X}(x_f)$ at $x = x_f$. By Eq. (36) we can obtain

$$\mathbf{X}^f = \mathbf{G}_K(\Delta x) \cdots \mathbf{G}_1(\Delta x)\mathbf{X}^0, \tag{44}$$

where $\mathbf{X}^f$ approximates the real $\mathbf{X}(x_f)$ within a certain accuracy depending on Δx. However, let us recall that each $\mathbf{G}_i$, $i = 1, \ldots, K$, is an element of the Lie group $SO_o(2n,1)$, and by the closure property of Lie group, $\mathbf{G}_K(\Delta x) \cdots \mathbf{G}_1(\Delta x)$ is also a Lie group denoted by $\mathbf{G}(x_f)$. Hence, we have

$$\mathbf{X}^f = \mathbf{G}(x_f)\mathbf{X}^0. \tag{45}$$

This is a one-step Lie-group transformation from $\mathbf{X}^0$ to $\mathbf{X}^f$.

Usually, it is very hard to find the exact solution of $\mathbf{G}(x_f)$; however, a numerical one may be obtained approximately without any difficulty. The most simple method to calculate $\mathbf{G}(x_f)$ is given by

$$\mathbf{G}(x_f) = \begin{bmatrix} \mathbf{I}_{2n} + \frac{(a-1)}{\|\mathbf{f}^0\|^2}\mathbf{f}^0(\mathbf{f}^0)^{\mathrm{T}} & \frac{b\mathbf{f}^0}{\|\mathbf{f}^0\|} \\ \frac{b(\mathbf{f}^0)^{\mathrm{T}}}{\|\mathbf{f}^0\|} & a \end{bmatrix}, \tag{46}$$

where

$$a := \cosh\left(\frac{x_f \|\mathbf{f}^0\|}{\|\mathbf{w}^0\|}\right), \tag{47}$$

$$b := \sinh\left(\frac{x_f \|\mathbf{f}^0\|}{\|\mathbf{w}^0\|}\right). \tag{48}$$

Here, we use the value of $\mathbf{w}^0 = \mathbf{w}(0)$ to calculate $\mathbf{G}(x_f)$. Then from Eqs. (45) and (46) we obtain a one-step GPS:

$$\mathbf{w}^f = \mathbf{w}^0 + \eta \mathbf{f}^0, \tag{49}$$

$$\|\mathbf{w}^f\| = a\|\mathbf{w}^0\| + \frac{b\mathbf{f}^0 \cdot \mathbf{w}^0}{\|\mathbf{f}^0\|}, \tag{50}$$

where

$$\eta = \frac{(a-1)\mathbf{f}^0 \cdot \mathbf{w}^0 + b\|\mathbf{w}^0\|\|\mathbf{f}^0\|}{\|\mathbf{f}^0\|^2}. \tag{51}$$

4 Identifying $c(t)$ and $k(t)$ by the LGEM

In this section we will start to estimate the time-dependent coefficient functions $c(t)$ and $k(t)$.

By applying the one-step GPS on Eqs. (23) and (24) from $x=0$ to $x=x_f$ we obtain two nonlinear equations for k_i and p_i:

$$u_i^f=u_i^0+\frac{\eta}{\Delta t}(u_{i+1}^0-u_i^0)-\eta p_i v_i^0+\eta y_1^i, \tag{52}$$

$$v_i^f=v_i^0+\frac{\eta}{\Delta t}(v_{i+1}^0-v_i^0)+\frac{\eta k_i}{p_i}u_i^0-\frac{\eta F_i}{p_i}+\eta y_2^i. \tag{53}$$

At the first glance, η in the above seems a nonlinear function of k_i and p_i as shown by Eq. (51). However, we will prove below that η is fully determined by x_f when using the proportionality of u_i^0, u_i^f, v_i^0 and v_i^f.

In order to solve k_i and p_i, let us return to Eq. (49):

$$\mathbf{f}^0=\frac{1}{\eta}(\mathbf{w}^f-\mathbf{w}^0). \tag{54}$$

Substituting it for $\mathbf{f}^0$ into Eq. (50) we obtain

$$\frac{\|\mathbf{w}^f\|}{\|\mathbf{w}^0\|}=a+\frac{b[\mathbf{w}^f-\mathbf{w}^0]\cdot\mathbf{w}^0}{\|\mathbf{w}^f-\mathbf{w}^0\|\|\mathbf{w}^0\|}, \tag{55}$$

where

$$a:=\cosh\left(\frac{x_f\|\mathbf{w}^f-\mathbf{w}^0\|}{\eta\|\mathbf{w}^0\|}\right), \tag{56}$$

$$b:=\sinh\left(\frac{x_f\|\mathbf{w}^f-\mathbf{w}^0\|}{\eta\|\mathbf{w}^0\|}\right). \tag{57}$$

Let

$$\cos\theta:=\frac{[\mathbf{w}^f-\mathbf{w}^0]\cdot\mathbf{w}^0}{\|\mathbf{w}^f-\mathbf{w}^0\|\|\mathbf{w}^0\|}, \tag{58}$$

$$S:=\frac{x_f\|\mathbf{w}^f-\mathbf{w}^0\|}{\|\mathbf{w}^0\|}, \tag{59}$$

and from Eqs. (55)-(57) it follows that

$$\frac{\|\mathbf{w}^f\|}{\|\mathbf{w}^0\|}=\cosh\left(\frac{S}{\eta}\right)+\cos\theta\sinh\left(\frac{S}{\eta}\right). \tag{60}$$

Upon defining

$$Z:=\exp\left(\frac{S}{\eta}\right), \tag{61}$$

from Eq. (60) we obtain a quadratic equation for Z:

$$(1+\cos\theta)Z^2-\frac{2\|\mathbf{w}^f\|}{\|\mathbf{w}^0\|}Z+1-\cos\theta=0. \tag{62}$$

On the other hand, by inserting Eq. (54) for $\mathbf{f}^0$ into Eq. (51) we obtain

$$\|\mathbf{w}^f-\mathbf{w}^0\|^2= (a-1)(\mathbf{w}^f-\mathbf{w}^0)\cdot\mathbf{w}^0+b\|\mathbf{w}^0\|\|\mathbf{w}^f-\mathbf{w}^0\|. \tag{63}$$

Dividing both sides by $\|\mathbf{w}^0\|\|\mathbf{w}^f-\mathbf{w}^0\|$ and using Eqs. (56)-(59) and (61) we obtain another quadratic equation for Z:

$$(1+\cos\theta)Z^2-2\left(\cos\theta+\frac{\|\mathbf{w}^f-\mathbf{w}^0\|}{\|\mathbf{w}^0\|}\right)Z +\cos\theta-1=0. \tag{64}$$

From Eqs. (62) and (64), the solution of Z is found to be

$$Z=\frac{(\cos\theta-1)\|\mathbf{w}^0\|}{\cos\theta\|\mathbf{w}^0\|+\|\mathbf{w}^f-\mathbf{w}^0\|-\|\mathbf{w}^f\|}, \tag{65}$$

and from Eq. (61) we obtain a closed-form solution of η:

$$\eta=\frac{x_f\|\mathbf{w}^f-\mathbf{w}^0\|}{\|\mathbf{w}^0\|\ln Z}. \tag{66}$$

Up to here we must point out that for a given x_f, η is fully determined by $\mathbf{w}^0$ and $\mathbf{w}^f$, which are supposed to be known. Therefore, the original nonlinear equation (53) becomes a linear equation for k_i and p_i.

By using Eqs. (11) and (12) we have

$$u_i^f=(1+x_f)u_i^0,\ \ v_i^f=(1+x_f)v_i^0, \tag{67}$$

and thus the vector $\mathbf{w}^f$ is proportional to $\mathbf{w}^0$ with a multiplier being $1+x_f$ larger than 1. Under this condition we have $\cos\theta=1$ and Z is given by

$$Z=1+x_f, \tag{68}$$

and hence from Eq. (66) we have

$$\eta=\frac{x_f^2}{\ln(1+x_f)}. \tag{69}$$

It is very surprising that η is a constant for a given x_f.

On the other hand, by using Eqs. (8) and (12) on Eq. (53) we have

$$\frac{x_f\dot{\phi}_i}{\eta p_i}=\frac{1}{\Delta t}\left(\frac{\dot{\phi}_{i+1}}{p_{i+1}}-\frac{\dot{\phi}_i}{p_i}\right)+\frac{k_i}{p_i}\phi_i-\frac{F_i}{p_i}+\frac{\dot{\phi}_i}{p_i}, \tag{70}$$

where $\phi_i = \phi(t_i)$ and $\dot{\phi}_i = \dot{\phi}(t_i)$ denote respectively the displacement and velocity at the i-th time point.

Now, multiplying both the sides of Eq. (70) by p_i and using Eq. (69) we can obtain

$$\frac{\dot{\phi}_i \ln(1+x_f)}{x_f} = \frac{1}{\Delta t}\left(\frac{p_i}{p_{i+1}}\dot{\phi}_{i+1} - \dot{\phi}_i\right) + k_i\phi_i - F_i + \dot{\phi}_i. \quad (71)$$

In order to obtain an uncoupled equations system for k_i and p_i, let us return to Eq. (1), from which by using Eqs. (11) and (7) we can derive a semi-discretization for u_i:

$$u_i'(x) = \frac{1}{(\Delta t)^2}[u_{i+1}(x) - 2u_i(x) + u_{i-1}(x)] + c_i\frac{u_{i+1}(x) - u_i(x)}{\Delta t} + k_i u_i(x) + h_i(x), \quad (72)$$

where $h_i(x) = \phi_i - (1+x)F_i$.

By using the one-step GPS for the above equation we can get

$$u_i^f = u_i^0 + \frac{\eta}{(\Delta t)^2}(u_{i+1}^0 - 2u_i^0 + u_{i-1}^0) + \eta c_i\frac{u_{i+1}^0 - u_i^0}{\Delta t} + \eta k_i u_i^0 + \eta h_i(0), \quad (73)$$

where η is defined as that in Eq. (69). After inserting Eqs. (11) and (7) for ϕ_i and Eq. (69) for η, it is not difficult to rewrite Eq. (73) as

$$k_i\phi_i = \frac{\phi_i \ln(1+x_f)}{x_f} - \frac{1}{(\Delta t)^2}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) - c_i\frac{\phi_{i+1} - \phi_i}{\Delta t} - \phi_i + F_i. \quad (74)$$

Inserting the above equation for $k_i\phi_i$ into Eq. (71) we can obtain

$$\frac{\dot{\phi}_i \ln(1+x_f)}{x_f} = \frac{1}{\Delta t}\left(\frac{p_i}{p_{i+1}}\dot{\phi}_{i+1} + \Delta t\dot{\phi}_i - \dot{\phi}_i\right) + \frac{\phi_i \ln(1+x_f)}{x_f} - \frac{1}{(\Delta t)^2}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) - c_i\frac{\phi_{i+1} - \phi_i}{\Delta t} - \phi_i. \quad (75)$$

With the aid of Eqs. (75) and (5) we can derive a nonlinear equation for c_i given by

$$A_i \exp\left[\frac{\Delta t}{2}(c_i + c_{i+1})\right] + B_i c_i = D_i, \quad (76)$$

where

$$A_i := \frac{\dot{\phi}_{i+1}}{\Delta t}, \quad (77)$$

$$B_i := -\frac{\phi_{i+1} - \phi_i}{\Delta t}, \quad (78)$$

$$D_i := \frac{\dot{\phi}_i \ln(1+x_f)}{x_f} - \dot{\phi}_i + \frac{\dot{\phi}_i}{\Delta t} - \frac{\phi_i \ln(1+x_f)}{x_f} + \frac{1}{(\Delta t)^2}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) + \phi_i. \quad (79)$$

Up to this point we have eventually obtained an uncoupled estimation equation for c_i.

Solving Eq. (76) for c_{i+1}, we have

$$c_{i+1} = \frac{2}{\Delta t}\log\left(\frac{D_i - B_i c_i}{A_i}\right) - c_i. \quad (80)$$

Starting from a given c_1 we can obtain c_i, $i = 2,\ldots,n$, by sequentially using the above equation, and inserting these c_i into Eq. (74) we can obtain k_i.

In the above we have used the vibration data of displacement and velocity as inputs in the estimations of c_i and k_i. If the data of velocity and acceleration are available we can replace the term $k_i\phi_i - F_i$ in Eq. (71) by $-\ddot{\phi}_i - \dot{\phi}_i$, and a similar derivation leads to Eq. (80) again. However, we have the same A_i, but B_i and D_i should be replaced by

$$B_i = -\dot{\phi}_i, \quad (81)$$

$$D_i = \frac{\dot{\phi}_i \ln(1+x_f)}{x_f} - \dot{\phi}_i + \frac{\dot{\phi}_i}{\Delta t} + \ddot{\phi}_i. \quad (82)$$

Therefore, we also have an estimation method based on the input data of velocity and acceleration.

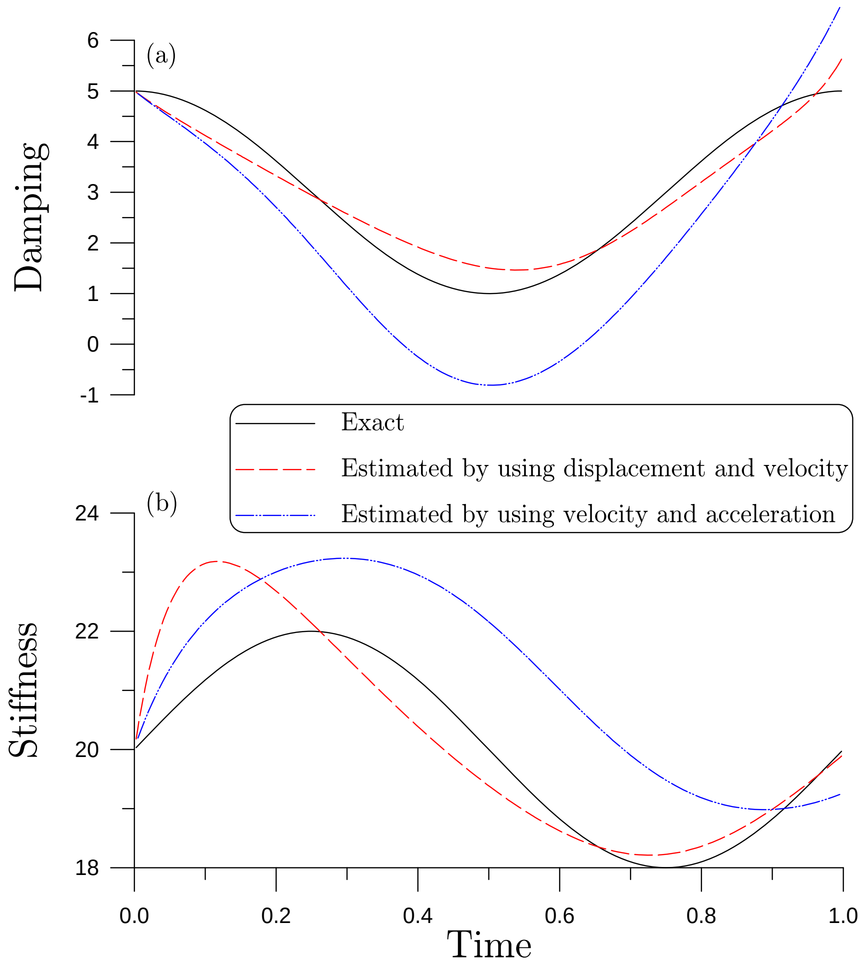

Figure 1: For Example 1 we comparing exact and estimated results by using displacement and velocity as input (dashed line) and velocity and acceleration as input (dashed-dotted line).

5 Numerical examples

5.1 Example 1

Let us consider

$$c(t) = 3 + 2\cos(2\pi t), \quad (83)$$

$$k(t) = 20 + 2\sin(2\pi t), \quad (84)$$

$$F(t) = F_0 + F_1 t. \quad (85)$$

In order to obtain the data of $\phi(t)$ and $\dot{\phi}(t)$ we are applied the fourth-order Runge-Kutta method (RK4) to Eqs. (1)-(3), where A_0 and B_0 are specified.

We first use the vibration data of displacement and velocity as inputs of the estimations of c_i and k_i. In this calculation we have fixed $\Delta t = 1/400$, $F_0 = 17$, $F_1 = 14$, $A_0 = 0.3$ $B_0 = 2$ and $x_f = 0.00001$. The profile of $c(t)$ is plotted in Fig. 1(a) by the dashed line, which is compared with the exact one plotted by the solid line. Then, the profile of $k(t)$ is plotted in Fig. 1(b) by the dashed line, which is compared with the exact one plotted by the solid line.

Next, we use the vibration data of velocity and acceleration as inputs of the estimations of c_i and k_i. In this calculation we have fixed $\Delta t = 1/200$, $F_0 = 14$, $F_1 = 17$, $A_0 = 0.5$ $B_0 = 1$ and $x_f = 0.0001$. The profile of $c(t)$ is plotted in Fig. 1(a) by the dashed-dotted line. Then, the profile of $k(t)$ is plotted in Fig. 1(b) by the dashed-dotted line. It can be seen that the estimation method based on the input data of displacement and velocity is feasible, which yields reasonable results. However,

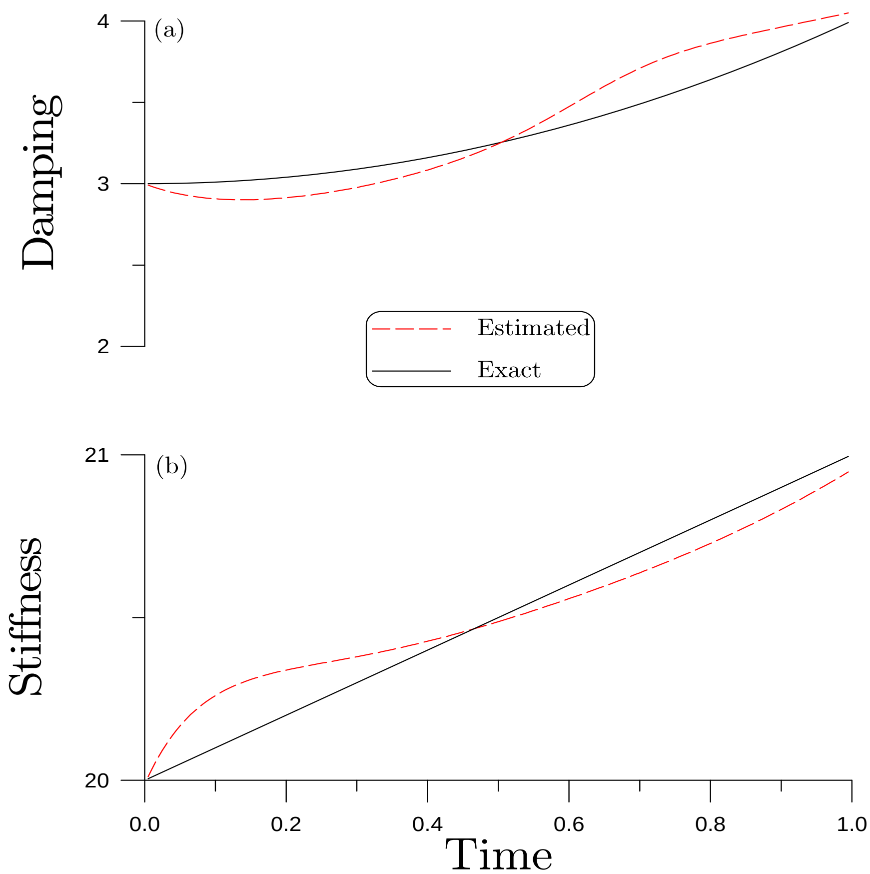

Figure 2: For Example 2 we comparing exact and estimated results by using displacement and velocity as input.

the use of velocity and acceleration data as inputs in the estimation may suffer the drawback of providing a non-physical result of negative c as shown in Fig. 1(a) by the dashed-dotted line.

5.2 Example 2

Then, we consider

$$c(t) = 3 + t^2, \tag{86}$$

$$k(t) = 20 + t. \tag{87}$$

For this example we use the following parameters $\Delta t = 1/200$, $F_0 = 40$, $F_1 = 28$, $A_0 = 1.6$, $B_0 = 3.5$ and $x_f = 0.05$ to estimate c_i and k_i, of which the maximum estimation error of c_i is about 0.235 as shown in Fig. 2(a), and the maximum estimation error of k_i is about 0.164 as shown in Fig. 2(b).

6 Conclusions

The inverse vibration problem of simultaneous estimation of both the damping and stiffness coefficients is rather difficult. No previous report in this issue is available. An initial success of the present paper is that we could offer an acceptable and simple method without any iteration to estimate both the damping and stiffness coefficients simultaneously. The key point hinges on three type transformations. By using the velocity and acceleration data as inputs is not suggested here from our numerical simulation. Instead of, when the displacement and velocity data are chosen as inputs, the estimation accuracy can be controlled within the first decimal point. However, there still leaves a large room to improve the present method to enhance the estimation accuracy.

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