

Controllability Conditions of Finite Oscillations of Hyper-Elastic Cylindrical Tubes Composed of a Class of Ogden Material Models

X.G. Yuan^{1,2}, R.J. Zhang³ and H.W. Zhang¹

Abstract: In this paper, the dynamic inflation problems are examined for infinitely long cylindrical tubes composed of a class of transversely isotropic incompressible Ogden material models. The inner surface of the tube is subjected to a class of periodic step radial pressures relating to time. The influences of various parameters, namely, the material parameters, the structure parameters and the applied pressures, on dynamic behaviors of the tube are discussed in detail. Significantly, for some given material parameters, it is proved that the motion of the tube would present a class of nonlinear periodic oscillations for any given pressures and the amplitude of oscillation is discontinuous for some special values of the given pressures. For other cases of material parameters, there exists a critical pressure such that the motion of the tube would also present nonlinear periodic oscillations if the given pressure does not exceed the critical value, however, the tube would inflate infinitely with the increasing time if the pressure exceeds the critical value. Finally, the case of periodic step pressures is considered and all controllability conditions for the finitely periodic oscillations of the cylindrical tube with time are presented by using the phase diagrams of the ordinary differential equation that governs the motion of the inner surface of the tube, especially for the neo-Hookean material, i.e., a special case of the incompressible Ogden material models. Meanwhile, some numerical examples are given.

Keyword: Cylindrical tube, dynamic inflation, transversely isotropic incompressible Ogden material, equilibrium point, nonlinear periodic oscillation

1 Introduction

There are several material groups such as elastomers, polymers, foams and biological tissues which can undergo large deformations without permanent set, and hence exhibit large nonlinear elastic behavior. Moreover, the mathematical theory of elasticity of materials subjected to large deformations is inherently nonlinear.

At present, a number of problems on the static deformations have been extensively investigated in the context of nonlinear theory of elasticity for both incompressible and compressible nonlinearly elastic bodies, which may be found in the monograph on Nonlinear Elasticity: Theory and Applications edited by Fu and Ogden (2001). In particular, inflation responses of spherical and cylindrical shells, membranes have been well studied (see the review article by Beatty (1987)). Non-homogeneous deformations such as bending, shearing, everting, straightening and stretching of nonlinearly elastic materials were examined by Carroll and Horgan (1990), Hill and Arrigo (1996), Haughton et al (2003), Ogden et al (2004), and so on. Cavitation in solid spheres and cylinders was firstly supplied by the theoretical work of Ball (1982). Thereafter, many significant works have been carried out. See the review articles, by Horgan and Polignone (1995) and by Yuan et al. (2005), for comprehensive reviews for both incompressible and compressible materials. While the investigations on static deformations are well understood, the analogous dynamic problems are relatively unexplored. The

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first investigation on the radial oscillations of some axisymmetric structures was undertaken by Knowles (1960, 1962). He respectively considered a cylindrical tube and a spherical shell composed of isotropic incompressible hyper-elastic materials, and reduced the equations of motion to second order ordinary differential equations. The finite oscillation of nonlinear elastic spherical shells was also examined by Guo and Sulecki (1963), Calderer (1983), and the dynamical mechanisms of the motion of the shells were analyzed. Moreover, dynamic inflation of hyper-elastic spherical membranes was studied by Veron et al (1999). Recently, the radial oscillations of thin cylindrical and spherical shells were investigated by Roussos and Mason (2005) by using the Lie point symmetry structures. Other aspects of nonlinear elastodynamics for hyper-elastic materials may be found in Chou-Wang and Horgan (1989), Dai et al (2002, 2006), Yuan et al (2006, 2007), and so on.

In applications, it is known that the loading forms acting on the structures are always dynamic loads such as periodic loads or step loads relating to time. The aim of this paper is to study dynamic inflation of infinitely long cylindrical tubes composed of a class of transversely isotropic incompressible Ogden material models, where the inner surface of the tube is subjected to a class of periodic step radial pressures relating to time. As a special case of Ogden material, the dynamic behaviors of the tube composed of the neo-Hookean material are also studied. Some interesting conclusions are obtained in this work. In Section 2, the mathematical model of the problem is proposed and a second order nonlinear ordinary differential equation that governs the dynamic inflation of the inner surface of the tube with time is presented. There are three parts in Section 3, in which the influences of various parameters, namely, the material parameters, the structure parameters and the applied pressures, on dynamic behaviors of the tube are discussed in detail. In Subsection 3.1, the dependence of the number of equilibrium points of the differential equations on all parameters is studied. In Subsection 3.2, the case of constant pressure that is

independent of time is considered, the existence conditions of the periodic solutions of the differential equation are proposed. In particular, for some material parameters, it is proved that the motion of the tube would present a class of nonlinear periodic oscillations for any given pressures and the amplitude of oscillation is discontinuous for some special values of the given pressures. For other cases of material parameters, there exists a critical pressure such that the motion of the tube would present nonlinear periodic oscillations as the given pressure does not exceed the critical value, however, the tube would inflate infinitely with the increasing time if the pressure exceeds the critical value. In Subsection 3.3, the case of periodic step pressures relating to time is studied and all controllability conditions for the nonlinearly periodic oscillations of the tube are presented by using the phase diagrams of the differential equation. Meanwhile, some numerical examples are also carried out.

2 Formulation and solutions

2.1 Mathematical model

Consider a homogeneous incompressible hyper-elastic material, whose mechanical response in plane strain is characterized by its strain energy density $W(\lambda_1, \lambda_2)$, and λ_1, λ_2 are the principal stretches of the deformation gradient tensor \mathbf{F} .

Here we are concerned with the dynamic inflation problems of an infinitely long cylindrical tube of such a material, where the inner surface of the tube is subjected to a class of periodic step radial pressures $\hat{p}(t)$ relating to time t , and the form of $\hat{p}(t)$ is taken as

$$\hat{p}(t) = \begin{cases} p_1, & t \in [2kT, 2kT + t_0), \\ p_2, & t \in [2kT + t_0, 2kT + t_0 + 2t_1), \\ p_1, & t \in [2kT + t_0 + 2t_1, 2(k+1)T) \end{cases} \quad (1)$$

In Eq.(1), $p_1, p_2 > 0$. Obviously, $\hat{p}(t)$ is a step function of period $T = 2t_0 + 2t_1$, ($k = 0, 1, 2, \dots$).

Under the assumption of radial symmetric deformation, the resulting deformation takes the point with Cartesian coordinates $(R \cos \Theta, R \sin \Theta)$ to the point $(r \cos \theta, r \sin \theta)$ at time t , moreover, we

have

$$r = r(R, t) > 0, \quad R_1 < R \leq R_2; \quad \theta = \Theta \quad (2)$$

where $r(R, t)$ is the radial deformation function to be determined, R_1 and R_2 denote the radii of the inner and the outer surfaces of the undeformed tube, respectively. The associated deformation gradient \mathbf{F} is written as

$$\mathbf{F} = \text{diag}(\lambda_r, \lambda_\theta) = \text{diag}(\partial r(R, t)/\partial R, r(R, t)/R) \quad (3)$$

where

$$\begin{aligned} \lambda_1 &= \lambda_r = \partial r(R, t)/\partial R, \\ \lambda_2 &= \lambda_\theta = r(R, t)/R \end{aligned} \quad (4)$$

are the radial and the circumference stretches, respectively.

It is known that the response of an elastic material can be described completely by its strain energy function. In the case of plane strain, the form is given by

$$W = \hat{W}(\mathbf{F}) = W(\lambda_r, \lambda_\theta) \quad (5)$$

In this work, assume that the cylindrical tube is composed of a class of transversely isotropic incompressible Ogden material models, and that the corresponding strain energy density function is given by

$$W(\lambda_r, \lambda_\theta) = \frac{\mu}{\alpha} [(\lambda_r^\alpha + \lambda_\theta^\alpha - 2) + \beta(\lambda_r^2 - 1)^2] \quad (6)$$

where μ , α and β are material constants, in which μ coincides with the usual infinitesimal shear modulus, β is a dimensionless material parameter measuring the degree of transverse isotropy of the material about the radial direction. Obviously, the case $\beta = 0$ corresponds to the isotropic Ogden material model first proposed by Ogden (1972), while the case $\alpha = 2$ coincides with the transversely isotropic neo-Hookean material model first proposed by Polignone and Horgan (1993).

For incompressible hyper-elastic materials, the incompressibility condition requires that $\det(\mathbf{F}) = \lambda_r \lambda_\theta = 1$, and from Eq.(4) we have

$$r = r(R, t) = (R^2 + r_1^2(t) - R_1^2)^{1/2}, \quad t \geq 0 \quad (7)$$

where $r_1(t)$ is the undetermined radial deformation function of the inner radius of the tube. It is easy to know that the motion of the tube can be completely described by $r_1(t)$.

Let $R = R_2$, this leads to

$$r_2 = r(R_2, t) = (R_2^2 + r_1^2(t) - R_1^2)^{1/2} \quad (8)$$

In the absence of body force, the differential equations that govern the radial motion of the cylindrical tube reduce to the single equation

$$\begin{aligned} & \frac{\partial \tau_{rr}(r, t)}{\partial R} \left(\frac{\partial r(R, t)}{\partial R} \right)^{-1} \\ & + \frac{1}{r(R, t)} [\tau_{rr}(r, t) - \tau_{\theta\theta}(r, t)] \\ & = \rho_0 \frac{\partial^2 r(R, t)}{\partial t^2}, \quad t \geq 0 \end{aligned} \quad (9)$$

where ρ_0 is the constant mass density of the material and

$$\begin{aligned} \tau_{rr}(r(R, t), t) &= \\ & \mu [\lambda_r^\alpha + 4(\beta/\alpha)\lambda_r^2(\lambda_r^2 - 1)] - p(r, t), \end{aligned} \quad (10a)$$

$$\tau_{\theta\theta}(r(R, t), t) = \mu \lambda_\theta^\alpha - p(r, t) \quad (10b)$$

are the principal components of the Cauchy stress tensor associated with the transversely isotropic Ogden material (6), and $p(r, t)$ is the hydrostatic pressure to be determined.

Since the inner surface of the cylindrical tube is subjected to the suddenly applied periodic step pressures $\hat{p}(t)$ given by Eq.(1) and the outer surface is traction-free, we have the following boundary conditions

$$\begin{aligned} \tau_{rr}(r(R_1, t), t) &= -\Delta p(t), \quad \tau_{rr}(r(R_2, t), t) = 0, \\ & t \geq 0 \end{aligned} \quad (11)$$

Assume that the sphere is in an undeformed state and at rest at time $t \leq 0$, so we have the initial conditions

$$r(R, 0) = R, \quad \dot{r}(R, 0) = 0 \quad (12)$$

Note. It is supposed that the dots over any letters denote derivatives with respect to t .

Thus, under the suddenly applied periodic step pressures $\hat{p}(t)$ given by Eq.(1), the mathematical model that governs the dynamic inflation of the tube is composed of Eqs.(6) ~ (10), the initial-boundary conditions (11) and (12).

2.2 Solutions

Differentiating with respect to t in Eq.(7), we deduce that $\partial^2 r / \partial t^2 = r^{-3} (r^2 - r_1^2) \dot{r}_1^2 + r^{-1} r_1 \ddot{r}_1$ and so

$$\frac{\partial^2 r}{\partial t^2} = \frac{\partial}{\partial r} \left((r_1 \ln r) \dot{r}_1 + (\ln r + \frac{1}{2} r^{-2} r_1^2) \dot{r}_1^2 \right) \quad (13)$$

Attentively, the term $\frac{\partial \tau_{rr}(r,t)}{\partial R} \left(\frac{\partial R(R,t)}{\partial R} \right)^{-1}$ in Eq.(9) can be written as $\partial \tau_{rr}(r,t) / \partial r$.

Substituting Eqs.(10a, b) into (9), then integrating it with respect to r from $r_1(t)$ to r_2 , and using the boundary conditions (11), we obtain

$$\begin{aligned} & \frac{1}{2} \rho_0 \left\{ r_1 \dot{r}_1 \ln \frac{r_2^2}{r_1^2} + \dot{r}_1^2 \left[\ln \frac{r_2^2}{r_1^2} + r_1^2 (r_2^{-2} - r_1^{-2}) \right] \right\} \\ & - \mu \int_{r_1}^{r_2} ((\lambda_r^\alpha - \lambda_\theta^\alpha) + 4(\beta/\alpha) \lambda_r^2 (\lambda_r^2 - 1)) \frac{dr}{r} \\ & - \Delta p(t) = 0 \quad (14) \end{aligned}$$

where r_2 is given by Eq.(8).

Remark. Similarly, $p(r,t)$ can be obtained by the above solving processes, i.e.,

$$\begin{aligned} p(r,t) &= \mu [a \lambda_r - b \lambda_r^{-1} + 2\alpha \lambda_r (\lambda_r - 1)] \\ &+ \mu \int_{r_1}^r ((\lambda_r^\alpha - \lambda_\theta^\alpha) + 4(\beta/\alpha) \lambda_r^2 (\lambda_r^2 - 1)) \frac{dr}{r} \\ &- \frac{1}{2} \rho_0 \left\{ r_1 \dot{r}_1 \ln \frac{r^2}{r_1^2} + \dot{r}_1^2 \left[\ln \frac{r^2}{r_1^2} + r_1^2 (r^{-2} - r_1^{-2}) \right] \right\} \quad (15) \end{aligned}$$

From Eqs.(7) and (12), the initial conditions become

$$r_1(0) = R_1, \dot{r}_1(0) = 0 \quad (16)$$

In sum, if there exists a solution $r_1(t)$ of Eq.(14) satisfying the initial conditions (16), then the motion of the tube can be completely described, that is to say, Eqs.(7) and (15) are solutions of the dynamical inflation problems of an infinitely long

cylindrical tube composed of the transversely isotropic incompressible Ogden material model (6).

Next we examine the dynamic properties of Eq.(14).

3 Nonlinear dynamic analyses of Eq.(14)

First of all, rewrite Eq.(7) as $R = (r^2 - r_1^2 + R_1^2)^{1/2}$ and introduce the following notation

$$\kappa = \kappa(r, r_1) = \left(1 - \frac{r_1^2 - R_1^2}{r^2} \right)^{-1/2} \quad (17)$$

this leads to $\lambda_r = \kappa^{-1}$ and $\lambda_\theta = \kappa$.

In what follows, it is convenient to introduce the dimensionless quantities

$$x(t) = r_1(t) / R_1, \quad \delta = R_2^2 / R_1^2 - 1 \quad (18)$$

In this case, we have some useful notations

$$\frac{r_2^2}{R_2^2} = \frac{\delta + x^2}{1 + \delta}, \quad \frac{r_1^2}{R_1^2} = 1 + \frac{\delta}{x^2}, \quad \frac{dr}{r} = \frac{1}{1 - \kappa^2} \frac{d\kappa}{\kappa} \quad (19)$$

Consequently, Eq.(14) can be rewritten as

$$\begin{aligned} & \frac{1}{2} \rho_0 R_1^2 x \ln \left(1 + \frac{\delta}{x^2} \right) \ddot{x} \\ & + \frac{1}{2} \rho R_1^2 \left[\ln \left(1 + \frac{\delta}{x^2} \right) - \frac{\delta}{x^2 + \delta} \right] \dot{x}^2 \\ & - \mu \int_x^{\left(\frac{x^2 + \delta}{1 + \delta} \right)^{1/2}} \left(\frac{\kappa^{-\alpha} - \kappa^\alpha}{\kappa(1 - \kappa^2)} + 4 \frac{\beta}{\alpha} \kappa^{-5} \right) d\kappa \\ & - \Delta p(t) = 0 \quad (20) \end{aligned}$$

and the initial conditions become

$$x(0) = 1, \dot{x}(0) = 0 \quad (21)$$

To better study the qualitative properties of the solutions of Eq.(20), we firstly consider the case $\Delta p(t) \equiv P$ in Eq.(1) which is independent of time t .

Further, let $y = \dot{x}$, then Eq.(20) is equivalent to the first order differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ A(x, y) \end{pmatrix} \quad (22)$$

where

$$A(x, y) = \frac{-C(x, \delta)y^2 - F(x, \delta, \alpha, \beta) + P}{B(x, \delta)}, \quad (23a)$$

$$B(x, \delta) = \frac{1}{2}\rho R_1^2 x \ln\left(1 + \frac{\delta}{x^2}\right), \quad (23b)$$

$$C(x, \delta) = \frac{1}{2}\rho R_1^2 \left[\ln\left(1 + \frac{\delta}{x^2}\right) - \frac{\delta}{x^2 + \delta} \right], \quad (23c)$$

and in (23a), we have

$$F(x, \delta, \alpha, \beta) = -\mu \int_x^{\left(\frac{x^2+\delta}{1+\delta}\right)^{1/2}} \left(\frac{\kappa^{-\alpha} - \kappa^\alpha}{\kappa(1-\kappa^2)} + 4\frac{\beta}{\alpha}\kappa^{-5} \right) d\kappa \quad (23d)$$

Obviously, the equilibrium point of Eq.(22) is $(x, y) = (\bar{x}, 0)$, where \bar{x} is a positive real solution of the following equation

$$P = F(x, \delta, \alpha, \beta) \quad (24)$$

However, whether \bar{x} exists or not depends exactly on the parameters δ, α, β and P .

In order to determine the stability of each equilibrium state, it requires studying the behaviors of system (22). The Jacobian matrix $[J]$ of system (22) at the equilibrium point $(\bar{x}, 0)$ is given by

$$[J]_{x=\bar{x}, y=0} = \begin{bmatrix} 0 & 1 \\ \frac{\partial A(\bar{x}, 0)}{\partial x} & 0 \end{bmatrix} \quad (25)$$

Moreover, it is easy to obtain the eigenvalues of the linearization equation of system (22), as follows,

$$\lambda_{1,2} = \pm \left[\frac{-F_x(\bar{x}, \delta, \alpha, \beta)}{B(\bar{x}, \delta)} \right]^{\frac{1}{2}} \quad (26)$$

3.1 Number of Equilibrium points

In this subsection, we will mainly discuss the number of equilibrium points of Eq.(22) for different values of δ, α, β and P .

(i) Interestingly, the case $\alpha = 2$ corresponds to the transversely isotropic neo-Hookean material model first proposed by Polignone and Horgan (1993), and some typical phenomena come into

being, as follows:

Firstly, $F(x, \delta, 2, \beta)$ given by Eq.(23d) has the following explicit expression

$$F(x, \delta, 2, \beta) = \frac{1}{2}\mu \left[\ln(1 + \delta) - \ln\left(1 + \frac{\delta}{x^2}\right) + \frac{\delta(x^2 - 1)}{x^4(x^2 + \delta)^2} (x^2(x^2 + \delta) + 2\beta x^2 + \beta\delta(x^2 + 1)) \right] \quad (27)$$

For any values of δ and β , we have $F(1, \delta, 2, \beta) = 0$, $\lim_{x \rightarrow 0^+} F(x, \delta, 2, \beta) = -\infty$ and $\lim_{x \rightarrow +\infty} F(x, \delta, 2, \beta) = (1/2)\mu \ln(1 + \delta)$, in other words, Eq.(27) has a horizontal asymptote, written as $P_a = (1/2)\mu \ln(1 + \delta)$.

Secondly, by using the equation $F_x(x, \delta, 2, \beta) = 0$, it is not difficult to show that the following conclusions are valid for any δ and for any $x > 0$.

Conclusion 1 (a) If $0 \leq \beta \leq 1/2$, we then have $F_x(x, \delta, 2, \beta) > 0$, that is to say, $F(x, \delta, 2, \beta)$ increases monotonically with $x \in (0, +\infty)$, $F(x, \delta, 2, \beta) < 0$ as $x \in (0, 1)$ and $F(x, \delta, 2, \beta) > 0$ as $x \in (1, +\infty)$, moreover, $P_a = (1/2)\mu \ln(1 + \delta)$ is the maximum of Eq.(27).

(b) If $\beta > 1/2$, we have $F_x(x, \delta, 2, \beta) > 0$ as $x \in (0, 1)$, $F_x(1, \delta, 2, \beta) = 2\mu\delta(1 + \beta)/(1 + \delta) > 0$ as $x = 1$ and $F_x(x, \delta, 2, \beta) < 0$ for sufficient large values of x , this means that there exists a unique value $x_m \in (1, +\infty)$ such that $F_x(x_m, \delta, 2, \beta) = 0$, namely, $P_m = F(x_m, \delta, 2, \beta)$ is the maximum of Eq.(27). Moreover, $F_x(x, \delta, 2, \beta) > 0$ as $x \in (0, x_m)$ and $F_x(x, \delta, 2, \beta) < 0$ as $x \in (x_m, +\infty)$.

For other cases $\alpha \neq 2$, however, the conclusions are quite different.

(ii) For $0 < \alpha < 2$, we have $\lim_{x \rightarrow 0^+} F(x, \delta, \alpha, \beta) = -\infty$, $F(1, \delta, \alpha, \beta) = 0$ and $\lim_{x \rightarrow +\infty} F(x, \delta, \alpha, \beta) = 0$.

Moreover, it can be numerically shown that Eq.(23d) has a maximum for the given values of δ, α and β .

(iii) For $\alpha > 2$, we have $\lim_{x \rightarrow 0^+} F(x, \delta, \alpha, \beta) = -\infty$, $F(1, \delta, \alpha, \beta) = 0$ and $\lim_{x \rightarrow +\infty} F(x, \delta, \alpha, \beta) = +\infty$.

Table 1: Maximums of Eq.(23d) for $\delta = 1$ and for different values of α, β .

| | $\alpha = 1$ | $\alpha = 1.5$ | $\alpha = 2$ |
|---------------|--------------|----------------|--------------|
| $\beta = 0$ | 0.1325 | 0.2138 | 0.3466 |
| $\beta = 0.5$ | 0.2084 | 0.2489 | 0.3466 |
| $\beta = 1.5$ | 0.4026 | 0.3623 | 0.3905 |
| $\beta = 2.5$ | 0.6036 | 0.4910 | 0.4746 |

It can be numerically shown that the following conclusion is valid.

Conclusion 2 For the given values of δ and α , there exists a critical value of β , written as β_c , such that $P = F(x, \delta, \alpha, \beta)$ increases monotonically if $\beta < \beta_c$ and has a local minimum and a local maximum if $\beta > \beta_c$.

The curves, which also describe the effects of material anisotropy on the number of equilibrium points, P vs x are respectively shown in Fig.1 for different values of β and for $\alpha = 2, \delta = 1$; in Fig.2 for different values of $0 < \alpha < 2, \beta$ and for $\delta = 1$; in Fig.3 for different values of β and for $\delta = 1, \alpha = 2.5$.

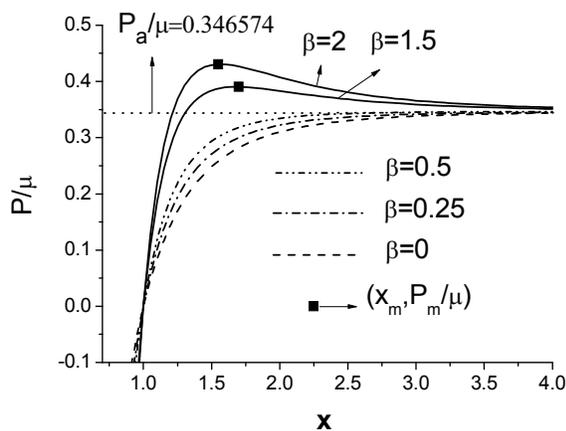


Figure 1: Curves of P/μ vs x for different values of β and for $\alpha = 2, \delta = 1$.

3.2 Dynamic inflation ($\phi \tilde{n}$): Constant pressure case

In this subsection, we examine the constant pressure case which is independent of time t , i.e., $p_1 = p_2 = P$.

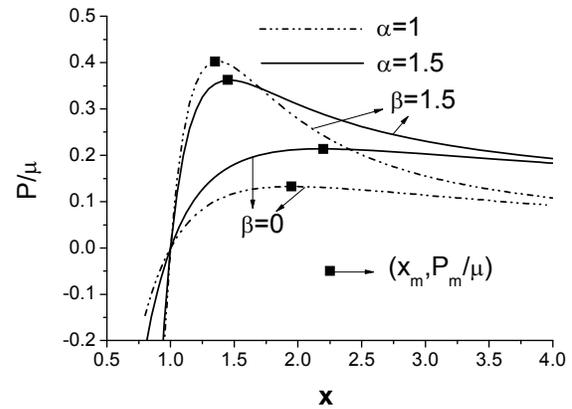


Figure 2: Curves of P/μ vs x for different values of $0 < \alpha < 2, \beta$ and for $\delta = 1$.

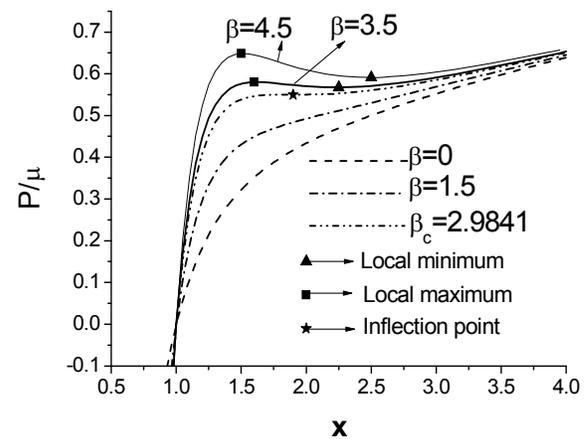


Figure 3: Curves of P/μ vs x for different values of β and for $\delta = 1, \alpha = 2.5$.

Combining the conclusions in Subsection 3.1, it is not difficult to show that the following conclusions are valid.

(i) $\alpha = 2$, i.e., the transversely isotropic neo-Hookean material model.

Conclusion 3 For any $x \in (1, +\infty)$ and for any values of $\delta > 0$, we have

(a) If $0 \leq \beta \leq 1/2$, there exists a critical pressure $P_a = (1/2)\mu \ln(1 + \delta)$ such that Eq.(22) has a unique equilibrium point as $0 \leq P < P_a$, written as $(x_1, 0)$, moreover, $(x_1, 0)$ is a center; Eq.(22) has no equilibrium point as $P \geq P_a$.

(b) If $\beta > 1/2$, there also exists a critical pressure P_m such that Eq.(22) has a unique equilibrium point as $P \in (0, P_a]$, written as $(x_2, 0)$, more-

over, $(x_2, 0)$ is a center; Eq.(22) has two equilibrium points as $P \in (P_a, P_m)$, written as, $(x_3, 0)$ and $(x_4, 0)$, where $x_3 < x_4$, moreover, $(x_3, 0)$ is a center and $(x_4, 0)$ is a saddle point; Eq.(22) has no equilibrium point any more as $P > P_m$. (cf. **Figure 1**).

Remark. Conclusions of the case $0 < \alpha < 2$ are similar to those of the case $\alpha = 2$.

Examples of phase diagrams of Eq.(22) for the given values of δ , α , β and P are respectively shown in Figs.4 and 5, in which $v = (\rho_0 R_1^2 / \mu) \dot{x}^2$.

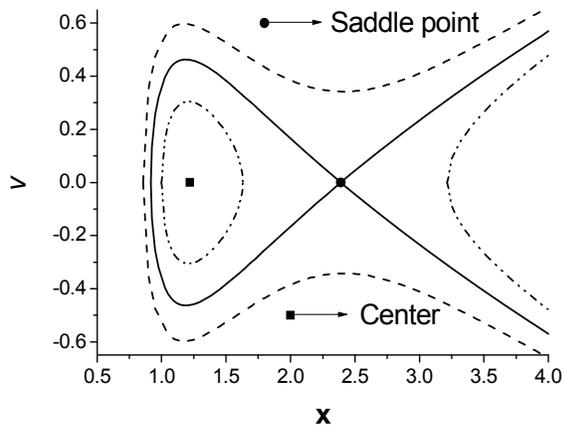


Figure 4: Phase diagrams of Eq.(22) satisfying different initial conditions for $P/\mu = 0.4$, $\delta = 1$, $\alpha = 2$ and $\beta = 2.5$.

It is worth noting that, as shown in Fig.5, there exists a critical value of P , written as $P_{cr} < P_m$, such that the motion trajectories of the solutions of Eq.(22) satisfying the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$ are close, convex and smooth curves if $0 < P \leq P_{cr}$, in other words, the dynamic inflation of the inner surface of the tube would present a class of nonlinearly periodic oscillations. However, the inflation of the tube will not be periodic with the increasing time t if $P > P_{cr}$. Interestingly, the inflation of the tube is at the critical state of periodic oscillation if $P = P_{cr}$.

(ii) $\alpha > 2$, i.e., another case of the transversely isotropic incompressible Ogden material model.

Conclusion 4 For the given values of $\delta > 0$, $\alpha > 2$ and for any $x \in (1, +\infty)$, there exists a critical

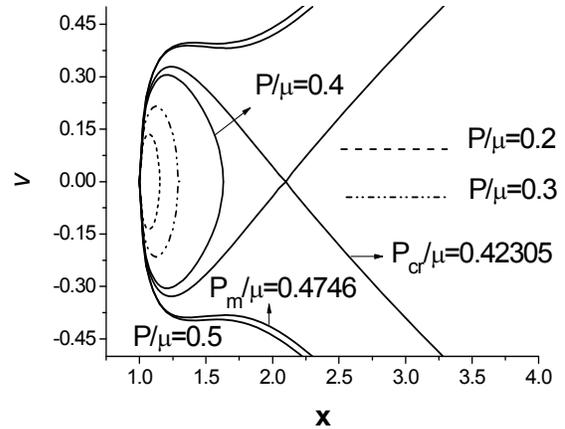


Figure 5: Dynamic inflation of cylindrical tubes composed of the transversely isotropic neo-Hookean material models: v vs x for $\delta = 1$, $\alpha = 2$, $\beta = 2.5$ and for different values of P/μ .

value of β , written as β_c such that

(a) If $0 \leq \beta \leq \beta_c$, Eq.(22) has a unique equilibrium point for any P , written as $(x_5, 0)$, moreover, $(x_5, 0)$ is a center.

(b) If $\beta > \beta_c$, Eq.(22) has a local minimum and a local maximum, respectively written as P' and P'' ($P' < P''$). Further, Eq.(22) has a unique equilibrium point as $P < P'$ and $P > P''$, written as $(x_6, 0)$, moreover, $(x_6, 0)$ is a center; Eq.(22) has three equilibrium points as $P' < P < P''$, respectively written as $(x_7, 0)$, $(x_8, 0)$ and $(x_9, 0)$, where $x_7 < x_8 < x_9$, moreover, $(x_7, 0)$ and $(x_9, 0)$ are centers, and $(x_8, 0)$ is a saddle point.

For the given parameters δ , α , β and P , examples of phase diagrams of Eq.(22) are respectively shown in Figs.6 and 7, in which $v = (\rho_0 R_1^2 / \mu) \dot{x}^2$.

It is also worth noting that, as shown in Fig.7, the dynamic inflation of the inner surface of the tube presents a class of nonlinearly periodic oscillations, moreover, the amplitude of oscillation increases gradually as P increases from 0 to P_b . However, the amplitude is discontinuous as P passes through P_b . Another interesting phenomenon occurs as $P = P_b$, namely, the phase diagram is a homoclinic orbit.

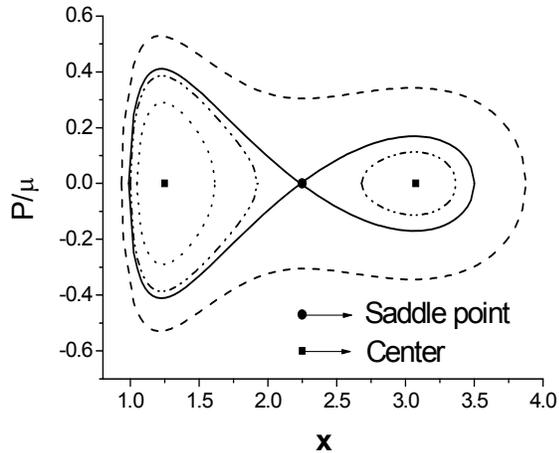


Figure 6: Phase diagrams of Eq.(22) satisfying different initial conditions for $P/\mu = 0.61$, $\delta = 1$, $\alpha = 2.5$ and $\beta = 5$.

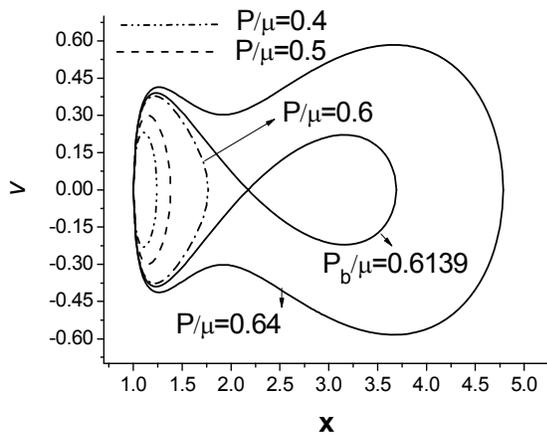


Figure 7: Dynamic inflation of cylindrical tubes composed of the transversely isotropic Ogden material models: v vs x for $\delta = 1$, $\alpha = 2.5$, $\beta = 5$ and for different values of P/μ .

3.3 Dynamic inflation (II): Periodic step pressures case

In this subsection, by using the phase diagrams of Eq.(22) given in Subsection 3.2, we examine the case of periodic step pressures given by Eq.(1).

In particular, we only study the case $\alpha = 2$ and the existence conditions of the periodic solutions of Eq.(22) satisfying the initial conditions (21). Discussions of other cases are similar.

(i) $0 < p_1, p_2 < P_{cr}$.

In this case, let \hat{T} and \hat{T}_1 be the periods of the solutions of Eq.(22) starting at $x(0) = 1, \dot{x}(0) = 0$ for the given values of p_1 and p_2 , respectively. m, n are positive integers,

(a) If $t_0 = m\hat{T}$ and $2t_1 = n\hat{T}_1$, we can conclude that Eq.(22) has periodic solutions of period T . This means that the inner surface of the tube oscillates periodically m times starting at $x(0) = 1, \dot{x}(0) = 0$ and $\Delta p(t) = p_1$ as $t \in [0, t_0)$, moreover, $x(t_0) = 1, \dot{x}(t_0) = 0$. The pressure is p_2 as $t \in [t_0, t_0 + 2t_1)$ and the inner surface of the tube oscillates periodically n times, moreover, $x(t_0 + 2t_1) = 1, \dot{x}(t_0 + 2t_1) = 0$. In succession, as $t \in [t_0 + 2t_1, T]$, the pressure is p_1 again, the inner surface of the tube also oscillates periodically m times. Further, in the following period T , the process will be the same as the previous process, see the close curves shown in Fig.5. Otherwise, if $2t_1 \neq n\hat{T}_1$, Eq.(22) has no periodic solutions of period T .

(b) If $m\hat{T} < t_0 < m\hat{T} + \hat{T}/2$, in other words, the inner surface motions from $x(0) = 1, \dot{x}(0) = 0$ and reaches to $x(t_0) = x_0, \dot{x}(t_0) = \dot{x}_0$ at time t_0 . The pressure is p_2 as $t \in [t_0, t_0 + 2t_1)$, interestingly, if $x(t_0 + 2t_1) = x_0$ and $\dot{x}(t_0 + 2t_1) = -\dot{x}_0$, then we can conclude that Eq.(22) has periodic solutions of period T . Since the pressure is p_1 again as $t \in [t_0 + 2t_1, T]$, the inner surface will reach to $x(2t_0 + 2t_1) = 1, \dot{x}(2t_0 + 2t_1) = 0$. In the following period T , the process will be the same as the previous process, as shown in Fig.8. In other cases, the solutions of Eq.(22) are no longer periodic.

(c) If $t_0 = m\hat{T} + \hat{T}/2$, i.e., the inner surface motions from $x(0) = 1, \dot{x}(0) = 0$ and reaches to $x(t_0) = \hat{x}, \dot{x}(t_0) = 0$ at time t_0 , while if $2t_1 = n\hat{T}_2$, where \hat{T}_2 is the oscillation period of the inner surface starting at $x(t_0) = \hat{x}, \dot{x}(t_0) = 0$ for the given values of p_2 , then we can also conclude that Eq.(22) has periodic solutions of period T , as shown in Fig.9. Otherwise, the solutions are no longer periodic.

(d) If $t_0 > m\hat{T} + \hat{T}/2$, it is numerical shown that Eq.(22) has no periodic solutions of period T for any values of t_1 .

(ii) For the case of $0 < p_1 < P_{cr}, p_2 > P_{cr}$ and for the case of $p_1, p_2 > P_{cr}$, Eq.(22) has only increasing solution with the infinitely increasing time t

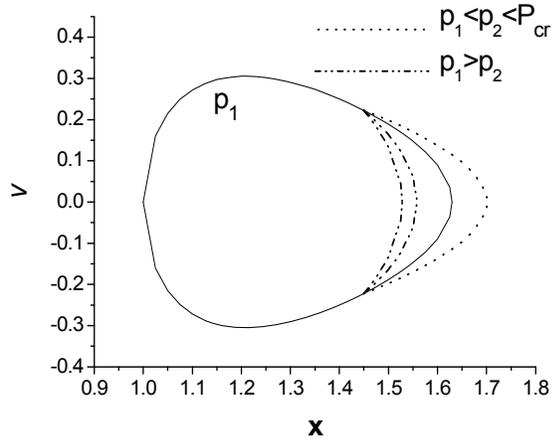


Figure 8: Phase diagrams of Eq.(22) for the cases of periodic oscillations of the tube as $m\hat{T} < t_0 < m\hat{T} + \hat{T}/2$, where $\delta = 1$, $\beta = 2.5$ and $p_1 = 0.4$.

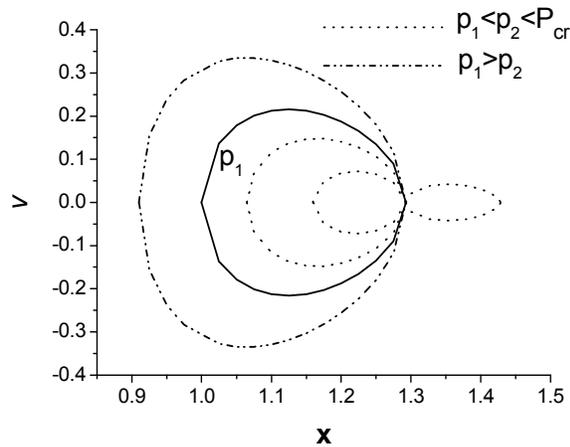


Figure 9: Phase diagrams of Eq.(22) for the cases of periodic oscillations as $t_0 = m\hat{T} + \hat{T}/2$, where $\delta = 1$, $\beta = 2.5$ and $p_1 = 0.3$.

for any values of t_0 and t_1 .

(iii) $p_1 > P_{cr}$ and $0 \leq p_2 \leq P_{cr}$.

In this case, an interesting phenomenon may appear, as shown in Fig.10, although the initial pressure p_1 exceeds the critical value P_{cr} , there exists another critical pressure, written as P_d , such that the periodic oscillations of the inner surface of the tube are also controllable if $p_2 < P_d$, otherwise, the inner surface will inflate infinitely if $p_2 > P_d$.

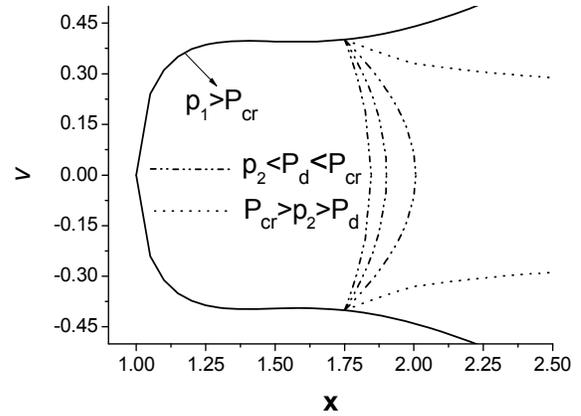


Figure 10: Phase diagrams of Eq.(22) for the cases of periodic oscillations as $t_0 = m\hat{T} + \hat{T}/2$, where $\delta = 1$, $\beta = 2.5$ and $p_1 = 0.48$.

4 Conclusions

In this work, the dynamic inflation problems of infinitely long cylindrical tubes composed of a class of transversely isotropic incompressible Ogden material models are investigated by studying the qualitative properties of the differential equation that governs the motion of the inner surface of the tube. The effects of all parameters on the finitely periodic oscillations of the tube are discussed in detail and all the controllability conditions for finitely periodic oscillations of the tube under both constant pressure and periodic step pressures are presented, particularly for the special case $\alpha = 2$, i.e., the neo-Hookean material. Some new phenomena are observed, such as:

- (i) For the case of $0 < \alpha \leq 2$, it is proved that there exists a critical pressure such that the dynamic inflation of the tube would present a class of nonlinear periodic oscillations as the given pressure does not exceed the critical value, however, the tube would inflate infinitely with the increasing time if the pressure exceeds the critical value. (cf. Conclusion 3)
- (ii) For the case of $\alpha > 2$, it is proved that the motion of the tube would present nonlinear periodic oscillations for any given pressures and the amplitude of oscillation is discontinuous in some cases. (cf. Conclusion 4)
- (iii) Under the periodic step pressures, the finitely

periodic oscillations of the tube are controllable if some necessary conditions are imposed, even though the initial pressure p_1 exceeds the critical value P_{cr} . (cf. Subsection 3.3)

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