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A Review on the Three-Dimensional Analytical Approaches of Multilayered and Functionally Graded Piezoelectric Plates and Shells

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Abstract: The article is to present an overview of various three-dimensional (3D) analytical approaches for the analysis of multilayered and functionally graded (FG) piezoelectric plates and shells. The reported 3D approaches in the literature are classified as four different approaches, namely, Pagano's classical approach, the state space approach, the series expansion approach and the asymptotic approach. Both the mixed formulation and displacement-based formulation for the 3D analysis of multilayered piezoelectric plates are derived. The analytical process, based on the 3D formulations, for the aforementioned approaches is briefly interpreted. The present formulations of multilayered piezoelectric plates can also be used for the analysis of FG piezoelectric plates, of which material properties are heterogeneous through the thickness coordinate, by artificially dividing the plate as *NL*-layered plates with constant coefficients in an average sense for each layer. The present formulations can also be extended to the ones of piezoelectric shells using the associated shell coordinates. A comprehensive comparison among the 3D results available in the literature using various approaches is made. For illustration, the through-thickness distributions of various field variables for the simply-supported, multilayered and FG piezoelectric plates are presented using the asymptotic approach and doubly checked with a newly-proposed meshless method. The literature dealing with the 3D analysis of multilayered and FG piezoelectric plates is surveyed and included. This review article contains 191 references.

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1 Introduction

Three-dimensional (3D) analysis of plates and shells made of a variety of advanced materials (or so-called smart materials) has attracted the researchers' attention for a long time. It is mainly due to the fact that 3D solutions of the benchmark problems may serve as a standard for assessing various approximate two-dimensional (2D) theories of plates and shells. These 3D solutions also provide as a reference for making the appropriate kinetics or kinematics assumptions prior to develop the advanced theories and numerical modeling of plates and shells. Hence, 3D analysis is inherently of much importance not only for academic interest but also for industrial applications.

The review literature on 2D theoretical methodologies and numerical modeling of multilayered composite elastic as well as piezoelectric plates/shells is surveyed and is tabulated in Table 1. It is shown that the number of review literature on laminated composite elastic plates/shells is much larger than that on multilayered piezoelectric plates/shells. After a close examination in the literature, we found that the review literature on 3D analysis of multilayered and functionally graded (FG) elastic/piezoelectric structures is scarce. Hence, the paper aims to making a comprehensive survey on the present topics to replenish this insufficiency.

In general there are four different approaches, namely Pagano's classical approach, the state space approach, the series expansion approach and the asymptotic approach, drawn from the literature to determine the 3D solutions of multilayered and FG piezoelectric (or elastic) plates and

Table 1: Partial list of references on reviews for the analysis of multilayered elastic and piezoelectric structures

Structures	Assessments	Literature
Multilayered composite plates	Mechanics	Noor, 1992
	Computational models	Liu and Soldatos, 2003; Noor and Burden, 1990a;
		Subha, Shashidharan, Savithri and Syam, 2007
	Mixed theories	Carrera, 2000a, 2003
	Displacement-based theories	Liu and Li, 1996;
		Matsunaga, 2002;
		Noor and Burton, 1989;
		Kant and Swaminathan, 2000
	Bending	Carrera and Ciuffreda, 2005
	Thermal	Carrera, 2000b
	Vibration	Carrera, 2004
Multilayered composite shells	Mechanics	Kapania, 1989;
		Noor, 1990
	3D dynamic analysis	Soldatos, 1994
	Computational models	Noor, and Burton, 1990b;
		Noor, Burton and Peters, 1991
	Static	Chandrashekhara and Pavan, 1995
	Nonlinear	Reddy and Chandrashekhara, 1987
	Thermal	Noor and Burton, 1992
	Buckling	Jaunky and Knight, 1999
Multilayered piezoelectric plates	Laminated plate theories	Gopinathan, Varadan and Varadan, 2000
	Computational models	Tang, Noor and Xu, 1996
	Thermal	Tauchert, Ashida and Noda, 1999
	Sensing, actuation and control	Rao and Sunar, 1994; Chee et al., 1998
Multilayered piezoelectric shells	Laminated shell theories	Kapuria, sengupta and Dumir, 1998;
		Carrera and Brischetto, 2007
	Computational models	Saravanos and Heyliger, 1999
	Nonlinear	Yu, 1995;
		Carrera and Parisch, 1997
Others	Nontraditional theories	Ambartsumian, 2002

shells. The pioneers who initiated the applications of various approaches to the structural behavior of plates and shells were mentioned in each coming paragraph. The existing articles on exact 3D analyses of multilayered and FG elastic/ piezoelectric structures were selectively mentioned in this section and comprehensively surveyed and tabulated in the coming sections, respectively.

The first approach is based on Pagano's study (Pagano, 1969) where a 3D problem of laminated composite plates has been reduced as a plane strain problem by considering an infinite plate with the identical configuration and loading conditions along one of in-plane coordinates which is so-called the cylindrical bending problem. The compatibility equation has been expressed as a fourth-order ordinary differential equation with constant coefficients for each individual layer and in terms of the thickness coordinate only by introducing an Airy stress function satisfying the stress equilibrium equations in the formulation. The 3D solution of laminated plates under cylindrical bending has been obtained by imposing the traction conditions on the bottom and top surfaces and the continuity conditions at the interfaces between adjacent layers. This approach has also been extended to the 3D structural behaviors of rectangular laminated plates (Pagano, 1970; Srinivas and Rao, 1970). However, Pagano's classical approach fails to exactly analyze the shell problems of which governing equations are a system of differential equations with variable coefficients. A successive approximation (SA) method has been proposed by Soldatos and Hadjigeorgiou (1990) to overcome the aforementioned straits. In the SA method, the shell is artificially divided into certain layers of smaller thickness so that one may reasonably approximate the system of thickness-varying differential equations to a system of thickness-invariant differential equations. That makes Pagano's classical approach feasible for the 3D analysis of shells. The SA method has been demonstrated to be practically an exact method in the sense that it can approximate the exact solution of relevant problems to any desired accuracy. Using the Pagano's classical approach in combination with the SA method and matching the interface displacement and stress continuity conditions, Bhimaraddi (1991, 1993) and Bhimaraddi and Chandrashekhara (1992) obtained the approximate 3D solutions for the static and dynamic responses of simply-supported, doubly curved shallow shells.

The second approach is based on the state space method which is also called the method of transfer (or propagator) matrix (Bufler, 1971) or the method of initial functions (Vlasov, 1957). In the approach, the sets of state variable equations with constant coefficients for plates and with variable coefficients for shells have been obtained by means of direct elimination. Using a modal matrix composed of the eigenvectors of a full coefficient matrix of the coupled system of state variable equations and performing a similarity transformation, one may transform the coupled system equations to certain uncoupled system equations. Afterwards, the unknowns can be independently determined. Again, for shell problems, the SA method has been used to reduce the system of thickness-varying equations to a series of thickness-invariant equations. By imposing the boundary conditions on the lateral surfaces of the plates or shells, one may determine the state variables through the thickness coordinate using the method of transfer matrix. A detailed description of the method of state space and its applications to the bending, vibration, buckling analyses of laminated composite plates and shells has been made (Soldatos and Hadjigeorgiou, 1990; Fan and Ye, 1990a, b; Fan and Zhang, 1992; Spencer et al., 1993; Ye and Soldatos, 1994a, b, 1995; Pan, 2003; Ye, 2003; Tarn and Wang, 2001, 2003, 2004).

The third approach is based on the series expansion method, i.e., the method of Frobenius, where the primary variables have been constructed in the forms of a power series in the thickness coordinate (Ren, 1987, 1989; Varadan and Bhasksr, 1991; Huang and Tauchert, 1991, 1992; Huang, 1995). Substituting these specific forms of primary variables into the system differential equations and equating the coefficients of various power terms to be zero, one may obtain an indicial equation and a recurrence relation. Again, imposing both the continuity conditions at interfaces between adjacent layers and traction conditions on the lateral surfaces, one may iteratively solve for the considered 3D problems. Kapuria et al. (1997a, b) and Xu and Noor (1996) proposed a modified Frobenius method for the 3D analysis of multilayered piezoelectric shells where the primary field variables have been constructed in the form of a product of an exponential function and a power series in the thickness coordinate. The method of modified Frobenius has been demonstrated to yield better convergence properties.

The fourth approach is based on the method of perturbation (Gol'denvaizer, 1961, 1963; Cicala, 1965; Rogers et al., 1992, 1995; Wang and Tarn, 1994; Wu, et al., 1996a, 1997, 2002). In the approach, a geometric perturbation parameter has been introduced in the asymptotic formulation. The 3D problems can be expanded as a series of 2D problems governed by the equations of classical plate or shell theories (CPT or CST). The 3D solutions can then be determined in a hierarchic and systematic manner. Studies of the dynamic analysis of nonhomogeneous plates by means of the method of perturbation were presented long ago (Widera, 1970; Johnson and Widera, 1971). In their formulation a straightforward expansion using a single time variable has been used and certain assumptions regarding the material compliances have been made. No numerical results have been given to demonstrate the applicability of their theories due to a rather complicated formulation. Asymptotic analysis for dynamic response of nonhomogeneous plates and shells is not just a matter of applying the standard perturbation method. This will lead to not only equations too cumbersome to be useful but also nonuniform expansions containing secular terms. The method of multiple scales was therefore introduced in the 3D asymptotic formulations of laminated plates and shells to eliminate the secular terms raised from the regular asymptotic expansions (Tarn and Wang, 1994; Wu et al., 1996b, 1998). The previous asymptotic approach was successfully extended to various benchmark problems for the bending, thermoelastic, vibration, buckling, dynamic instability and nonlinear analyses of laminated composite elastic plates and shells (Tarn and Wang, 1995; Tarn, 1996, 1997; Wu and Chi, 1999, 2004, 2005; Wu and Hung, 1999; Wu and Lo, 2000; Wu and Chiu, 2001, 2002; Wu and Chen, 2001).

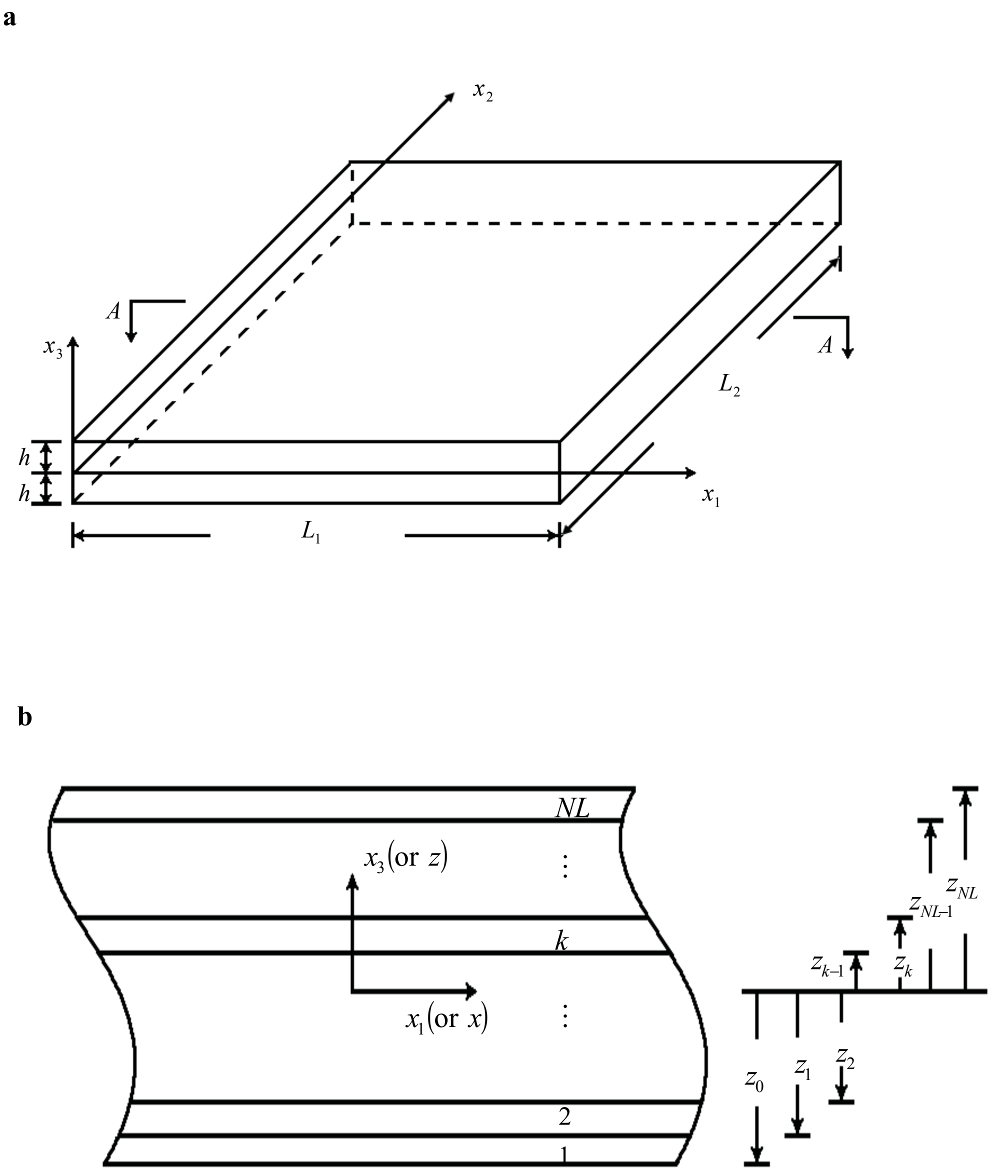
The extension of aforementioned approaches to 3D analysis of multilayered and FG piezoelectric structures was completely collected and tabulated in the following sections. Several representative articles dealing with 3D numerical modeling of multilayered and FG piezoelectric (or elastic) structures were also mentioned. For illustration the mathematical equations, based on the mixed formulation and the displacement-based formulation, for the 3D analysis of multilayered piezoelectric plates were derived. The analytical process using various approaches was briefly interpreted. Comparisons of the 3D results of elastic and electric field variables obtained from a variety of approaches were also presented.

2 Basic equations of 3D Piezoelectricity

As shown in Fig. 1, we consider a simply-supported, multilayered piezoelectric plate of which thickness is $2h$. The in-plane dimensions in x_1 and x_2 directions are L_1 and L_2.

The linear constitutive equations valid for the nature of symmetry class of the piezoelectric material are given by

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{23}\\ \tau_{13}\\ \tau_{12}\end{Bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{13}&0&0&0\\ c_{12}&c_{22}&c_{23}&0&0&0\\ c_{13}&c_{23}&c_{33}&0&0&0\\ 0&0&0&c_{44}&0&0\\ 0&0&0&0&c_{55}&0\\ 0&0&0&0&0&c_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{Bmatrix}-\begin{bmatrix}0&0&e_{31}\\ 0&0&e_{32}\\ 0&0&e_{33}\\ 0&e_{24}&0\\ e_{15}&0&0\\ 0&0&0\end{bmatrix}\begin{Bmatrix}E_1\\ E_2\\ E_3\end{Bmatrix}\quad(1)$$



Section A-A

Figure 1: (a) The geometry and coordinates of a piezoelectric plate; (b) The configuration of section A-A and the dimensionless thickness coordinate.

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{24} & 0 & 0 \\ e_{31} & e_{32} & e_{33} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} + \begin{bmatrix} \eta_{11} & 0 & 0 \\ 0 & \eta_{22} & 0 \\ 0 & 0 & \eta_{33} \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}, \quad (2)$$

where σ_1, σ_2, σ_3, τ_{13}, τ_{23}, τ_{12} and ε_1, ε_2, ε_3, γ_{13}, γ_{23}, γ_{12} denote the stress and strain components, respectively; (D_1, D_2, D_3) and (E_1, E_2, E_3) denote the components of electric displacement and electric field, respectively. c_{ij}, e_{ij} and η_{ij} are the elastic coefficients, piezoelectric and dielectric permeability coefficients, respectively, relative to the geometrical axes of the plate. The material properties are considered to be layerwise constants.

The kinematic equations in terms of the curvilinear coordinates x_1, x_2 and x_3 are

$$\varepsilon_1 = u_{1,1}, \quad \varepsilon_2 = u_{2,2}, \quad \varepsilon_3 = u_{3,3}$$

$$\gamma_{13} = u_{1,3} + u_{3,1}, \quad \gamma_{23} u_{2,3} + u_{3,2},$$

$$\gamma_{33} = u_{3,3} + u_{3,3} \tag{3}$$

in which u_1, u_2 and u_3 are the displacement components.

The stress equilibrium equations without body forces are given by

$$\sigma_{1,1} + \tau_{12,2} + \tau_{13,3} = 0, \tag{4}$$

$$\tau_{12,1} + \sigma_{2,2} + \tau_{23,3} = 0, \tag{5}$$

$$\tau_{13,1} + \tau_{23,2} + \sigma_{3,3} = 0. \tag{6}$$

The equation of electrostatics is

$$D_{1,1} + D_{2,2} + D_{3,3} = 0. \tag{7}$$

The relations between the electric field and electric potential are

$$E_1 = -\Phi_{,1}, \quad E_2 = -\Phi_{,2}, \quad E_3 = -\Phi_{,3}, \tag{8}$$

where Φ denotes the electric potential.

The boundary conditions of the problem are specified as follows:

On the lateral surface the transverse load and the electric potential (or the normal electric displacement) are prescribed and given by

$$\begin{bmatrix}\tau_{13} & \tau_{23} & \sigma_3 & \phi\end{bmatrix} = \begin{bmatrix}0 & 0 & \overline{q}^{\pm} & \overline{\phi}^{\pm}\end{bmatrix}$$

on $\zeta = \pm h$ (for closed-circuit surface conditions), (9)

$$\begin{bmatrix}\tau_{13} & \tau_{23} & \sigma_3 & D_3\end{bmatrix} = \begin{bmatrix}0 & 0 & \overline{q}_3^{\pm} & \overline{D}_3^{\pm}\end{bmatrix}$$

on $z = \pm h$(for open-circuit surface conditions). (10)

The edge boundary conditions of the shells are considered as fully simple supports with electrically grounded and are given by

$$sigma_1 = u_2 = u_3 = \phi = 0 \text{ at } x_1 = 0 \text{ and } x_1 = L_1 \tag{11a}$$

$$\sigma_2 = u_1 = u_3 = \phi = 0 text at x_2 = 0 \text{ and } x_2 = L_2 \tag{11b}$$

It is noted that Eqs. (1)–(11) represent twenty-two basic equations in twenty-two unknowns with a set of appropriate boundary conditions for the 3D analysis of a simply-supported, multilayered piezoelectric plate.

3 Nondimensionalization

For making the calculation more efficient and preventing from the ill-conditioned of system matrix, we select a set of dimensionless variables to normalize the coordinates and the variables of electric and elastic fields. The dimensionless variables are given as

$$x = x_1/L, \quad y = x_2/L, \quad z = x_3/h,$$

$$u = u_1/h, \quad v = u_2/h, \quad w = u_3/L,$$

$$\sigma_x = L\sigma_1/(hQ), \quad \sigma_y = L\sigma_2/(hQ),$$

$$\tau_{xy} = L\tau_{12}/(hQ), \quad \tau_{xz} = L^2\tau_{13}/h^2Q,$$

$$\tau_{yz} = L^2\tau_{23}/h^2Q, \quad D_x = hD_1/(Le),,$$

$$D_y = hD_2/(Le), \quad D_z = LD_3/(he),$$

$$\phi = Le\Phi/(h^2Q), \quad \sigma_z = L^3\sigma_3/(h^3Q) \tag{12}$$

where L denotes a typical in-plane dimension of the plate and is taken to be $L = \sqrt{L_1 L_2}$ in the paper; $-1 \le z \le 1$; e and Q stand for a reference piezoelectric and elastic modulus; Q is taken as $Q = (1/2h)\int_{-h}^{h} c_{33} dx_3$.

4 Direct Elimination

4.1 The mixed formulation

In the mixed formulation, the elastic displacements (u, v, w), transverse stresses (τ_{xz}, τ_{yz}, σ_z), the electric potential (ϕ) and the normal electric displacement (D_z) are regarded as the primary variables. Other fourteen variables are the dependent variables and can be determined by the primary variables. By means of introducing the set of dimensionless coordinates and variables (Eq. (12)) and then performing the direct elimination, we may rewrite the basic 3D piezoelectricity equations (Eqs. (1)–(8)) as a system of eight partially differential equations in terms of eight primary variables as follows:

$$\begin{Bmatrix} u_{,z} \\ v_{,z} \\ \sigma_{z,z} \\ D_{z,z} \\ \tau_{xz,z} \\ \tau_{yz,z} \\ w_{,z} \\ \phi_{,z} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 & d_{17} & d_{18} \\ 0 & 0 & 0 & 0 & 0 & d_{26} & d_{27} & d_{28} \\ 0 & 0 & 0 & 0 & d_{17} & d_{27} & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{18} & d_{28} & 0 & d_{48} \\ d_{51} & d_{52} & d_{53} & d_{54} & 0 & 0 & 0 & 0 \\ d_{61} & d_{62} & d_{63} & d_{64} & 0 & 0 & 0 & 0 \\ d_{53} & d_{63} & d_{73} & d_{74} & 0 & 0 & 0 & 0 \\ d_{54} & d_{64} & d_{74} & d_{84} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ \sigma_z \\ D_z \\ \tau_{xz} \\ \tau_{yz} \\ w \\ \phi \end{Bmatrix}, \quad (13)$$

where

$$d_{15} = Qh^2/c_{55}L^2, \quad d_{17} = -\partial_x,$$

$$d_{18} = -Qe_{15}h^2\partial_x/c_{55}eL^2, \quad d_{26} = Qh^2/c_{44}L^2,$$

$$d_{27} = -\partial_y, \quad d_{28} = -Qe_{24}h^2\partial_y/c_{44}eL^2,$$

$$d_{48} = (c_{55}^{-1}e_{15}^2 + \eta_{11})(Qh^2/e^2L^2)\partial_{xx} + (c_{44}^{-1}e_{24}^2 + \eta_{22})(Qh^2/e^2L^2)\partial_{yy},$$

$$d_{51} = -(\tilde{Q}_{11}\partial_{xx} + \tilde{Q}_{66}\partial_{yy}),$$

$$d_{52} = -(\tilde{Q}_{12} + \tilde{Q}_{66})\partial_{xy}, \quad d_{53} = -a_1(h^2/L^2)\partial_x,$$

$$d_{54} = -b_1(e/Q)\partial_x; \quad d_{61} = (\tilde{Q}_{21} + \tilde{Q}_{66})\partial_{xy},$$

$$d_{62} = -(\tilde{Q}_{66}\partial_{xx} + \tilde{Q}_{22}\partial_{yy}), \quad d_{63} = -a_2(h^2/L^2)\partial_y,$$

$$d_{64} = -b_2(e/Q)\partial_y, \quad d_{73} = \overline{\eta}Q(h^4/L^4),$$

$$d_{74} = \overline{e}eh^2/L^2; \quad d_{84} = -\overline{c}e_0^2/Q,$$

$$\overline{c} = c_{33}/\left(e_{33}^2 + \eta_{33}c_{33}\right),$$

$$\overline{e} = e_{33}/\left(e_{33}^2 + \eta_{33}c_{33}\right),$$

$$\overline{\eta} = \eta_{33}/\left(e_{33}^2 + \eta_{33}c_{33}\right),$$

$$Q_{ij} = c_{ij} - a_j c_{i3} - b_j e_{3i}\ (i, j = 1, 2, 6),$$

$$\tilde{Q}_{ij} = Q_{ij}/Q,$$

$$a_i = (e_{33}e_{3i} + \eta_{33}c_{i3}) / \left(e_{33}^2 + \eta_{33}c_{33}\right),$$

$$b_i = (e_{33}c_{i3} - c_{33}e_{3i}) / \left(e_{33}^2 + \eta_{33}c_{33}\right).$$

The dimensionless form of boundary conditions of the problem are specified as follows:

On the lateral surface the transverse load and electric potential (or the normal electric displacement) are given by

$$[\tau_{xz} \quad \tau_{yz} \quad \sigma_z \quad \phi] = \left[0 \quad 0 \quad (\overline{q}_3^{\pm}L^3/h^3Q) \quad (\overline{\Phi}^{\pm}Le/h^2Q)\right] \text{ on } z = \pm 1$$

(for closed-circuit surface conditions), (14)

$$[\tau_{xz} \quad \tau_{yz} \quad \sigma_z \quad D_z] = \left[0 \quad 0 \quad (\overline{q}_3^{\pm}L^3/h^3Q) \quad (\overline{D}_z^{\pm}L/he)\right] \text{ on } z = \pm 1$$

(for open-circuit surface conditions). (15)

At the edges, the boundary conditions are given by

$$\sigma_x = v = w = \phi = 0 \text{ at } x = L_1/L; \quad (16a)$$

$$\sigma_y = u = w = \phi = 0 \text{ at } y = 0 \text{ and } y = L_2/L. \quad (16b)$$

The method of double Fourier series expansion is firstly applied to transform the system of partial differential equations (Eq. (13)) into a system of ordinary differential equations. In order to satisfy the edge boundary conditions, we express the primary variables in the following form

$$(u,\tau_{xz}) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(u_{mn}(z),\tau_{xzmn}(z))\cos\tilde{m}x\sin\tilde{n}y, \quad (17)$$

$$(v,\tau_{yz}) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(v_{mn}(z),\tau_{yzmn}(z))\sin\tilde{m}x\cos\tilde{n}y, \quad (18)$$

$$(w,\sigma_z,\phi,D_z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(w_{mn}(z),\sigma_{zmn}(z),\phi_{mn}(z),D_{zmn}(z)) \sin\tilde{m}x\sin\tilde{n}y, \quad (19)$$

where $\tilde{m} = m\pi L/L_1$ and $\tilde{n} = n\pi L/L_2$; m and n are positive integers.

For brevity, the symbols of summation are omitted in the following derivation. Substituting Eqs. (17)–(19) in the basic 3D equations (Eq. (13)), we may yield the resulting equations as follows:

$$\frac{d}{dz}\mathbf{F}(z) = \mathbf{D}\mathbf{F}(z), \quad (20)$$

where $\mathbf{F}(z)$ is called the state vector of the plate and given as $\mathbf{F}(z) = \{u_{mn}\ v_{mn}\ \sigma_{zmn}\ D_{zmn}\ \tau_{xzmn}\ \tau_{yzmn}\ w_{mn}\ \phi_{mn}\}^T$; $\mathbf{D}$ is the relevant coefficient matrix which is a constant coefficient matrix for plates and a variable coefficient matrix for shells.

Similarly, the dimensionless dependent variables of in-plane stresses and in-plane electric displacements can be expressed in terms of the primary variables as follows:

$$(\sigma_x,\sigma_y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\sigma_{xmn}(z),\sigma_{ymn}(z))\sin\tilde{m}x\sin\tilde{n}y, \quad (21)$$

$$\tau_{xy} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tau_{xymn}(z)\cos\tilde{m}x\cos\tilde{n}y, \quad (22)$$

$$D_x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}D_{xmn}(z)\cos\tilde{m}x\sin\tilde{n}y, \quad (23)$$

$$D_y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}D_{ymn}(z)\sin\tilde{m}x\cos\tilde{n}y, \quad (24)$$

where

$$\begin{Bmatrix}\sigma_{xmn}\\ \sigma_{ymn}\\ \tau_{xymn}\end{Bmatrix} = \begin{bmatrix} l_{11} & l_{12}\\ l_{21} & l_{22}\\ l_{31} & l_{32}\end{bmatrix}\begin{Bmatrix}u_{mn}\\ v_{mn}\end{Bmatrix} + \begin{bmatrix}l_{13}\\ l_{23}\\ 0\end{bmatrix}\sigma_{zmn} + \begin{bmatrix}l_{14}\\ l_{24}\\ 0\end{bmatrix}D_{zmn},$$

$$\begin{Bmatrix}D_{xmn}\\ D_{ymn}\end{Bmatrix} = \begin{bmatrix}l_{41} & 0\\ 0 & l_{52}\end{bmatrix}\begin{Bmatrix}\sigma_{xzmn}\\ \sigma_{yzmn}\end{Bmatrix} + \begin{bmatrix}l_{43}\\ l_{53}\end{bmatrix}\phi_{mn},$$

l_{ij} are the relevant coefficients and given in Appendix A.

Eq. (20) represents a system of eight simultaneously linear first-order ordinary differential equations in terms of eight primary variables. Various approaches have been applied to determine the primary variables in the elastic and electric fields in the literature. Once these primary variables are determined, the dependent variables can then be calculated using Eqs. (21)–(24).

4.2 The displacement-based formulation

In the displacement-based formulation, the elastic displacements (u, v, w) and the electric potential (ϕ) are regarded as the primary variables. Similarly, by means of introducing the set of dimensionless coordinates and variables and then performing the double Foruier series expansion and the direct elimination, we may rewrite the basic 3D piezoelectricity equations (Eqs. (1)–(8)) as a system of four ordinally differential equations in terms of four primary variables as follows:

$$\mathbf{P}\frac{d^2\mathbf{G}}{dz^2} + \mathbf{Q}\frac{d\mathbf{G}}{dz} + \mathbf{R}\mathbf{G} = \mathbf{0}, \quad (25)$$

where

$$\mathbf{G} = \{u_{mn} \quad v_{mn} \quad w_{mn} \quad \phi_{mn}\}^T,$$

$$\mathbf{P} = \begin{bmatrix} p_{11} & 0 & 0 & 0\\ 0 & p_{22} & 0 & 0\\ 0 & 0 & p_{33} & p_{34}\\ 0 & 0 & p_{34} & p_{44}\end{bmatrix},$$

$$\mathbf{Q}=\begin{bmatrix}0 & 0 & q_{13} & q_{14}\\ 0 & 0 & q_{23} & q_{24}\\ q_{13} & q_{23} & 0 & 0\\ q_{14} & q_{24} & 0 & 0\end{bmatrix},$$

$$\mathbf{R}=\begin{bmatrix}r_{11} & r_{12} & 0 & 0\\ r_{12} & r_{22} & 0 & 0\\ 0 & 0 & r_{33} & r_{34}\\ 0 & 0 & r_{34} & r_{44}\end{bmatrix},$$

$$p_{11}=c_{55}/Q,\quad p_{22}=c_{44}/Q,\quad p_{33}=c_{33}L^2/Qh^2,$$
$$p_{34}=-e_{33}/e,\quad p_{44}=\left(\eta_{33}Qh^2\right)/\left(e^2L^2\right),$$
$$q_{13}=\tilde{m}(c_{13}+c_{55})/Q,$$
$$q_{14}=\tilde{m}(e_{31}+e_{15})h^2/\left(eL^2\right),$$
$$q_{23}=\tilde{n}(c_{23}+c_{44})/Q,$$
$$q_{24}=\tilde{n}(e_{32}+e_{24})h^2/\left(eL^2\right),$$
$$r_{11}=-\left(\tilde{m}^2c_{11}+\tilde{n}^2c_{66}\right)h^2/\left(QL^2\right),$$
$$r_{12}=-\tilde{m}\tilde{n}(c_{12}+c_{66})h^2/\left(QL^2\right),$$
$$r_{22}=-(\tilde{m}^2c_{66}+\tilde{n}^2c_{22})h^2/\left(QL^2\right),$$
$$r_{33}=\left(\tilde{m}^2c_{55}+\tilde{n}^2c_{44}\right)/Q,$$
$$r_{34}=(\tilde{m}^2e_{15}+\tilde{n}^2e_{24})h^2/\left(eL^2\right),$$
$$r_{44}=-(\tilde{m}^2\eta_{11}+\tilde{n}^2\eta_{22})Qh^4/\left(e^2L^4\right).$$

The dependent variables can be determined by the previous primary variables using Eqs. (1)–(8), (17)–(19) and Eqs. (21)–(24), and they are given by

$$\begin{Bmatrix}\sigma_{xmn}\\ \sigma_{ymn}\\ \tau_{xymn}\end{Bmatrix}= \begin{bmatrix}\bar{l}_{11} & \bar{l}_{12}\\ \bar{l}_{21} & \bar{l}_{22}\\ \bar{l}_{31} & \bar{l}_{32}\end{bmatrix}\begin{Bmatrix}u_{mn}\\ v_{mn}\end{Bmatrix}+\begin{bmatrix}\bar{l}_{13}\\ \bar{l}_{23}\\ 0\end{bmatrix}\frac{dw_{mn}}{dz}+\begin{bmatrix}\bar{l}_{14}\\ \bar{l}_{24}\\ 0\end{bmatrix}\frac{d\phi_{mn}}{dz}, \tag{26}$$

$$\begin{Bmatrix}\tau_{xzmn}\\ \tau_{yzmn}\end{Bmatrix}= \begin{bmatrix}\bar{l}_{41} & 0\\ 0 & \bar{l}_{52}\end{bmatrix}\begin{Bmatrix}\frac{du_{mn}}{dz}\\ \frac{dv_{mn}}{dz}\end{Bmatrix}+\begin{bmatrix}\bar{l}_{43}\\ \bar{l}_{53}\end{bmatrix}w_{mn}+\begin{bmatrix}\bar{l}_{44}\\ \bar{l}_{54}\end{bmatrix}\phi_{mn}, \tag{27}$$

$$\begin{Bmatrix}D_{xzmn}\\ D_{yzmn}\end{Bmatrix}= \begin{bmatrix}\bar{l}_{61} & 0\\ 0 & \bar{l}_{72}\end{bmatrix}\begin{Bmatrix}\frac{du_{mn}}{dz}\\ \frac{dv_{mn}}{dz}\end{Bmatrix}+\begin{bmatrix}\bar{l}_{63}\\ \bar{l}_{73}\end{bmatrix}w_{mn}+\begin{bmatrix}\bar{l}_{64}\\ \bar{l}_{74}\end{bmatrix}\phi_{mn}, \tag{28}$$

$$\begin{Bmatrix}\sigma_{zmn}\\ D_{zmn}\end{Bmatrix}= \begin{bmatrix}\bar{l}_{81} & \bar{l}_{82}\\ \bar{l}_{91} & \bar{l}_{92}\end{bmatrix}\begin{Bmatrix}u_{mn}\\ v_{mn}\end{Bmatrix}+\begin{bmatrix}\bar{l}_{83}\\ \bar{l}_{93}\end{bmatrix}\frac{dw_{mn}}{dz}+\begin{bmatrix}\bar{l}_{84}\\ \bar{l}_{94}\end{bmatrix}\frac{d\phi_{mn}}{dz}, \tag{29}$$

where $\bar{l}_{ij}$ are the relevant coefficients and are given in Appendix B.

5 3D analytical approaches

5.1 The Pagano's classical approach

Based on the Pagano's classical approach, the primary variables in the displacement-based formulation are assumed as

$$\mathbf{G}(z)=\overline{\mathbf{G}}e^{\lambda z}. \tag{30}$$

Substituting Eq. (30) into Eq. (25) leads to an eighth-order polynomial equation as

$$A_1\lambda^8+A_2\lambda^6+A_3\lambda^4+A_4\lambda^2+A_5=0, \tag{31}$$

where $A_i \quad (i=1-5)$ are the relevant coefficients of the polynomial.

The roots of Eq. (31) must be functions of the material properties and the laminate geometry. The roots can therefore be real, imaginary or complex. By using the material properties of a typical k-th layer, one may obtain the eigenvalues $\lambda_i^{(k)}$ $(i=1-8)$ and their corresponding eigenvectors $\overline{\mathbf{G}}_i^{(k)}$ (i=1–8) from Eqs. (31) and (30). The general solutions of primary unknowns of k-th layer can then be written by

$$\mathbf{G}^{(k)}(z)=\sum_{i=1}^{8}\alpha_i^{(k)}\overline{\mathbf{G}}_i^{(k)}e^{\lambda_i^{(k)}z}$$
$$(k=1,2,\ldots,NL;\ z_{k-1}\le z\le z_k), \tag{32}$$

where $\alpha_i^{(k)}$ (i=1–8 and k=1–NL) are the undetermined coefficients. For a NL-layered plate, totally there are ($8\times NL$) undetermined coefficients in Eq. (32).

It is noted that the eigenvalues in Eq. (32) can be obtained in a certain pair of complex conjugate as $\lambda_{1,2} = Re(\overline{\lambda}) \pm Im(\overline{\lambda})$ and their corresponding eigenvectors are $\overline{\mathbf{G}}_{1,2} = Re(\overline{\mathbf{G}}) \pm Im(\overline{\mathbf{G}})$ where $\overline{\lambda}$ is a certain complex eigenvalue and its corresponding eigenvector is $\overline{\mathbf{G}}$. There is nothing wrong with those complex-valued solutions (i.e., $\overline{\mathbf{G}}_{1,2}$). However, we may prefer to replace them with two linearly independent solutions involving only real-valued quantities and they are given by

$$\overline{\mathbf{G}}_1 = e^{Re(\overline{\lambda})z} \left(Re(\overline{\mathbf{G}})\cos(Im(\overline{\lambda})z) - Im(\overline{\mathbf{G}})\sin(Im(\overline{\lambda})z)\right), \tag{33a}$$

$$\overline{\mathbf{G}}_2 = e^{Re(\overline{\lambda})z} \left(Re(\overline{\mathbf{G}})\sin(Im(\overline{\lambda})z) + Im(\overline{\mathbf{G}})\cos(Im(\overline{\lambda})z)\right). \tag{33b}$$

The problem also supply with ($8\times NL$) conditions for determining $\alpha_i^{(k)}$ in Eq. (32). There are four boundary conditions at each lateral surface (i.e., either Eqs. (14) or (15)) and eight interfacial continuity conditions at interfaces between adjacent layers such that

$$\mathbf{F}^{(k)}(z = z_k) = \mathbf{F}^{(k+1)}(z = z_k) \quad (k = 1,2,\ldots,(NL-1)), \tag{34}$$

where z_k denotes the dimensionless thickness coordinate measured from the mid-plane of the plate to the top surface of the k^{th}-layer (see Fig. 1); $\mathbf{F}^{(k)}$ can be calculated from $\mathbf{G}^{(k)}$.

The complementary solutions of primary variables of each layer can be determined by imposing the applied lateral surface conditions and the continuity conditions at interfaces between adjacent layers. Afterwards, the dependent variables can be obtained using Eqs. (26)-(29).

The Pagano's classical approach originally proposed for the 3D analyses of laminated composite elastic plates was extendedly used for a variety of 3D structural analyses of multilayered and FG piezoelectric plates and shells. The static behaviors of simply-supported laminated piezoelectric cylinders, plates and strips were studied by Heyliger (1994, 1997a, b) and Heyliger and Brooks (1996), respectively. The free vibration and cylindrical bending vibration of multilayered piezoelectric plates were analyzed by Heyliger and Saravanos (1995) and Heyliger and Brooks (1995), respectively. A partial publication list of relevant studies using the Pagano's classical approach is classified and tabulated in Table 2.

5.2 *The state space approach*

On the basis of the mixed formulaion, one may express the primary variables of the typical k^{th}-layer as

$$\frac{d}{dz}\mathbf{F}(z) = \overline{\mathbf{D}}_k\mathbf{F}(z) \quad z_{k-1} \le z \le z_k, \tag{35}$$

where $\overline{\mathbf{D}}_k$ is a 8x8 constant coefficient matrix for the k^{th}-layer; z_{k-1} denotes the dimensionless thickness coordinate measured from the mid-plane of the plate to the bottom surface of the k^{th}-layer (see Fig. 1).

With a known state vector at the bottom surface of the k^{th}-layer ($\mathbf{F}_{k-1}$), the solution of equation (35) is

$$\begin{aligned}\mathbf{F}(z) &= e^{\overline{\mathbf{D}}_k(z-z_{k-1})}\mathbf{F}_{k-1} \\ &= \mathbf{M}_k e^{\Lambda_k(z-z_{k-1})}\mathbf{M}_k^{-1}\mathbf{F}_{k-1}\end{aligned}, \tag{36}$$

where $\mathbf{M}_k$ is the modal matrix of $\overline{\mathbf{D}}_k$ consisting of eight independent eigenvectors; $\mathbf{F}_{k-1} = \mathbf{F}(z = z_{k-1})$; $e^{\Lambda_k z}$ is a 8x8 diagonal matrix and given by $e^{\Lambda_k z} = \begin{bmatrix} e^{\lambda_1 z} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 z} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_8 z}\end{bmatrix}$ in which $\lambda_1, \lambda_2, \ldots, \lambda_8$ are the set of eigen values of $\overline{\mathbf{D}}_k$.

According to Eq. (36), the state vector within the k^{th}-layer can be determined. The state vector of the top surface of the k^{th}-layer is then determined as follows.

$$\mathbf{F}_k = \mathbf{R}_k\mathbf{F}_{k-1}, \tag{37}$$

where $\mathbf{F}_k = \mathbf{F}(z_k)$; $\mathbf{R}_k = \mathbf{M}_k e^{\Lambda_k h_k}\mathbf{M}_k^{-1}$.

Table 2: Partial list of references on 3D analysis of single-layer homogeneous, multilayered and FG piezoelectric plates using Pagano's classical approach

Problems	Literature
Static:	Ray, Rao and Samanta, 1992, 1993; Ray, Bhattacharya and Samanta, 1993;
	Heyliger, 1994; Heyliger and Brooks, 1996; Heyliger, 1997a, b
Thermal:	Ootao and Tanigawa, 2000a, b
Vibration:	Heyliger and Brooks, 1995; Heyliger and Saravanos, 1995; Batra and Liang, 1997;
	Ray, Bhattacharya and Samanta, 1998; Vel, Mewer and Batra, 2004; Cupial, 2005
Others:	Kapuria, Dumir and Sengupta, 1999

By analogy, the state vectors between the top and bottom surfaces of the shell are linked by

$$\begin{aligned}\mathbf{F}_{NL} &= \mathbf{R}_{NL}\mathbf{F}_{(NL-1)} \\ &= \mathbf{R}_{NL}\mathbf{R}_{(NL-1)}\ldots\mathbf{R}_1\mathbf{F}_0\end{aligned}. \tag{38}$$

By defining a symbol of consecutive multiplication, we rewrite Eq. (38) in the form of

$$\mathbf{F}_{NL} = \left(\prod_{k=1}^{NL}\mathbf{R}_k\right)\mathbf{F}_0, \tag{39}$$

where $\prod_{k=1}^{NL}\mathbf{R}_k = \mathbf{R}_{NL}\mathbf{R}_{(NL-1)}\ldots\mathbf{R}_2\mathbf{R}_1$.

Equation (39) represents a set of simultaneous algebraic equations. After imposing the boundary conditions prescribed on the lateral surfaces, we can determine the other unknowns in state vectors of lateral surfaces of the plate. After then, the state vector through the thickness coordinate of the shell can be obtained by

$$\begin{aligned}\mathbf{F}(z) &= \mathbf{M}_k e^{\Lambda_k(z-z_{k-1})}\mathbf{M}_k^{-1}\mathbf{F}_{k-1} \\ &= \mathbf{M}_k e^{\Lambda_k(z-z_{k-1})}\mathbf{M}_k^{-1}\left(\prod_{i=1}^{k-1}\mathbf{R}_i\right)\mathbf{F}_0\end{aligned}. \tag{40}$$

Equations (39)-(40) provide the 3D solutions of a system of the state equations.

The state space approach in conjunction with the method of transfer matrix has been used for a variety of 3D structural analyses of multilayered and FG piezoelectric plates and shells. Vel and Batra (2001a, b) presented exact solutions for the cylindrical bending of laminated plates and for rectangular sandwich plates, respectively, with embedded piezoelectric shear actuators. Exact solutions for the static, piezothermoelectric and vibration analyses of the simply-supported, FG piezoelectric plates were presented by Zhong and Shang (2003, 2005) and Zhong and Yu (2006), respectively. Chen et al. (2001a, b) studied the static and free vibration problems of a fluid-filled piezoceramic hollow sphere. A partial publication list of relevant studies is classified and tabulated in Table 3.

5.3 *The series expansion approach*

As we previously mentioned, the systems of differential equations derived from both the displacement-based formulation and the mixed formulation are the ones with constant coefficients for plates and with variable coefficients for shells. Pagano's classical approach fails to exactly analyze the system of thickness-varying differential equations. Hence, the series expansion method becomes a feasible analytical approach for shell problems.

5.3.1 *Frobenius series expansion*

For the Frobenius series expansion approach (Huang and Tauchert, 1992), the primary variables in the displacement-based formulation are expressed in the following form

$$\mathbf{G}(z) = z^p \sum_{q=0,1,2}^{\infty} \overline{\mathbf{G}}_q z^q. \tag{41}$$

Substituting Eq. (41) into Eq. (25), and equating the coefficients of the smallest power to zero lead

Table 3: Partial list of references on 3D analysis of single-layer homogeneous, multilayered and FG piezoelectric structures using the state space approach

Problems		Literature
Static	(Plates):	Brissaud, 1996; Lee and Jiang, 1996; Bisegna and Maceri, 1996;
		Ding, Xu and Guo, 1999; Benjeddou and Deü , 2001; Vel and Batra, 2001a, b; Zhong and Shang,
		2003; Chen, Cai, Ye and Wang, 2004;
		Chen, Ying, Cai and Ye, 2004; Lu, Lee and Lu, 2005, 2006
	(Shells):	Chen, Ding and Xu, 2001a; Wu and Liu, 2007
Thermal	(Plates):	Xu, Noor and Tang, 1995; Tarn, 2002; Zhong and Shang, 2005
	(Shells):	Xu and Noor (1996)
Vibration	(Plates):	Yang, Batra and Liang, 1994; Batra, Liang and Yang, 1996;
		Xu, Noor and Tang, 1997; Ding, Xu and Chen, 2000; Chen and Ding, 2002; Deü and Benjeddou,
		2005; Zhong and Yu, 2006
	(Shells):	Chen and Wang, 2002
Buckling	(Plates):	Kapuria and Achary, 2004
Others	(Plates):	Kapuria and Achary, 2005; Sheng, Wang and Ye, 2007
	(Shells):	Chen, Ding and Xu, 2001b; Wang and Zhong, 2003; Chen, Bian and Ding, 2004

to an indicial equation of the differential equations (Eq. (25) and a recurrence relation relating $(\overline{\mathbf{G}}_q,(q=1,2,\ldots,\infty))$ to $\overline{\mathbf{G}}_0$. The indicial equation is used to determine the values of p where p_i $(i=1-8)$. Hence, the complementary solution of Eq. (25) for a typical k^{th}-layer can then be written as

$$\mathbf{G}^{(k)}(z)=\sum_{i=1}^{8}B_i^{(k)}z^{p_i^{(k)}}\sum_{q=0,1,2}^{\infty}\overline{\mathbf{G}}_q^{(k)}z^q. \tag{42}$$

Again, the unknown coefficients $B_i^{(k)}$ $(i=1,2,\ldots,8;k=1,2,\ldots,NL)$are determined from the generalized traction conditions on the lateral surface conditions (Eqs. (14)–(15)) and continuity conditions at the interfaces between adjacent layers (Eq. (34)).

5.3.2 *Modified Frobenius series expansion*

For the modified Frobenius series expansion approach (Kapuria et al., 1997a, b; Xu and Noor, 1996), the primary variables in the mixed formulation are expressed in the following form

$$\mathbf{F}(z)=e^{pz}\sum_{q=0,1,2}^{\infty}\overline{\mathbf{F}}_q z^q. \tag{43}$$

Substituting Eq. (43) into Eq. (20), and equating the coefficients of each power of z to zero lead to a characteristic equation solving for p where p_i $(i=1-8)$. The complementary solution of Eq. (20) for a typical k^{th}-layer is the sum of the eight solutions for the eight values of p_i $(i=1-8)$ and can then be written as

$$\mathbf{F}^{(k)}(z)=\sum_{i=1}^{8}C_i^{(k)}e^{p_i^{(k)}z}\sum_{q=0,1,2}^{\infty}\overline{\mathbf{F}}_q z^q. \tag{44}$$

Again, the unknown coefficients $C_i^{(k)}$ $(i=1,2,\ldots,8;k=1,2,\ldots,NL)$ are determined from the generalized traction conditions on the lateral surface conditions and continuity conditions at the interfaces between adjacent layers.

The modified Frobenius method has been used for a variety of 3D structural analyses of multilayered and FG piezoelectric plates and shells. Kapuria et al. (1997a) presented exact solutions for the

Table 4: Partial list of references on 3D analysis of single-layer homogeneous, multilayered and FG piezoelectric shells using the series expansion approach

Problems	Literature
Static:	Chen, Shen and Wang, 1996; Dumir, Dube and Kapuria, 1997;
	Kapuria, Dumir and Sengupta, 1997a; Kapuria, Sengupta and Dumir, 1997a;
	Heyliger, 1997a; Chen, Shen and Liang, 1999
Thermal:	Dube, Kapuria and Dumir, 1996a, b; Kapuria, Sengupta and Dumir, 1997b;
	Kapuria, Dumir and Sengupta, 1997b; Wu, Shen and Chen, 2003
Vibration:	Chen and Shen, 1998; Hussein and Heyliger, 1998
Others:	Chen, Ding and Liang, 2001

coupled electro-elastic analysis of a piezoelectric cylindrical shell under axisymmetric load. Kapuria, Sengupta and Dumir (1997b) and Kapuria, Dumir and Sengupta (1997b) studied for the 3D piezothermoelastic analyses of simply-supported, multilayered piezoelectric cylindrical shells under axisymmetric and nonaxisymmetric thermo-electric loads, respectively. A partial publication list of relevant studies using the series expansion approach is classified and tabulated in Table 4.

5.4 The asymptotic approach

On the basis of the asymptotic approach, a perturbation parameter $\in$ ($\in^2 = h/L$) is introduced in the mixed formulation. The primary variables are then asymptotically expanded in the powers ε^2 as follows (Nayfeh, 1981):

$$f(x,y,z,\varepsilon) = f^{(0)}(x,y,z) + \varepsilon^2 f^{(1)}(x,y,z) + \varepsilon^4 f^{(2)}(x,y,z) + \ldots. \quad (45)$$

Substituting Eq. (45) into the basic 3D equations (Eq. (20)), collecting coefficients of equal powers of ε and performing the successive integration to the resulting equations of various order problems through the thickness coordinate, we finally obtain the recursive sets of governing equations of various order problems as follows:

5.4.1 Plates with open-circuit surface conditions

In the cases of plates with open-circuit surface conditions, the governing equations of various order problems are (Wu and Syu, 2007)

$$K_{11}u^k + K_{12}v^k + K_{13}w^k + K_{14}D_z^k = f_{1k}(1), \quad (46)$$

$$K_{21}u^k + K_{22}v^k + K_{23}w^k + K_{24}D_z^k = f_{2k}(1), \quad (47)$$

$$K_{31}u^k + K_{32}v^k + K_{33}w^k + K_{34}D_z^k = f_{3k}(1) - mf_{1k}(1) - nf_{2k}(1), \quad (48)$$

$$K_{41}u^k + K_{42}v^k + K_{43}w^k + K_{44}D_z^k = f_{4k}(1), \quad (49)$$

where K_{ij} (i,j=1-4) are the relevant differential operators; f_{ik} ($k = 0,1,2,\ldots$) are the revelant functions and $f_{10} = f_{20} = 0$, $f_{30} = \overline{q}_z^+ - \overline{q}_z^-$, $f_{40} = \overline{D}_z^+ - \overline{D}_z^-$.

The boundary conditions for various order problems are specified as follows:

On the lateral surface the transverse loads and normal electric displacement are given by

$$\begin{bmatrix}\tau_{xz}^{(0)} & \tau_{yz}^{(0)} & \sigma_z^{(0)} & D_z^{(0)}\end{bmatrix} = \begin{bmatrix}0 & 0 & \overline{q}_z^{\pm} & \overline{D}_z^{\pm}\end{bmatrix} \quad \text{on } z = \pm 1; \quad (50a)$$

$$\begin{bmatrix}\tau_{xz}^{k} & \tau_{yz}^{k} & \sigma_z^{(k)} & D_z^{(k)}\end{bmatrix} = [0 \quad 0 \quad 0 \quad 0] \quad (k = 1,2,3,\ldots) \text{ on } z = \pm 1; \quad (50b)$$

Along the edges, the boundary conditions are given by

$$\sigma_x^{(k)} = v^{(k)} = w^{(k)} = \phi^{(k)} = 0 \quad (k = 1,2,3,\ldots) \quad \text{at } x = 0 \text{ and } y = L_1/L. \quad (51a)$$

$$\sigma_y^{(k)} = u^{(k)} = w^{(k)} = \phi^{(k)} = 0 \quad (k = 1,2,3,\ldots) \quad \text{at } y = 0 \text{ and } y = L_2/L. \quad (51b)$$

5.4.2 Plates with closed-circuit surface conditions

In the cases of plates with closed-circuit surface conditions, the governing equations of various order problems are (Wu and Syu, 2007)

$$L_{11}u^k + L_{12}v^k + L_{13}w^k + L_{14}\phi^k = g_{1k}(1), \quad (52)$$

$$L_{21}u^k + L_{22}v^k + L_{23}w^k + L_{24}\phi^k = g_{2k}(1), \quad (53)$$

$$L_{31}u^k + L_{32}v^k + L_{33}w^k + L_{34}\phi^k = g_{3k}(1) - mg_{1k}(1) - ng_{2k}(1), \quad (54)$$

$$L_{41}u^k + L_{42}v^k + L_{43}w^k + L_{44}\phi^k = g_{4k}(1), \quad (55)$$

where L_{ij} (i, j=1–4) are the relevant differential operators; g_{ik} ($k = 0, 1, 2, \ldots$) are the revelant functions and $g_{10} = g_{20} = 0$, $g_{30} = \overline{q}_z^+ - \overline{q}_z^-$, $g_{40} = \overline{\phi}^+ - \overline{\phi}^-$.

The boundary conditions for various order problem are specified as follows:

On the lateral surface the transverse loads and electric potential are given by

$$\begin{bmatrix}\tau_{xz}^{(0)} & \tau_{yz}^{(0)} & \sigma_z^{(0)} & \phi^{(0)}\end{bmatrix} = \begin{bmatrix}0 & 0 & \overline{q}_z^{\pm} & \overline{\phi}^{\pm}\end{bmatrix} \quad \text{on } z = \pm 1; \quad (56a)$$

$$\begin{bmatrix}\tau_{xz}^{(k)} & \tau_{yz}^{(k)} & \sigma_z^{(k)} & \phi^{(k)}\end{bmatrix} = \begin{bmatrix}0 & 0 & \overline{q}_z^{\pm} & \overline{\phi}^{\pm}\end{bmatrix} \quad \text{on } z = \pm 1; \quad (56b)$$

Along the edges, the boundary conditions are given by

$$\sigma_x^{(k)} = v^{(k)} = w^{(k)} = \phi^{(k)} = 0 \quad (k = 1, 2, 3, \ldots) \quad \text{at } x = 0 \text{ and } x = L_1/L, \quad (57a)$$

$$\sigma_y^{(k)} = u^{(k)} = w^{(k)} = \phi^{(k)} = 0 \quad (k = 1, 2, 3, \ldots) \quad \text{at } y = 0 \text{ and } y = L_2/L. \quad (57b)$$

By observation of the governing equations of various order problems, we found that the differential operators among the various order problems remain identical and the nonhomogeneous terms of higher-order problems can be calculated from the lower-order solutions. Hence, it is shown that the solution process of the leading-order problem can be repeatedly applied to the higher-order problems. The asymptotic solutions can be determined order-by-order in a hierarchic and consistent manner.

Recently, the asymptotic approach has been used for the 3D static analyses of laminated piezoelectric plates (Cheng and Batra, 2000a, 2000b), of laminated piezoelectric shells (Wu, Lo and Chao, 2005; Wu and Syu, 2006; Wu, Syu and Lo, 2007). The approach was further applied to determine exact solutions for the static and cylindrical bending vibration analyses of FG piezoelectric cylindrical shells (Wu and Syu, 2007; Wu and Tsai, 2008). A partial publication list of relevant studies using the asymptotic approach is classified and tabulated in Table 5.

5.5 Supplementary remarks

5.5.1 Material properties

Without loss of the generality, the material properties can be considered as heterogeneous varying through the thickness coordinate in the aforementioned formulations. The structural behavior of single-layer homogeneous, multilayered and FG plates and shells can be studied by assuming appropriate material–property variations through the thickness coordinate. The 3D analysis of several types of FG plates and shells studied in the literature with specific material-property variations are given as follows:

Type 1–single-layer homogeneous plates and shells.

For a Type 1 pate or shell, the material properties are assumed as homogeneous, independent upon the thickness coordinate, and are given by

$$m_{ij}(z) = m_{ij}, \quad (58)$$

where $m_{ij} = c_{ij}$, e_{ij}, η_{ij}, etc.

Type 2–multilayered plates and shells.

For a Type 2 plate or shell, the material properties are assumed to be layerwise Heaviside functions

Table 5: Partial list of references on 3D analysis of single-layer homogeneous, multilayered and FG piezoelectric structures using the asymptotic approach

Problems		Literature
Static	(Plates):	Cheng, Lim and Kitipornchai, 1999, 2000; Cheng and Batra, 2000a;
		Kalamkarov and Kolpakov, 2001
	(Shells):	Cheng, Reddy, 2002; Wu, Lo and Chao, 2005; Wu and Syu, 2006;
		Wu and Syu, 2007; Wu, Syu and Lo, 2007
Thermal	(Plates):	Cheng and Batra, 2000b
Vibration	(Plates):	Cheng and Reddy, 2003
	(Shells):	Wu and Lo, 2006; Wu and Tsai, 2008

and are given by

$$m_{ij}(z) = \sum_{k=1}^{NL} m_{ij}^{(k)} \left[H(z - z_{k-1}) - H(z - z_k) \right], \quad (59)$$

where $H(z)$ is the Heaviside function; z_{k-1} and z_k denote the dimesionless thickness coordinate measured from the middle surface of the structure to the bottom and top surfaces of the k^{th}-layer.

Type 3–exponent-law varied, functionally graded plates and shells.

For a Type 3 plate or shell, the material properties are assumed to obey the identical exponent-law varied exponentially with the thickness coordinate and are given by

$$m_{ij} = m_{ij}^{(b)} e^{\kappa_p [(z+1)/2]}, \quad (60)$$

where the superscript b in the parentheses denotes the bottom surface; κ_p is the material-property gradient index which represents the degree of the material gradient along the thickness and can be determined by the values of the material properties at the top and bottom surfaces, i.e.,

$$\kappa_p = \ln \frac{m_{ij}^{(t)}}{m_{ij}^{(b)}}, \quad (61)$$

where the superscript t in the parentheses denotes the top surface. $\ln(z)$ denotes the natural logarithm function of z which is the inverse of the exponential function, e^z.

Type 4–power-law varied, functionally graded plates and shells.

For a Type 4 plate or shell, the material properties are assumed to obey the identical power-law distribution of the volume fractions of the constituents and are given by

$$m_{ij}(z) = m_{ij}^{(t)} \Gamma(z) + m_{ij}^{(b)} \left[1 - \Gamma(z) \right], \quad (62)$$

where $\Gamma(z)$ denotes the volume fraction and is defined as $\Gamma(z) = \left(\frac{z+1}{2}\right)^{k_p}$. k_p is the power-law exponent which represents the degree of the material gradient along the thickness. As k_p=0 and $k_p = \infty$, the present FG plate (or shell) reduces to a homogeneous piezoelectric plate (or shell) with material properties $m_{ij}^{(t)}$ and $m_{ij}^{(b)}$, respectively. As k_p=1, it represents that the material properties of the FG plate (or shell) linearly varied through the thickness coordinate.

The solution process for the aforementioned 3D approaches will become unfeasible (such as the Pagano's classical and state space approaches) or become inconvenient (such as the series expansion approach), as a system of differential equations with variable coefficients resulting from the initial curvature of shells or from the variations of material properties is treated. A successive approximation method, originally proposed by Soldatos and Hadjigeorgiou (1990), has been commonly used to make the aforementioned approaches feasible. In the SA method, the FG plate (or shell) is artificially divided into a *NL*-layered of small thickness and of homogeneous material properties. For a typical k^{th}-layer, the material properties $m_{ij}^{(k)}$ are determined in a thickness average sense and given as

$$m_{ij}^{(k)} = \frac{1}{h_k}\int_{z_{k-1}}^{z_k} m_{ij}(z)dz. \tag{63}$$

The SA method has been demonstrated that it can approximate the exact solutions of relevant problems of FG piezoelectric shells to any desired accuracy (Wu and Liu, 2007).

Unlike the Pagano's classical, state space and series expansion approaches existing the previously mentioned straits, the interlaminar field variables in the asymptotic formulations are derived as a definite integration through the thickness direction and can be directly determined without using the SA method. The asymptotic approach has been applied for the static and dynamic analyses of FG piezoelectric shells (Wu and Syu, 2007; Wu and Tsai, 2008). The asymptotic solutions have been demonstrated to rapidly converge and to be in excellent agreement with the relevant 3D solutions available in the literature (Zhong and Shang, 2003; Zhong and Yu, 2006).

5.5.2 *Coupled magneto-electro-elastic effects*

Recently, the state space and asymptotic approaches have also been extendedly used for the 3D coupled analysis of multilayered and FG magneto-electro-elastic plates and shells. Pan and Heyliger (2002, 2003) and Pan and Han (2005) presented the exact solution for the static and free vibration analyses of multilayered rectangular plate made of FG, anisotropic and linear magneto-electro-elastic materials using the state space approach. A Pseudo-Stroh formalism combined with the propagator matrix method for the 3D analyses was developed and applied to two FG sandwich plates made of piezoelectric $BaTiO_3$ and magnetostrictive $CoFe_2O_4$. It is shown that the coupled magneto-electro-elastic effects has remarkable influence on the static and dynamic behaviors of the FG plate. Chen et al. (2005) studied on the free vibration of non-homogeneous transversely isotropic magneto-electro-elastic plates using the state space approach. Other relevant studies using the state space approach can be found in the literature (Lee and Chen, 2003; Wang et al., 2003; Guan and He, 2006; Chen et al., 2007).

Using the asymptotic approach, Wu and his colleagues presented the 3D solutions for the static and free vibration analyses of doubly curved, FG magneto-electro-elastic shells (Wu and Tsai, 2007, 2008; Tsai et al., 2008; Tsai and Wu, 2008). The presented formulations can be reduced to those of FG magneto-electro-elastic cylindrical shells and plates by letting the corresponding curvature radius zero. Parametric studies for both the coupling magneto-electro-elastic effect and the influence of the gradient index of material properties on the structural behavior of various shells have been presented.

5.5.3 *Approximate 3D numerical modeling and methodologies*

Several approximate 3D numerical modeling and methodologies have been proposed for the analysis of multilayered and FG elastic, piezoelectric and magneto-electro-elastic plates/shells with a variety of edge boundary conditions. They are selectively reported as follows.

Ramirez and his colleagues proposed a discrete layer approach in combination with the Ritz method (or a finite element method) for the approximate 3D static and free vibration analyses of multilayered and FG elastic, piezoelectric and magneto-electro-elastic plates (Heyliger et al., 1994; Heyliger and Ramirez, 2000; Ramirez et al., 2006a, b). The discrete layer scheme has been demonstrated to be not limited to specific boundary conditions and gradation functions.

Shu et al. (2003) recently proposed a local RBF-based differential quadrature (DQ) method. In the method, the conventional DQ method is combined with the radial basis functions as the trial functions in the DQ scheme. The local RBF-based DQ method has been successfully applied to study the incompressible flows in the steady and unsteady regions (Shu et al., 2005), to solve 2D incompressible Navier-Stokes equations (Shu et al., 2003) and to solve 3D incompressible viscous flows with curved boundary (Shan et al., 2008). Liew et al. (2005) and Zhang et al. (2006) developed the 3D differential quadrature formulations for the coupled thermo-electro-elastic and piezoelectric analyses of multilayered plates, respectively. In their analyses, the approximate 3D semi-analytical formulations have been demonstrated to be in good agreement with the available 3D solutions in the literature. In addition, the approximate 3D solutions of the plates with various boundary conditions were also presented. In conjunction with the methods of perturbation and differential quadrature, Wu and Wu (2000) and Wu and Tsai (2004) presented the asymptotic DQ solutions for the free vibration analysis of laminated conical shells and for the bending analysis of functionally graded annular spherical shells, respectively.

Cheung and Jiang (2001) and Akhras and Li (2007) studied the 3D static analysis and 3D static, vibration and stability analyses of piezoelectric composite plates, respectively, using a finite layer method. In the finite layer method, the field variables are considered to be separable and expanded as the Fourier functions in the in-plane coordinates and as the polynomials in the thickness coordinate. Thus, the 3D analysis is transformed into a series of one-dimensional analysis. It is demonstrated that the approximate 3D solution is accurate and the proposed method is efficient for computation. On the basis of the Hellinger–Reissner (H–R) principle, Wu et al. (2001) proposed an asymptotic finite strip method (FSM) for the analysis of doubly curved laminated shells. In their analysis, recursive sets of element characteristic equations of first-order deformation theory have been obtained. It is demonstrated that the asymptotic FSM converges rapidly not only in mesh refinement and but also in higher-order modifications.

Sladek et al. (2006, 2007) proposed a meshless local Petrov-Galerkin method for the coupled analysis of thermo-piezoelectricity and plane piezoelectricity. Hesthaven and Warburton (2004) proposed a high-order accurate method for the time-domain solution of 3D electromagnetics. Wu et al. (2008) proposed a differential reproducing kernel particle (DRKP) method for the analysis of multilayered elastic and piezoelectric plates. In their analysis, the Euler-Lagrange equations of 3D piezoelectricity and possible boundary conditions are derived using the stationary principle of H–R energy functional of the plate. A point collocation method, based on the DRKP approximants, has been formulated for the approximate 3D analysis of multilayered piezoelectric plates under electro-mechanical loads.

6 Illustrative Examples

6.1 Multilayered hybrid piezoelectric and elastic strips

A simply-supported, two-layer laminate, composed of [PZT-4/ceramics] with equal thickness layers and with open-circuit surface conditions, has been studied using the Pagano's classical approach (Heyliger and Brooks, 1996) and the asymptotic approach (Wu, Syu and Lo, 2007). The dimensions of length and total thickness of the strip are L_2=0.1m and $2h$=0.01m. The material properties of piezoelectric and ceramics layers are given in Table 6. The cylindrical bending type of either mechanical load or electric potential (i.e., $\overline{q}_3^+(x_2) = \sin(\pi x_2/L_2)$ or $\overline{\Phi}^+(x_2) = \sin(\pi x_2/L_2)$) is applied on the top surface of the strip. The 3D solutions of elastic and electric field components at the middle surface of each layer and at interfaces between layers, obtained by the Pagano's classical approach and the asymptotic approach are quoted and given in Tables 7–8. It is shown that the 3D solutions obtained from these two different approaches are consistent.

6.2 Multilayered hybrid piezoelectric and elastic plates

The coupled analysis of simply-supported, multilayered hybrid piezoelectric and elastic plates, i.e., [PZT-4/0°/90°/90°/0°/PZT-4] laminates, with closed-circuit and open-circuit surface conditions are studied using the asymptotic approach (Wu et al., 2005) and a newly-proposed meshless DRKP method (Wu et al., 2008) for double checking. The plates are composed of a [0°/90°/90°/0°] laminated composite elastic plate bounded with the piezoelectric layers (PZT-4) on the outer surfaces of the laminate. The applied electro-mechanical loads on lateral surfaces are given as follows:

Table 6: Elastic, piezoelectric and dielectric properties of composite and piezoelectric materials

Moduli	Ceramics	PZT-4	Fiber-reinforced composite	PVDF
$c_{11}\,(Gpa)$	138.28	139.021	134.855	3.0
c_{22}	138.28	139.021	14.352	3.0
c_{33}	128.07	115.449	14.352	3.0
c_{12}	32.359	77.848	5.156	1.5
c_{13}	27.821	74.328	7.133	1.5
c_{23}	27.821	74.328	5.156	1.5
c_{44}	53.5	25.6	3.606	0.75
c_{55}	53.5	25.6	5.654	0.75
c_{66}	53.0	30.6	5.654	0.75
$e_{24}\ \left(C/m^2\right)$	2.96	12.72	0.000	0.0
e_{15}	2.96	12.72	0.000	0.0
e_{31}	0.8	-5.2	0.000	-0.15e-02
e_{32}	0.8	-5.2	0.000	0.285e-01
e_{33}	6.88	15.08	0.000	-0.15e-01
$\eta_{11}\ (F/m)$	1.7885e-09	1.306e-08	0.3099e-10	0.1062e-09
η_{22}	1.7885e-09	1.306e-08	0.2656e-10	0.1062e-09
η_{33}	1.60257e-09	1.151e-08	0.2656e-10	0.1062e-09

For the closed-circuit surface conditions,

Case 1:

$$\begin{aligned}&\overline{q}_3^+(x_1,x_2) = q_0\sin(\pi x_1/L_1)\sin(\pi x_2/L_2),\\&\overline{q}_3^-(x_1,x_2) = \overline{\Phi}^+(x_1,x_2) = \overline{\Phi}^-(x_1,x_2) = 0,\end{aligned} \tag{64}$$

Case 2:

$$\begin{aligned}&\overline{\Phi}^+(x_1,x_2) = \Phi_0\sin(\pi x_1/L_1)\sin(\pi x_2/L_2),\\&\overline{\Phi}^-(x_1,x_2) = \overline{q}_3^+(x_1,x_2) = \overline{q}_3^-(x_1,x_2) = 0;\end{aligned} \tag{65}$$

For the open-circuit surface conditions,

Case 3:

$$\begin{aligned}&\overline{q}_3^+(x_1,x_2) = q_0\sin(\pi x_1/L_1)\sin(\pi x_2/L_2),\\&\overline{q}_3^-(x_1,x_2) = \overline{D}_3^+(x_1,x_2) = \overline{D}_3^-(x_1,x_2) = 0,\end{aligned} \tag{66}$$

Case 4:

$$\begin{aligned}&\overline{D}_3^+(x_1,x_2) = D_0\sin(\pi x_1/L_1)\sin(\pi x_2/L_2),\\&\overline{D}_3^-(x_1,x_2) = \overline{q}_3^+(x_1,x_2) = \overline{q}_3^-(x_1,x_2) = 0\end{aligned} \tag{67}$$

The material properties of PZT-4 and composite layers are given in Table 6 (Heyliger, 1994). The dimensions of length and total thickness of the

Table 7: The elastic and electric field variables in a two-layer hybrid piezoelectric plate under the cylindrical bending type of mechanical load

x_3	Approaches	$u_2(0,x_3)$x10^{13}	$u_3(\frac{L}{2},x_3)$x10^{10}	$\Phi(\frac{L}{2},x_3)$x10^4	$\sigma_2(\frac{L}{2},x_3)$	$\tau_{23}(0,x_3)$	$\sigma_3(\frac{L}{2},x_3)$	$D_3(\frac{L}{2},x_3)$x10^{10}
h	Asymptotic approach (Wu et al., 2007)	-170.415	1.05613	0.00000	57.8904	0.00000	1.000000	-2.21653
	Pagano's classical approach (Heyliger and Brooks, 1996)	-170.406	1.05609	0.00000	57.8914	0.00000	1.000000	-2.21625
0.5h	Asymptotic approach (Wu et al., 2007)	-88.8861	1.06109	10.5729	30.1903	3.45473	0.850098	-2.68015
	Pagano's classical approach (Heyliger and Brooks, 1996)	-88.8804	1.06105	10.5763	30.1904	3.45477	0.850095	-2.67988
0	Asymptotic approach (Wu et al., 2007)	-9.13402	1.06295	14.0662	2.93275 (3.77196)	4.75387	0.513739	-3.73427
	Pagano's classical approach (Heyliger and Brooks, 1996)	-9.13120	1.06291	14.0706	2.93185 (3.77076)	4.75387	0.513734	-3.73402
-0.5h	Asymptotic approach (Wu et al., 2007)	72.3128	1.06267	8.14370	-30.2007	3.71709	0.163625	-3.87361
	Pagano's classical approach (Heyliger and Brooks, 1996)	72.3126	1.06263	8.14620	-30.2007	3.71705	0.163623	-3.87336
-h	Asymptotic approach (Wu et al., 2007)	154.769	1.06116	0.00000	-64.5538	0.00000	0.00000	-3.94362
	Pagano's classical approach (Heyliger and Brooks, 1996)	154.765	1.06112	0.00000	-64.5526	0.00000	0.00000	-3.94337

Table 8: The elastic and electric field variables in a two-layer hybrid piezoelectric plate under the cylindrical bending type of electric potential

x_3	Approaches	$u_2(0,x_3)$x10^{11}	$u_3(\frac{L}{2},x_3)$ x 10^{10}	$\Phi(\frac{L}{2},x_3)$	$\sigma_2(\frac{L}{2},x_3)$	$\tau_{23}(0,x_3)$	${}_3(\frac{L}{2},x_3)$x10^2	$D_3(\frac{L}{2},x_3)$x10^7
h	Asymptotic approach (Wu et al., 2007)	-17.2285	2.21653	1.00000	98.0664	0.00000	1.000000	-4.38171
	Pagano's classical approach (Heyliger and Brooks, 1996)	-17.2277	2.21625	1.00000	98.0706	0.00000	1.000000	-4.38016
0.5h	Asymptotic approach (Wu et al., 2007)	-11.6682	2.37365	0.935608	-38.9881	2.31023	-16.1157	-3.91986
	Pagano's classical approach (Heyliger and Brooks, 1996)	-11.6676	2.37336	0.935611	-38.9872	2.31044	-16.1166	-3.91847
0	Asymptotic approach (Wu et al., 2007)	-6.21174	2.49838	0.874731	-175.591 (135.720)	-6.11243	-8.20466	-3.48676
	Pagano's classical approach (Heyliger and Brooks, 1996)	-6.21137	2.49809	0.874736	-175.593 (135.710)	-6.11227	-8.20718	-3.48550
-0.5h	Asymptotic approach (Wu et al., 2007)	-3.85896	2.74246	0.436357	38.9437	0.743825	7.90414	-3.45511
	Pagano's classical approach (Heyliger and Brooks, 1996)	-3.85882	2.74217	0.436359	38.9426	0.743546	7.90257	-3.45386
-h	Asymptotic approach (Wu et al., 2007)	-1.52083	2.98145	0.00000	-58.0171	0.00000	0.00000	-3.44464
	Pagano's classical approach (Heyliger and Brooks, 1996)	-1.52091	2.98116	0.00000	-58.0090	0.00000	0.00000	-3.44340

plate are $L_1 = L_2 = L$ and $S = L/2h = 4, 10, 20$. The thickness ratio of each layer is PZT-4 layer: 0°-layer: 90°-layer: 90°-layer: 0°-layer: PZT-4 layer =0.2h: 0.4h: 0.4h: 0.4h: 0.4h: 0.2h. A set of normalized elastic and electric variables are given as follows:

For the loading conditions of Cases 1 and 3,

$$\begin{aligned}
&(\overline{u},\overline{w}) = (u_1,u_3)c^*/q_0\,(2h)\,,\\
&(\overline{\tau}_{xz},\overline{\sigma}_z) = (\tau_{13},\sigma_3)/q_0,\\
&\overline{\phi} = \Phi e^*/q_0\,(2h)\,,\\
&\overline{D}_z = D_3 c^*/(q_0 e^*);
\end{aligned} \tag{68}$$

For the loading conditions of Case 2,

$$\begin{aligned}
&(\overline{u},\overline{w}) = (u_1,u_3)c^*/(\Phi_0 e^*),\\
&(\overline{\tau}_{xz},\overline{\sigma}_z) = (\tau_{13},\sigma_3)\,(2h)\,/(\Phi_0 e^*),\\
&\overline{\phi} = \Phi/\Phi_0,\\
&\overline{D}_z = D_3 c^*\,(2h)\,/\Phi_0\,(e^*)^2\,;
\end{aligned} \tag{69}$$

For the loading conditions of Case 4,

$$\begin{aligned}
&(\overline{u},\overline{w}) = (u_1,u_3)e^*/(2hD_0),\\
&(\overline{\tau}_{xz},\overline{\sigma}_z) = (\tau_{13},\sigma_3)e^*/(D_0 c^*),\\
&\overline{\phi} = \Phi\,(e^*)^2\,/(2hD_0 c^*),\\
&\overline{D}_z = D_3/D_0;
\end{aligned} \tag{70}$$

where $c^* = 10^{10}\text{N/m}^2$, $e^* = 10\text{C/m}^2$, $q_0 = 1\text{N/m}^2$, $\phi_0 = 1\text{V}$, $D_0 = 1\text{C/m}^2$.

Figures 2–3 and 4–5 present the through-thickness distributions of various elastic and electric variables of the [PZT-4/0°/90°/90°/0°/PZT-4] laminated plates under loading conditions of Cases 1-2 and 3-4, respectively. It is shown that the transverse shear stresses produced in the plate decrease as the plates become thicker for the applied mechanical load cases (Cases 1 and 3); contrarily, they increase as the plates become thicker for the applied electric load cases (Cases 2 and 4). The maximum transverse shear stresses occur in the composite material layer for the applied mechanical load cases; they, however, occur at interfaces between elastic and piezoelectric layers for the applied electric load cases. The distributions of the elastic displacements through the thickness coordinate are merely linear functions for the applied mechanical load cases; they, however, reveal approximately layerwise linear or higher-order polynomial functions for the applied electric load cases. It is observed that the through-thickness distributions of elastic and electric variables reveal large difference between the applied mechanical load cases and the applied electric load cases.

6.3 Single-layer piezoelectric shells

The coupled cylindrical bending analysis of simply-supported, homogeneous piezoelectric cylindrical shells with closed-circuit surface conditions has been studied using the state space approach (Wu and Liu, 2007), the series expansion approach (Dumir et al., 1997) and the asymptotic approach (Wu and Syu, 2007). The configuration and coordinates (x, θ, r) of the cylindrical shell are given in Fig. 6 where $a_\theta = R_\theta \theta_\alpha$. R_θ and a_θ denote the curvature radius to the middle surface and the curvilinear dimension in circumferential direction, respectively. $2h$ and θ_α denote the total thickness and the angle between two edges of the shell, respectively. The shells are considered to be composed of polyvinyledence fluoride (PVDF) polarized along the radial direction. The elastic, piezoelectric and dielectric properties of PVDF material are given in Table 6. The cylindrical bending types of applied mechanical load and applied electric potential (i.e., $\overline{q}_3^+(\theta) = q_0 \sin(\pi\theta/\theta_\alpha)$ and $\overline{\Phi}^+(\theta) = \phi_0 \sin(\pi\theta/\theta_\alpha)$) are applied on the top surface of the shells. The dimensionless variables are denoted as the same forms of those in the Reference (Dumir et al., 1997) and given as follows:

For the cases of applied mechanical load,

$$(\overline{u}_\theta,\overline{u}_r) = \frac{100Y_r}{2hS_R^4\,|q_0|}\,(u_\theta,u_r)\,,$$

$$\begin{aligned}
&(\overline{\sigma}_x,\overline{\sigma}_\theta,\overline{\sigma}_r,\overline{\tau}_{\theta r})\\
&\qquad = \left(\sigma_x/S_R^2,\sigma_\theta/S_R^2,\sigma_r,\tau_{\theta r}/S_R\right)/\,|q_0|\,,
\end{aligned}$$

$$\left(\overline{D}_\theta,\overline{D}_r\right) = (D_\theta,D_r)\,/\,|d_1|\,S_R\,|q_0|\,,$$

$$\overline{\phi} = |d_1|\,Y_r\Phi/2hS_R^2\,|q_0|\,, \tag{71}$$

and $S_R^2 = R_\theta/2h$, $Y_r = 2.0GPa$, $d_1 = -30\text{x}10^{-12}CN^{-1}$;

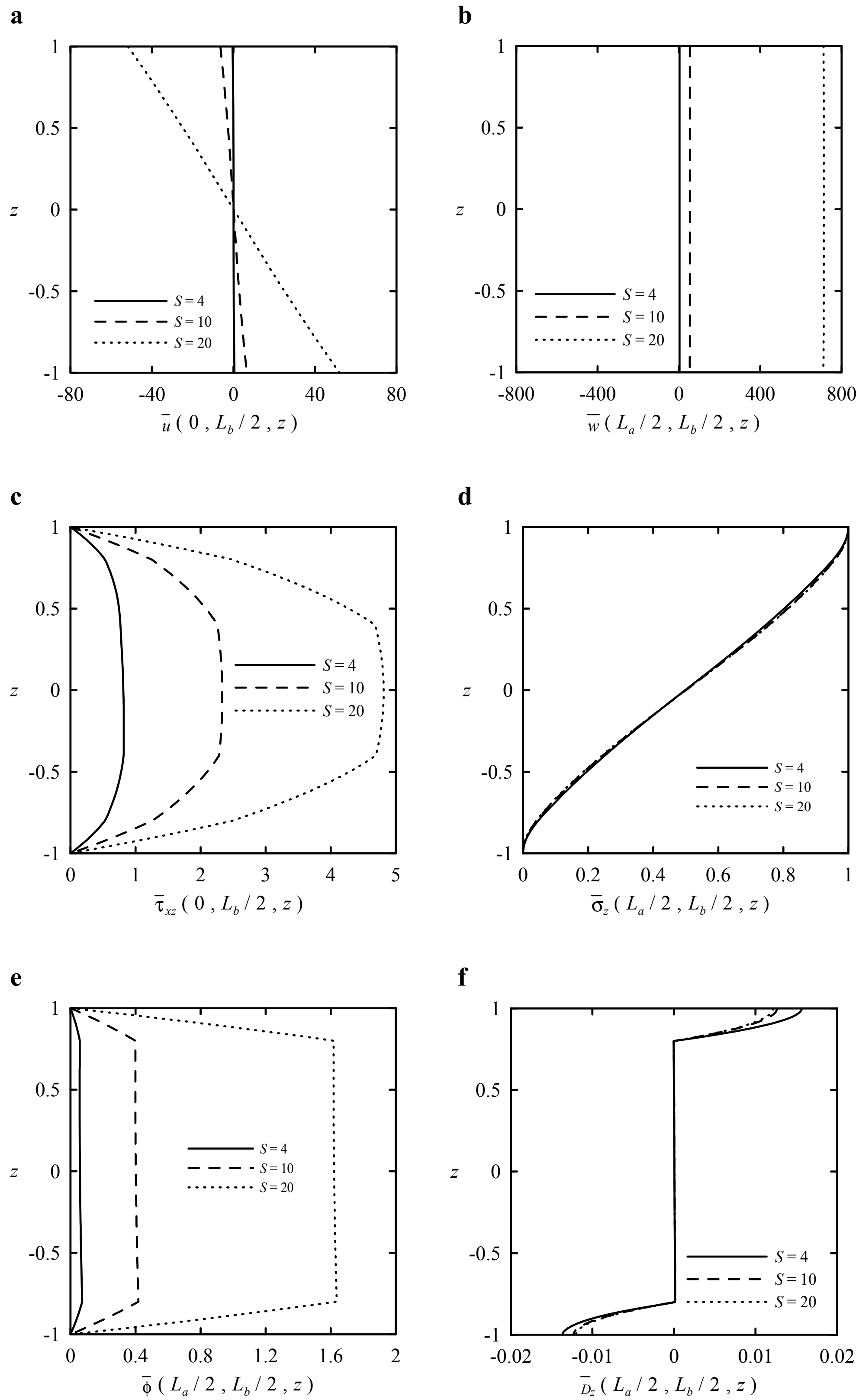


Figure 2: The through-thickness distributions of various field variables in a laminated [PZT-4/0°/90°/90°/0°/PZT-4] plate with closed-circuit surface conditions (Case 1).

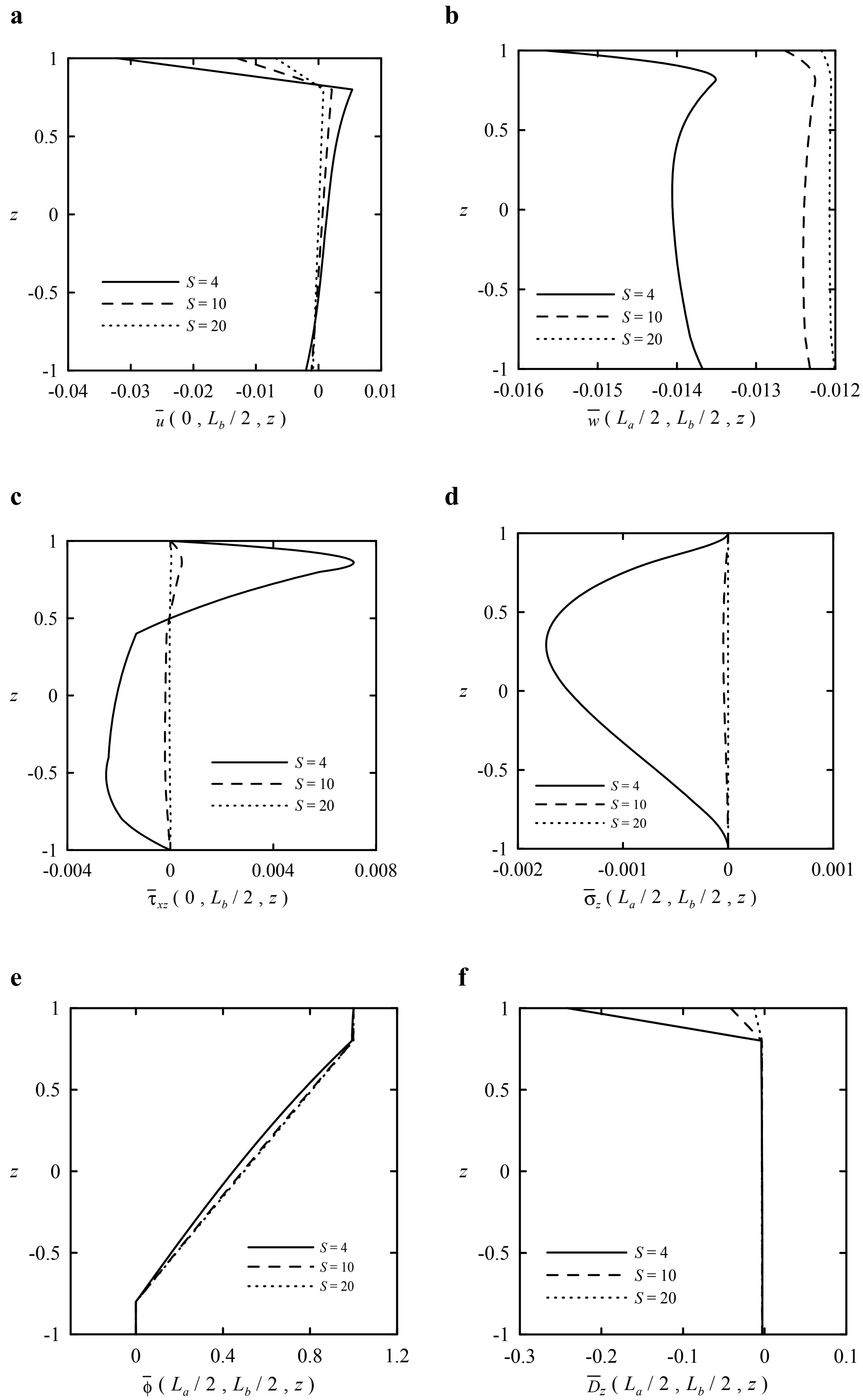


Figure 3: The through-thickness distributions of various field variables in a laminated [PZT-4/0°/90°/90°/0°/PZT-4] plate with closed-circuit surface conditions (Case 2).

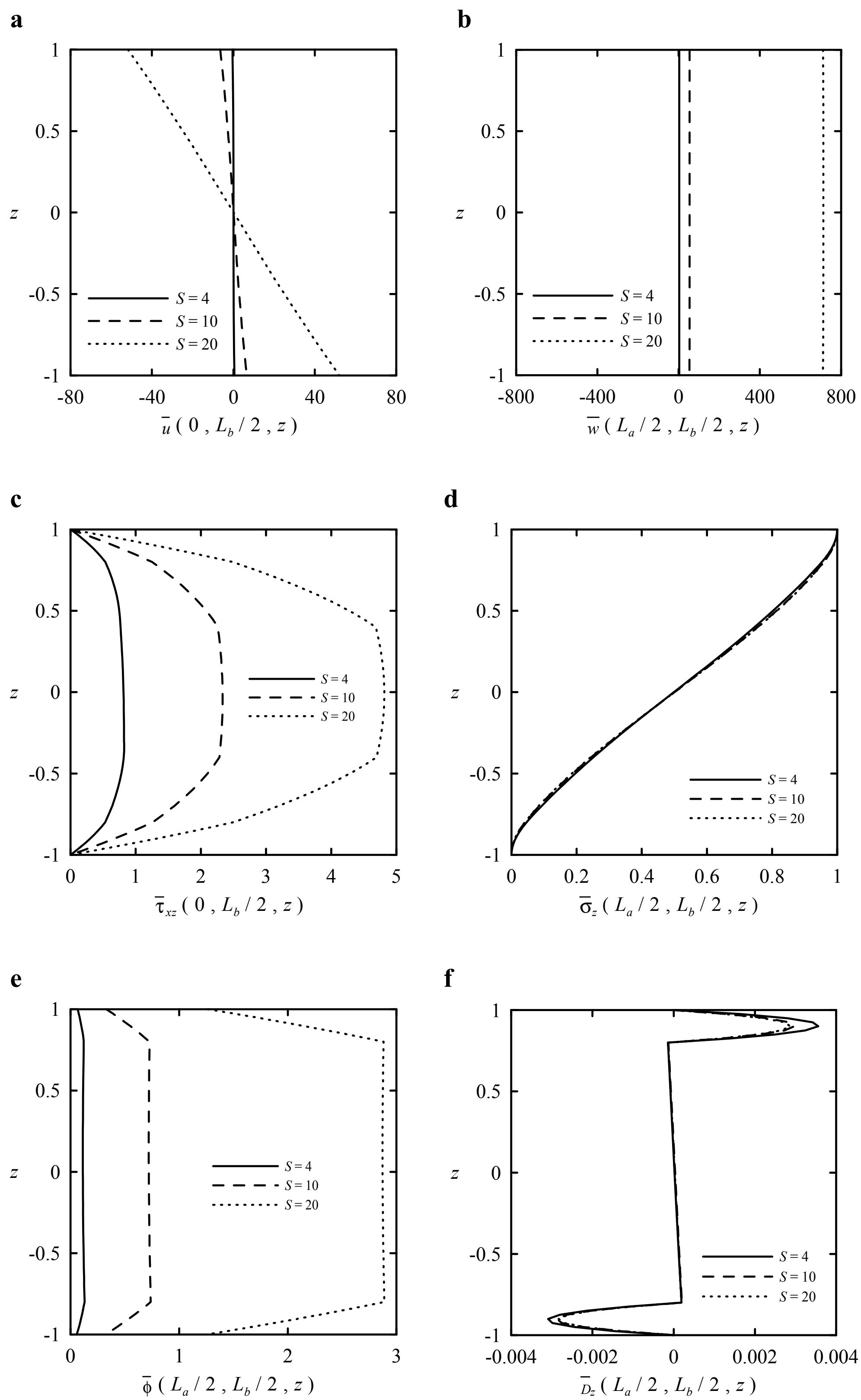


Figure 4: The through-thickness distributions of various field variables in a laminated [PZT-4/0°/90°/90°/0°/PZT-4] plate with closed-circuit surface conditions (Case 3).

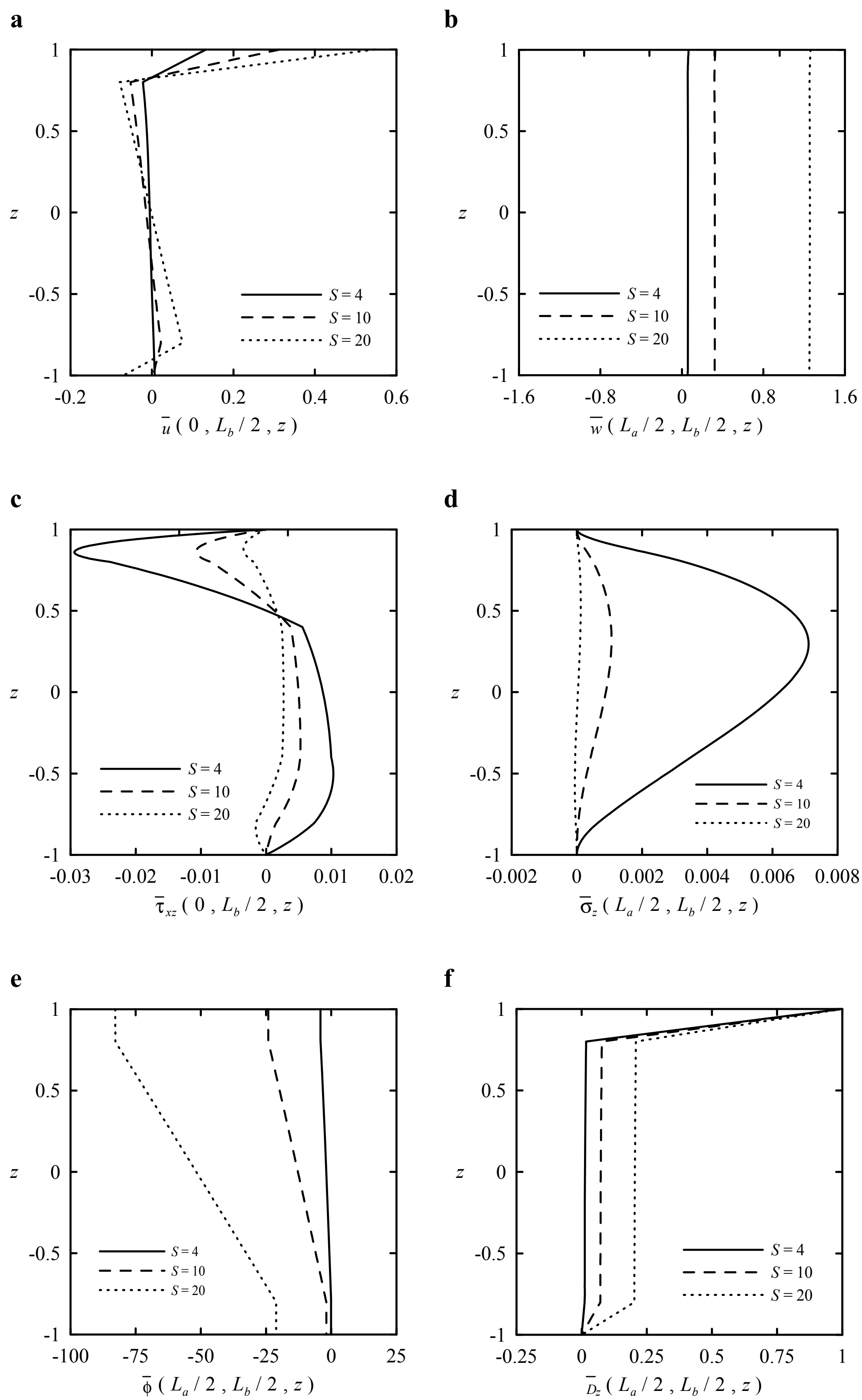


Figure 5: The through-thickness distributions of various field variables in a laminated [PZT-4/0°/90°/90°/0°/PZT-4] plate with closed-circuit surface conditions (Case 4).

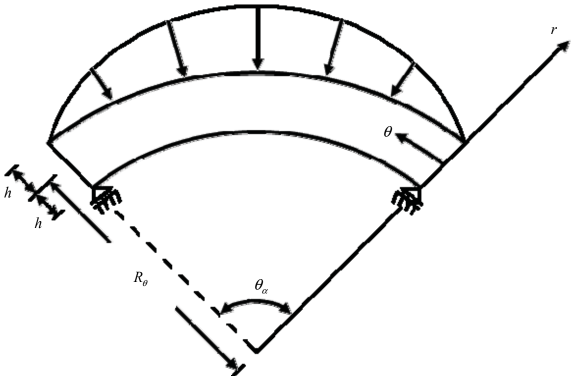


Figure 6: The geometry and coordinates of a piezoelectric strip.

For the cases of applied electric potential,

$$(\overline{u}_\theta, \overline{u}_r) = \frac{100}{|d_1| S_R |\phi_0|} (u_\theta, u_r),$$

$$\begin{aligned}
&(\overline{\sigma}_x, \overline{\sigma}_\theta, \overline{\sigma}_r, \overline{\tau}_{\theta r}) \\
&\qquad = \left(\sigma_x, S_R^2\sigma_\theta, S_R^3\sigma_r, S_R^3\tau_{\theta r}\right) 2h/Y_r |d_1| |\phi_0|, \\
&\left(\overline{D}_\theta, \overline{D}_r\right) = (S_R D_\theta, D_r) 2h/|d_1|^2 Y_r |\phi_0|, \\
&\overline{\phi} = \Phi/|\phi_0|. \qquad (72)
\end{aligned}$$

Tables 9–10 show the 3D solutions of elastic and electric field variables at crucial positions in the cylindrical shells under the applied mechanical load and the applied electric potential, respectively. The geometric parameters are taken as $R_\theta/2h$= 4, 10, 100 and $\theta_\alpha = \pi/3$. Again it is shown that the 3D solutions of the piezoelectric cylindrical shells obtained using the the state space approach, the series expansion approach and the asymptotic approach are consistent.

6.4 Functionally graded piezoelectric plates

The coupled analysis of simply-supported, FG piezoelectric plates with closed-circuit and open-circuit surface conditions are studied by the asymptotic approach (Wu and Syu, 2007) and a meshless DRKP method (Wu et al., 2008) for double checking. Firures 7–10 show the through-thickness distributions of mechanical and electric variables for the moderately shells ($R/2h$=10) under the loading conditions of Cases 1-4, respectively. The material properties are assumed to obey the identical exponent-law varied exponentially with the thickness coordinate and are given as Eq. (60). The material properties of PZT-4 are used as the reference material properties (Table 6) and placed on the bottom surface (i.e., $c_{ij}^{(b)}, e_{ij}^{(b)}, \eta_{ij}^{(b)}$). According to Eq. (61), the ratio of material properties between top surface and bottom surface is

$$\frac{c_{ij}^{(t)}}{c_{ij}^{(b)}} = \frac{e_{ij}^{(t)}}{e_{ij}^{(b)}} = \frac{\eta_{ij}^{(t)}}{\eta_{ij}^{(b)}} = e^{\kappa_p}, \qquad (73)$$

where κ_p are considered to be -3.0, -1.5, 0.0, 1.5, 3.0 so that the ratio of material properties between top surface and bottom surface is approximately from 0.05 to 20. In addition, the present results for FG piezoelectric plates with a particular value of $\kappa_p = 0$ may reduce to the results of single-layer homogeneous piezoelectric plates.

The electric and elastic field variables are normalized as the identical forms of Eqs. (68)–(70). The influence of material-property gradient index on the mechanical and electric variables is studied. Figures 7(a)-(d) and 9(a)-(d) show that the through-thickness distributions of mechanical variables in the FG plates with closed-circuit and open-circuit surface conditions are almost identical as the mechanical load is applied. Figures 8(c)–(d) and 10(c)–(d) show that the through-thickness distributions of transverse stresses change dramatically as the index κ_p becomes a positive value for the loading conditions of Case 2; conversely, they change dramatically as the index κ_p becomes a negative value for the loading condition of Case 4. The distributions of transverse stresses across the thickness coordinate in the loading conditions of Cases 2 and 4 are higher-degree polynomials and are back and forth among the positive and negative values. It is also shown from Figs. 7(e)–(f) that the distributions of normal electric displacement through the thickness coordinate are approximately linear functions and parabolic functions for electric potential in the loading condition of Case 1. The through-thickness distributions of electric potential and normal electric displacement in Figs. 8(e)–(f) and Figs 10(e)–(f) are shown to be different patterns between homogeneous piezoelectric

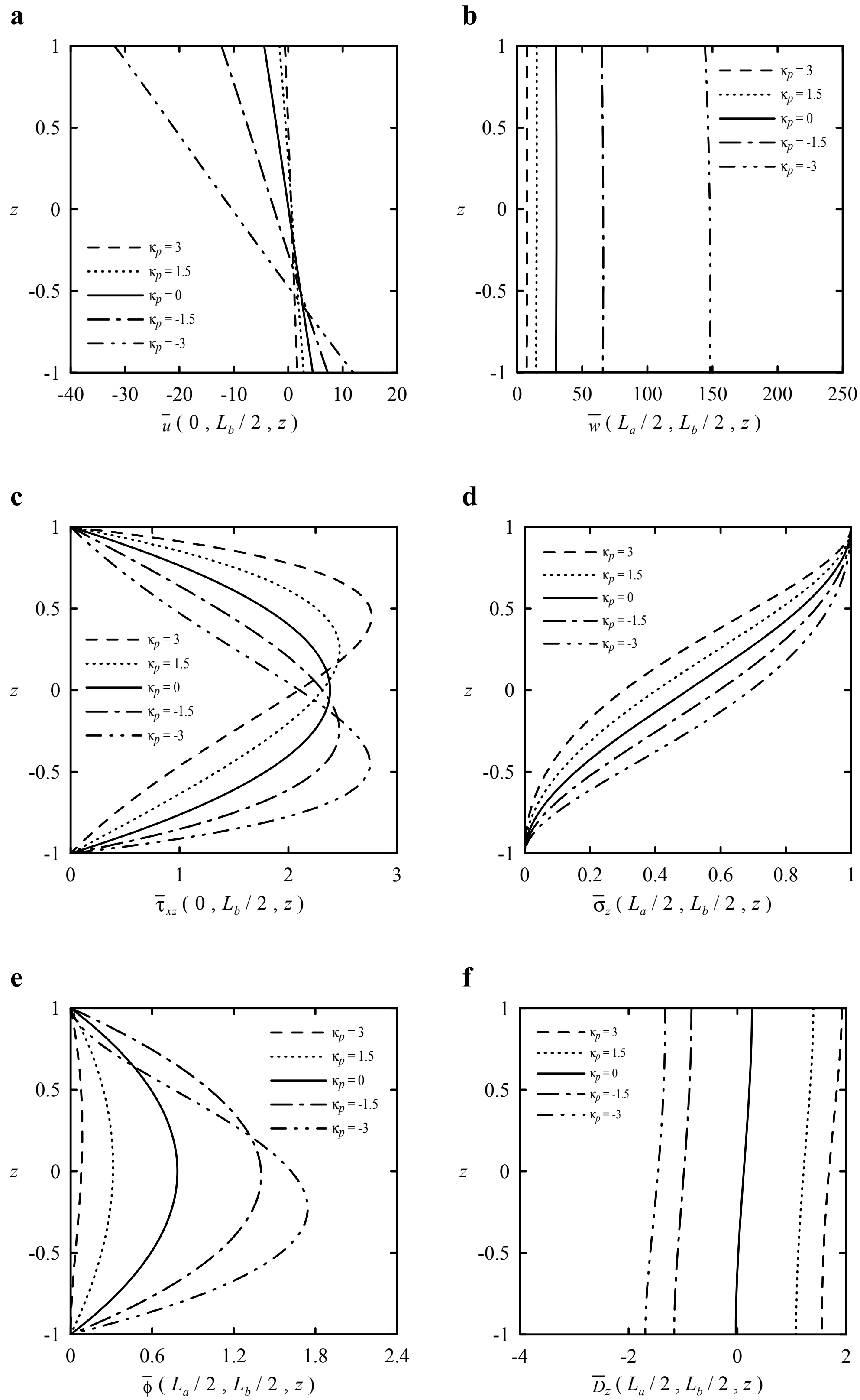


Figure 7: The through-thickness distributions of various field variables in a functionally graded piezoelectric plate with closed-circuit surface conditions (Case 1).

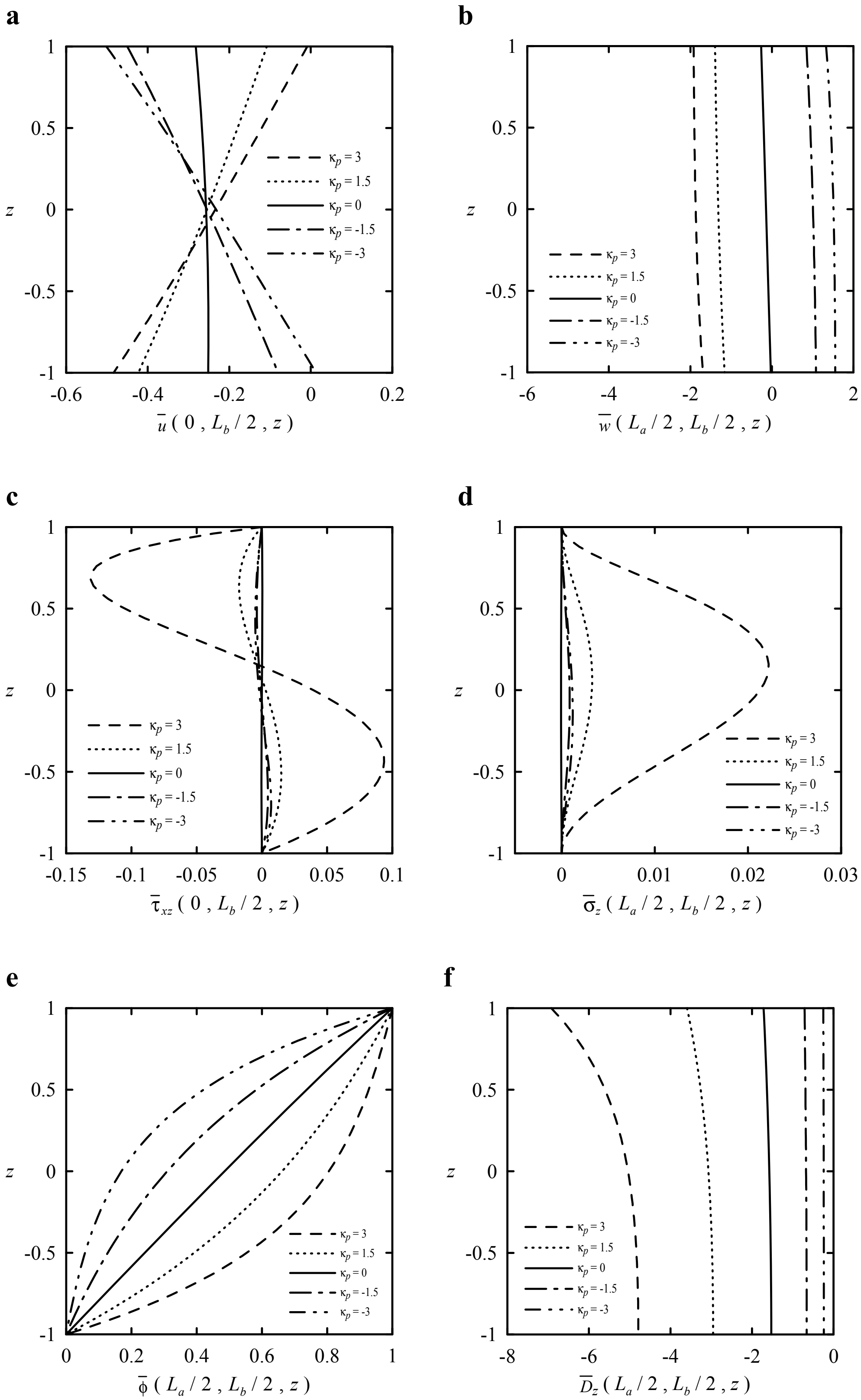


Figure 8: The through-thickness distributions of various field variables in a functionally graded piezoelectric plate with closed-circuit surface conditions (Case 2).

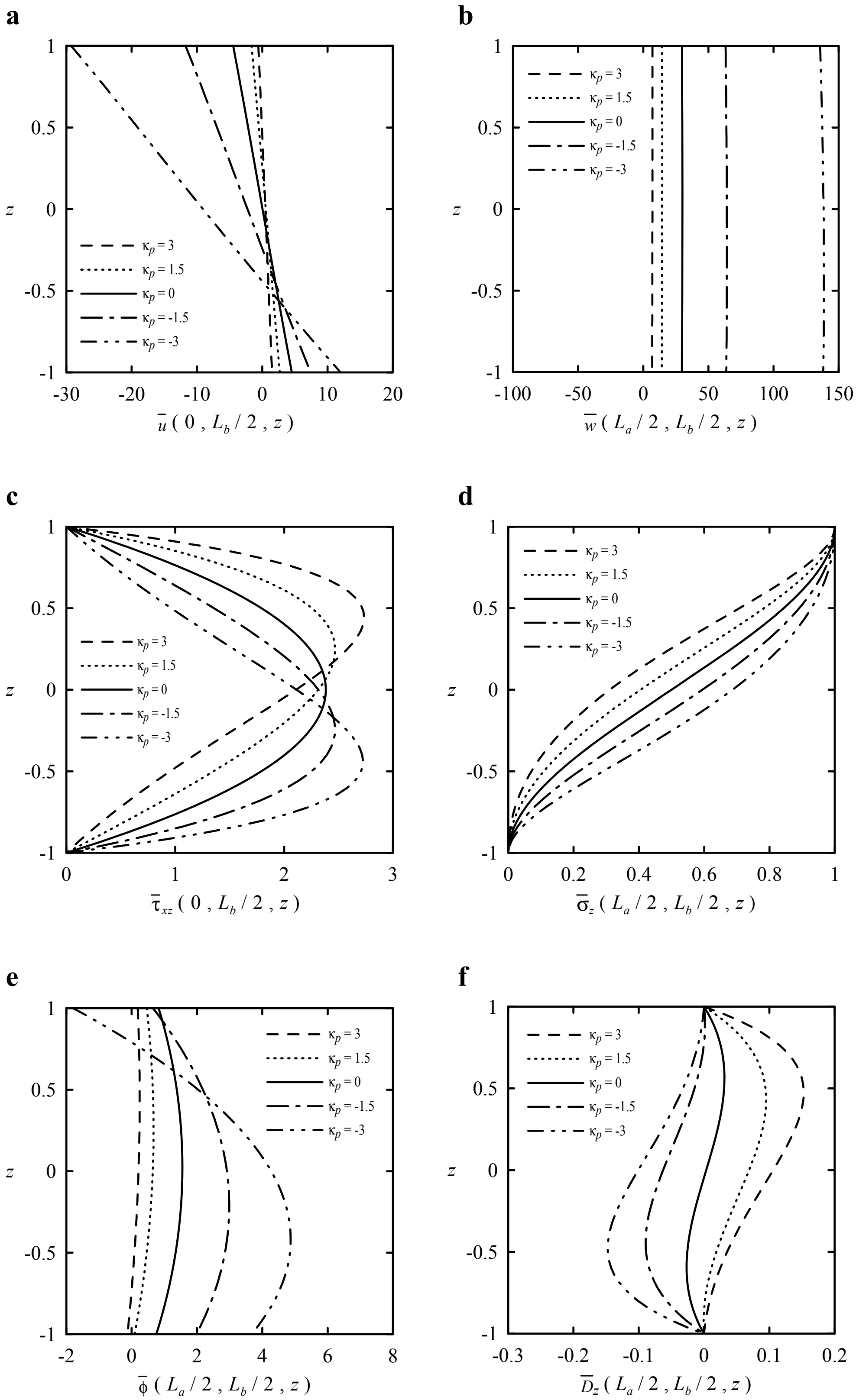


Figure 9: The through-thickness distributions of various field variables in a functionally graded piezoelectric plate with closed-circuit surface conditions (Case 3).

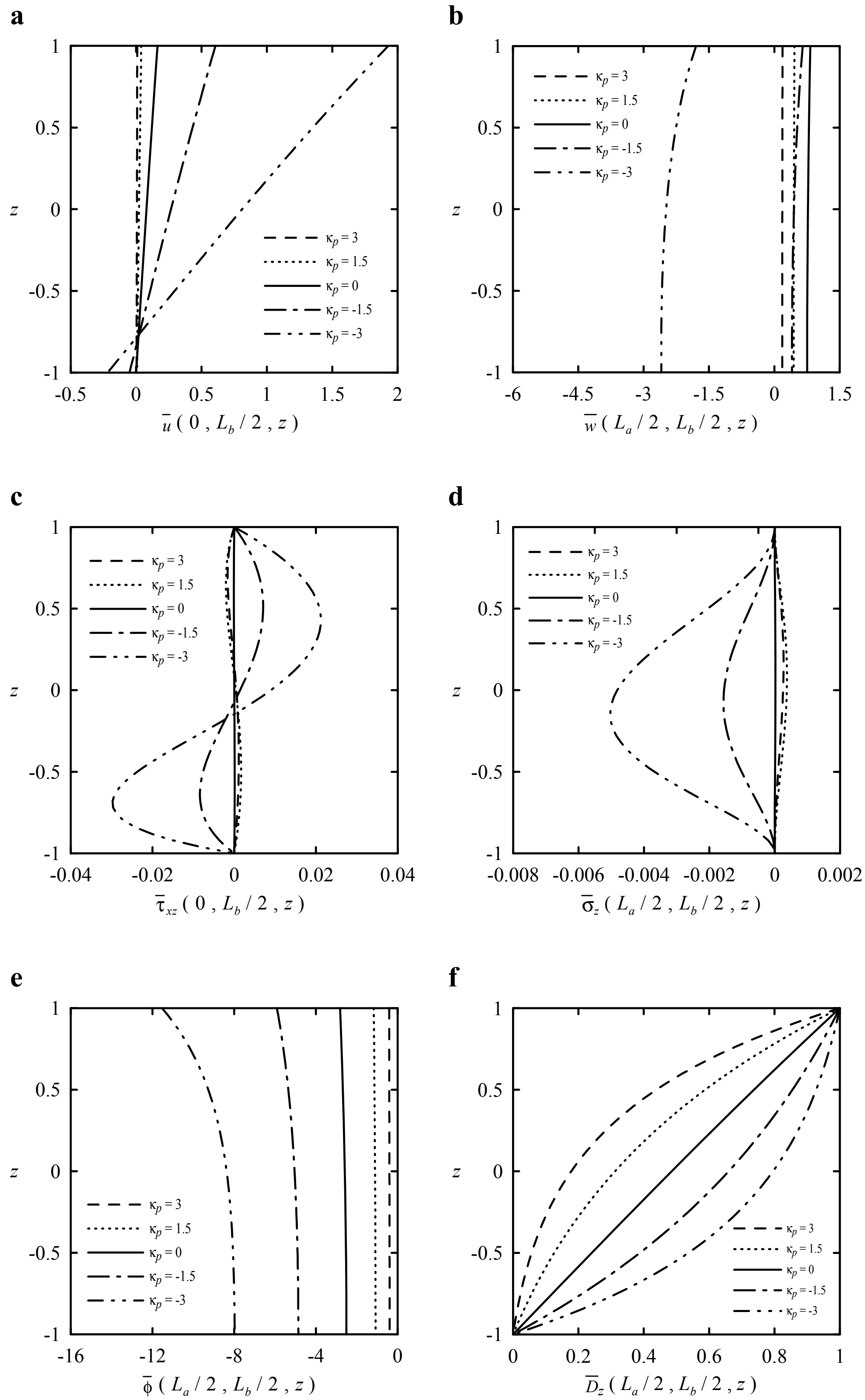


Figure 10: The through-thickness distributions of various field variables in a functionally graded piezoelectric plate with closed-circuit surface conditions (Case 4).

Table 9: Mechanical and electric components at the crucial positions in single-layer piezoelectric cylindrical shells under the cylindrical bending type of mechanical load

$R_\theta/2h$	Approaches	$\bar{u}_\theta(0,+h)$	$\bar{u}_r\left(\frac{\theta_\alpha}{2},0\right)$	$\bar{\sigma}_x\left(\frac{\theta_\alpha}{2},+h\right)$	$\bar{\sigma}_\theta\left(\frac{\theta_\alpha}{2},+h\right)$	$\bar{\sigma}_r\left(\frac{\theta_\alpha}{2},0\right)$	$\bar{\tau}_{\theta r}(0,0)$	$10^3\bar{\phi}\left(\frac{\theta_\alpha}{2},0\right)$	$10\bar{D}_r\left(\frac{\theta_\alpha}{2},+h\right)$
4	The state space approach (Wu and Liu, 2007)	-0.8806	-21.1031	-0.2747	-0.7279	0.2170	-0.6238	2.4428	0.0214
	The series expansion approach (Dumir et al., 1997)	-0.8806	-21.10	-0.275	-0.7597	0.217	-0.6238	2.443	0.0213
	The asymptotic approach (Wu and Syu, 2007)	-0.8806	-21.1029	-0.2750	-0.7597	0.2170	-0.6238	2.4428	0.0213
10	The state space approach (Wu and Liu, 2007)	-3.5723	-17.6845	-0.2546	-0.7460	1.4195	-0.5893	2.5600	-0.1574
	The series expansion approach (Dumir et al., 1997)	-3.572	-17.68	-0.2548	-0.7514	1.420	-0.5893	2.560	-0.1575
	The asymptotic approach (Wu and Syu, 2007)	-3.5723	-17.6844	-0.2548	-0.7514	1.4195	-0.5893	2.5600	-0.1575
100	The state space approach (Wu and Liu, 2007)	-5.2969	-16.5535	-0.2510	-0.7499	18.3389	-0.5653	2.5035	-0.2040
	The series expansion approach (Dumir et al., 1997)	-5.297	-16.55	-0.2510	-0.7500	18.34	-0.5653	2.504	-0.2041
	The asymptotic approach (Wu and Syu, 2007)	-5.2969	-16.5535	-0.2510	-0.7500	18.3389	-0.5653	2.5035	-0.2041

plates (κ_p=0) and FG piezoelectric plates in the cases of applied electric loads (Cases 2 and 4). By observation through Figs. 7–10, we found that the distributions of mechanical and electric variables through the thickness coordinate in FG piezoelectric plates reveal different patterns from homogeneous piezoelectric plates. Hence, it is suggested that an advanced 2D theory may be necessary to be developed for the analysis of FG piezoelectric plates, especially when the plates are subjected to electric loads.

7 Conclusions

Various 3D coupled analyses of multilayered and functionally graded piezoelectric plates/shells in the literature are surveyed. The theoretical methodologies for those analyses are classified as four different approaches and are briefly interpreted. For comparison purposes, several available 3D results obtained form these four approaches are quoted and presented. It is pleased to see that the 3D solutions obtained from different approaches are demonstrated to be consistent. The existing 3D solutions may serve as a standard for assessing a variety of 2D theories of piezoelectric plates. The through-thickness distributions of various field variables in the multilayered and FG piezoelectric plates under electromechanical loads are presented using an asymptotic approach and doubly checked using a meshless DRKP method. These distributions may serve as a reference for making the appropriate kinetic or kinematics assumptions in advance, as an advanced 2D theory is to be developed. Totally there are 191 references included in the present review article. The authors hope the present review article may provide a comprehensive aspect for the theoretical methodologies of 3D analysis of plates/shells in the existing literature. However, this survey is inevitable to miss some scientists who might make significant contributions on the present subject since the authors' academic information might be lacking. The authors hope to express their apologies at not being able to include all the relevant publishing papers on the present

Table 10: Mechanical and electric components at the crucial positions in single-layer piezoelectric cylindrical shells under the cylindrical bending type of electric potential

$R_\theta/2h$	Approaches	$\bar{u}_\theta(0,+h)$	$\bar{u}_r\left(\frac{\theta_\alpha}{2},0\right)$	$\bar{\sigma}_x\left(\frac{\theta_\alpha}{2},+h\right)$	$\bar{\sigma}_x\left(\frac{\theta_\alpha}{2},-h\right)$	$\bar{\sigma}_\theta\left(\frac{\theta_\alpha}{2},-h\right)$	$\bar{\tau}_{\theta r}\left(0,-\frac{h}{2}\right)$	$\bar{\phi}\left(\frac{\theta_\alpha}{2},0\right)$	$\bar{D}_r\left(\frac{\theta_\alpha}{2},+h\right)$
4	The state space approach (Wu and Liu, 2007)	-28.7436	11.8943	-0.1311	-0.1363	-1.5552	0.3924	-0.4978	62.7680
	The series expansion approach (Dumir et al., 1997)	-28.74	11.90	-0.1307	-0.1359	-1.536	0.3925	-0.4978	62.77
	The asymptotic approach (Wu and Syu, 2007)	-28.7436	11.8952	-0.1307	-0.1359	-1.5361	0.3925	-0.4978	62.7679
10	The state space approach (Wu and Liu, 2007)	-26.4865	6.3387	-0.1026	-0.1087	-1.5189	0.3982	-0.5070	58.9644
	The series expansion approach (Dumir et al., 1997)	-26.49	6.340	-0.1024	-0.1086	-1.410	0.3984	-0.5070	58.96
	The asymptotic approach (Wu and Syu, 2007)	-26.4863	6.3397	-0.1024	-0.1086	-1.4705	0.3984	-0.5070	58.9643
100	The state space approach (Wu and Liu, 2007)	-25.9902	2.5221	-0.0996	-0.1005	-1.8661	0.3928	-0.5012	59.9193
	The series expansion approach (Dumir et al., 1997)	-25.99	2.523	-0.0996	-0.1005	-1.411	0.3953	-0.5012	59.92
	The asymptotic approach (Wu and Syu, 2007)	-25.9899	2.5231	-0.0996	-0.1005	-1.4108	0.3953	-0.5012	59.9193

subject.

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Appendix A

The relevant coefficients of l_{ij} in Eqs. (21)-(24) are given by

$$l_{11} = -\tilde{m}\tilde{Q}_{11}, \quad l_{12} = -\tilde{n}\tilde{Q}_{12}, \quad l_{13} = a_1 h^2/L^2,$$

$$l_{14} = b_1 e/Q, \quad l_{21} = -\tilde{m}\tilde{Q}_{21}, \quad l_{22} = -\tilde{n}\tilde{Q}_{22},$$

$$l_{23} = a_2 h^2/L^2, \quad l_{24} = b_2 e/Q, \quad l_{31} = \tilde{n}\tilde{Q}_{66},$$

$$l_{31} = \tilde{m}\tilde{Q}_{66}, \quad l_{41} = e_{15} Q h^2/(e c_{55} L^2),$$

$$l_{43} = -\tilde{m}(c_{55}^{-1} e_{15}^2 + \eta_{11}) Q h^2/(e^2 L^2),$$

$$l_{52} = e_{24} Q h^2/(e c_{44} L^2),$$

$$l_{53} = -\tilde{n}(c_{44}^{-1} e_{24}^2 + \eta_{22}) Q h^2/(e^2 L^2).$$

Appendix B

The relevant coefficients of $\bar{l}_{ij}$ in Eqs. (26)-(29) are given by

$$\bar{l}_{11} = -\tilde{m}\tilde{c}_{11}, \quad \bar{l}_{12} = -\tilde{n}\tilde{c}_{12}, \quad \bar{l}_{13} = \tilde{c}_{13} L^2/h^2,$$

$$\bar{l}_{14} = e_{31}/e, \quad \bar{l}_{21} = -\tilde{m}\tilde{c}_{12}, \quad \bar{l}_{22} = -\tilde{n}\tilde{c}_{22},$$

$$\bar{l}_{23} = \tilde{c}_{23} L^2/h^2, \quad \bar{l}_{24} = e_{32}/e, \quad \bar{l}_{31} = \tilde{n}\tilde{c}_{66},$$

$$\bar{l}_{32} = \tilde{m}\tilde{c}_{66}, \quad \bar{l}_{41} = \tilde{c}_{55} L^2/h^2, \quad \bar{l}_{43} = \tilde{m}\tilde{c}_{55} L^2/h^2,$$

$$\bar{l}_{44} = \tilde{m} e_{15}/e, \quad \bar{l}_{52} = \tilde{c}_{44} L^2/h^2, \quad \bar{l}_{53} = \tilde{n}\tilde{c}_{44} L^2/h^2,$$

$$\bar{l}_{54} = \tilde{n} e_{24}/e, \quad \bar{l}_{61} = e_{15} h/(eL),$$

$$\bar{l}_{63} = \tilde{m} e_{15} h/(eL), \quad \bar{l}_{64} = -\tilde{m}\eta_{11} h^3 Q/(e^2 L^3),$$

$$\bar{l}_{72} = e_{24} h/(eL), \quad \bar{l}_{73} = \tilde{n} e_{24} h/(eL),$$

$$\bar{l}_{74} = -\tilde{n}\eta_{11} h^3 Q/(e^2 L^3),$$

$$\bar{l}_{81} = -\tilde{m}\tilde{c}_{13} L^2/h^2, \quad \bar{l}_{82} = -\tilde{n}\tilde{c}_{23} L^2/h^2,$$

$$\bar{l}_{83} = \tilde{c}_{33} L^4/h^4, \quad \bar{l}_{84} = e_{33} L^2/(e h^2),$$

$$\bar{l}_{91} = -\tilde{m} e_{31}/e, \quad \bar{l}_{92} = -\tilde{n} e_{32}/e,$$

$$\bar{l}_{93} = e_{33} L^2/(e h^2), \quad \bar{l}_{94} = -\eta_{33} Q/e^2,$$

$$\tilde{c}_{ij} = c_{ij}/Q.$$