

A Fictitious Time Integration Method for Solving Delay Ordinary Differential Equations

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Abstract: A new numerical method is proposed for solving the delay ordinary differential equations (DODEs) under multiple time-varying delays or state-dependent delays. The finite difference scheme is used to approximate the ODEs, which together with the initial conditions constitute a system of nonlinear algebraic equations (NAEs). Then, a Fictitious Time Integration Method (FTIM) is used to solve these NAEs. Numerical examples confirm that the present approach is highly accurate and efficient with a fast convergence.

Keywords: Delay ordinary differential equations, Multiple time-varying delays, State-dependent delays, Fictitious Time Integration Method (FTIM)

1 Introduction

The time delay is frequently encountered in various electronic implementation of neural networks, such as, Hopfield neural networks, cellular neural networks, and bi-directional associative memory networks. The existence of time-delay is a source of oscillation and instability of neural networks. Therefore, the research of the dynamical characteristics of neural networks with time delays is an important topic in the neural networks theory. Considerable efforts have been devoted to the analysis of the stability in signal and image processing, artificial intelligence, industrial automation, and other fields [Baldi and Atiya (1994); Cao (2000); Gopalsamy and He (1994); Xu et al. (2005); Liao and Wang (2000)].

The most works on delay ordinary differential equations (DODEs) have dealt with the stability analysis problem. In this paper we propose a new method for the

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numerical solution of the following multiple time-varying delays ODEs:

$$\dot{x}_i(t) = F_i(t, x_1(t), \dots, x_m(t), x_1(t - \tau_{i1}), \dots, x_m(t - \tau_{im})), \tau_{ik} \geq 0, i, k = 1, \dots, m, \quad (1)$$

$$x_i(t) = \phi_i(t), \quad -\tau \leq t \leq 0, \quad (2)$$

where $\tau = \max\{\tau_{ij}(t), t \geq 0\}$, and $\phi_i(t)$ are called the initial values of $x_i(t)$. The initial values are specified in a time-span $[-\tau, 0]$, rather than that at a single initial time $t = 0$.

Complicated situations in which the delay depends on the unknown functions have been proposed in the mathematical modelings of many different fields in recent years. These equations are usually called equations with state-dependent delay. Many works related to this topic have been published, such as in classical electrodynamics [Driver (1963)], in population models [Bélair (1991)], in models of commodity price fluctuations [Mackey (1989)], in models of blood cell productions [Mallet-Paret and Nussbaum (1989)], and in models of boundary layers [Mallet-Paret and Nussbaum (1992)]. Differential equations with state-dependent delay have also been the subject of several mathematical works. Alt (1979) proved the existence and periodicity for some state-dependent delay differential equation. Arino et al. (1998) have proven also the existence of oscillatory and periodic solutions for some state-dependent delay differential equations arising from population dynamics. Bélair (1991) has proven the stability of some state-dependent models arising from epidemic problems. Ait Dads and Ezzinbi (2002) have studied the existence and uniqueness of bounded solutions for state-dependent DODEs.

Apart from the above time-dependent DODEs, in this paper we also provide a numerical solution of the following state-dependent DODEs:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{F}(t, \mathbf{x}(t), \mathbf{x}(t - \rho(\mathbf{x}_t))), \quad t \geq 0, \\ \mathbf{x}_0 &= \varphi, \end{aligned} \quad (3)$$

where φ is a given function in the space of continuous functions from $[-\tau, 0]$ to \mathbb{R}^m . This space is denoted by $C = C([-\tau, 0]; \mathbb{R}^m)$. For every $t \geq 0$, the history function $\mathbf{x}_t \in C$ is defined by

$$\mathbf{x}_t(\theta) = \mathbf{x}(t + \theta), \quad \theta \in [-\tau, 0]. \quad (4)$$

The function \mathbf{F} is continuous from $\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$ to \mathbb{R}^m , ρ is a positive bounded continuous function on C , and τ is the maximal delay defined by

$$\tau = \sup_{\varphi \in C} \rho(\varphi). \quad (5)$$

The standard argument for uniqueness [Mallet-Paret et al. (1994)] cannot be applied to the following example:

$$\begin{aligned} \dot{x}(t) &= x(t+1-x(t)), \quad t \in [0, 1], \\ x(t) &= \sqrt{|t|} + 1, \quad t \in [-1, 0]. \end{aligned} \quad (6)$$

Eq. (6) has two solutions, namely,

$$x_1(t) = 1 + t + \frac{t^2}{4}, \quad x_2(t) = 1 + t, \quad t \in [0, 1]. \quad (7)$$

Such a situation of non-uniqueness may complicate the calculations of state-dependent DODEs.

The Runge-Kutta codes for delay ODEs have been written by Ooppelstrup (1976), Oberle and Pesch (1981), and Bellen and Zennaro (1985). However, these methods are complex, and the extensions to time-varying delay and state-dependent delay are not straightforward.

This paper is arranged as follows. In Section 2 we transform the above DODEs into the nonlinear algebraic equations (NAEs) by using the finite difference approximations, wherein we explain a mathematical basis of a fictitious time integration method (FTIM) for solving NAEs. In Section 3 we use some numerical examples to demonstrate the efficiency of the new method of FTIM. Then, we draw conclusions in Section 4.

2 A fictitious time integration method

When solving DODEs, one of the basic requirements is the storage of sufficient back information, so that the method can evaluate the delay term when it is required at some point $t < T$, where T is a final time. The amount of information to be stored at each time step depends on the method for approximating the delay term, but the interval on which the information is to be stored and the quantities to be stored on that interval should be flexible and adaptable for each problem, depending on the nature of the delay term and accuracy required. If the delay term falls at some point $t < 0$, then the initial conditions must be used. The delay argument may fall in the current step because it is smaller than the stepsize or may even vanish, we call this type of delay a small delay or when the delay vanishes we call it vanishing delay. These types of delays are handled by either restricting the stepsize to be smaller than the delays or using an extrapolation technique.

2.1 Finite difference equations

We divide the time interval of $[0, T]$ into $m_2 - 1$ subintervals by using a constant time-step length $\Delta t = T/(m_2 - 1)$. At a temporal grid point $t_j = (j - 1)\Delta t$, x_i^j is used to approximate the true value of $x_i(t_j)$.

Let m_1 be an integer large enough, such that $\tau_0 = -m_1\Delta t \leq -\tau$. In the time interval of $[\tau_0, T]$ we have collocated totally $m_1 + m_2$ grid points. For each time delay $\tau_{ik}(t_j)$, we can check the location of $t_j - \tau_{ik}(t_j)$. If $t_\ell \leq t_j - \tau_{ik}(t_j) \leq t_{\ell+1}$ for an ℓ , where $t_\ell = \tau_0 + (\ell - 1)\Delta t$, then we employ a linear interpolation to approximate $x_i(t_j - \tau_{ik}(t_j))$ by

$$\bar{x}_i^j = x_i^\ell + \frac{t_j - \tau_{ik}(t_j) - t_\ell}{\Delta t} (x_i^{\ell+1} - x_i^\ell). \quad (8)$$

Therefore, from Eqs. (1), (2) and (8) by using a finite difference scheme we can derive the following equations:

$$\frac{x_i^j - x_i^{j-1}}{\Delta t} - F_i(t_j, x_1^j, \dots, x_m^j, \bar{x}_1^j, \dots, \bar{x}_m^j) = 0, \quad 2 \leq j \leq m_2, \quad (9)$$

$$x_i^j = \phi_i(t_j), \quad -\tau \leq t_j \leq 0. \quad (10)$$

2.2 Transformation into an ODEs system

Eq. (9) constitutes a system of $n = m \times (m_2 - 1)$ nonlinear algebraic equations (NAEs), which can be used to solve the n unknowns of x_i^j , $i = 1, \dots, m$, $j = 2, \dots, m_2$.

In order to apply our new method to solve the system of NAEs, let us demonstrate it by using a single NAE:

$$F(x) = 0, \quad (11)$$

where we only have an independent variable x . We transform it into a first-order ODE by introducing a fictitious time-like variable ξ into the following transformation of variables from x to y :

$$y(\xi) = (1 + \xi)^\gamma x. \quad (12)$$

Here, γ is a positive constant, and ξ is a variable which is independent of x ; hence, $y' = dy/d\xi = \gamma(1 + \xi)^{\gamma-1}x$. If $v \neq 0$, Eq. (11) is equivalent to

$$0 = -vF(x). \quad (13)$$

Adding $y' = \gamma(1 + \xi)^{\gamma-1}x$ to Eq. (13) we obtain

$$y' = \gamma(1 + \xi)^{\gamma-1}x - \nu F(x). \tag{14}$$

By using Eq. (12) we can derive

$$y' = \frac{\gamma y}{1 + \xi} - \nu F\left(\frac{y}{(1 + \xi)^\gamma}\right). \tag{15}$$

Multiplying Eq. (15) by an integrating factor of $1/(1 + \xi)^\gamma$ we can obtain

$$\frac{d}{d\xi} \left(\frac{y}{(1 + \xi)^\gamma} \right) = -\frac{\nu}{(1 + \xi)^\gamma} F\left(\frac{y}{(1 + \xi)^\gamma}\right). \tag{16}$$

Further using $y/(1 + \xi)^\gamma = x$, leads to

$$x' = -\frac{\nu}{(1 + \xi)^\gamma} F(x). \tag{17}$$

Therefore, we have transformed the algebraic Eq. (11) into a first-order nonautonomous ODE. Under certain condition we expect that the solution of Eq. (17) starting from an initial guess of $x(0)$ can approximate the true solution x of Eq. (11).

The above idea was first proposed by Liu (2008a) to treat an inverse Sturm-Liouville problem by transforming an ODE into a PDE. Then, Liu (2008b, 2008c, 2008d), and Liu, Chang, Chang and Chen (2008) extended this idea to develop new methods for estimating parameters in the inverse vibration problems. Liu and Atluri (2008a) have employed the technique of FTIM to solve a large system of nonlinear algebraic equations, and showed that high performance can be achieved by using the FTIM. More recently, Liu (2008e) has used the FTIM technique to solve the nonlinear complementarity problems originated from the obstacle problems of elliptic type PDE. Numerical results appeared at there are very well. Then, Liu (2008f) used the FTIM to solve the boundary value problems of elliptic type partial differential equations. Liu and Atluri (2008b) also employed this technique of FTIM to solve mixed-complementarity problems and optimization problems. Then, Liu and Atluri (2008c) using the technique of FTIM solved the inverse Sturm-Liouville problem, for specified eigenvalues. Recently, Liu and Atluri (2009) used the FTIM-based method to study the filtering effect of FTIM by using the different time-like function $q(t)$ to solve the ill-posed linear algebraic equations system. They showed that when $q(t) = 1/(1 + t)^\gamma$, $0 < \gamma \leq 1$ is used, the filtering effect of the FTIM is better than that of the Tikhonov filter.

Now, applying Eq. (17) to Eq. (9) we can obtain

$$\frac{dx_i^j}{d\xi} = -\frac{\nu}{(1 + \xi)^\gamma} \left[\frac{x_i^j - x_i^{j-1}}{\Delta t} - F_i(t_j, x_1^j, \dots, x_m^j, \bar{x}_1^j, \dots, \bar{x}_m^j) \right]. \tag{18}$$

The same idea can be used to solve Eq. (3). In Section 3.6 we use a definite example to write the FTIM for the state-dependent DODE.

Liu (2009) has employed the FTIM to solve m -point boundary value problems of ODEs, and a higher-dimensional first-order ODEs system obtained from the wave equation of Euler-Bernoulli beam by subjecting to a three-point boundary value. Because in the FTIM one does not need to inverse the resulting nonlinear algebraic equations, this method is very effective to find the numerical solutions of m -point boundary value problems of nonlinear ODEs. Similarly, the same merit of the FTIM can be employed here to find the numerical solutions of delayed ODEs, no matter they are time-varying delays or state-dependent delays, and no matter they are linear or nonlinear.

2.3 The GPS for ODEs system

We can write Eq. (18) as

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}, \xi), \quad \mathbf{x} \in \mathbb{R}^n, \quad (19)$$

where $\mathbf{x} = (x_1, \dots, x_n)^T$.

Group-preserving scheme (GPS) can preserve the internal symmetry group of the considered ODE system. Although we do not know previously the symmetry group of differential equations system, Liu (2001) has embedded it into an augmented differential system, which concerns with not only the evolution of state variables themselves but also the evolution of the magnitude of the state variables vector. Let us note that

$$\|\mathbf{x}\| = \sqrt{\mathbf{x}^T \mathbf{x}} = \sqrt{\mathbf{x} \cdot \mathbf{x}}, \quad (20)$$

where the dot between two n -dimensional vectors denotes their inner product. Taking the derivatives of both the sides of Eq. (20) with respect to ξ , we have

$$\frac{d\|\mathbf{x}\|}{d\xi} = \frac{(\mathbf{x}')^T \mathbf{x}}{\sqrt{\mathbf{x}^T \mathbf{x}}}. \quad (21)$$

Then, by using Eqs. (19) and (20) we can derive

$$\frac{d\|\mathbf{x}\|}{d\xi} = \frac{\mathbf{f}^T \mathbf{x}}{\|\mathbf{x}\|}. \quad (22)$$

It is interesting that Eqs. (19) and (22) can be combined together into a simple matrix equation:

$$\frac{d}{d\xi} \begin{bmatrix} \mathbf{x} \\ \|\mathbf{x}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n \times n} & \frac{\mathbf{f}(\mathbf{x}, \xi)}{\|\mathbf{x}\|} \\ \frac{\mathbf{f}^T(\mathbf{x}, \xi)}{\|\mathbf{x}\|} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \|\mathbf{x}\| \end{bmatrix}. \quad (23)$$

It is obvious that the first row in Eq. (23) is the same as the original equation (19), but the inclusion of the second row in Eq. (23) gives us a Minkowskian structure of the augmented state variables of $\mathbf{X} := (\mathbf{x}^T, \|\mathbf{x}\|)^T$, which satisfies the cone condition:

$$\mathbf{X}^T \mathbf{g} \mathbf{X} = 0, \tag{24}$$

where

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & -1 \end{bmatrix} \tag{25}$$

is a Minkowski metric, and \mathbf{I}_n is the identity matrix of order n . In terms of $(\mathbf{x}, \|\mathbf{x}\|)$, Eq. (24) becomes

$$\mathbf{X}^T \mathbf{g} \mathbf{X} = \mathbf{x} \cdot \mathbf{x} - \|\mathbf{x}\|^2 = \|\mathbf{x}\|^2 - \|\mathbf{x}\|^2 = 0. \tag{26}$$

It follows from the definition given in Eq. (20), and thus Eq. (24) is a natural result. Consequently, we have an $n + 1$ -dimensional augmented system:

$$\mathbf{X}' = \mathbf{A} \mathbf{X} \tag{27}$$

with a constraint (24), where

$$\mathbf{A} := \begin{bmatrix} \mathbf{0}_{n \times n} & \frac{\mathbf{f}(\mathbf{x}, \xi)}{\|\mathbf{x}\|} \\ \frac{\mathbf{f}^T(\mathbf{x}, \xi)}{\|\mathbf{x}\|} & 0 \end{bmatrix}, \tag{28}$$

satisfying

$$\mathbf{A}^T \mathbf{g} + \mathbf{g} \mathbf{A} = \mathbf{0}, \tag{29}$$

is a Lie algebra $so(n, 1)$ of the proper orthochronous Lorentz group $SO_o(n, 1)$. This fact prompts us to devise the group-preserving scheme (GPS), whose discretized mapping \mathbf{G} must exactly preserve the following Lie-group properties:

$$\mathbf{G}^T \mathbf{g} \mathbf{G} = \mathbf{g}, \tag{30}$$

$$\det \mathbf{G} = 1, \tag{31}$$

$$G_0^0 > 0, \tag{32}$$

where G_0^0 is the 00-th component of \mathbf{G} .

Although the dimension of the new system is raised one more, it has been shown that the new system permits a GPS given as follows:

$$\mathbf{X}_{k+1} = \mathbf{G}(k) \mathbf{X}_k, \tag{33}$$

where \mathbf{X}_k denotes the numerical value of \mathbf{X} at ξ_k , and $\mathbf{G}(k) \in SO_o(n, 1)$ is the group value of \mathbf{G} at ξ_k . If $\mathbf{G}(k)$ satisfies the properties in Eqs. (30)-(32), then \mathbf{X}_k satisfies the cone condition in Eq. (24).

The Lie group can be generated from $\mathbf{A} \in so(n, 1)$ by an exponential mapping,

$$\mathbf{G}(k) = \exp[h\mathbf{A}(k)] = \begin{bmatrix} \mathbf{I}_n + \frac{(a_k-1)}{\|\mathbf{f}_k\|^2} \mathbf{f}_k \mathbf{f}_k^T & \frac{b_k \mathbf{f}_k}{\|\mathbf{f}_k\|} \\ \frac{b_k \mathbf{f}_k^T}{\|\mathbf{f}_k\|} & a_k \end{bmatrix}, \quad (34)$$

where

$$a_k := \cosh\left(\frac{h\|\mathbf{f}_k\|}{\|\mathbf{x}_k\|}\right), \quad (35)$$

$$b_k := \sinh\left(\frac{h\|\mathbf{f}_k\|}{\|\mathbf{x}_k\|}\right), \quad (36)$$

and $h = \xi_{k+1} - \xi_k$ is a constant step length of the fictitious time ξ .

Substituting Eq. (34) for $\mathbf{G}(k)$ into Eq. (33), we obtain

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \eta_k \mathbf{f}_k, \quad (37)$$

$$\|\mathbf{x}_{k+1}\| = a_k \|\mathbf{x}_k\| + \frac{b_k}{\|\mathbf{f}_k\|} \mathbf{f}_k \cdot \mathbf{x}_k, \quad (38)$$

where

$$\eta_k := \frac{b_k \|\mathbf{x}_k\| \|\mathbf{f}_k\| + (a_k - 1) \mathbf{f}_k \cdot \mathbf{x}_k}{\|\mathbf{f}_k\|^2}. \quad (39)$$

This scheme is group properties preserved for all $h > 0$, and is called the group-preserving scheme.

2.4 Numerical procedure

Starting from an initial value of $\mathbf{x}(0)$, we can employ the above GPS to integrate Eq. (18) from $\xi = 0$ to a selected final time ξ_f . In the numerical integration process we can check the convergence of x_i at the k - and $k + 1$ -steps by

$$\sum_{i=1}^n (x_i^{k+1} - x_i^k)^2 \leq \varepsilon^2, \quad (40)$$

where ε is a selected convergence criterion. If at a time $\xi_0 \leq \xi_f$ the above criterion is satisfied, then the solution of x_i is obtained. In practice, if a suitable ξ_f is selected we find that the numerical solution also approaches very well to the true solution, even the above convergence criterion is not satisfied.

3 Numerical examples

In order to assess the performance of the newly developed method let us investigate the following examples.

3.1 Example 1

Let us consider a single constant delay ODE:

$$\begin{cases} \dot{x}(t) = -x(t-1) & t > 0, \\ x(t) = t & t \in [-1, 0]. \end{cases} \quad (41)$$

The exact solution up to $t \leq 2$ is

$$x(t) = \begin{cases} \frac{1}{2} - \frac{(t-1)^2}{2} & t \in [0, 1], \\ \frac{(t-2)^3}{6} - \frac{t}{2} + \frac{7}{6} & t \in [1, 2]. \end{cases} \quad (42)$$

We calculate the first case using the following parameters: $m_2 = 151$, $h = 0.01$, $\gamma = 1$, $\nu = 5$, and $\varepsilon = 10^{-4}$. The initial guess of x^j is given by $x^j = 0$. Through 413 steps the solution is obtained. By comparing with the exact solution in Fig. 1(a), we can see that the numerical result is good with the maximum error 8.3×10^{-3} .

A similar case is a two constant delays ODE:

$$\begin{cases} \dot{x}(t) = x(t-2) + 2x(t-3) & t > 0, \\ x(t) = t & t \in [-3, 0]. \end{cases} \quad (43)$$

The exact solution up to $t \leq 4$ is

$$x(t) = \begin{cases} \frac{3t^2}{2} - 8t & t \in [0, 2], \\ \frac{3t^3}{2} - 24t^2 + 97t - 120 & t \in [2, 4]. \end{cases} \quad (44)$$

The exact solutions of the above two cases are obtained by an interval-by-interval integration; however, when the terminal time is large, this method is rather cumbersome.

We calculate the second case using the following parameters: $m_2 = 401$, $h = 0.1$, $\gamma = 1$, $\nu = 0.001$, and $\varepsilon = 10^{-3}$. Through 107 steps the solution is obtained. From the comparison with the exact solution in Fig. 1(b), it can be seen that the numerical result is good with the maximum error 0.2. This case is more difficult than the first case, because the solution exhibits a discontinuous slope at $t = 2$.

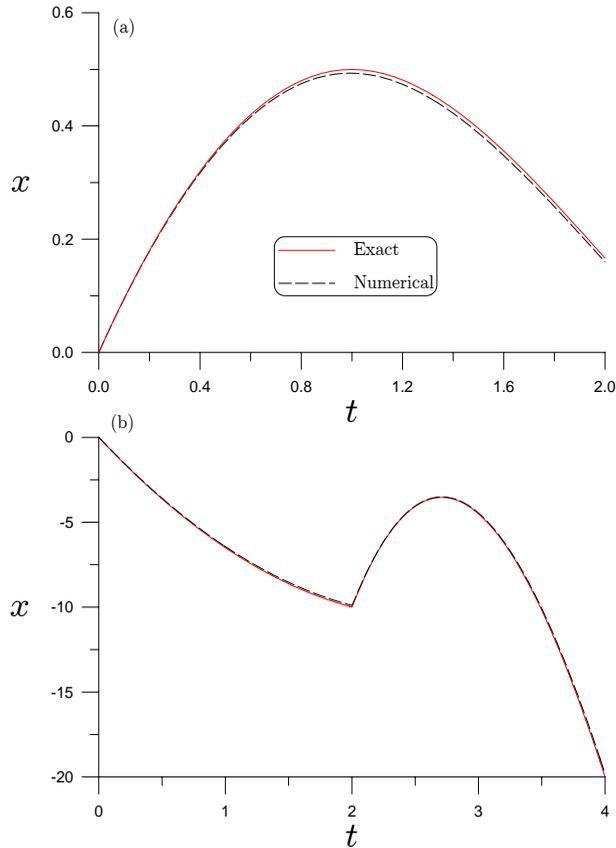


Figure 1: Displaying the numerical and exact solutions of Example 1: (a) case 1, and (b) case 2.

3.2 Example 2

In this case we consider a time delay cellular neural network model:

$$\begin{cases} \dot{x}(t) = -x(t) + y(t) + y(t - \tau), & y(t) = \frac{1}{2}(|x(t) + 1| - |x(t) - 1|) & t > 0, \\ x(t) = 0.5 & & t \leq 0. \end{cases} \quad (45)$$

We calculate this problem using the following parameters: $\tau = 1.5\pi$, $T = 5\pi$, $m_2 = 201$, $h = 0.01$, $\gamma = 1$, $\nu = 10$, and $\varepsilon = 10^{-5}$. Through about 200 steps the solutions are obtained. Time delay case and no time delay case are compared in Fig. 2. It can be seen that both cases tend to a stable value of 2.

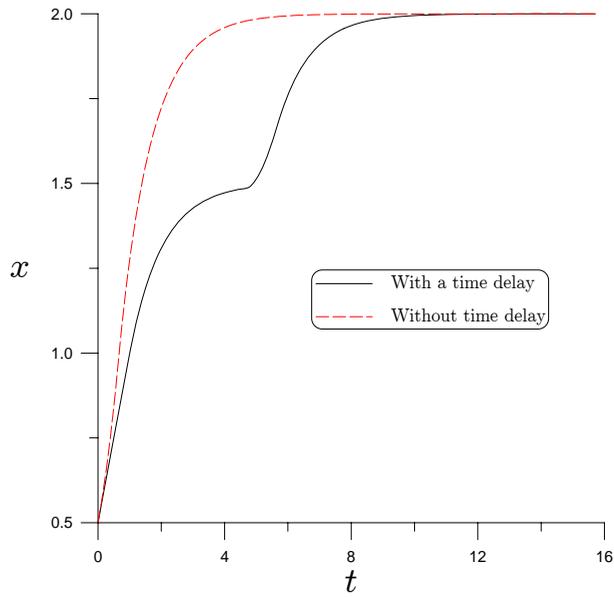


Figure 2: For Example 2 showing a time delay and a no-time delay numerical solutions.

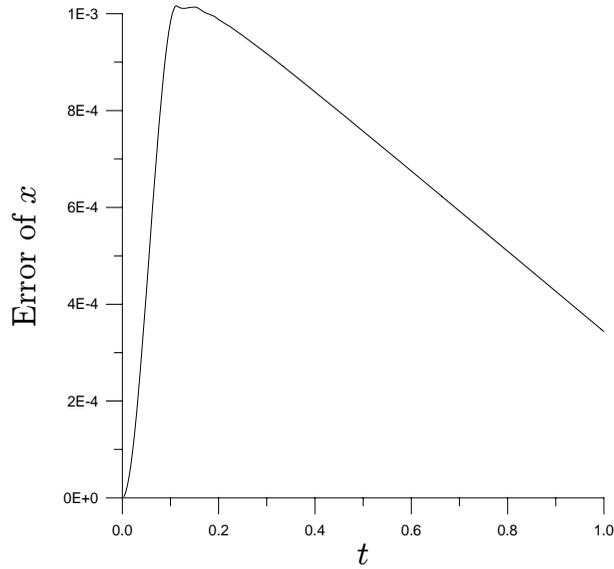


Figure 3: Displaying the numerical error of Example 3.

3.3 Example 3

We consider a time-varying DODE:

$$\begin{cases} \dot{x}(t) = x(t - \tau(t)), & \tau(t) = t + \frac{t^2}{4} & t \in [0, 1], \\ x(t) = \sqrt{|t|} + 1 & & t \in [-1, 0], \end{cases} \quad (46)$$

of which the exact solution is

$$x(t) = 1 + t + \frac{t^2}{4}. \quad (47)$$

By applying the FTIM we use the following parameters: $m_2 = 301$, $h = 0.001$, $\gamma = 1$, $\nu = 4$, and $\varepsilon = 10^{-7}$. Through 316 steps the solution is obtained, whose numerical error is plotted in Fig. 3, of which the maximum error is 10^{-3} .

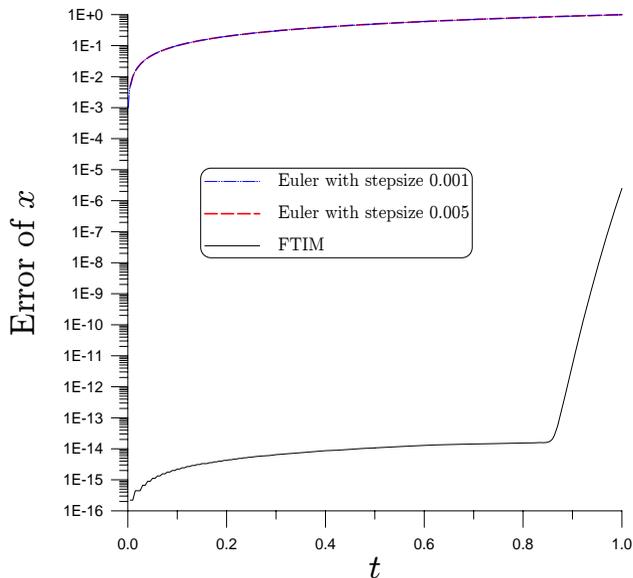


Figure 4: For Example 4 comparing the numerical errors obtained by the FTIM and the Euler method.

3.4 Example 4

Next we consider a state-dependent DODE:

$$\begin{cases} \dot{x}(t) = x(t - \rho), & \rho(x(t)) = x(t) - 1, & t \in [0, 1], \\ x(t) = \sqrt{|t|} + 1 & & t \in [-1, 0]. \end{cases} \quad (48)$$

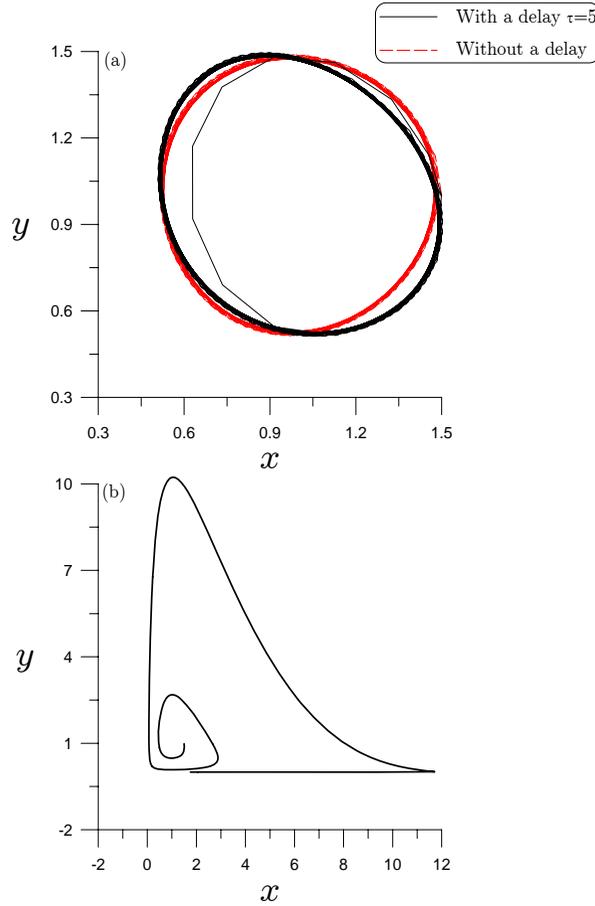


Figure 5: For Example 5 (a) comparing the numerical solutions of delayed and non-delayed cases, and (b) showing the numerical result obtained by the Euler method.

By applying the FTIM we use the following parameters: $m_2 = 201$, $h = 0.001$, $\gamma = 1$, $\nu = 4$, and $\varepsilon = 10^{-6}$. Through 346 steps the solution is obtained, whose numerical error is plotted in Fig. 4 by comparing with the exact solution $1 + t$, of which the maximum error is 2.5×10^{-6} .

For the purpose of comparison we also apply the Euler method to the above equation by taking the state-dependent delay into account. Using the stepsizes 0.001 and 0.005 the errors are the same, and they are much larger than that obtained by the above FTIM. We found that when stepsize is more small, the Euler method is unstable.

3.5 Example 5

Now, let us apply the above FTIM to a two-dimensional predator-prey equation of Lotka-Volterra type with a constant delay:

$$\dot{x}(t) = -x(t) + x(t - \tau)y(t - \tau), \quad (49)$$

$$\dot{y}(t) = y(t) - x(t)y(t), \quad (50)$$

where x is the population of predator and y is the population of prey. Liu (2006) has calculated the above model without time delay, where the modified group preserving scheme is used to preserve the invariant of the above model.

In Fig. 5(a) we compare the orbits of (x, y) for the delay case with $\tau = 5$ and the case without considering time delay. Both cases tend to periodic solutions; however, when the delay is large this system may be unstable.

For the purpose of comparison we also apply the GPS to Eqs. (49) and (50), and take the delay into account. The result is shown in Fig. 5(b). The constant delay is taken to be $\tau = 0.5$, and the stepsize used in the GPS is 0.04. Even for a small delay, the conventional numerical integration method, like as the GPS, is easily tending to unstable, giving an incorrect solution as shown in Fig. 5(b). The correct one is a periodic orbit. Conversely, the FTIM even under a large time spacing with 0.5 and a large time delay with $\tau = 5$, the stable periodic solution as shown in Fig. 5(a) by the solid line is also available. Under the parameters used in Fig. 5(a), we found that the GPS cannot be applied to obtain the periodic solution.

3.6 Example 6

We consider a state-dependent DODE:

$$\begin{cases} \dot{x}(t) = -F(x(t)) + F(x(t - r)), & r = r(x(t - \tau)) & t > 0, \\ x(t) = \phi(t) & & t \leq 0. \end{cases} \quad (51)$$

By applying the FTIM we first need to check the position of $t_j - \tau$. If $t_\ell \leq t_j - \tau \leq t_{\ell+1}$ for an ℓ , we employ a linear interpolation to approximate $x(t_j - \tau)$ by

$$\bar{x}^j = x^\ell + \frac{t_j - \tau - t_\ell}{\Delta t} (x^{\ell+1} - x^\ell). \quad (52)$$

Then, inserting the above \bar{x}^j into the function r we can get

$$\bar{r}^j = r(\bar{x}^j). \quad (53)$$

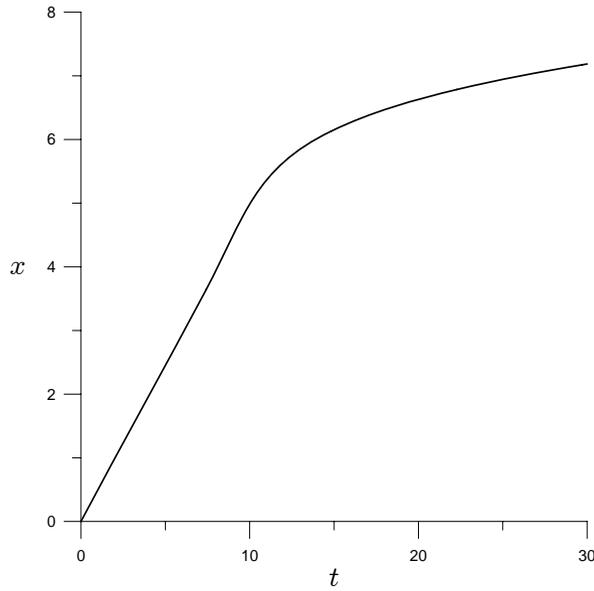


Figure 6: For Example 6 showing the numerical solution obtained by the FTIM.

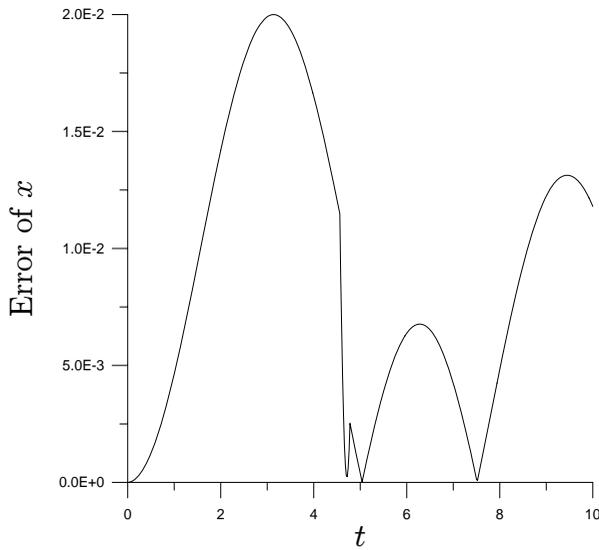


Figure 7: Displaying the numerical error of Example 7.

Second, we need to check the position of $t_j - \bar{r}^j$. If $t_\ell \leq t_j - \bar{r}^j \leq t_{\ell+1}$, we employ

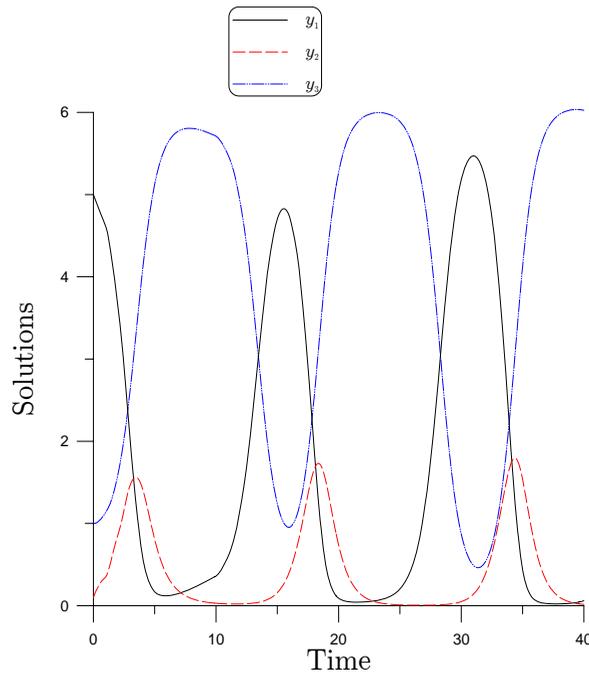


Figure 8: For Example 8 showing the numerical solutions obtained by the FTIM.

a linear interpolation to approximate $x(t_j - \bar{r}^j)$ by

$$\hat{x}^j = x^\ell + \frac{t_j - \bar{r}^j - t_\ell}{\Delta t} (x^{\ell+1} - x^\ell). \tag{54}$$

From Eqs. (51) and (54) by using a finite difference scheme we can derive the following equation:

$$\frac{x^j - x^{j-1}}{\Delta t} + F(x^j) - F(\hat{x}^j) = 0. \tag{55}$$

Then we apply the GPS to integrate the following equation:

$$\frac{dx^j}{d\xi} = \frac{-v}{(1 + \xi)^\gamma} \left[\frac{x^j - x^{j-1}}{\Delta t} + F(x^j) - F(\hat{x}^j) \right]. \tag{56}$$

We fix $F(x) = \exp(x)$ and $r(x) = 1 + \sin x$. By applying the FTIM we use the following parameters: $m_2 = 401$, $h = 0.001$, $\gamma = 1$, $v = 0.01$, and $T = 30$. The variation of x with respect to t is plotted in Fig. 6.

3.7 Example 7

Next, we consider a state-dependent DODE:

$$\begin{cases} \dot{x}(t) = \cos(t)x(x(t) - 2) & t \in [0, 10], \\ x(t) = 1 & t \in [-1, 0]. \end{cases} \quad (57)$$

It has a closed-form solution $x(t) = 1 + \sin t$.

By applying the FTIM we use the following parameters: $m_2 = 501$, $h = 0.01$, $\gamma = 0.5$, $\nu = 2$, and $\varepsilon = 10^{-4}$. Through 798 steps the solution is obtained, whose numerical error is plotted in Fig. 7 by comparing with the exact solution $1 + \sin t$, of which the maximum error is 2×10^{-2} .

3.8 Example 8

The following example is borrowed from Hairer et al. (1993):

$$\begin{aligned} y_1'(x) &= -y_1(x)y_2(x-1) + y_2(x-10), \\ y_2'(x) &= y_1(x)y_2(x-1) - y_2(x), \\ y_3'(x) &= y_2(x) - y_2(x-10). \end{aligned} \quad (58)$$

The solutions, under the initial phases $y_1(x) = 5$, $y_2(x) = 0.1$, and $y_3(x) = 1$ for $x \leq 0$, are plotted in Fig. 8. These results are match very well with that of Hairer et al. (1993).

4 Conclusions

The multiple time-varying delays and state-dependent delays ODEs are discretized by the finite difference method together with a linear interpolation technique to treat the delay term. The present paper simply transformed the resulting nonlinear algebraic equations into an evolutionary system of equations by introducing a fictitious time, and had adding a coefficient ν to enhance the stability of numerical integration of the resulting ODEs and to speed up the convergence of numerical solutions. Several numerical examples were worked out. Some are compared with exact solutions, revealing that high accuracy can be achieved by the FTIM. The conventional Euler method and the GPS, by taking the delay into account, were easily tending to unstable and gave incorrect numerical solutions. In contrast, the FTIM is easy implementation and efficient.

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