An Analysis of the Heat Conduction Problem for Plates with the Functionally Graded Material Using the Hybrid Numerical Method

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Abstract: A heat conduction analysis of the functionally graded material (FGM) plates has been investigated based on the hybrid numerical method (HNM). HNM combines the layer element method with the method of Fourier transforms and proves to be efficient and reliable. The FGM plates are infinite large and the material properties vary continuously through thickness. The heat source continually acted one the FGM plates. The temperature distribution of the FGM plates is obtained in different time and different position. Some useful results for heat conduction problems are shown in figures. This article applies HNM to heat conduction firstly and provides us a new way for studying the heat conduction problems.

Keywords: Functionally graded material; heat conduction; hybrid numerical method

1 Introduction

Functionally graded material (FGM) has been developed as a new material that is adaptable for a super-high-temperature environment. Temperature change often represents a significant factor, and often induces the failure of FGM. So investigating the heat problems of the FGM plates is very necessary.

Researchers have presented many methods to investigate heat problems of FGM. Ootao and Tanigawa (2002) is concerned with the theoretical treatment of transient thermal stress problem involving an angle-ply laminated cylindrical panel consisting of an oblique pile of layers having orthotropic material properties due to

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nonuniform heat supply in the circumferential direction. M.Savoia and J.N.Reddy (1995) present results of the stress analysis of multilayered plates subject to thermal and mechanical loads in the context of the three-dimensional quassi-static theory of thermoelasticity. Cheng and Batra (2000), Reddy and Cheng (2001) have used the asymptotic expansion method to study the three-dimensional thermoelastic deformations of functionally graded elliptic and rectangular plates, respectively. Vel and Batra (2002,2003) have presented exact three-dimensional solutions for the steady-state and quasi-static transient thermoelastic response of functionally graded thick plates with an arbitrary variation of material properties in the thickness direction. Qian et al. (2004), Qian and Batra (2004) have obtained results for the steady-state and transient thermoelastic response of functionally graded plates using the meshless local Petrov-Galerkin method that agrees with the exact solution of Vel and Batra (2002,2003). Sladek et al. (2008a, 2008b) applied a meshless local Petrov-Galerkin (MLPG) method into solving problems of Reissner-Mindlin shells under thermal loading and also proposed a meshless method based on the local Petrov-Galerkin approach to obtain the solution of steady-state and transient heat conduction problems in a continuously nonhomogeneous anisotropic medium. Chein-Shan Liu (2008) treat inverse thermal stress problems using the Lie-group shooting method through an internal temperature measurement. Lutz and Zimmerman presented the exact solutions for one-dimensional thermal stresses of functionally graded cylinder (1996) and sphere (1999) whose elastic moduli and thermal expansion coefficients vary linearly along radius.

Recently, G.R.Liu (2002) proposes an efficient and reliable method called HNM. This method has been applied to solve displacement response problems and proved to be validity. To the author's knowledge, HNM for heat conduction problem of FGM plate has not been reported. In this paper, the analysis of the heat conduction problem for plates with the functionally graded material is presented using HNM.

2 Heat conduction formulation based on HNM

Consider a functionally graded material that has nonhomogeneous thermal and mechanical properties along the thickness direction as shown in Fig.1. Its thickness is represented by H. Here, assume that the plate is initially at zero temperature and is suddenly heated from the middle line y axis of the upper surface by the surrounding medium with relative heat transfer coefficient α , and assume that the heat source is throughout the y-direction. Then this problem is reduced a two-dimensional heat conduction problem in x - z plane. The temperature of the surrounding medium is denoted by a function T_f and assume its lower surface (z = 0) is held zero temperature.



Figure 1: Analysis model and coordinate system

2.1 HNM for heat conduction problem

The thermal conductivity of the plate is assumed to take the following continuous form

$$\lambda(z) = \lambda_l + \frac{\lambda_u - \lambda_l}{H} z \quad 0 \le z \le H \tag{1}$$

Where λ_u and λ_l are the thermal conductivity of upper and lower surfaces, respectively.

Specific heat c and density ρ are also assumed to be the continuous linear form just like the thermal conductivity. The heat conduction equation involving an internal heat source is taken as

$$\frac{\partial}{\partial x} [\lambda(z)\frac{\partial T}{\partial x}] + \frac{\partial}{\partial z} [\lambda(z)\frac{\partial T}{\partial z}] + q_{\nu} = c\rho \frac{\partial T}{\partial t}$$
(2)

The temperature field within a layer element is approximated as

$$T = \mathbf{N}(z)\mathbf{\Phi}(x,t) \tag{3}$$

where N(z) is a shape function matrix, and here the quadratic shape function is used as

$$\mathbf{N}(z) = \left[(1 - 3\bar{z} + 2\bar{z}^2) \ 4(\bar{z} - \bar{z}^2) \ (2\bar{z}^2 - \bar{z}) \right] \tag{4}$$

in which $\overline{z} = z/h$, *h* is thickness of a layer element. In Eq. (3), Φ is a matrix consisting of nodal line temperatures, which are a function of the coordinate *x* and time *t*, and is denoted as

$$\mathbf{\Phi} = \left\{ \begin{array}{c} \Phi_l \\ \Phi_m \\ \Phi_u \end{array} \right\} \tag{5}$$

where the subscript l, m, u denotes lower, middle, upper nodal lines of element. The initial condition and thermal boundary conditions are expressed as follows

$$T|_{t=0} = 0$$
 (6)

$$z = H; \ -\lambda(z) \frac{\partial T}{\partial z} \Big|_{\Gamma} = \alpha (T_f - T) \Big|_{\Gamma}$$
(7)

where Γ denotes the boundary of the plate.

Using the method of weighted residual for Eq. (2) and Eq. (7), there have

$$\int_{0}^{h} \mathbf{N}^{\mathrm{T}} \left[c\rho \frac{\partial T}{\partial t} - \left[\frac{\partial}{\partial x} (\lambda(z) \frac{\partial T}{\partial x}) + \frac{\partial}{\partial z} (\lambda(z) \frac{\partial T}{\partial z}) \right] - q_{\nu} \right] dz + \lambda(z) \frac{\partial T}{\partial z} \Big|_{\Gamma} + \alpha (T - T_{f}) \Big|_{\Gamma} = 0 \quad (8)$$

Performing lengthy and simple operations, we obtain

$$\mathbf{C}\dot{\mathbf{\Phi}} + \mathbf{K}_D \mathbf{\Phi} = \mathbf{Q} \tag{9}$$

Where

$$\mathbf{C} = \int_0^h \mathbf{N}^{\mathrm{T}} c \boldsymbol{\rho} \mathbf{N} dz \tag{10}$$

$$\mathbf{K}_{D} = -\int_{0}^{h} \mathbf{N}^{T} \lambda(z) \mathbf{N} dz \cdot \frac{\partial^{2}}{\partial x^{2}} + \left(\lambda(z) \frac{\partial \mathbf{N}}{\partial z}\Big|_{\Gamma} + \alpha \mathbf{N}|_{\Gamma} - \int_{0}^{h} \mathbf{N}^{T} \lambda(z) \frac{\partial^{2} \mathbf{N}}{\partial z^{2}} dz\right)$$
(11)

$$\mathbf{Q} = \int_0^h \mathbf{N}^{\mathrm{T}} q_\nu dz + \alpha T_f \big|_{\Gamma}$$
(12)

Substituting Eq. (4) into Eq. (10) and integrating along z, we obtain

$$\mathbf{C} = \frac{c\rho h}{30} \begin{bmatrix} 4 & 2 & -1\\ 2 & 16 & 2\\ -1 & 2 & 4 \end{bmatrix}$$
(13)

Performing simple operation to Eq. (11), \mathbf{K}_D can be written as

$$\mathbf{K}_D = -\mathbf{A}_1 \cdot \frac{\partial^2}{\partial x^2} + \mathbf{A}_0 \tag{14}$$

where

$$\mathbf{A}_{1} = \frac{h\lambda(z)}{30} \begin{bmatrix} 4 & 2 & -1\\ 2 & 16 & 2\\ -1 & 2 & 4 \end{bmatrix}$$
(15)

$$\mathbf{A}_{0} = \frac{\lambda(z)}{3h} \begin{bmatrix} 7 & -8 & 1\\ -8 & 16 & -8\\ 1 & -8 & 7 \end{bmatrix} + \alpha \begin{bmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1 \end{bmatrix}$$
(16)

After lengthy and tedious to Eq. (12), we have

$$\mathbf{Q} = \frac{q_{\nu}h}{6} \begin{bmatrix} 1\\4\\1 \end{bmatrix} + \alpha T_f \big|_{\Gamma}$$
⁽¹⁷⁾

Assembling all elements, the Eq. (9) becomes as

$$\mathbf{C}_{s}\dot{\mathbf{\Phi}}_{s} + \mathbf{K}_{Ds}\mathbf{\Phi}_{s} = \mathbf{Q}_{s} \tag{18}$$

where

$$\mathbf{K}_{Ds} = -\mathbf{A}_{1s} \cdot \frac{\partial^2}{\partial x^2} + \mathbf{A}_{0s} \tag{19}$$

where the subscript *D* denotes the matrix \mathbf{K}_s is a differential operator matrix. Matrices \mathbf{C}_s , $\mathbf{A}_{is}(i = 1, 2)$, \mathbf{Q}_s are obtained by assembling \mathbf{C} , $\mathbf{A}_i(i = 1, 2)$, \mathbf{Q} for all the elements. Because the quadratic shape function is used and the three nodal lines in one element, the dimension of \mathbf{C}_s , $\mathbf{A}_{is}(i = 1, 2)$ matrices are $M \times M(M = 2N + 1)$, the dimension of \mathbf{Q}_s matrices is $M \times 1(M = 2N + 1)$, where *N* is the number of the layer element.

2.2 Fourier Transform

The Fourier transform of spatial to ζ_x domain can be defined as follows

$$\tilde{\mathbf{\Phi}}_{s}(\zeta_{x},t) = \int_{-\infty}^{\infty} \mathbf{\Phi}_{s}(x,t) e^{i\zeta_{x}x} dx$$
⁽²⁰⁾

The application of the Fourier transform to Eq. (16) leads to a set of system equations as follows

$$\mathbf{C}_s \tilde{\mathbf{\Phi}}_s + \mathbf{K}_s \tilde{\mathbf{\Phi}}_s = \tilde{\mathbf{Q}}_s \tag{21}$$

where $\dot{\Phi}_s$, $\tilde{\Phi}_s$ and $\tilde{\mathbf{Q}}_s$ are the Fourier transform of $\dot{\Phi}_s$, Φ_s and \mathbf{Q}_s , respectively. Matrix \mathbf{K}_s is given by

$$\mathbf{K}_s = \zeta_x^2 \mathbf{A}_{1s} + \mathbf{A}_{0s} \tag{22}$$

Which is a constant matrix for given ζ_x .

2.3 Temperature in ζ_x Domain

Assume the corresponding homogeneous solution of Eq. (21) as

$$\tilde{\mathbf{\Phi}}_{s} = \mathbf{\psi} e^{-\omega t} \tag{23}$$

Instituting Eq. (23) into Eq. (21), obtain

$$[\mathbf{K}_s - \boldsymbol{\omega} \mathbf{C}_s] \boldsymbol{\psi} = 0 \tag{24}$$

Its eigenvalue equation is obtained as

$$|\mathbf{K}_s - \boldsymbol{\omega} \mathbf{C}_s| = 0 \tag{25}$$

From Eq.(25) and (24), eigenvalues $\omega_m (m = 1, 2, ..., M)$ and corresponding eigenvectors $\boldsymbol{\psi}_m^R$ can be obtained.

Transformed $\tilde{\mathbf{\Phi}}_s$ and $\tilde{\mathbf{Q}}_s$ of Eq. (21) can be expressed as series of eigenvectors

$$\tilde{\boldsymbol{\Phi}}_{s} = \sum_{m=1}^{M} a_{m} \boldsymbol{\psi}_{m}^{R}$$
⁽²⁶⁾

$$\tilde{\mathbf{Q}}_{s} = \sum_{m=1}^{M} b_{m} \mathbf{C}_{s} \boldsymbol{\psi}_{m}^{R}$$
(27)

where a_m, b_m are functions of time.

Substitution of Eqs. (26) and (27) into (19) leads to the following equation

$$\dot{a}_m + \omega_m a_m = b_m \tag{28}$$

In which b_m can be rewrite by Eq. (27) as follows

$$b_m = \frac{\boldsymbol{\psi}_m^L \tilde{\boldsymbol{Q}}_s}{\boldsymbol{\psi}_m^L \boldsymbol{C}_s \boldsymbol{\psi}_m^R} \tag{29}$$

Where superscript L and R represent left eigenvector and right eigenvector, respectively.

For a given $\tilde{\mathbf{Q}}_s$, b_m can be obtained.

From initial condition (6), we have

$$a_m|_{\tau=0} = 0 \tag{30}$$

Solving Eq. (28) for a_m , we obtain

$$a_m = e^{-\omega_m t} \left(\int \frac{\boldsymbol{\psi}_m^L \tilde{\mathbf{Q}}_s}{\boldsymbol{\psi}_m^L \mathbf{C}_s \boldsymbol{\psi}_m^R} e^{\omega_m t} dt + O \right)$$
(31)

When the internal heat source is not considered, \mathbf{Q} can be rewritten as

$$\mathbf{Q}_s = \alpha T_f \big|_{\Gamma} \tag{32}$$

Here assume that \mathbf{Q}_s has the following form

$$\mathbf{Q}_s = \alpha T_{0f} H(t) \big|_{\Gamma s} \tag{33}$$

where H(t) is the Heaviside function as

$$H(t) = \begin{cases} 1 & t > 0 \\ 0 & t \le 0 \end{cases}$$
(34)

 T_{0f} is a constant of media temperature.

Applying the Fourier transform to Eq. (33), we have

$$\tilde{\mathbf{Q}}_{s} = \alpha \tilde{T}_{0f} H(t) \big|_{\Gamma s}$$
(35)

Instituting Eq. (35) into Eq. (31), a_m becomes as

$$a_m = e^{-\omega_m t} \left(\int \frac{\boldsymbol{\Psi}_m^L \, \alpha \tilde{T}_{0f} H(t) \big|_{\Gamma s}}{\mathbf{C}_c} e^{\omega_m t} dt + O \right) \tag{36}$$

In which

$$\mathbf{C}_c = \boldsymbol{\psi}_m^L \mathbf{C}_s \boldsymbol{\psi}_m^R \tag{37}$$

Instituting Eq. (30) into Eq. (36), then into Eq. (26), the temperature matrix in the Fourier transform domain is given as

$$\tilde{\mathbf{\Phi}}_{s} = \sum_{m=1}^{M} \frac{\mathbf{\psi}_{m}^{L} \, \alpha \tilde{T}_{0f} (1 - e^{-\omega_{m}t}) \mathbf{\psi}_{m}^{R} \big|_{\Gamma s}}{\omega_{m} \mathbf{C}_{c}} \tag{38}$$

2.4 Temperature in Space-Time Domain

Performing the inverse Fourier transform of Eq.(38), the temperature in the spacetime domain can be expressed as

$$\mathbf{\Phi}_{s}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\mathbf{\Phi}}_{s}(\zeta_{x},t) e^{-i\zeta_{x}x} d\zeta_{x}$$
(39)

The integration in Eq. (39) can be carried out using fast Fourier transform (FFT) techniques.

3 Results and Discussions

In computation, a plate is divided into ten layer elements along thickness direction. In each element, three node lines are used. Time interval is 1 second. The time domain is divided into 200 steps. The wave number domain is taken as 0 to 24π and is divided into 1024 steps. Space domain is taken as 0 to 42.7 and is divided into 1024 steps. Ceramic ZrO₂ and metal Ti - 6Al - 4V are taken in investigation. The material properties of ZrO₂ and Ti - 6Al - 4V are shown in table 1. Upper and lower surface consist of ZrO₂ and Ti - 6Al - 4V, respectively. The characteristic of middle material varies continuous along thickness direction.

Material	Thermal conductivity	Specific heat	Density
	$\lambda(\mathbf{W} \cdot \mathbf{m}^{-1} \cdot \mathbf{K}^{-1})$	$J \cdot kg^{-1} \cdot K^{-1}$	$\rho(\text{kg}\cdot\text{m}^{-3})$
ZrO ₂	2.09	456.7	5331
Ti - 6Al - 4V	7.50	537.0	4420

Table 1: Material properties of ZrO₂ and Ti - 6Al - 4V

In computation, the following dimensionless parameters are introduced

$$\overline{T} = T/T_0, \quad \overline{t} = t/t_0, \quad \overline{x} = x/H, \quad \overline{\lambda} = \lambda/\lambda_0, \quad \overline{c} = c/c_0,$$

 $\overline{\rho} = \rho/\rho_0, \quad \overline{\alpha} = \alpha/\alpha_0.$

The temperatures distribution of the FGM plate subjected to the steady surrounding heat of Eq. (33) are computed using the HNM at different time step. The temperature distributions of four positions are shown from Fig. 2 to Fig. 5. The four positions are upper surface, 0.1 position from upper surface, middle surface, lower surface, respectively. Four observation times are considered, *t*1 time, *t*2 time, *t*3 time and *t*4 time. $t1 = 6.02 \times 10^{-3}$ s, $t2 = 2.11 \times 10^{-2}$ s, $t3 = 1.01 \times 10^{-1}$ s, t4 = 0.50s. From fig. 2, it can be seen, the longer time, the higher temperature; the nearer heat source, the higher temperature. It agrees with the practice. We also find the temperature is very high in the centre of the plate, the temperature decreases rapidly when far from centre, that is the heat source is in centre. After the 5 position, the temperature reaches at quite lower temperature, to the 15 position, the temperature near to zero. The temperature is zero at Infinite far position. When time reaches at t3, the temperature has been steady state distribution. It can be seen from figure, two curves are almost coincident and hard to distinguish at t3 time and t4 time.

From fig.2 to fig.5, we can see that far from heat source in the thickness direction, the temperature distribution is decrease at the same time. Around heat source, the temperature for upper surface is 0.52 at t4 time, the temperature for lower surface is only 0.0008 at the same time. We also find that there are no four temperature distribution lines in Fig.4 and Fig.5. That is because heat conduction hasn't reach there and 0 temperature distribution at t1 time and t2 time.

The temperature distribution of upper surface of 1^{st} element (n1), 2^{nd} element (n2), 6^{th} element (n3) and 10^{th} element (n4) is obtained in fig.6 at t2 time. It is clear shown that the temperature of near upper surface is more than the one of lower at same time. The temperature of 10^{th} element (n4) is still zero because the temperature haven't conducted to the 10^{th} element at t2 time.

We plot the relation chart of temperature in the direction thick at x^2 observation at 1 time as shown in Figure 7. It shows that the temperature of upper surface of the FGM plate is highest. The temperature decreases long the thick direction to lower surface. It is the inversely-proportional relationship between the temperature and thick. The temperature almost reaches to zero after 8 element number. This accords with practical case well.

Temperature distribution of time histories of FGM plate in the different observation position subjected to a line steady surrounding heat supply are shown in Figure 8. Four observation positions are chosen for studying. They are x1 = 0.167, x2 = 0.375, x3 = 0.792 and x4 = 2.042, respectively. From figure it clearly shown that the nearer heat source, the higher temperature. This result agrees with the above results. When the surrounding temperature acts on the plate, the temperature hoists very fast. The temperatures reach to the balance after 0.1, from now on the system reach to the heat balance station.

4 Conclusions

In the present article, the heat conduction of FGM plate has been investigated using the HNM. The heat balances of the FGM along the thickness direction and



Figure 2: Temperature distribution of the upper surface of FGM plate in different times subjected to steady heat supply



Figure 3: Temperature distribution of 0.1 position from upper surface of FGM plate in different times subjected to steady heat supply



Figure 4: Temperature distribution of the middle surface of FGM plate in different times subjected to steady heat supply



Figure 5: Temperature distribution of the lower surface of FGM plate in different times subjected to steady heat supply



Figure 6: Temperature distribution of FGM plate in different element at t^2 time subjected to steady heat supply



Figure 7: Temperature distribution of the thick direction of FGM plate at 1 time at x^2 position subjected to steady heat supply



Figure 8: Temperature distribution of time histories of FGM plate in the different observation position subjected to steady heat supply

length direction are shown. This study applies HNM into heat conduction and obtained useful result. The application domain of HNM is extends to heat domain and presents a new way to investigate the heat problem.

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