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Boundary Particle Method with High-Order Trefftz Functions

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Abstract: This paper presents high-order Trefftz functions for some commonly used differential operators. These Trefftz functions are then used to construct boundary particle method for solving inhomogeneous problems with the boundary discretization only, i.e., no inner nodes and mesh are required in forming the final linear equation system. It should be mentioned that the presented Trefftz functions are nonsingular and avoids the singularity occurred in the fundamental solution and, in particular, have no problem-dependent parameter. Numerical experiments demonstrate the efficiency and accuracy of the present scheme in the solution of inhomogeneous problems.

Keywords: High-order Trefftz functions, boundary particle method, inhomogeneous problems, meshfree

1 Introduction

Since the first paper on Trefftz method was presented by Trefftz (1926), its mathematical theory was extensively studied by Herrera (1980) and many other researchers. In 1995 a special issue on Trefftz method, was published in the journal of Advances in Engineering Software for celebrating its 70 years of development [Kamiya and Kita (1995)]. Qin (2000, 2005) presented an overview of the Trefftz finite element and its application in various engineering problems. The Trefftz method employs T-complete functions, which satisfies the governing differential operators and is widely applied to potential problems [Cheung, Jin and Zienkiewicz (1989)], two-dimensional elastic problems [Zielinski and Zienkiewicz (1985)], transient heat conduction [Jirousek and Qin (1996)], viscoelasticity prob-

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lems [Cozzano and Rodríguez(2007)] and free vibration analysis [Lee and Chen (2009)] and so on.

Studies on collocation Trefftz methods (CTM) can be found in [Jin, Cheung and Zienckiewicz (1993); Jin and Cheung (1999)], which belongs to the boundary meshless approaches including method of fundamental solutions (MFS) [Chen, Golberg and Hon (1998); Fairweather and Karageorghis (1998); Gu, Young and Fan (2009); Hu, Young and Fan (2008); Liu (2008a); Liu, Yeih and Atluri (2009); Marin (2008, 2009); Marin and Karageorghis (2009); Wei, Hon and Ling (2007)], boundary knot method (BKM) [Chen and Tanaka (2002); Wang, Ling and Chen (2009)], meshless regularized integral equation method [Liu (2007a, 2007b)], regularized meshless method (RMM) [Chen, Kao, Chen (2009); Song and Chen (2009); Young, Chen, Lee (2005); Young, Chen, Chen and Kao(2007); Young, Chen, Liu, Shen and Wu (2009)], modified method of fundamental solution (MMFS) [Šarler (2008)], and singular boundary method (SBM) [Chen (2009); Chen, Fu and Wei(2009)]. More interestingly, Chen, Wu, Lee and Chen (2007) proved that Trefftz method is equal to the MFS when applying them to Laplace and biharmonic problems. Li, Lu, Huang and Cheng (2006) performed a systematic comparison among the Trefftz, collocation and other boundary methods. Lately, Li, Lu, Huang and Cheng (2009) also investigated error analysis of trefftz methods. They concluded that CTM is the simplest approach with the best numerical accuracy and stability. On the other hand, Liu (2007c, 2008b) presented a modified CTM to solve the boundary value problem of the homogeneous differential equation. However, the above-mentioned methods require particular solution techniques to handle inhomogeneous problems. Kita, Ikeda and Kamiya (2005) used the polynomial function to derive the particular solution of Poisson equation, in which inner nodes are required to evaluate the particular solution.

Recently, Chen, Fu and Jin (2009) developed a boundary particle method (BPM), whose key point is an improved multiple reciprocity method, called the recursive composite multiple reciprocity method (RC-MRM). The RC-MRM employs the high-order composite differential operator to solve a wide variety of inhomogeneous problems using boundary-only collocation nodes, and significantly reduces computational cost via a recursive algorithm. Chen and his coworkers apply either high-order nonsingular general solutions or singular fundamental solutions in the RC-MRM to solve the second or the fourth order elliptic partial differential equations [Fu and Chen (2009); Fu, Chen and Yang (2009)]. Chen and Fu (2009) also applied the RC-MRM to inverse Cauchy problems.

The RC-MRM requires constructing high-order solutions of the differential operators, whereas there are no suitable RBF-based (radial basis function) nonsingular general solutions to calculate Laplacian problems. Some techniques have been proposed to overcome this drawback, such as the nonsingular harmonic function [Chen, Fu and Jin (2009); Hon and Wu (2000)] and the Bessel function with a small parameter [Chen (2001)]. However, both of the two approaches involvs a problem-dependent parameter, like the shape parameter in the well-known MQ function [Kansa (1990)]. To remove this disadvantage, the paper proposes a family of high-order Trefftz functions for Laplacian, Helmholtz and Biharmonic operators. It is found that the high-order Trefftz functions is very suitable for implementing the recursive algorithm to reduce computational cost.

This study uses the Trefftz method coupled with the RC-MRM to solve a broader territory of inhomogeneous problems. The remaining part of the paper is organized as follows. Section 2 proposes high-order Trefftz functions for some commonly used differential operators. Section 3 describes the Trefftz method coupled with the recursive composite multiple reciprocity method for inhomogeneous problems, followed by Section 4 to numerically examine the efficiency and stability of the present scheme. Finally, Section 5 presents some conclusions and a few opening issues.

2 High-order Trefftz functions

This section presents high-order Trefftz functions for some commonly used differential operators. By definition, the Trefftz solution u^T of a differential operator $L\{\}$ have to satisfy

$$L\left\{u^T\right\} = 0 \tag{1}$$

The solution satisfying Eq. (1) is called the zero-order Trefftz function, while the nth order Trefftz functions are defined as

$$L^n\left\{u^T\right\} = 0 \tag{2}$$

where $L^n\{\}$ denotes the$n$th order operator of $L\{\}$, namely, $L^1\{\} = L\{L\{\}\}$, $L^n\{\} = L\{L^{n-1}\{\}\}$. Making use of the computer algebraic package 'Maple', we constructed and proved the high-order Trefftz functions of the following typical differential operators.

2.1 High-order Trefftz functions of Laplacian operator Δ^{n+1}

In sections 2.1-2.3, we firstly, consider a two-dimensional problem with the domain

$$\Omega = \{(r,\theta)\,|0 \le r < R,\ 0 \le \theta \le 2\pi\} \tag{3}$$

For Laplace equation $\Delta u = 0$, when n=0, its Trefftz function are given in the literature as

$$1, r^m \cos(m\theta), r^m \sin(m\theta) \;\; m = 1,2,\ldots, \;\; (r,\theta) \in \Omega \tag{4}$$

which are known as zero-order Trefftz function u^{T0}. The corresponding nth order Trefftz function u^{Tn} is presented as follows

$$A_n r^{2n}, A_n r^{m+2n} \cos(m\theta), A_n r^{m+2n} \sin(m\theta) \;\; m = 1,2,\ldots, \;\; (r,\theta) \in \Omega \tag{5}$$

where $A_n = \frac{A_{n-1}}{4n(m+n)}, A_0 = 1$.

2.2 High-order Trefftz functions of Helmholtz operator $\left(\Delta + \lambda^2\right)^{n+1}$

In the Helmholtz operator, $\lambda > 0$ is a real number and assume that λ^2 is not an eigenvalue of Laplace operator. Then the zero-order Trefftz function u^{T0} can be written as

$$J_0(\lambda r), J_0(\lambda r)\cos(m\theta), J_0(\lambda r)\sin(m\theta) \;\; m = 1,2,\ldots, \;\; (r,\theta) \in \Omega \tag{6}$$

where J_0 is the Bessel function of the first kind. And the following functions are the corresponding nth order Trefftz functions u^{Tn}.

$$\begin{aligned} &A_n(\lambda r)^n J_n(\lambda r), A_n(\lambda r)^n J_{m+n}(\lambda r)\cos(m\theta), \\ &A_n(\lambda r)^n J_{m+n}(\lambda r)\sin(m\theta), \; m = 1,2,\ldots, \;\; (r,\theta) \in \Omega \end{aligned} \tag{7}$$

where $A_n = \frac{A_{n-1}}{2n\lambda^2}, A_0 = 1$.

2.3 High-order Trefftz functions of modified Helmholtz operator $\left(\Delta - \lambda^2\right)^{n+1}$

In the modified Helmholtz operator, λ is again a real number and $\lambda > 0$. Then the zero-order Trefftz function u^{T0} can be given in the form

$$I_0(\lambda r), I_0(\lambda r)\cos(m\theta), I_0(\lambda r)\sin(m\theta) \;\; m = 1,2,\ldots, \;\; (r,\theta) \in \Omega \tag{8}$$

where I_0 is the Bessel and Hankel functions with a purely imaginary argument. The corresponding nth order Trefftz function u^{Tn} is presented as follows

$$\begin{aligned} &A_n(\lambda r)^n I_n(\lambda r), A_n(\lambda r)^n I_{m+n}(\lambda r)\cos(m\theta), \\ &A_n(\lambda r)^n I_{m+n}(\lambda r)\sin(m\theta), \; m = 1,2,\ldots, \;\; (r,\theta) \in \Omega \end{aligned} \tag{9}$$

where $A_n = \frac{A_{n-1}}{2n\lambda^2}, A_0 = 1$.

2.4 Three dimensional problems

Now consider a sphere

$$S = \{(r,\phi,\theta)\,|0 \le r < R, 0 \le \phi \le \pi, 0 \le \theta \le 2\pi\} \tag{10}$$

where r,ϕ,θ denote the spherical coordinates.

For the Laplacian operator Δ^{n+1}, the nth order Trefftz function u^{Tn} can be represented as

$$\begin{aligned} &A_n r^{2n} P_\nu^0(\cos\phi), A_n r^{m+2n} P_\nu^m(\cos\phi)\cos(m\theta), \\ &A_n r^{m+2n} P_\nu^m(\cos\phi)\sin(m\theta),\ m = 1,2,\ldots,\ (r,\theta) \in S \end{aligned} \tag{11}$$

where $A_n = \frac{A_{n-1}}{4n\left(m+n+\frac{1}{2}\right)}, A_0 = 1, P_\nu^m$ is the associated Legendre polynomial function, and the function parameters $\nu = 0,1,2,\ldots; -\nu-1 < m < \nu+1$.

For the Helmholtz operator $\left(\Delta+\lambda^2\right)^{n+1}$, the nth order Trefftz function u^{Tn} can be represented as

$$\begin{aligned} &A_n(\lambda r)^{n-\frac{1}{2}} J_n(\lambda r) P_\nu^0(\cos\phi), A_n(\lambda r)^{n-\frac{1}{2}} J_{m+n}(\lambda r) P_\nu^m(\cos\phi)\cos(m\theta), \\ &A_n(\lambda r)^{n-\frac{1}{2}} J_{m+n}(\lambda r) P_\nu^m(\cos\phi)\sin(m\theta), m = 1,2,\ldots,\ (r,\theta) \in S \end{aligned} \tag{12}$$

where $A_n = \frac{A_{n-1}}{2n\lambda^2}, A_0 = 1$.

For the modified Helmholtz operator $\left(\Delta-\lambda^2\right)^{n+1}$, the nth order Trefftz function u^{Tn} can be represented as

$$\begin{aligned} &A_n(\lambda r)^{n-\frac{1}{2}} I_n(\lambda r) P_\nu^0(\cos\phi), A_n(\lambda r)^{n-\frac{1}{2}} I_{m+n}(\lambda r) P_\nu^m(\cos\phi)\cos(m\theta), \\ &A_n(\lambda r)^{n-\frac{1}{2}} I_{m+n}(\lambda r) P_\nu^m(\cos\phi)\sin(m\theta), m = 1,2,\ldots,\ (r,\theta) \in S \end{aligned} \tag{13}$$

where $A_n = \frac{A_{n-1}}{2n\lambda^2}, A_0 = 1$.

All of the above high-order Trefftz functions have an important relationship for constructing a recursive iteration algorithm to reduce computational cost dramatically:

$$L\left\{u^{Tn}\right\} = u^{T(n-1)}. \tag{14}$$

3 Collocation Trefftz method with RC-MRM

This section introduces the collocation Trefftz method (CTM) coupled with the recursive composite multiple reciprocity method. To this end, consider following equations

$$\Re\{u\} = f(x), \quad x \in \Omega, \tag{15}$$

$$u(x) = R(x), \quad x \subset \Gamma_D, \tag{16}$$

$$\frac{\partial u(x)}{\partial n} = N(x), \quad x \subset \Gamma_N, \tag{17}$$

where $\Re$ is a differential operator, x means a multi-dimensional independent variable, Γ_D and Γ_N are the Dirichlet and Neumann boundary parts, respectively, and n the unit outward normal. The solution of Eqs. (15) and (16) can be assumed in the form

$$u = u_h + u_p, \tag{18}$$

where u_h and u_p are the homogeneous and the particular solutions, respectively. The particular solution u_p satisfies

$$\Re\{u_p\} = f(x), \tag{19}$$

but does not necessarily satisfy boundary conditions. In contrast, the homogeneous solution has to satisfy not only the corresponding homogeneous equation

$$\Re\{u_h\} = 0 \tag{20}$$

but also boundary conditions

$$u_h(x) = R(x) - u_p(x), \quad x \subset \Gamma_D, \tag{21a}$$

$$\frac{\partial u_h(x)}{\partial n} = N(x) - \frac{\partial u_p(x)}{\partial n}, \quad x \subset \Gamma_N. \tag{21b}$$

This study employs an improved MRM, RC-MRM, to evaluate the particular solution. Unlike in the standard MRM, the annihilating differential operator in the RC-MRM is not necessary the governing equation operator $\Re\{\}$ in the MRM [Nowak and Neves (1994)]. Instead, a composite differential operator can be chosen to realize the basic assumption of the MRM, vanishing inhomogeneous term $f(x)$ in equation (18) by iterative differentiations

$$\lim_{n\to\infty} L_n \ldots L_2 L_1 \{f(x)\} \to 0, \tag{22}$$

where $L_1, L_2, \ldots, L_n$ are differential operators of the same or different kinds. Under the assumption that the annihilation (21) is finite order or is truncated at certain order N, the representation can be modified as

$$L_N \cdots L_2 L_1 \Re\{u\} \cong 0, \;\; x \in \Omega. \tag{23}$$

According to Eq. (22), the original inhomogeneous governing equation (18) can be transformed into the following high-order homogeneous equation.

$$\begin{cases} L_N \ldots L_2 L_1 \Re u(x) = 0 & x \in \Omega \\ \vdots & \\ L_2 L_1 \Re u(x) = L_2 L_1 \left(f(x)\right) & x \in \partial\Omega \\ L_1 \Re u(x) = L_1 \left(f(x)\right) & x \in \partial\Omega \\ \Re u(x) = f(x) & x \in \partial\Omega \end{cases} \tag{24}$$

Thus the corresponding particular solution $u_p(x)$ can be evaluated by the high-order homogeneous equation (23). Then the foregoing inhomogeneous problem is equal to solve two homogeneous equations (19), (20) and (23).

For solving the homogeneous equations, it is necessary to select the corresponding Trefftz functions satisfying the corresponding governing equation. Then the homogeneous solution can be expressed as follows

$$u = \sum_{i=1}^{\infty} \bar{a}_i u_i^T, \tag{25}$$

where $\bar{a}_i$ are the true expansion coefficients. Choosing finite terms in (24) gives the approximate solution

$$\tilde{u} = \sum_{i=1}^{TN} a_i u_i^T, \tag{26}$$

where the coefficients a_i are an approximation of $\bar{a}_i$. Since we have *TN* unknowns, it requires choosing *NN*($\geq$*TN*) boundary collocation points, and enforces the functions (25) to satisfy the boundary conditions at those boundary points. This paper adopts *NN*=*TN* for constructing the square interpolation matrix. Then, the successively boundary condition equations can be written by

$$b^0 = \begin{cases} u_h^0(x_i) = R(x_i) - u_p^0(x_i) & x \in \Gamma_D, \\ \frac{\partial u_h^0(x_j)}{\partial n} = N(x_j) - \frac{\partial u_p^0(x_j)}{\partial n} & x \in \Gamma_N, \end{cases} \tag{27a}$$

$$b^1 = \begin{cases} L^0\left\{u_h^1(x_i)\right\} = f(x_i) - L^0\left\{u_p^1(x_i)\right\} \\ \frac{\partial L^0\left\{u_h^1(x_j)\right\}}{\partial n} = \frac{\partial\left(f(x_j) - L^0\left\{u_p^1(x_j)\right\}\right)}{\partial n} \end{cases}, \tag{27b}$$

$$b^n = \begin{cases} L^{(n-1)}\left\{u_h^n(x_i)\right\} = L^{(n-2)}\left\{f(x_i)\right\} - L^{(n-1)}\left\{u_p^n(x_i)\right\} \\ \frac{\partial L^{(n-1)}\left\{u_h^n(x_j)\right\}}{\partial n} = \frac{\partial\left(L^{(n-2)}\left\{f(x_j)\right\} - L^{(n-1)}\left\{u_p^n(x_j)\right\}\right)}{\partial n} \end{cases}, \quad n = 2, 3, \ldots, N,$$

(27c)

where $L^0\{f(x)\}=\Re\{f(x)\}$, $L^{(n-1)}\{f(x)\}=L_{n-1}\cdots L_2L_1\Re\{f(x)\}$. The homogeneous solution u_h^n in (26) is the homogeneous solution of a high-order composite operator $L^{(n-1)}\{\}$. Then, we construct the approximate representation of the inhomogeneous problem

$$\tilde{u}=\sum_{n=0}^{N}\sum_{k=1}^{NN}\alpha_k^n u_k^{Tn}. \tag{28}$$

where α_k^n are the expansion coefficients of the n-order homogeneous solution, and u^{Tn} the nth-order Trefftz solution satisfying operator $L^{(n)}\{\}$. Substituting (27) into the successive equations (26) to obtain discretization RC-MRM equations, the corresponding interpolation matrix can be written in following form

$$\begin{pmatrix} \mathbf{A}_{00} & \mathbf{A}_{01} & \cdots & \mathbf{A}_{0N} \\ & \mathbf{A}_{11} & \cdots & \mathbf{A}_{1N} \\ & & \vdots & \vdots \\ & & & \mathbf{A}_{NN} \end{pmatrix} \begin{pmatrix} \alpha^0 \\ \alpha^1 \\ \vdots \\ \alpha^N \end{pmatrix} = \begin{pmatrix} \mathbf{b}^0 \\ \mathbf{b}^1 \\ \vdots \\ \mathbf{b}^N \end{pmatrix}, \tag{29}$$

According to Eq. (14), when a differential operator occurs repetitively, then their respective interpolation matrices are the same by choosing the above high-order Trefftz functions. In case that the operators $L_1=L_2=\Re$, then we have the following useful property $\mathbf{A}_{11}$=$\mathbf{A}_{22}$ irrespective of the boundary conditions. For the Dirichlet boundary condition, we have $\mathbf{A}_{00}$=$\mathbf{A}_{11}$=$\mathbf{A}_{22}$, $\mathbf{A}_{01}$=$\mathbf{A}_{12}$. Then, the inhomogeneous solution can be represented in the recursive form, where all interpolation matrixes are the same in terms of Eq. (28), namely,

$$A\alpha_k^n=b^n, \quad n=N,N-1,\ldots,1,0. \tag{30}$$

Here the RC-MRM interpolation matrix A in Eq. (29) remains unchanged for all iterative steps, while the right-hand side term depends on the *nth*-order particular solution and the differentiated inhomogeneous term as shown in Eq. (26). The solution procedure can be schematized as a recursive process:

$$\alpha^N\rightarrow\alpha^{N-1}\rightarrow\cdots\rightarrow\alpha^0. \tag{31}$$

Correspondingly, in terms of the representation (29), we actually have a solution process of high-towards-lower order homogeneous solutions

$$u_h^N\rightarrow u_h^{N-1}\rightarrow\cdots\rightarrow u_h^0. \tag{32}$$

It should be pointed out that throughout the solution procedure of the present recursive composite multiple reciprocity method, we do not use any inner nodes in the solution of inhomogeneous problems. The algorithm is particularly novel to use the composite high-order differential operator to smooth out inhomogeneous term without extra computing efforts. It is noted that the present approach also can be viewed as the boundary particle method with the high-order Trefftz functions.

4 Numerical results and discussions

In this section we present numerical examples to illustrate the efficiency of the proposed method for several 2D inhomogeneous boundary value problems.

The average relative error $rerr(u)$ and maximum absolute error $merr(u)$ defined as follows are used to measure the accuracy of the numerical results

$$rerr(u) = \sqrt{\frac{\sum_{k=1}^{NT} (u_k - \tilde{u}_k)^2}{\sum_{k=1}^{NT} (u_k)^2}} \tag{33a}$$

$$merr(u) = \max_k |u_k - \tilde{u}_k| \tag{33b}$$

where u_k and $\tilde{u}_k$ are the analytical and numerical solutions evaluated at x_k, respectively, and NT is the total number of evaluated points.

Example 1. Dirichlet Poisson problem with square domain

Consider a Poisson problem with the forcing term$f(x) = 8x_1$ in square domain $[-0.5, 0.5] \times [-0.5, 0.5]$. Its analytical solution is given by

$$u(x) = x_1 \left(x_1^2 + x_2^2\right) + e^{3x_1} \cos(3x_2) \tag{34}$$

whose boundary conditions can easily be derived accordingly.

Fig.1a shows that the numerical accuracy with respect to the number of boundary collocation points *NN*. Roughly speaking, the numerical accuracy first enhances rapidly with an increase in the value of *NN,* and then further increase of *NN* would not gain much improvement in accuracy. It's observed that tens of collocation points (e.g. 57) suffice extreme accuracy. Fig.1b displays the condition number of the RC-MRM interpolation matrix against the number of boundary points *NN*, where condition number *Cond* is defined as the ratio of the largest and smallest singular value.

The profile of the exact solution and errors *err(u)* are shown in Fig. 2, where *err(u)* is defined as pointwise differences between the numerical and exact solutions. The

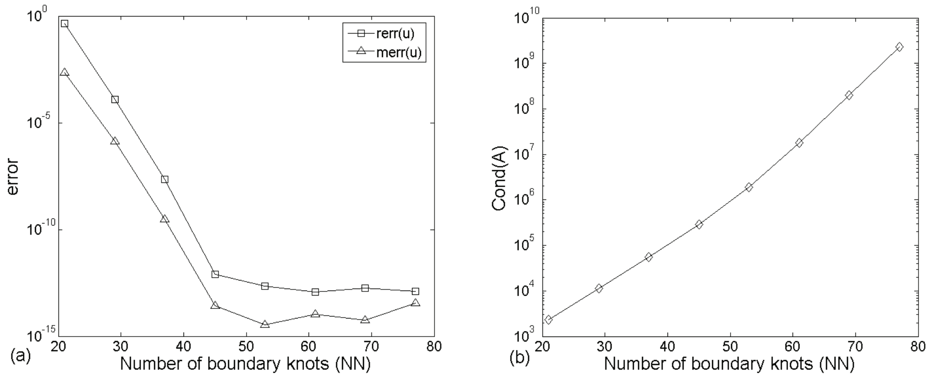

Figure 1: (a) variation of error with respect to *NN* and (b) the condition number of the RC-MRM interpolation matrix.

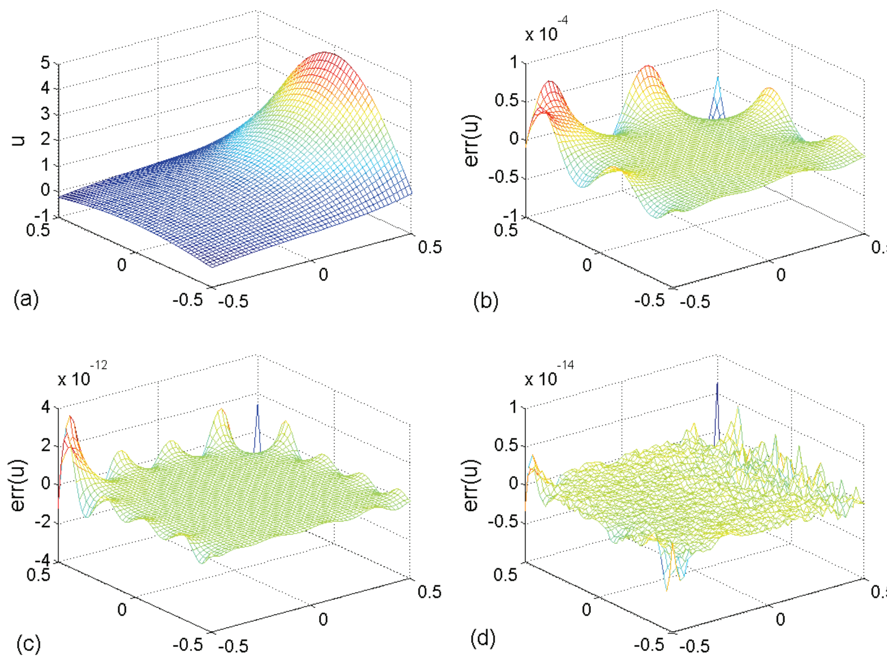

Figure 2: The analytical solution profile (a) of Example 1, and its error surfaces of the numerical results, obtained using (b) 25, (c) 41, and (d) 57 collocation points, respectively.

error surfaces were obtained using at 51×51 knots uniformly-spacing over the unit square. Figs. 2b-2d show that the errors are far pronounced on the boundary, and the maximum error appears occurring around the corners. It's interesting to note that 25 collocation points yield acceptable results, and a few more points (e.g. 41) improve the accuracy remarkably.

Example 2. Dirichlet Poisson problem with irregular domain

In this example, we investigate an irregular domain problem with the forcing term $f(x) = 6x_1$. The analytical solution is given by

$$u(x) = x_1^3 + e^{x_2}(\sin(x_1) + \cos(x_1)) \tag{35}$$

whose boundary conditions can easily be derived accordingly.

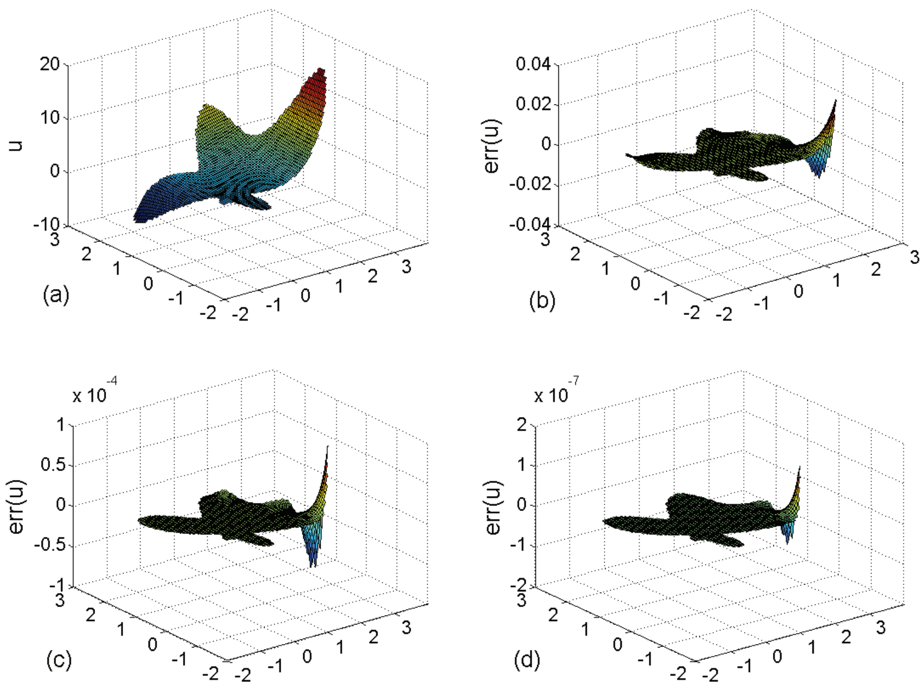

Figure 3: The solution profile (a) of Example 2 and its error surfaces for the numerical results obtained using (b) 17, (c) 25 and (d) 33 collocation points.

Fig. 3 shows the profile of the exact solution and absolute errors for Example 2. It shows that the proposed method could work well with irregular domain problem. Figs. 3b-3d present that the errors are far pronounced on the boundary. Table 1 displays the numerical results obtained using various numbers of boundary collocation points. It is observed that the errors decrease with an increase in the number of points, i.e. *NN*. For *NN*>33, little accuracy improvement can be achieved.

Table 1: Numerical results of Example 2 with various numbers of boundary collocation points.

NN	$rerr(u)$	$merr(u)$	$Cond(A)$
17	9.4559e-04	2.2800e-02	1.7892e+05
21	1.5988e-05	1.6000e-03	5.7041e+06
25	4.9567e-07	8.0766e-05	1.1834e+08
29	3.8655e-08	7.1699e-06	9.8859e+08
33	4.7425e-10	1.0994e-07	4.3696e+10
37	1.0539e-05	2.4000e-03	5.1742e+14
41	3.2464e-08	7.9615e-06	2.4259e+12

Example 3. Dirichlet Helmholtz problem with circular domain

In the third example, we consider a Helmholtz problem with the force term$f(x) = 2\cos(x_1)\cos(x_2)$ in a disc of radii $r = 2$. The analytical solution is given by

$$u(x) = x_1\sin(x_1)\cos(x_2) \tag{36}$$

whose boundary conditions can easily be derived accordingly.

Table 2: Numerical results for Example 3 with various numbers of boundary collocation points.

NN	$rerr(u)$	$merr(u)$	$Cond(A)$
17	6.9100e-02	9.7474e-04	4.3309e+18
25	1.3885e-05	4.1378e-07	1.5497e+22
33	5.4888e-08	1.6807e-09	9.4222e+25
41	1.5012e-10	4.1078e-15	3.4742e+30
49	2.4685e-10	6.0472e-15	1.2074e+35

Table 2 displays the numerical results obtained using various numbers of boundary collocation points. It is observed that the errors decrease with an increase in the number of points. For *NN*>41, little accuracy improvement can be achieved. The solution profile its error surfaces are given in Fig. 4. When 21 and 31 boundary collocation points are used, the error surfaces oscillate less dramatically, and the maximum error appears to occur on the boundary. However, when 41 nodes are employed, the errors seem quite uniform through the whole computational domain.

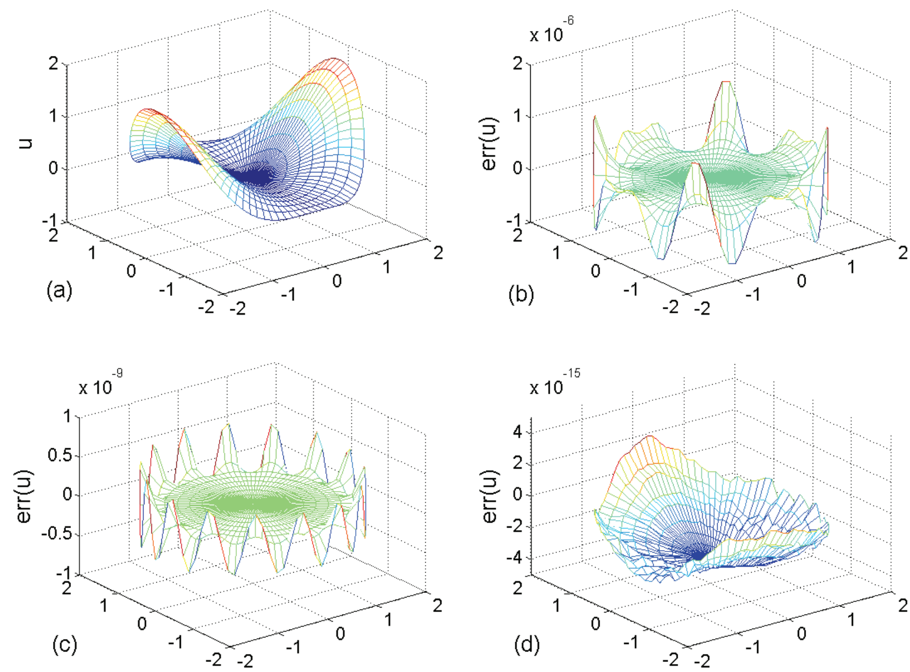

Figure 4: The solution profile (a) of Example 3 and its error surfaces for the numerical results obtained using (b) 21, (c) 31 and (d) 41 collocation points.

5 Conclusions

This paper proposes a family of high-order Trefftz functions for some typical differential operators, and applies them to collocation Trefftz method in conjunction with the recursive composite multiple reciprocity method, which also can be viewed as a kind of boundary particle method using high-order Trefftz functions. Numerical verification shows that the present numerical scheme produces an accurate and stable numerical solution. It should be mentioned that the present scheme solves the inhomogeneous problems without using any inner nodes, which is far more attractive than the other existing numerical methods in the solution of some problems.

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