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An Analytical Method for Computing the One-Dimensional Backward Wave Problem

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Abstract: The present paper reveals a new computational method for the ill-posed backward wave problem. The Fourier series is used to formulate a first-kind Fredholm integral equation for the unknown initial data of velocity. Then, we consider a direct regularization to obtain a second-kind Fredholm integral equation. The termwise separable property of kernel function allows us to obtain an analytical solution of regularization type. The sufficient condition of the data for the existence and uniqueness of solution is derived. The error estimate of the regularization solution is provided. Some numerical results illustrate the performance of the new method.

Keywords: Wave equation, Backward wave problem, Fourier series, Regularization solution, Error estimate, Separable kernel

1 Backward wave problem

The wave motions appeared in many engineering problems, such as stress wave in solids, wave propagation in fluids, scattering problems of electromagnetic waves, as well as the sound wave propagation in media. There were many available methods for solving wave equations [Young and Ruan (2005); Shu, Wu and Wang (2005); Godinho, Tadeu and Amado Mendes (2007); Ma (2007); Gu, Young and Fan (2009)].

For the one-dimensional wave equation the direct problem is to solve

$$u_{tt} = c^2 u_{xx}, \;\; 0 < x < \ell, \;\; 0 < t < T, \tag{1}$$

$$u(0,t) = 0, \;\; u(\ell,t) = 0, \;\; 0 \le t \le T, \tag{2}$$

$$u(x,0) = f(x), \;\; u_t(x,0) = g(x), \;\; 0 \le x \le \ell. \tag{3}$$

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Recently, Young, Gu and Fan (2009), and Gu, Young and Fan (2009) proposed by using a singularity-free Euler-Lagrangian method of fundamental solutions to solve this sort problem.

We consider the backward wave problem (BWP) by replacing Eq. (3) with the following conditions:

$$u(x,0) = f(x), \ \ u(x,T) = h_1(x), \ \ 0 \le x \le \ell. \tag{4}$$

From the view of wave control, we formulate an inverse wave problem by solving the problem that given what initial condition of $u_t(x,0)$ the wave at a time T will be vibrating with the desired quantity $h_1(x)$. The method used at this time was still of the least-square/shooting type and was taking advantage of the control formalism and methodology [Bardos and Rauch (1994); Yang (2006)].

Suppose that $v(x,t)$ is another solution of Eqs. (1), (2) and (4). Define the difference $w(x,t) = u(x,t) - v(x,t)$. For the following equations

$$w_{tt} = c^2 w_{xx}, \ \ 0 < x < \ell, \ \ 0 < t < T,$$
$$w(0,t) = 0, \ \ w(\ell,t) = 0, \ \ 0 \le t \le T,$$
$$w(x,0) = 0, \ \ w(x,T) = 0, \ \ 0 \le x \le \ell,$$

Bourgin and Duffin (1939) and Abdul-Latif and Diaz (1971) have proved that $w(x,t) \equiv 0$ if and only if cT/ℓ=irrational. It means that the BWP has a unique solution only when cT/ℓ=irrational. But when cT/ℓ=rational, the uniqueness is not satisfied.

Eqs. (1), (2) and (4) are known to be ill-posed, since the existence and uniqueness of solution may fail. Fox and Pucci (1958) have studied the existence question of this problem carefully. Levine and Vessella (1985) have studied the existence and continuous dependence on data. The literature in this area is quite meager. However, a recent paper by Lesnic (2002) is devoting to the computation of BWP by using the Adomain decomposition method. He found that for the direct problem the convergence of the Adomain decomposition method is faster than that for the backward problem.

The BWP as pointed out by Ames and Straugham (1997) has important applications in geophysics and optimal control theory. In this paper we propose a different approach to the BWP by searching a sufficient condition on the data $f(x)$ and $h_1(x)$, such that the BWP has a solution. Moreover, we provide an analytical method to search a regularization solution, and also include some numerical results which illustrate the performance of the new method.

In this paper, we cast the BWP into the first-kind Fredholm integral equation, and then we propose a Lavrentiev type regularization to transform it into the second-kind Fredholm integral equation. By utilizing the separating characteristic property of kernel function and eigenfunctions expansion techniques we can derive a closed-form regularization solution of the second-kind Fredholm integral equation. This method was first used by Liu (2007a) to solve a direct problem of elastic torsion in an arbitrary plane domain, where it was called a meshless regularized integral equation method. Then, Liu (2007b, 2007c) extended it to solve the Laplace direct problem in arbitrary plane domains. A similar second-kind Fredholm integral equation regularization method was used to treat the inverse problems; Liu, Chang and Chiang (2008) have applied the new method to determine the geometrical shape of a constant temperature curve, Liu, Chang and Chang (2009) used it to calculate the backward in time advection-dispersion equation, Liu (2009a) has employed this new method to solve the Robin problem in the Laplace equation, and Liu (2009b) has employed it to solve the backward heat conduction problem. The present method would provide us a semi-analytical solution, and renders a more compendious numerical implementation than other schemes to solve the BWP.

2 The Fredholm integral equations

By utilizing the technique of separation of variables we are easy to write a series expansion of $u(x,t)$ satisfying Eqs. (1)-(3):

$$u(x,t) = \sum_{k=1}^{\infty} \left(a_k \cos \frac{kc\pi t}{\ell} + b_k \sin \frac{kc\pi t}{\ell} \right) \sin \frac{k\pi x}{\ell}, \tag{5}$$

where

$$a_k = \frac{2}{\ell} \int_0^\ell \sin \frac{k\pi\xi}{\ell} f(\xi) d\xi, \tag{6}$$

$$b_k = \frac{2}{kc\pi} \int_0^\ell \sin \frac{k\pi\xi}{\ell} g(\xi) d\xi. \tag{7}$$

By imposing the final time condition (4) to Eq. (5) we obtain

$$u(x,T) = \sum_{k=1}^{\infty} \left(a_k \cos \frac{kc\pi T}{\ell} + b_k \sin \frac{kc\pi T}{\ell} \right) \sin \frac{k\pi x}{\ell} = h_1(x). \tag{8}$$

Substituting Eq. (7) for b_k into Eq. (8), it follows that

$$\int_0^\ell K(x,\xi) g(\xi) d\xi = h(x), \tag{9}$$

where

$$K(x,\xi) := \frac{2}{c\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin \frac{kc\pi T}{\ell} \sin \frac{k\pi x}{\ell} \sin \frac{k\pi\xi}{\ell} \tag{10}$$

is a kernel function, and

$$h(x) := h_1(x) - \sum_{k=1}^{\infty} a_k \cos\frac{kc\pi T}{\ell} \sin\frac{k\pi x}{\ell} \tag{11}$$

is a known function.

In order to recover $g(x)$ from the data $h_1(x)$ given at a final time T we have to solve the first-kind Fredholm integral equation (9). On the other hand, we suppose that cT/ℓ is not a positive integer, hence $K(x,\xi) \neq 0$, and Eq. (9) has a solution for $g(x)$.

We assume that there exists a regularization parameter α, such that Eq. (9) can be replaced by a regularized form:

$$\alpha g(x) + \int_0^{\ell} K(x,\xi) g(\xi) d\xi = h(x), \tag{12}$$

which is a second-kind Fredholm integral equation.

About the first-kind Fredholm integral equation (9), Liu and Atluri (2009a,2009b), and Liu, Yeih and Atluri (2009) have developed several novel numerical methods to solve it.

3 Two-point boundary value problem

We assume that the kernel function in Eq. (12) can be approximated by m terms with

$$K(x,\xi) = \frac{2}{c\pi} \sum_{k=1}^{m} \frac{1}{k} \sin\frac{kc\pi T}{\ell} \sin\frac{k\pi x}{\ell} \sin\frac{k\pi\xi}{\ell}. \tag{13}$$

The above kernel is termwise separable, which is also called the degenerate kernel or the Pincherle-Goursat kernel. In the later, we can replace m by ∞ again, after performing all the derivations of required equations.

By the inspection of Eq. (13) we have

$$K(x,\xi) = \mathbf{P}(x) \cdot \mathbf{Q}(\xi), \tag{14}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are m-dimensional vectors given by

$$\mathbf{P} := \frac{2}{c\pi} \begin{bmatrix} \sin\frac{c\pi T}{\ell} \sin\frac{\pi x}{\ell} \\ \frac{1}{2}\sin\frac{2c\pi T}{\ell} \sin\frac{2\pi x}{\ell} \\ \vdots \\ \frac{1}{m}\sin\frac{mc\pi T}{\ell} \sin\frac{m\pi x}{\ell} \end{bmatrix}, \quad \mathbf{Q} := \begin{bmatrix} \sin\frac{\pi\xi}{\ell} \\ \sin\frac{2\pi\xi}{\ell} \\ \vdots \\ \sin\frac{m\pi\xi}{\ell} \end{bmatrix}, \tag{15}$$

and the dot between $\mathbf{P}$ and $\mathbf{Q}$ denotes the inner product, which is sometimes written as $\mathbf{P}^{\mathrm{T}}\mathbf{Q}$, where the superscript T signifies the transpose.

With the aid of Eq. (14), Eq. (12) can be decomposed as

$$\alpha g(x) + \int_0^x \mathbf{P}^{\mathrm{T}}(x)\mathbf{Q}(\xi)g(\xi)d\xi + \int_x^\ell \mathbf{P}^{\mathrm{T}}(x)\mathbf{Q}(\xi)g(\xi)d\xi = h(x). \tag{16}$$

When we define

$$\mathbf{u}_1(x) := \int_0^x \mathbf{Q}(\xi)g(\xi)d\xi, \tag{17}$$

$$\mathbf{u}_2(x) := \int_\ell^x \mathbf{Q}(\xi)g(\xi)d\xi, \tag{18}$$

Eq. (16) can be expressed as

$$\alpha g(x) + \mathbf{P}^{\mathrm{T}}(x)[\mathbf{u}_1(x) - \mathbf{u}_2(x)] = h(x). \tag{19}$$

Taking the differentials of Eqs. (17) and (18) with respect to x we can obtain

$$\mathbf{u}_1'(x) = \mathbf{Q}(x)g(x), \tag{20}$$

$$\mathbf{u}_2'(x) = \mathbf{Q}(x)g(x). \tag{21}$$

Inserting Eq. (19) for $g(x)$ into the above two equations, leads to

$$\alpha\mathbf{u}_1'(x) = \mathbf{Q}(x)\mathbf{P}^{\mathrm{T}}(x)[\mathbf{u}_2(x) - \mathbf{u}_1(x)] + h(x)\mathbf{Q}(x), \;\; \mathbf{u}_1(0) = \mathbf{0}, \tag{22}$$

$$\alpha\mathbf{u}_2'(x) = \mathbf{Q}(x)\mathbf{P}^{\mathrm{T}}(x)[\mathbf{u}_2(x) - \mathbf{u}_1(x)] + h(x)\mathbf{Q}(x), \quad \mathbf{u}_2(\ell) = \mathbf{0}, \tag{23}$$

where the last two conditions follow from Eqs. (17) and (18) readily. The above two equations constitute a two-point boundary value problem.

4 An analytical solution

In this section we will find an analytical solution of $g(x)$. From Eqs. (20) and (21) it can be seen that $\mathbf{u}_1' = \mathbf{u}_2'$, which means that

$$\mathbf{u}_1 = \mathbf{u}_2 + \mathbf{c}, \tag{24}$$

where $\mathbf{c}$ is a constant vector to be determined below. By using the final condition in Eq. (23) we find that

$$\mathbf{u}_1(\ell) = \mathbf{u}_2(\ell) + \mathbf{c} = \mathbf{c}. \tag{25}$$

Substituting Eq. (24) into Eq. (22) we have

$$\alpha \mathbf{u}_1'(x) = -\mathbf{Q}(x)\mathbf{P}^{\mathrm{T}}(x)\mathbf{c} + h(x)\mathbf{Q}(x), \ \ \mathbf{u}_1(0) = \mathbf{0}. \tag{26}$$

Integrating and using the condition $\mathbf{u}_1(0) = \mathbf{0}$, one has

$$\mathbf{u}_1(x) = \frac{-1}{\alpha}\int_0^x \mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi\mathbf{c} + \frac{1}{\alpha}\int_0^x h(\xi)\mathbf{Q}(\xi)d\xi. \tag{27}$$

Taking $x = \ell$ in the above equation and imposing condition (25) one obtains a governing equation for $\mathbf{c}$:

$$\left(\alpha \mathbf{I}_m + \int_0^\ell \mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi\right)\mathbf{c} = \int_0^\ell h(\xi)\mathbf{Q}(\xi)d\xi. \tag{28}$$

It is straightforward to write

$$\mathbf{c} = \left(\alpha \mathbf{I}_m + \int_0^\ell \mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi\right)^{-1}\int_0^\ell h(\xi)\mathbf{Q}(\xi)d\xi. \tag{29}$$

On the other hand, from Eqs. (19) and (24) it follows that

$$\alpha g(x) = h(x) - \mathbf{P}(x)\cdot\mathbf{c}. \tag{30}$$

Inserting Eq. (29) into the above equation we obtain

$$\alpha g(x) = h(x) - \mathbf{P}(x)\cdot\left(\alpha \mathbf{I}_m + \int_0^\ell \mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi\right)^{-1}\int_0^\ell h(\xi)\mathbf{Q}(\xi)d\xi. \tag{31}$$

Due to the orthogonality of

$$\int_0^\ell \sin\frac{j\pi\xi}{\ell}\sin\frac{k\pi\xi}{\ell}d\xi = \frac{\ell}{2}\delta_{jk}, \tag{32}$$

where δ_{jk} is the Kronecker delta, the $m\times m$ matrix can be written as

$$\int_0^\ell \mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi = \frac{\ell}{c\pi}\mathrm{diag}\left[\sin\frac{c\pi T}{\ell}, \frac{1}{2}\sin\frac{2c\pi T}{\ell}, \ldots, \frac{1}{m}\sin\frac{mc\pi T}{\ell}\right], \tag{33}$$

where diag means that the matrix is a diagonal matrix.

Inserting Eq. (33) into Eq. (31) we thus obtain

$$\begin{aligned} g(x) =& \frac{1}{\alpha}h(x) \\ & - \frac{1}{\alpha}\mathbf{P}^{\mathrm{T}}(x)\mathrm{diag}\left[\frac{1}{\alpha + \frac{\ell}{c\pi}\sin\frac{c\pi T}{\ell}}, \frac{1}{\alpha + \frac{\ell}{2c\pi}\sin\frac{2c\pi T}{\ell}}, \ldots, \frac{1}{\alpha + \frac{\ell}{mc\pi}\sin\frac{mc\pi T}{\ell}}\right] \\ & \int_0^\ell h(\xi)\mathbf{Q}(\xi)d\xi. \end{aligned} \tag{34}$$

While we use Eq. (15) for **P** and **Q**, we can get

$$g(x) = \frac{1}{\alpha}h(x) - \frac{2}{\alpha c\pi}\sum_{k=1}^{\infty}\frac{\frac{1}{k}\sin\frac{kc\pi T}{\ell}}{\alpha+\frac{\ell}{kc\pi}\sin\frac{kc\pi T}{\ell}}\int_0^\ell \sin\frac{k\pi x}{\ell}\sin\frac{k\pi\xi}{\ell}h(\xi)d\xi, \tag{35}$$

where the summation upper bound m is now replaced by ∞, because our argument is independent of m.

At this moment it is impossible to take the limit of $\alpha = 0$ in Eq. (35). In order to get a formula where the limit of $\alpha = 0$ can be realized, we insert Eq. (35) for $g(x)$ into Eq. (7) and utilize Eq. (32) to get

$$b_k^\alpha = \frac{2}{kc\pi\alpha+\ell\sin\frac{kc\pi T}{\ell}}\int_0^\ell \sin\frac{k\pi\xi}{\ell}h(\xi)d\xi. \tag{36}$$

For a given $h(x)$, through some integrals one may employ the above equation to calculate b_k^α. However, in order to distinct it from the b_k given in Eq. (7) we add a superscript α to stress that they are regularized coefficients.

If b_k^α is available we can calculate $u(x,t)$ at any time $t < T$ by

$$u^\alpha(x,t) = \sum_{k=1}^{\infty}\left(a_k\cos\frac{kc\pi t}{\ell} + b_k^\alpha\sin\frac{kc\pi t}{\ell}\right)\sin\frac{k\pi x}{\ell}, \tag{37}$$

where a_k is still defined by Eq. (6). Eqs. (37), (6) and (36) constitute an analytical solution of the BWP. In order to distinct it from the exact solution $u(x,t)$ we use the symbol $u^\alpha(x,t)$ to denote it to be a regularization solution.

5 Two main results

In the previous section we have derived a regularization solution $u^\alpha(x,t)$ of Eqs. (1), (2) and (4) under the regularized format (12) with a regularization parameter $\alpha > 0$. According to these results we can prove the following theorems.

Theorem 1: The backward wave problem (1), (2) and (4) has a unique solution if the ratio cT/ℓ is irrational and the function $h(x)$ satisfies the following condition:

$$\sum_{k=1}^{\infty}\frac{k^2}{\sin^2\frac{kc\pi T}{\ell}}\left(\int_0^\ell \sin\frac{k\pi\xi}{\ell}h(\xi)d\xi\right)^2 < \infty. \tag{38}$$

Proof: Taking $\alpha = 0$ in Eq. (36) and inserting it into Eq. (37) we have a formal exact solution of Eqs. (1), (2) and (4):

$$u(x,t) = \sum_{k=1}^{\infty}\left(a_k\cos\frac{kc\pi t}{\ell} + b_k^\star\sin\frac{kc\pi t}{\ell}\right)\sin\frac{k\pi x}{\ell}, \tag{39}$$

where

$$b_k^\star = \frac{2}{\ell \sin \frac{kc\pi T}{\ell}} \int_0^\ell \sin \frac{k\pi\xi}{\ell} h(\xi) d\xi. \tag{40}$$

Since the ratio cT/ℓ is irrational, $\sin(kc\pi T/\ell) \neq 0$, and the above $b_k^\star$ is well-defined for all positive integer k.

By applying the Parseval's equality to the Fourier sine series of $u_t(x,0) = g(x) \in L^2(0,\ell)$:

$$g(x) = \sum_{k=1}^{\infty} b_k^\star \frac{kc\pi}{\ell} \sin \frac{k\pi x}{\ell}, \tag{41}$$

which is obtained from Eq. (39) by taking its differential with respect to time and then inserting $t = 0$, we obtain

$$\|g(x)\|_{L^2(0,\ell)} = \sum_{k=1}^{\infty} \frac{4k^2c^2\pi^2}{\ell^4 \sin^2 \frac{kc\pi T}{\ell}} \left(\int_0^\ell \sin \frac{k\pi\xi}{\ell} h(\xi) d\xi \right)^2.$$

By using $\|g(x)\|_{L^2(0,\ell)} < \infty$ and dividing it by the factor $4c^2\pi^2/\ell^4$ we obtain Eq. (38). This ends the proof. $\square$

However, we require $|\sin(kc\pi T/\ell)| \geq C_0 > 0$ for some positive constant C_0, in order to avoid the zero of $\sin(kc\pi T/\ell)$, which means that cT/ℓ cannot be a positive integer. This is true since cT/ℓ is irrational. Because of

$$k^2 \leq \frac{k^2}{\sin^2 \frac{kc\pi T}{\ell}},$$

from Eq. (38) we can also obtain

$$\sum_{k=1}^{\infty} k^2 \left(\int_0^\ell \sin \frac{k\pi\xi}{\ell} h(\xi) d\xi \right)^2 < \infty. \tag{42}$$

Now, we can prove the following theorem.

Theorem 2: For the backward wave problem (1), (2) and (4), if the ratio cT/ℓ is irrational and the function $h(x)$ satisfies condition (42), and there exists a $C_0 > 0$ such that $|\sin(kc\pi T/\ell)| \geq C_0 > 0$ and moreover,

$$\frac{4c^2\pi^2}{\rho \ell^4 C_0^4} \sum_{k=1}^{\infty} k^2 \left(\int_0^\ell \sin \frac{k\pi\xi}{\ell} h(\xi) d\xi \right)^2 := M^2 < \infty, \tag{43}$$

then for any $t \in [0,T)$ the regularization solution $u^\alpha(x,t)$ satisfies the following error estimation:

$$\|u^\alpha(x,t) - u(x,t)\|_{L^2(0,\ell)} \le \alpha M. \tag{44}$$

Proof: From Eqs. (37), (36), (39) and (40) it follows that

$$u(x,t) - u^\alpha(x,t) = \sum_{k=1}^{\infty} c_k \sin\frac{kc\pi t}{\ell} \sin\frac{k\pi x}{\ell}, \tag{45}$$

where

$$c_k = \frac{2\alpha}{\ell \sin\frac{kc\pi T}{\ell}[\alpha + \frac{\ell}{kc\pi}\sin\frac{kc\pi T}{\ell}]} \int_0^\ell \sin\frac{k\pi\xi}{\ell} h(\xi)d\xi. \tag{46}$$

Therefore, by using $|\sin(kc\pi T/\ell)| \ge C_0$ we have

$$\|u(x,t) - u^\alpha(x,t)\|^2_{L^2(0,\ell)} \le \frac{4\alpha^2}{\ell^2 C_0^2} \sum_{k=1}^{\infty} \frac{1}{[\alpha + \frac{\ell}{kc\pi}\sin\frac{kc\pi T}{\ell}]^2} \left(\int_0^\ell \sin\frac{k\pi\xi}{\ell} h(\xi)d\xi\right)^2. \tag{47}$$

For the term

$$\left[\alpha + \frac{\ell}{kc\pi}\sin\frac{kc\pi T}{\ell}\right]^2$$

we need to consider the possibility that $\sin(kc\pi T/\ell)$ may be negative. It is constrained by $\sin(kc\pi T/\ell) \ge -C_0$, such that

$$\left[\alpha + \frac{\ell}{kc\pi}\sin\frac{kc\pi T}{\ell}\right]^2 \ge \alpha^2 - 2\alpha\frac{\ell C_0}{kc\pi} + \frac{\ell^2 C_0^2}{(kc\pi)^2}.$$

There exists an ρ with $1 > \rho > 0$, such that, when $\alpha \le (1-\sqrt{\rho})\ell C_0/(kc\pi)$, the following inequality holds:

$$\left[\alpha + \frac{\ell}{kc\pi}\sin\frac{kc\pi T}{\ell}\right]^2 \ge \alpha^2 - 2\alpha\frac{\ell C_0}{kc\pi} + \frac{\ell^2 C_0^2}{(kc\pi)^2} \ge \frac{\rho\ell^2 C_0^2}{(kc\pi)^2} > 0.$$

Then, from Eq. (47) we have the following estimation:

$$\|u(x,t) - u^\alpha(x,t)\|^2_{L^2(0,\ell)} \le \alpha^2 \frac{4c^2\pi^2}{\rho\ell^4 C_0^4} \sum_{k=1}^{\infty} k^2 \left(\int_0^\ell \sin\frac{k\pi\xi}{\ell} h(\xi)d\xi\right)^2. \tag{48}$$

By condition (42) the summation term is finite, and we may let

$$M^2 = \frac{4c^2\pi^2}{\rho\ell^4 C_0^4}\sum_{k=1}^{\infty} k^2 \left(\int_0^{\ell} \sin\frac{k\pi\xi}{\ell} h(\xi) d\xi\right)^2 < \infty. \tag{49}$$

Therefore, by Eqs. (48) and (49) we complete the proof. □

The above theorems are crucial to identify that the proposed regularization is workable.

6 Numerical examples

In order to compare our numerical result with that obtained by Lesnic (2002), let us first consider Example 1 of one-dimensional BWP with the solution $u(x,t) = \sin x \sin t$. We have $c = 1$, $\ell = \pi$ and

$$u(x,0) = f(x) = 0, \ \ u(x,T) = h_1(x) = \sin x \sin T, \ \ 0 < x < \pi. \tag{50}$$

It is known that the above BWP has a unique solution if and only if the ratio π/T is an irrational number.

Because of $f(x) = 0$, from Eq. (6) we have $a_k = 0$, and thus by Eqs. (11) and (50), $h(x) = \sin x \sin T$. Inserting it into Eq. (36) we obtain

$$b_k^\alpha = \frac{2\sin T}{k\pi\alpha + \pi\sin(kT)} \int_0^{\pi} \sin(k\xi)\sin\xi d\xi. \tag{51}$$

By using Eq. (32) it is only that

$$b_1^\alpha = \frac{\sin T}{\alpha + \sin T} \tag{52}$$

is non-zero, and all the other coefficients are $b_k^\alpha = 0, \ \ k \geq 2.$ Hence, by Eq. (37) we get a regularization solution,

$$u^\alpha(x,t) = b_1^\alpha \sin t \sin x = \frac{\sin T}{\alpha + \sin T} \sin t \sin x. \tag{53}$$

It is interesting that the above solution is identical to the exact solution when $\alpha = 0$.

Let $T = 1$. We have compared the exact solution at $t = 0.5$ with the regularization solutions with $\alpha = 10^{-5}, 10^{-10}$ in Fig. 1(a). However, these curves are almost coincident without showing difference. Therefore, we display the errors in Fig. 1(b). It can be seen that the errors are smaller than α used. Lesnic (2002) has applied the Adomain decomposition method to this problem. However, the solution given

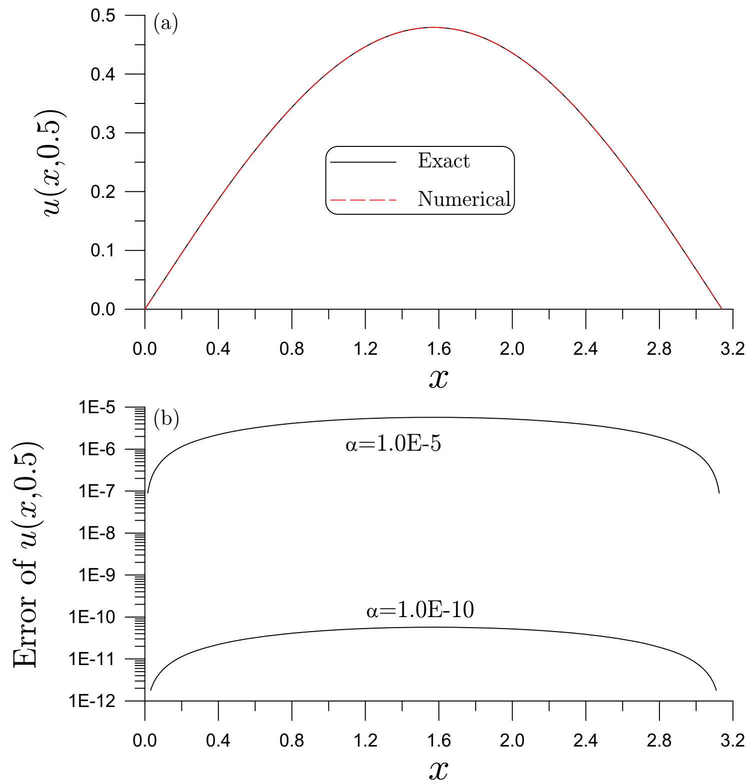

Figure 1: For the backward wave problem we have compared regularized and exact solutions in (a) and the numerical errors in (b) for example 1.

by Lesnic is convergent very slowly, about requiring 1000 terms, and even so, the Lesnic's solution is not accurate than our results.

Next, for Example 2 we compute a more complex problem with zero initial displacement $f(x) = 0$ and a discontinuous initial velocity:

$$u_t(x,0) = \begin{cases} x, & \text{for } 0 \le x \le \frac{\ell}{4}, \\ 0, & \text{for } \frac{\ell}{4} < x \le \ell. \end{cases} \tag{54}$$

The exact solution is given by

$$u(x,t) = \frac{2\ell^2}{c\pi^2} \sum_{k=1}^{\infty} \left[\frac{1}{k^2\pi} \sin\frac{k\pi}{4} - \frac{1}{4k} \cos\frac{k\pi}{4} \right] \sin\frac{k\pi x}{\ell} \sin\frac{kc\pi t}{\ell}. \tag{55}$$

The above is not a backward problem. However, we use it to provide the final

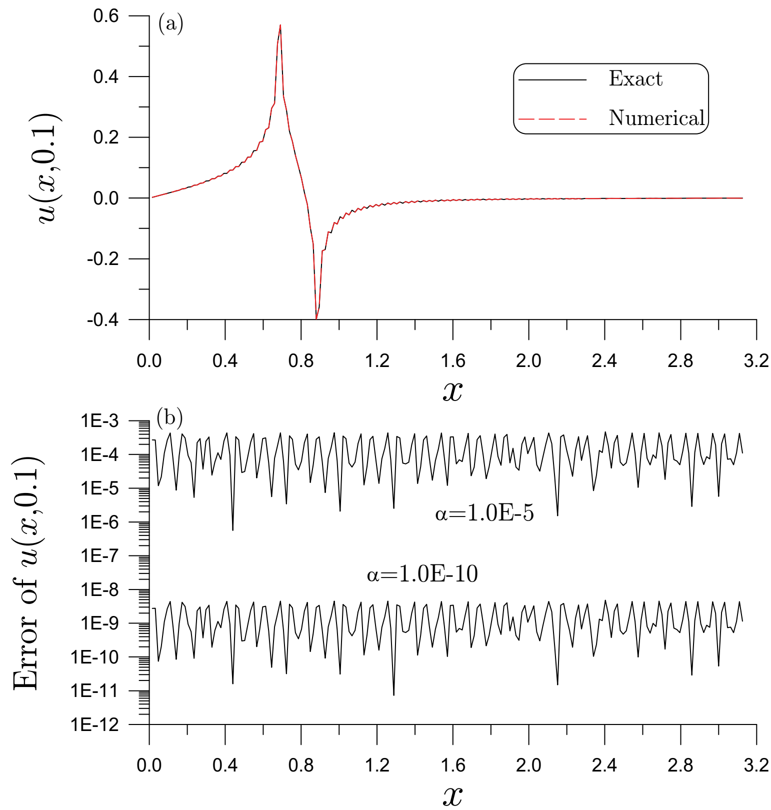

Figure 2: For example 2 we have compared regularized and exact solutions at time $t = 0.1$ in (a) and the numerical errors in (b).

condition at a time T:

$$h(x) = h_1(x) = u(x,T) = \frac{2\ell^2}{c\pi^2} \sum_{k=1}^{\infty} \left[\frac{1}{k^2\pi} \sin\frac{k\pi}{4} - \frac{1}{4k} \cos\frac{k\pi}{4} \right] \sin\frac{k\pi x}{\ell} \sin\frac{kc\pi T}{\ell}. \tag{56}$$

The difficulty of this problem is originated from that we use a smooth final data to retrieve a non-smooth initial data.

In the above we have set $h(x) = h_1(x)$, because of $f(x) = 0$, and hence $a_k = 0$. Substituting Eq. (56) for $h(x)$ into Eq. (36) we obtain

$$b_k^\alpha = \frac{2\ell^3}{c\pi^2[kc\pi\alpha + \ell \sin\frac{kc\pi T}{\ell}]} \left[\frac{1}{k^2\pi} \sin\frac{k\pi}{4} - \frac{1}{4k} \cos\frac{k\pi}{4} \right] \sin\frac{kc\pi T}{\ell}. \tag{57}$$

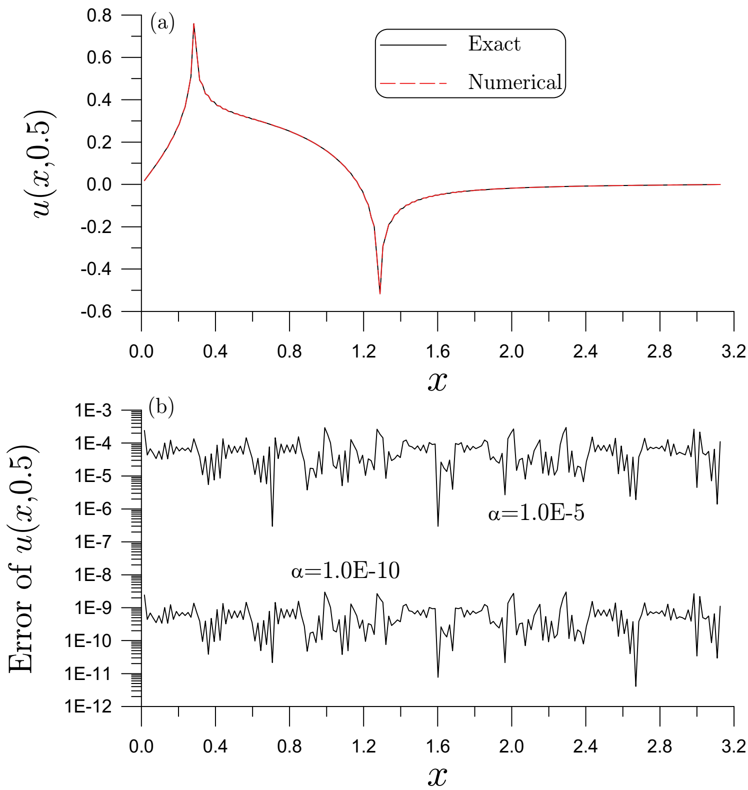

Figure 3: For example 2 we have compared regularized and exact solutions at time $t = 0.5$ in (a) and the numerical errors in (b).

At the same time the regularization solution is given by

$$u^\alpha(x,t) = \sum_{k=1}^{\infty} b_k^\alpha \sin\frac{kc\pi t}{\ell} \sin\frac{k\pi x}{\ell}. \tag{58}$$

Let $c = 1$, $\ell = \pi$ and $T = 1$. We have compared the exact solution at $t = 0.1$ with the regularization solutions with $\alpha = 10^{-5}, 10^{-10}$ in Fig. 2(a). In practice, the data are obtained by taking the sum of the first two hundred terms from Eqs. (55) and (58), which guarantees the convergence of the series. However, the curves in Fig. 2(a) are almost coincident without showing difference, and we display the errors in Fig. 2(b). It can be seen that the errors are very small.

Similarly, we have compared the exact solution at $t = 0.5$ with the regularization solutions with $\alpha = 10^{-5}, 10^{-10}$ in Fig. 3(a). However, these curves are almost coincident without showing difference. Therefore, we display the errors in Fig. 3(b).

It can be seen that the errors are very small. From Figs. 2 and 3, it can be seen that the initial discontinuity of velocity is propagated and the amplitude of discontinuity is amplified, replaced the initial vertical jump by a sharp transition thin layer and then the layer being expanded gradually.

7 Conclusions

The present paper has addressed a new idea to solve the backward wave problem by a linear integral equation. The ill-posedness of this problem is fully reflected in the first-kind Fredholm integral equation. Fortunately, by employing the Fourier series expansion technique and a termwise separable property of kernel function, an analytical regularization solution for approximating the exact solution can be derived exactly. The regularization solution was shown to be convergent to the exact solution and the error estimation was provided. The numerical examples have shown that the new method could provide very excellent numerical results.

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