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A Fictitious Time Integration Method for the Burgers Equation

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Abstract: When the given input data are corrupted by an intensive noise, most numerical methods may fail to produce acceptable numerical solutions. Here, we propose a new numerical scheme for solving the Burgers equation forward in time and backward in time. A fictitious time τ is used to transform the dependent variable $u(x,t)$ into a new one by $(1+\tau)u(x,t) =: v(x,t,\tau)$, such that the original Burgers equation is written as a new parabolic type partial differential equation in the space of (x,t,τ). A fictitious damping coefficient can be used to strengthen the stability in the numerical integration of a semi-discretized ordinary differential equations set on the spatial-temporal grid points. Even for a very large final time and under a large noise, the present Fictitious Time Integration Method (FTIM) can be used to retrieve the initial data very well. When the FTIM is used to solve the direct problems of Burgers equation, with a large Reynolds number and the input data being noised seriously, we can still reconstruct the solution rather accurately. This result however cannot be achieved by other conventional numerical methods. It is interesting that both the forward and backward problems of Burgers equation can be unifiedly treated by the FTIM.

Keywords: Backward Burgers equation, Forward Burgers equation, Fictitious Time Integration Method (FTIM), Seriously noised effect, Large Reynolds number

1 Introduction

In this paper we are concerned with the numerical solutions of a backward in time and a forward in time Burgers equation:

$$u_t + uu_x = \frac{1}{R}u_{xx} + H(x,t), \quad a < x < b, \quad 0 < t < T, \tag{1}$$

$$u(a,t) = u_a(t), \quad u(b,t) = u_b(t), \quad 0 \le t \le T, \tag{2}$$

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$$u(x,T) = f(x), \quad a \le x \le b, \quad \text{(Backward Problem)}, \tag{3}$$

$$u(x,0) = f(x), \quad a \le x \le b, \quad \text{(Direct Problem)}, \tag{4}$$

where R is the Reynolds number characterizing the viscosity of fluid. Given a velocity function $f(x)$ at a final time $t = T$, the backward problem is retrieving the past history and the initial profile of fluid velocity. When the forward in time Burgers equation is considered, we use the initial condition $u(x,0) = f(x)$ in Eq. (4), and the boundary conditions in Eq. (2) to obtain the numerical solution. In this paper the data of $f(x)$, $u_a(t)$ and $u_b(t)$ are allowed to be noised like as $\hat{f}(x) = f(x) + sR$, where R are random numbers in $[-1,1]$, and s is a level of noise intensity.

Burgers equation is an ideally suited test problem for the numerical solution of partial differential equation (PDE), because it is nonlinear and there is a parameter R, which can be selected to change the equation from predominantly parabolic with R finite to predominantly hyperbolic with $R = \infty$. Usually, the latter problem is much more difficult to be numerically solved than the former one.

Burgers' equation has been of considerable physical interest because it is an appropriate simplification of the Navier-Stokes equations, and is also the governing equation for a number of one-dimensional flow systems, including the convection and diffusion of heat, weak shock propagation, compressible turbulence, and continuum traffic simulation.

The Burgers equation was first appeared in a paper by Bateman (1915) and was named after Burgers (1948,1974). The behavior of Burgers equation exhibits a delicate balance between advection and diffusion. Moreover, it is one of the few nonlinear partial differential equations that exact and complete solutions are known in terms of the initial values [Cole (1951); Hopf (1950)]. In the past several decades there were much studies on the numerical solutions of Burgers' equation, for example, Fletcher (1983), Basdevant, Deville and Haldenwang (1986), Arina and Canuto (1993), Özis and Özdes (1996), Hon and Mao (1998), Kutluay, Bahadir and Özdes (1999), Lin and Zhou (2001), Wei and Gu (2002), Özis, Aksan and Özdes (2003), Young (2005), Young, Hu, Fan and Chen (2006), and Liu (2006a).

For the direct problems, the conventional numerical methods have some drawbacks. First, the time stepsize is constrained by some inequalities for the reason of numerical stability. Second, when the initial data are corrupted by noise, the error will propagate to pollute the whole solutions because these numerical methods are integrated in the direction of time. Examples 1 and 2 in Section 3 will demonstrate those phenomena. In this paper, one of our purposes is developing a new numerical method of fictitious time integration method (FTIM), which not only allows much larger time stepsize, but also can avoid the numerical solutions polluted by the noisy disturbance on the input data. It would be very interesting that the present approach is performed much better than other numerical methods from the aspects of stability and accuracy, when the input data are corrupted by noise.

It is known that the backward problems are essentially unstable, if one attempts to integrate the Burgers equation in a reversal time direction. The present approach of FTIM however will lead to a new integration method in a fictitious time direction. Therefore, we can not only avoid the above-mentioned instability, but also provide a high performance technique to tackle of the severely ill-posed backward in time Burgers equation. It would be seen that the new FTIM is highly stable, insensitive to the disturbance on final time data, and highly accurate. Indeed, the FTIM would render a more laconic numerical process than other methods to resolve the backward problems.

While most papers are concerned with the numerical integrations of the forward problems of Burgers equation, there are only a few papers which are devoted to the backward problems of Burgers equation, for example, Carasso (1977), and Marbán and Palencia (2002). Liu (2006b) has developed a one-step backward group preserving scheme to calculate the backward in time Burgers equation. However, the final time is not allowed too large. Here we will propose a new numerical scheme for solving the Burgers equation backward in time, allowing a large final time and a large noise imposing on the input data.

Numerical schemes adopted for backward problems are usually implicit. The explicit schemes that have been applied to solving the backward problems are apparently not very effective. As mentioned by Ames and Epperson (1997), because the backward problems are ill-posed, they are necessarily ill-conditioned from a numerical point of view, and the problem must be regularized before any approximation can be constructed. Although, most people assert that the backward problems are impossible to solve using the classical numerical methods and require a special regularization technique, we shall show that the new FTIM can resolve this ill-posed problem without resorting on regularization technique.

The idea by introducing a fictitious time was first proposed by Liu (2008a) to treat an inverse Sturm-Liouville problem by transforming an ODE into a PDE. Then, Liu and his coworkers [Liu (2008b, 2008c, 2008d); Liu, Chang, Chang and Chen (2008)] extended this idea to develop new methods for estimating parameters in the inverse vibration problems. Liu and Atluri (2008a) have employed the technique of FTIM to solve a large system of nonlinear algebraic equations, and showed that high performance can be achieved by using the FTIM. More recently, Liu (2008e) has used the FTIM technique to solve the nonlinear complementarity problems, whose numerical results are very well. Then, Liu (2008f) used the FTIM to solve the boundary value problems of elliptic type partial differential equations. Liu and Atluri (2008b) also employed this technique of FTIM to solve mixed-complementarity problems and optimization problems. Then, Liu and Atluri (2008c) using the technique of FTIM solved the inverse Sturm-Liouville problem, for specified eigenvalues.

This paper is organized as follows. In Section 2 we introduce the concept of fictitious time integration method. Section 2.1 devotes to a spatial transformation of the Burgers equation when the Reynolds number is large. In Section 2.2 we transform the original Burgers equation into another parabolic PDE by using the fictitious time. In Section 2.3 we transform these PDEs into the ODEs in terms of the fictitious time-like variable by discretizing the quantity on the spatial-temporal points. The group-preserving scheme (GPS) is introduced in Section 2.4, and the numerical procedures are described in Section 2.5. Numerical examples for direct problems and backward problems are set in Section 3. Then, we give the conclusions in Section 4.

2 A fictitious time integration method

2.1 Spatial Transformation

It is known that the Burgers equation is hard to be numerically solved when the Reynolds number is very large. We can consider a scalar transformation of x-coordinate by

$$y=\frac{1}{2A}\ln\frac{b-a+(x-a)\tanh A}{b-a-(x-a)\tanh A}. \tag{5}$$

When $A=0$, we can derive $x=a+(b-a)y$ by a limiting process.

From Eqs. (1) and (5) it follows that

$$u_t+\frac{\tanh A}{(b-a)A[1-\tanh^2(Ay)]}uu_y=\frac{\tanh^2 A}{(b-a)^2RA^2[1-\tanh^2(Ay)]^2}u_{yy} \\ +\frac{2\tanh^2 A\tanh(Ay)}{(b-a)^2RA[1-\tanh^2(Ay)]^2}u_y+H(y,t), \tag{6}$$

where $H(y,t)$ is defined in the interval of $0\leq y\leq 1$. At the same time, the boundary conditions are defined by $u(0,t)=u_a(t)$ and $u(1,t)=u_b(t)$. When we discretize the above equation by a finite difference with a uniform spacing length of y-coordinate, the transformation in Eq. (5) can accumulate much grid points in the region where the solution appears large variation viewed in the x-coordinate, and place a small number of grid points in the region where the solution does not change rapidly.

2.2 Transformation into a new PDE

2.2.1 Transformation into a new PDE for direct problem

We propose the following transformation:

$$v(y,t,\tau) = (1+\tau)u(y,t), \tag{7}$$

and introduce a fictitious viscosity damping coefficient $\nu > 0$ in Eq. (6):

$$\nu\left[\frac{\tanh^2 A}{(b-a)^2RA^2[1-\tanh^2(Ay)]^2}u_{yy} + \frac{2\tanh^2 A\tanh(Ay)}{(b-a)^2RA[1-\tanh^2(Ay)]^2}u_y + H(y,t)\right.$$
$$\left. -u_t - \frac{\tanh A}{(b-a)A[1-\tanh^2(Ay)]}uu_y\right] = 0. \tag{8}$$

Multiplying the above equation by $1+\tau$ and using Eq. (7) we have

$$\nu\left[\frac{\tanh^2 A}{(b-a)^2RA^2[1-\tanh^2(Ay)]^2}v_{yy} + \frac{2\tanh^2 A\tanh(Ay)}{(b-a)^2RA[1-\tanh^2(Ay)]^2}v_y + (1+\tau)H(y,t)\right.$$
$$\left. -v_t - \frac{\tanh A}{(b-a)A[1-\tanh^2(Ay)](1+\tau)}vv_y\right] = 0. \tag{9}$$

Recalling that $\partial v/\partial\tau = u(y,t)$ by Eq. (7), adding it on both sides of the above equation, and replacing u by $v/(1+\tau)$, we can change Eq. (6) into a new type parabolic PDE for v:

$$\frac{\partial v}{\partial \tau} = \nu\left[\frac{\tanh^2 A}{(b-a)^2RA^2[1-\tanh^2(Ay)]^2}v_{yy} + \frac{2\tanh^2 A\tanh(Ay)}{(b-a)^2RA[1-\tanh^2(Ay)]^2}v_y\right.$$
$$\left. +(1+\tau)H(y,t) - v_t - \frac{vv_y\tanh A}{(b-a)A[1-\tanh^2(Ay)](1+\tau)}\right] + \frac{v}{1+\tau}. \tag{10}$$

This equation is subjected to the following conditions:

$$v(0,t,\tau) = (1+\tau)u_a(t), \;\; v(1,t,\tau) = (1+\tau)u_b(t), \;\; 0 \le t \le T, \tag{11}$$

$$v(y,0,\tau) = (1+\tau)f(y), \;\; 0 \le y \le 1. \tag{12}$$

2.2.2 Transformation into a new PDE for backward problem

Similarly, we propose the following transformation for the backward problem:

$$v(x,t,\tau) = (1+\tau)u(x,t), \tag{13}$$

and introduce a fictitious damping coefficient $\nu > 0$ in Eq. (1):

$$\nu\left[\frac{1}{R}u_{xx} + H(x,t) - u_t - uu_x\right] = 0. \tag{14}$$

By using Eq. (13) we have

$$\frac{\nu}{1+\tau}\left[\frac{1}{R}v_{xx}+(1+\tau)H(x,t)-v_t-\frac{vv_x}{1+\tau}\right]=0. \tag{15}$$

Recalling that $\partial v/\partial\tau = u(x,t)$ by Eq. (13), and adding it on both sides of the above equation we obtain

$$\frac{\partial v}{\partial\tau}=\frac{\nu}{1+\tau}\left[\frac{1}{R}v_{xx}+(1+\tau)H(x,t)-v_t-\frac{vv_x}{1+\tau}\right]+u. \tag{16}$$

Then, by using $u=v/(1+\tau)$ we can change Eq. (1) into a new type parabolic PDE for v:

$$\frac{\partial v}{\partial\tau}=\frac{\nu}{1+\tau}\left[\frac{1}{R}v_{xx}+(1+\tau)H(x,t)-v_t-\frac{vv_x}{1+\tau}\right]+\frac{v}{1+\tau}. \tag{17}$$

Upon using

$$\frac{\partial}{\partial\tau}\left(\frac{v}{1+\tau}\right)=\frac{v_\tau}{1+\tau}-\frac{v}{(1+\tau)^2},$$

Eq. (17), after multiplying the integrating factor $1/(1+\tau)$ on both sides, can be further reduced to

$$\frac{\partial}{\partial\tau}\left(\frac{v}{1+\tau}\right)=\frac{\nu}{1+\tau}\left[\frac{1}{R(1+\tau)}v_{xx}+H(x,t)-\frac{v_t}{1+\tau}-\frac{vv_x}{(1+\tau)^2}\right]. \tag{18}$$

Now, by using $v/(1+\tau)=u$ again, we can change Eq. (1) into a new type parabolic PDE for u:

$$u_\tau=\frac{\nu}{1+\tau}\left[\frac{1}{R}u_{xx}+H(x,t)-u_t-uu_x\right]. \tag{19}$$

Here, we must stress that u is an unknown function with $u=u(x,t,\tau)$, subjecting to the constraints in Eqs. (2) and (3) for all $\tau\geq 0$, and $u(x,t,\tau=0)$ is given initially by a guess.

2.3 Semi-discretizations

2.3.1 Semi-discretization for the direct problem

Let $v_i^j(\tau):=v(y_i,t_j,\tau)$ be a numerical value of v at the grid point (y_i,t_j) and at a fictitous time τ. Applying a semi-discretization to the PDE in Eq. (10) yields a coupled system of ordinary differential equations (ODEs):

$$\frac{d}{d\tau}v_i^j=\nu\left[\frac{\tanh^2 A}{(b-a)^2RA^2[1-\tanh^2(Ay_i)]^2(\Delta y)^2}(v_{i+1}^j-2v_i^j+v_{i-1}^j)\right.$$

$$+\frac{2\tanh^2 A\tanh(Ay_i)}{2(b-a)^2RA[1-\tanh^2(Ay_i)]^2\Delta y}(v^j_{i+1}-v^j_{i-1})+(1+\tau)H_{ij}-\frac{1}{\Delta t}(v^j_i-v^{j-1}_i)$$
$$-\frac{\tanh A}{2(b-a)A[1-\tanh^2(Ay_i)](1+\tau)\Delta y}v^j_i(v^j_{i+1}-v^j_{i-1})\Bigg]+\frac{v^j_i}{1+\tau}, \tag{20}$$

where $\Delta y = 1/(m_1+1)$ and $\Delta t = T/m_2$ are uniform grid lengths in the y and t directions, and m_1 and m_2 are respectively the numbers of subintervals in each direction. For a short notation, we also use $H_{ij} = H(y_i,t_j)$.

2.3.2 *Semi-discretization for the backward problem*

Let $u^j_i(\tau) := u(x_i,t_j,\tau)$ be a numerical value of u at the grid point (x_i,t_j) and at a fictitous time τ. Applying a semi-discretization to the above PDE in Eq. (19) yields

$$\frac{d}{d\tau}u^j_i = \frac{\nu}{1+\tau}\left(\frac{1}{R(\Delta x)^2}[u^j_{i+1}-2u^j_i+u^j_{i-1}]-\frac{1}{\Delta t}[u^{j+1}_i-u^j_i]\right.$$
$$\left.-\frac{1}{2\Delta x}u^j_i[u^j_{i+1}-u^j_{i-1}]+H_{ij}\right), \tag{21}$$

where $\Delta x = (b-a)/(m_1+1)$ and $H_{ij} = H(x_i,t_j)$.

2.4 *The GPS for differential equations system*

Upon letting $\mathbf{u} = (u_{1,1},u_{1,2},\ldots,u_{m_1,m_2})^{\mathrm{T}}$ and $\mathbf{f}$ denoting a vector with the ij-th component being the right-hand side of Eq. (21) we can write it as a vector form:

$$\mathbf{u}' = \mathbf{f}(\mathbf{u},\tau),\ \ \mathbf{u}\in\mathbb{R}^n,\ \ \tau\in\mathbb{R}, \tag{22}$$

where $\mathbf{u}'$ denotes the differential of $\mathbf{u}$ with respect to τ, and $n = m_1m_2$ is the number of total grid points inside the domain $\Omega = (a,b)\times[0,T)$.

Group-preserving scheme (GPS) can preserve the internal symmetry group of the considered ODE system. Although we do not know previously the symmetry group of differential equations system, Liu (2001) has embedded it into an augmented differential system, which concerns with not only the evolution of state variables themselves but also the evolution of the magnitude of the state variables vector. Let us note that

$$\|\mathbf{u}\| = \sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{u}} = \sqrt{\mathbf{u}\cdot\mathbf{u}}, \tag{23}$$

where the superscript T signifies the transpose, and the dot between two n-dimensional vectors denotes their inner product. Taking the derivatives of both the sides of Eq. (23) with respect to τ, we have

$$\frac{d\|\mathbf{u}\|}{d\tau} = \frac{(\mathbf{u}')^{\mathrm{T}}\mathbf{u}}{\sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{u}}}. \tag{24}$$

Then, by using Eqs. (22) and (23) we can derive

$$\frac{d\|\mathbf{u}\|}{d\tau} = \frac{\mathbf{f}^{\mathrm{T}}\mathbf{u}}{\|\mathbf{u}\|}. \tag{25}$$

It is interesting that Eqs. (22) and (25) can be combined together into a simple matrix equation:

$$\frac{d}{d\tau}\begin{bmatrix} \mathbf{u} \\ \|\mathbf{u}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n\times n} & \frac{\mathbf{f}(\mathbf{u},\tau)}{\|\mathbf{u}\|} \\ \frac{\mathbf{f}^{\mathrm{T}}(\mathbf{u},\tau)}{\|\mathbf{u}\|} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{u} \\ \|\mathbf{u}\| \end{bmatrix}. \tag{26}$$

It is obvious that the first row in Eq. (26) is the same as the original equation (22), but the inclusion of the second row in Eq. (26) gives us a Minkowskian structure of the augmented state variables of $\mathbf{X} := (\mathbf{u}^{\mathrm{T}}, \|\mathbf{u}\|)^{\mathrm{T}}$, which satisfies the cone condition:

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = 0, \tag{27}$$

where

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & -1 \end{bmatrix} \tag{28}$$

is a Minkowski metric, and $\mathbf{I}_n$ is the identity matrix of order n. In terms of $(\mathbf{u}, \|\mathbf{u}\|)$, Eq. (27) becomes

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = \mathbf{u}\cdot\mathbf{u} - \|\mathbf{u}\|^2 = \|\mathbf{u}\|^2 - \|\mathbf{u}\|^2 = 0. \tag{29}$$

It follows from the definition given in Eq. (23), and thus Eq. (27) is a natural result. Consequently, we have an $n+1$-dimensional augmented system:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} \tag{30}$$

with a constraint (27), where

$$\mathbf{A} := \begin{bmatrix} \mathbf{0}_{n\times n} & \frac{\mathbf{f}(\mathbf{u},\tau)}{\|\mathbf{u}\|} \\ \frac{\mathbf{f}^{\mathrm{T}}(\mathbf{u},\tau)}{\|\mathbf{u}\|} & 0 \end{bmatrix}, \tag{31}$$

satisfying

$$\mathbf{A}^{\mathrm{T}}\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}, \tag{32}$$

is a Lie algebra $so(n,1)$ of the proper orthochronous Lorentz group $SO_o(n,1)$. This fact prompts us to devise the group-preserving scheme (GPS), whose discretized mapping $\mathbf{G}$ must exactly preserve the following properties:

$$\mathbf{G}^{\mathrm{T}}\mathbf{g}\mathbf{G} = \mathbf{g}, \tag{33}$$

$$\det \mathbf{G} = 1, \tag{34}$$

$$G_0^0 > 0, \tag{35}$$

where G_0^0 is the 00-th component of $\mathbf{G}$.

Although the dimension of the new system is raised one more, it has been shown that the new system permits a GPS given as follows [Liu (2001)]:

$$\mathbf{X}_{\ell+1} = \mathbf{G}(\ell)\mathbf{X}_\ell, \tag{36}$$

where $\mathbf{X}_\ell$ denotes the numerical value of $\mathbf{X}$ at τ_ℓ, and $\mathbf{G}(\ell) \in SO_o(n,1)$ is the group value of $\mathbf{G}$ at τ_ℓ. If $\mathbf{G}(\ell)$ satisfies the properties in Eqs. (33)-(35), then $\mathbf{X}_\ell$ satisfies the cone condition in Eq. (27).

The Lie group can be generated from $\mathbf{A} \in so(n,1)$ by an exponential mapping,

$$\mathbf{G}(\ell) = \exp[\Delta\tau\mathbf{A}(\ell)] = \begin{bmatrix} \mathbf{I}_n + \frac{(a_\ell - 1)}{\|\mathbf{f}_\ell\|^2}\mathbf{f}_\ell\mathbf{f}_\ell^{\mathrm{T}} & \frac{b_\ell \mathbf{f}_\ell}{\|\mathbf{f}_\ell\|} \\ \frac{b_\ell \mathbf{f}_\ell^{\mathrm{T}}}{\|\mathbf{f}_\ell\|} & a_\ell \end{bmatrix}, \tag{37}$$

where

$$a_\ell := \cosh\left(\frac{\Delta\tau\|\mathbf{f}_\ell\|}{\|\mathbf{u}_\ell\|}\right), \tag{38}$$

$$b_\ell := \sinh\left(\frac{\Delta\tau\|\mathbf{f}_\ell\|}{\|\mathbf{u}_\ell\|}\right). \tag{39}$$

Substituting Eq. (37) for $\mathbf{G}(\ell)$ into Eq. (36), we obtain

$$\mathbf{u}_{\ell+1} = \mathbf{u}_\ell + \eta_\ell \mathbf{f}_\ell, \tag{40}$$

$$\|\mathbf{u}_{\ell+1}\| = a_\ell\|\mathbf{u}_\ell\| + \frac{b_\ell}{\|\mathbf{f}_\ell\|}\mathbf{f}_\ell \cdot \mathbf{u}_\ell, \tag{41}$$

where

$$\eta_\ell := \frac{b_\ell\|\mathbf{u}_\ell\|\|\mathbf{f}_\ell\| + (a_\ell - 1)\mathbf{f}_\ell \cdot \mathbf{u}_\ell}{\|\mathbf{f}_\ell\|^2} \tag{42}$$

is an adaptive factor. From $\mathbf{f}_\ell \cdot \mathbf{u}_\ell \geq -\|\mathbf{f}_\ell\|\|\mathbf{u}_\ell\|$ we can prove that

$$\eta_\ell \geq \left[1-\exp\left(-\frac{\Delta\tau\|\mathbf{f}_\ell\|}{\|\mathbf{u}_\ell\|}\right)\right]\frac{\|\mathbf{u}_\ell\|}{\|\mathbf{f}_\ell\|} > 0, \ \ \forall \Delta\tau > 0. \tag{43}$$

This scheme is group properties preserved for all $\Delta\tau > 0$, and is called the group-preserving scheme (GPS).

2.5 Numerical procedures

Starting from an initial value of $u_i^j(0)$, we can employ the GPS to integrate Eq. (21) from $\tau = 0$ to a selected final time τ_f. In the numerical integration process we can check the convergence of u_i^j at the ℓ- and $\ell+1$-steps by

$$\sqrt{\sum_{i,j=1}^{m_1,m_2}[u_i^j(\ell+1)-u_i^j(\ell)]^2} \leq \varepsilon, \tag{44}$$

where ε is a selected convergent criterion. If at a time $\tau_0 \leq \tau_f$ the above criterion is satisfied, then the solution of u is obtained.

In practice, if a suitable τ_f is selected we find that the numerical solution is also approached very well to the true solution, even the above convergent criterion is not satisfied. The viscosity coefficient ν introduced in Eq. (21) can strengthen the stability of numerical integration. We should emphasize that the present method is a new fictitious time integration method (FTIM). Because it does not need to face the nonlinearity in the spatial domain, this new FTIM can calculate the backward in time Burgers equation very stably and effectively without needing of any iteration and regularization technique. Below we give numerical examples to display some advantages of the present FTIM.

When the initial data $u(x_i,0)$ are recovered by the above method, we can obtain the whole solution $u(x,t)$ in the problem domain by applying the GPS to integrate the following discretized equations:

$$\dot{u}_i = \frac{1}{R(\Delta x)^2}(u_{i+1}-2u_i+u_{i-1}) - \frac{1}{2\Delta x}u_i(u_{i+1}-u_{i-1}) + H(x_i,t), \tag{45}$$

where $u_i = u(x_i,t)$ is a function of time t.

For the direct problem calculated by the FTIM, the convergence criterion is given similarly by Eq. (44). When v_i^j are obtained at a fictitious time τ_0, the solutions of u_i^j are given by

$$u_i^j = \frac{v_i^j}{1+\tau_0}. \tag{46}$$

3 Numerical examples

When the input data are contaminated by random noise, we are concerned with the stability of the FTIM, which is investigated by adding the different levels of random noise on the data by $f(x_i)+sR(i)$, where we use the function RANDOM_NUMBER given in Fortran to generate the noisy data $R(i)$, and $R(i)$ are random numbers in $[-1,1]$.

3.1 Direct Problem: Example 1

Let us first consider the Burgers equation (1) with $H=0$ and under the following boundary conditions and initial condition:

$$u(0,t)=\frac{1}{1+\exp[-Rt/4]},$$
$$u(1,t)=\frac{1}{1+\exp[R/2-Rt/4]},$$
$$u(x,0)=\frac{1}{1+\exp[Rx/2]},\quad 0\leq x\leq 1. \tag{47}$$

The exact solution [Byrne and Hindmarsh (1987)] is given by

$$u(x,t)=\frac{1}{1+\exp[Rx/2-Rt/4]}. \tag{48}$$

In the case with $R=1$ and $T=1$ we consider the effect of noise on the numerical solutions by adding a noise with $s=0.05$ both on initial condition and two boundary conditions. We first apply the GPS to Eq. (45) to calculate the numerical solution by using $\Delta x=1/20$ and $\Delta t=1/800$. The numerical errors being the differences of numerical solutions and exact solutions are plotted in the top of Fig. 1. It can be seen that due to the noise effect the disturbance is propagated to the whole solutions with random errors distributed in the whole space of problem domain.

Then, we apply the FTIM to this direct problem by letting $m_1=19$, $m_2=50$, $\Delta\tau=0.01$, and $\nu=0.1$, where a noise with $s=0.05$ is adding both on the initial condition and two boundary conditions. The numerical errors are plotted in the bottom of Fig. 1. It can be seen that the numerical errors are smaller than 0.04 even under a large noise. Moreover, it does not like the above solution given by the GPS; here, no random errors are distributed in the whole space of problem domain.

The conventional numerical methods, such as the above GPS, will propagate the error appeared in the initial data to pollute the all solutions of $u(x,t)$, $t>0$, because they are integrated in the t-direction. However, for the FTIM the above problem can be overcome, because the FTIM is integrated in the τ-direction, and the given

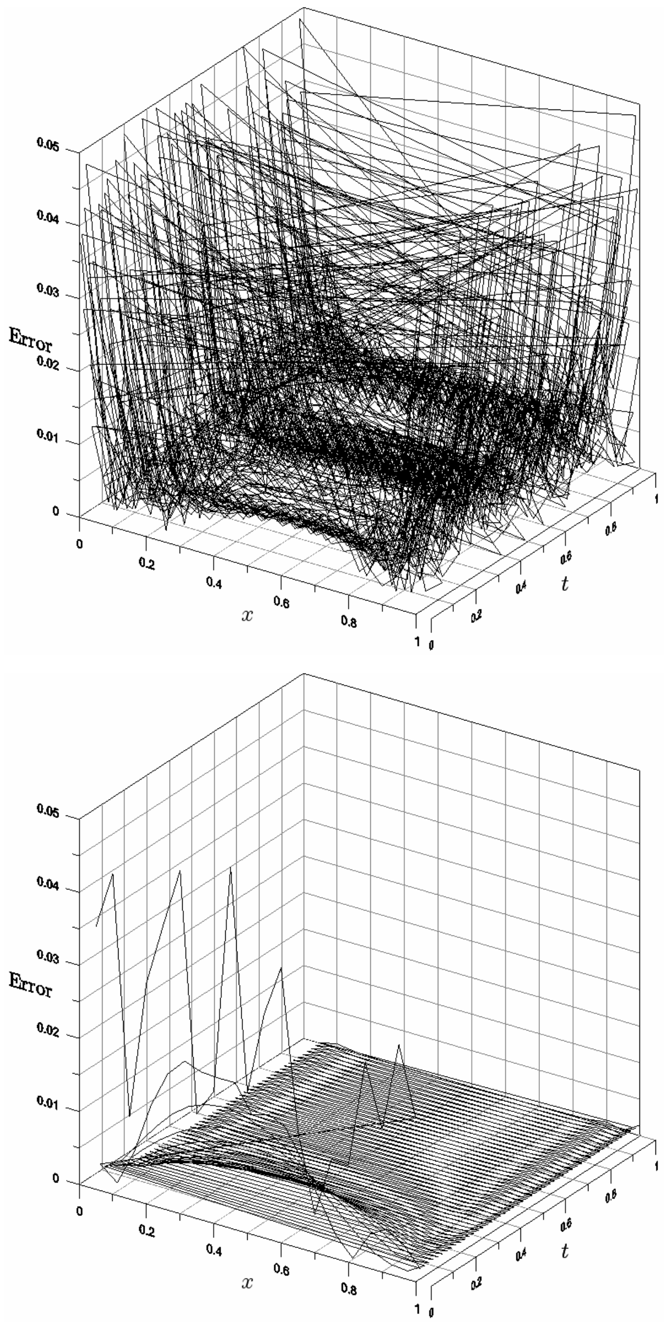


Figure 1: The numerical errors of Example 1 for direct problem by the GPS (top) and by the FTIM (bottom).

data influence just a few of the many differential equations. The FTIM is very robust against disturbance, and the accuracy is much better than that calculated by the GPS.

3.2 Direct Problem: Example 2

For the Burgers equation with $H = 0$ and under the following boundary conditions and initial condition:

$$u(0,t) = u(1,t) = 0,$$
$$u(x,0) = \sin \pi x, \tag{49}$$

the exact solution can be obtained by transforming them through the Hopf-Cole transformation [Cole (1951); Hopf (1950)]. However, when R is large over 100, the computation by means of exact solution is not practical due to the slow convergence of the Fourier series. In this sense, a numerical method that can treat the computations of Burgers equation with large R and under noise becomes significant.

We apply the FTIM to this example by comparing the numerical solutions without considering noise and a noise with $s = 0.1$ in Fig. 2. In this case we use $R = 10000$, $m_1 = 149$, $m_2 = 30$, $T = 1$, and $\nu = 0.01$. It can be seen that the numerical solutions are kept very well even under a very large noise.

In order to appreciate that the present approach of FTIM is insensitive to the noise, we compute this example by applying the GPS as reported by Liu (2006a) to Eq. (6) under the following parameters: $R = 10000$, $\Delta y = 0.01$, $\Delta t = 10^{-4}$, $T = 1$, and $s = 0.1$. From Fig. 3 it can be seen that the numerical solutions are seriously distorted by the random noise.

3.3 Backward Problem: Example 3

We consider a Burgers equation with a spatial-temporal-dependent source:

$$u_t + uu_x = u_{xx} - \frac{1}{2}e^{-2t}\sin(2x), \;\; 0 < x < 1, \;\; 0 < t < T,$$
$$u(0,t) = e^{-t}, \;\; u(1,t) = e^{-t}\cos 1, \;\; 0 \le t \le T,$$
$$u(x,T) = e^{-T}\cos x, \;\; 0 \le x \le 1. \tag{50}$$

The source term is chosen such that $u(x,t) = e^{-t}\cos x$ is a solution of the above equations.

In the calculations we fix the initial guess of u_i^j by $u_i^j(0) = 1$, and the other parameters used are $m_1 = 49$, $\Delta\tau = 0.001$, $\nu = 0.1$ and $\varepsilon = 10^{-3}$. Three final data

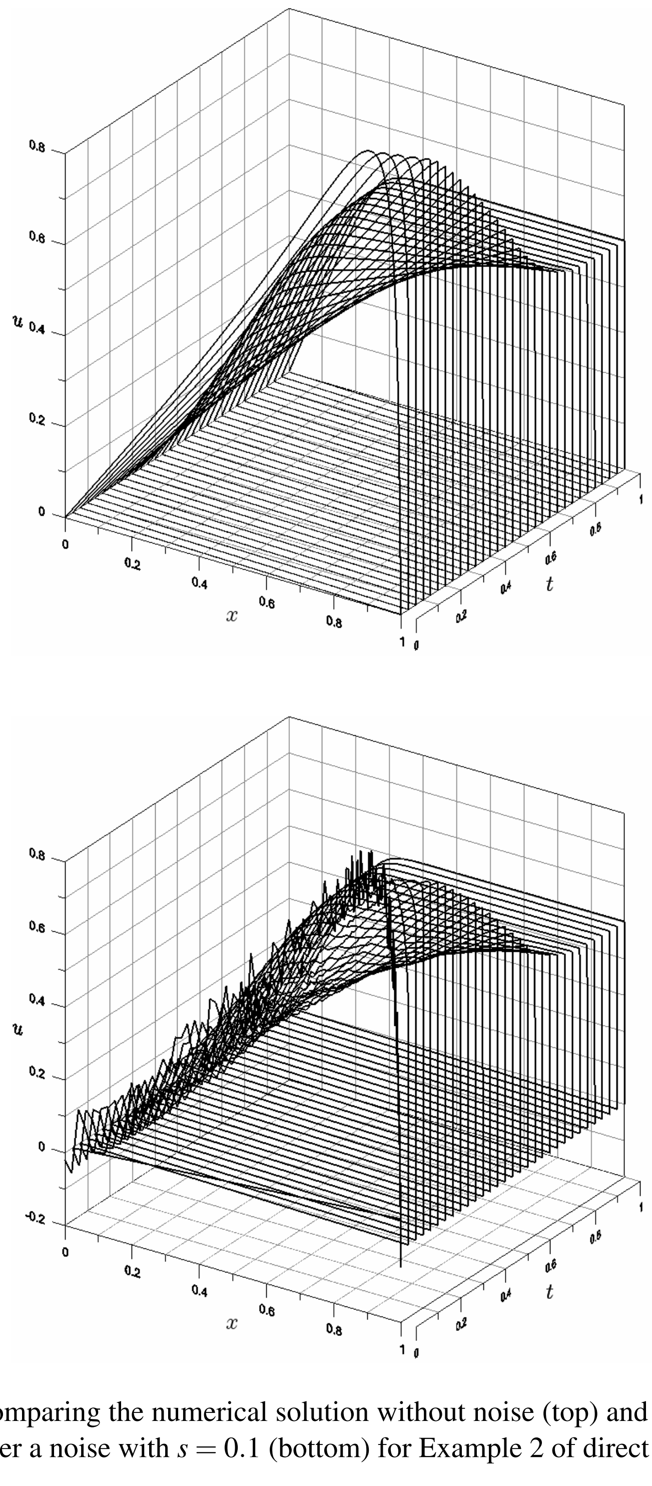


Figure 2: Comparing the numerical solution without noise (top) and the numerical solution under a noise with $s = 0.1$ (bottom) for Example 2 of direct problem.

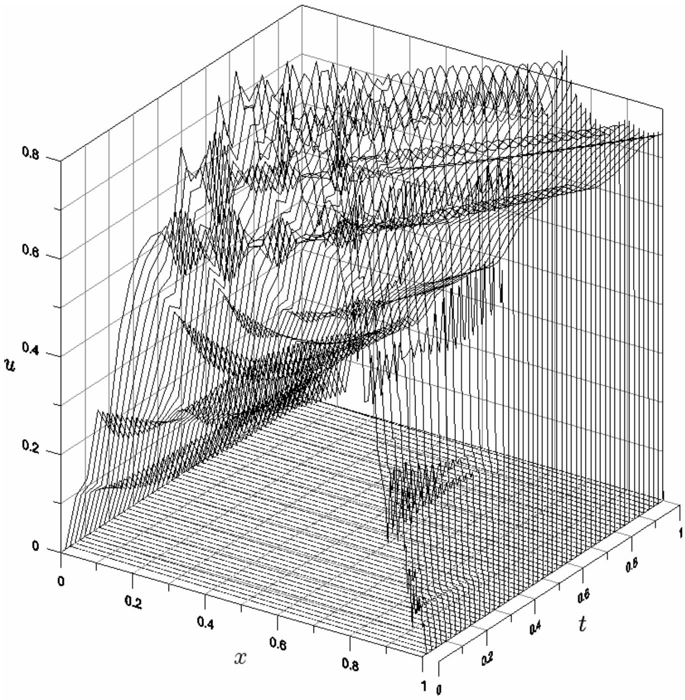


Figure 3: For Example 2 directly applying the GPS under a noised initial condition, given a seriously distorted numerical solution with random errors.

$f(x) = e^{-2}\cos x,\ e^{-5}\cos x,\ e^{-10}\cos x$ are considered, which are under the noises of $s = 0.01,\ 0.1,\ 0.5$. $m_2 = 10,\ 10,\ 20$ are used for these three cases. In Fig. 4 we compare the recovered initial data with the exact one. Even under very large noise, the recovered results are very well. Especially, for the last case when the noise to signal ratio is large up to about 10^4, we can still recover the initial data with an accuracy about in the order of 10^{-3}. In Fig. 5 we show the numerical errors in the domain of Ω for the above three cases. The maximum errors are smaller than 0.006.

The computational examples supported that we may use a FTIM to compute the backward in time Burgers problem. On the other hand, there are two reasons for a FTIM: (a) the FTIM is insensitive to the noise disturbance on the final time data, because for the FTIM the final time data influence just a few of the many differential equations; (b) there has no error propagation in the FTIM, because the FTIM is integrated in the τ-direction, not in the t-direction.

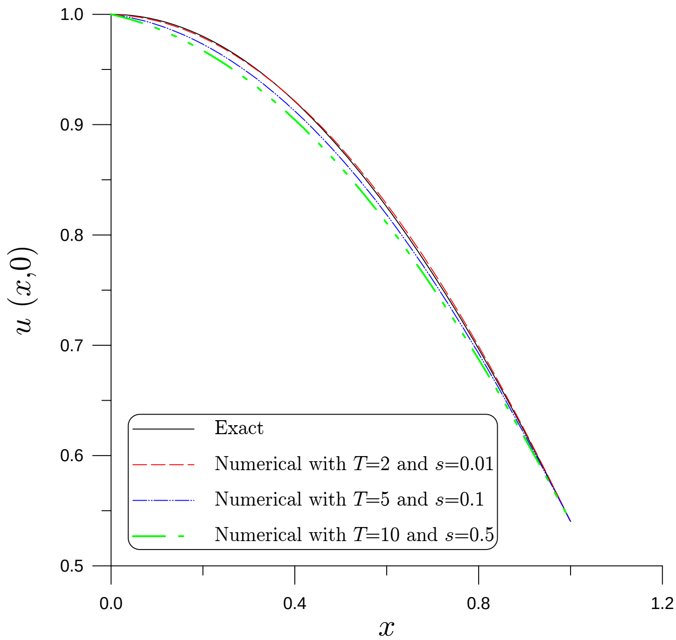


Figure 4: Comparing the recovered initial data with the exact one under different terminal times and noises for Example 3 of the backward problem.

3.4 Backward Problem: Example 4

For the Burgers equation (1) with $H = 0$ and under the following boundary conditions and initial condition:

$$u(0,t) = u(1,t) = e^{-t} - e^{-2t},$$
$$u(x,0) = \sin \pi x, \tag{51}$$

we first apply the GPS to integrate the discretized equation (45) to obtain the needed final data.

We consider two cases of $(T,s) = (1,0.01)$ and $(T,s) = (5,0.1)$. The parameters used in the FTIM are $m_1 = 19$, $m_2 = 10$ for the first case, $m_1 = 19$ and $m_2 = 20$ for the second case, $\Delta\tau = 10^{-5}$, $\varepsilon = 10^{-5}$, and $\nu = 0.01$. For the first case the exact solution of $u(x,t)$, which is obtained by using the GPS, is compared with the recovered solution in Fig. 6. In Fig. 7 we compare the exact solution and the recovered solution for the second case. It can be seen that the data are recovered very well.

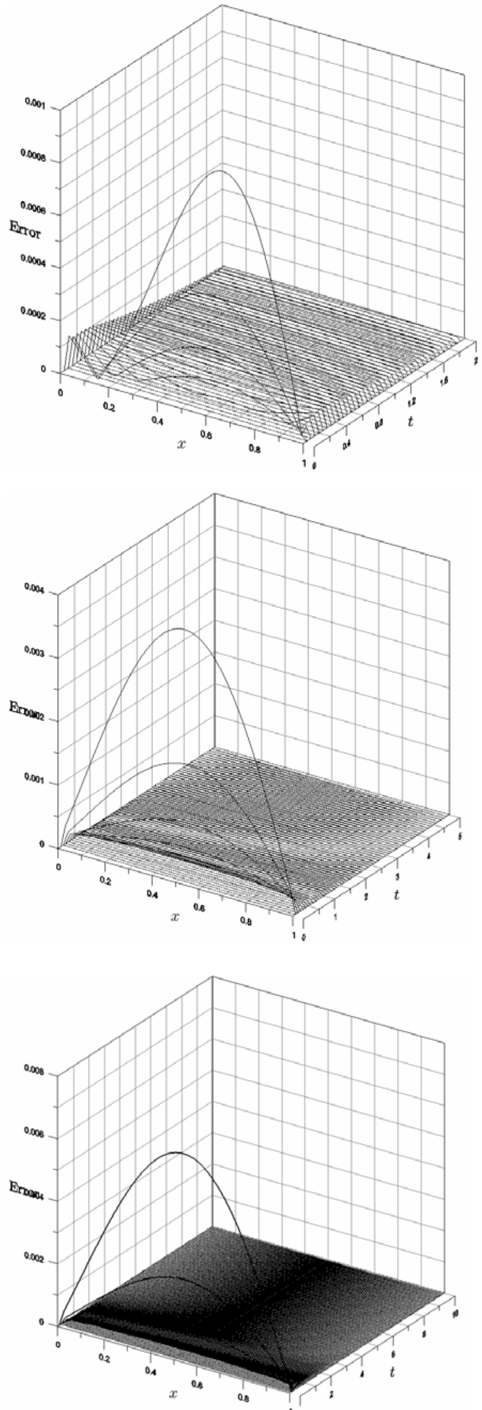


Figure 5: Errors of recovered data for Example 3 under three different conditions.

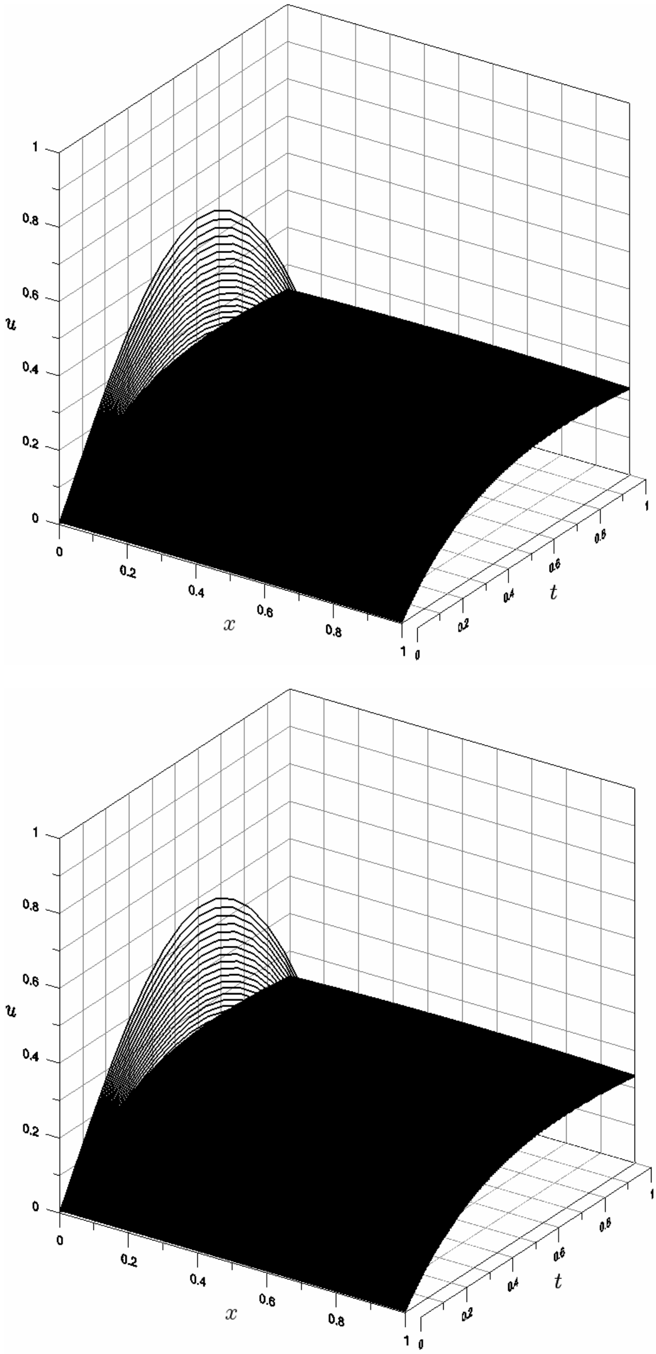
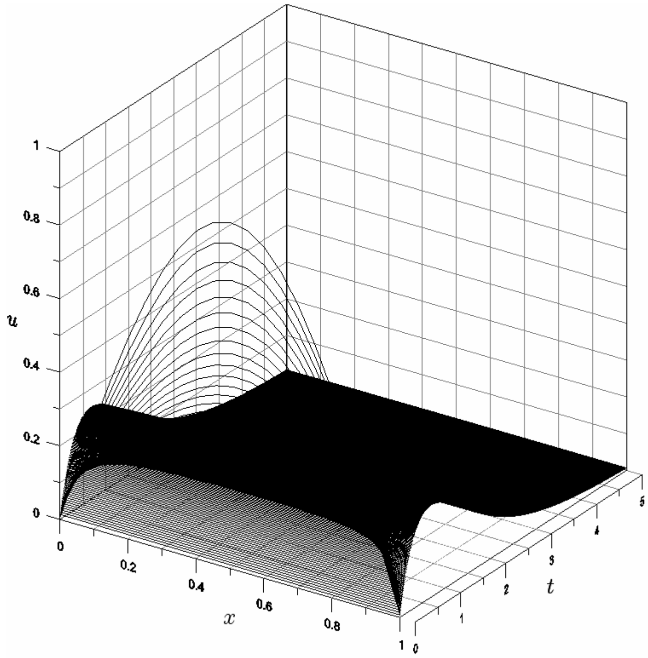


Figure 6: Comparing the exact solution (top) and the recovered solution (bottom) for the first case of Example 4.



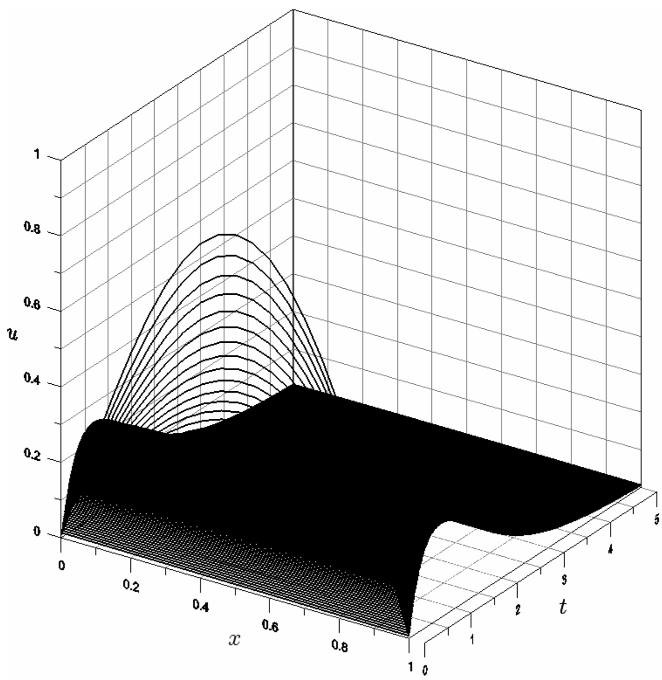


Figure 7: Comparing the exact solution (top) and the recovered solution (bottom) for the second case of Example 4.

3.5 Backward Problem: Example 5

For the Burgers equation (1) with $H = 0$ and under the following boundary conditions and initial condition:

$$u(0,t) = u(1,t) = e^{-t} - e^{-2t},$$
$$u(x,0) = 2x, \text{ if } 0 \leq x \leq 0.5, \ \ u(x,0) = 2-2x, \text{ if } 0.5 < x \leq 1, \tag{52}$$

we apply the GPS to integrate the discretized equation (45) to obtain the needed final data.

We consider one case of $(T,s) = (10,0.1)$. The parameters used in the FTIM are $m_1 = 19$, $m_2 = 40$, $\Delta\tau = 10^{-5}$, $\varepsilon = 10^{-5}$, and $\nu = 0.01$. In Fig. 8 we compare the exact solution and the recovered solution. It can be seen that the data are recovered very well.

4 Conclusions

In this paper, we have transformed the original Burgers equation into another parabolic type PDE in a one dimension higher space by introducing a fictitious time-like variable, and adding a fictitious viscous damping coefficient to enhance the stability of numerical integration of the discretized equations by employing the GPS. For the direct problems, the FTIM is little influenced by the noise, which adding even on all initial and boundary data. By using the FTIM, we can calculate the backward problems and retrieve the initial data very well with a high order accuracy. Several numerical examples of the backward problems were worked out, which show that our proposed approach is applicable to the seriously ill-posed backward in time Burgers problems. Under a noised final data the FTIM was also robust enough to retrieve the initial data. More significantly, we can unifiedly approach both the direct and backward problems very well, without resorting on any regularization technique to get rid of the ill-posed behavior.

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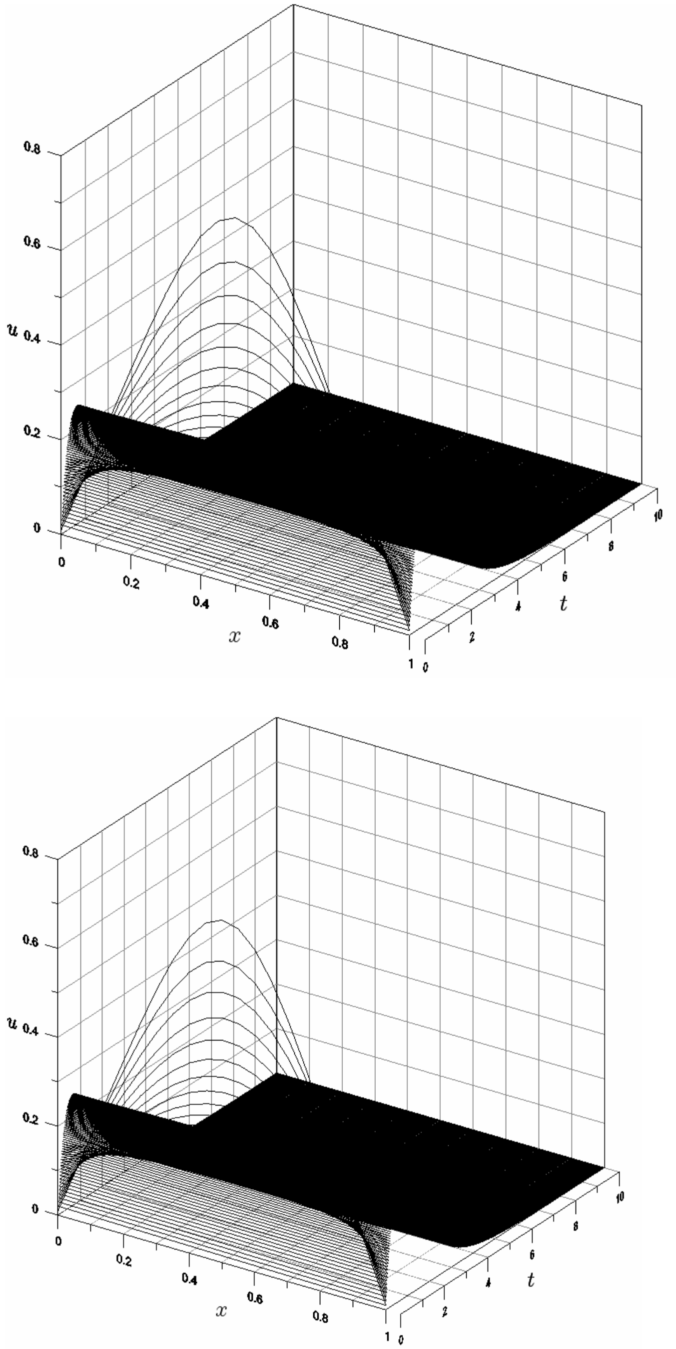

Figure 8: Comparing the exact solution (top) and the recovered solution (bottom) for Example 5.

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