The Fictitious Time Integration Method to Solve the Space- and Time-Fractional Burgers Equations

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Abstract: We propose a simple numerical scheme for solving the space- and time-fractional derivative Burgers equations: $D_t^\alpha u + \varepsilon u u_x = \nu u_{xx} + \eta D_x^\beta u$, $0 < \alpha, \beta \le 1$, and $u_t + D_*^\beta (D_*^{1-\beta} u)^2/2 = \nu u_{xx}$, $0 < \beta \le 1$. The time-fractional derivative $D_t^\alpha u$ and space-fractional derivative $D_x^\beta u$ are defined in the Caputo sense, while $D_*^\beta u$ is the Riemann-Liouville space-derivative. A fictitious time τ is used to transform the dependent variable $u(x,t)$ into a new one by $(1+\tau)^\gamma u(x,t) =: v(x,t,\tau)$, where $0 < \gamma \le 1$ is a parameter, such that the original equation is written as a new functional-differential type partial differential equation in the space of (x,t,τ). When the group-preserving scheme is used to integrate these equations under a semi-discretization of $u(x,t,\tau)$ at the spatial-temporal grid points, we can achieve rather accurate solutions.

Keywords: Fractional Burgers equation, Fictitious time integration method (FTIM), Caputo derivative, Riemann-Liouville derivative, Group-preserving scheme

1 Introduction

In this paper we are concerned with the numerical solution of a fractional Burgers equation:

$$D_t^\alpha u + \varepsilon u u_x = \nu u_{xx} + \eta D_x^\beta u, \ \ a < x < b, \ \ 0 < t < T, \tag{1}$$

$$u(a,t) = u_a(t), \ \ u(b,t) = u_b(t), \ \ 0 \le t \le T, \tag{2}$$

$$u(x,0) = f(x), \ \ a \le x \le b, \tag{3}$$

where $0 < \alpha, \beta \le 1$. The time-fractional derivative $D_t^\alpha u$ and space-fractional derivative $D_x^\beta u$ are defined in the Caputo sense. When $\alpha = 1$, $\varepsilon = 1$, and $\eta = 0$ we recover to the usual Burgers equation.

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The fractional advection-diffusion equation is recovered when $\varepsilon = 0$, which is known as the anomalous subdiffusion equation [Gu, Zhuang and Liu (2010)]. The phenomenon of anomalous subdiffusion has a broad application. It has been studied as a complicated dynamical system [Ye and Ding (2009); Kou, Yan and Liu (2009)], and had an extensive application in the fields such as semi-conductor, porous media, life science, and economy finance, etc. The anomalous diffusion is different from the normal diffusion. In the normal diffusion, particle motion is a Brownian motion, whose Green function is the Gaussian distribution, and the mean square displacement is a linear function of time, while particle diffusion can be described by the traditional second order advection-diffusion equation. The anomalous diffusion is essentially one kind of non-locality non-Markovian motion, so the time-space relativity must be taken into account. The particle motion is not a Brownian type, and the mean square displacement is not a linear function of time.

Burgers' equation has been of considerable physical interest because it is an appropriate simplification of the Navier-Stokes equations, and is also the governing equation for a number of one-dimensional flow systems, including the convection and diffusion of heat, weak shock propagation, compressible turbulence, and continuum traffic simulation.

The Burgers equation was first appeared in a paper by Bateman (1915) and was named after Burgers (1948, 1974). The behavior of Burgers equation exhibits a delicate balance between advection and diffusion. Moreover, it is one of the few nonlinear partial differential equations that exact and complete solutions are known in terms of the initial values through the Cole-Hopf transformation [Cole (1951); Hopf (1950)].

Besides the generalization in Eq. (1), we also consider the following nonlinear fractional Burgers equation:

$$u_t + \frac{1}{2}D_*^\beta (D_*^{1-\beta}u)^2 = \nu u_{xx}, \ 0 < \beta \leq 1, \tag{4}$$

where the space-fractional derivative $D_*^\beta u$ is defined in the Riemann-Liouville sense. When $\beta = 1$, we recover to the usual Burgers equation. As pointed out by Miskinis (2002), the above generalization of the Burgers equation has two important advantages: (a) the effect of nonlinearity and nonlocality is concentrated in one term, and (b) a fractional generalization of the Hopf and Cole transformation is allowed.

In recent years, it has turned out that many phenomena in engineering sciences can be well described by the models using the mathematical tools from fractional calculus. For example, the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow [He (1999)]. The space-fractional Burgers equation describes the physical processes of

unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe. The fractional derivative results from the memory effect of the wall friction through the boundary layer. The same form can be found in other systems such as shallow-water waves and waves in bubbly liquids [Momani (2006)]. Biler, Funaki and Woyczynski (1998) studied the local and global in time solutions to a class of multidimensional generalized Burgers-type equations with a fractal power of the Laplacian in the principal part and with general algebraic nonlinearity.

Previously, Liu (2006a) has computed the Burgers equation by using the group preserving scheme, and Liu (2006b) also developed a backward group preserving scheme to compute the backward in time problem of Burgers equation. Liu has utilized a simple fictitious time integration method (FTIM) to compute both the forward and backward in time problems of Burgers equation. Liu has found that the FTIM is robust against the noise. In this paper, our purpose is developing a new and simple numerical method of fictitious time integration method (FTIM) to solve the fractional Burgers equations, which allows much larger temporal and spatial grid sizes. It would be very interesting that the present approach is performed much better than other numerical methods from the aspects of stability and accuracy. The idea by introducing a fictitious time was first proposed by Liu (2008a) to treat an inverse Sturm-Liouville problem by transforming an ODE into a PDE. Then, Liu and his coworkers [Liu (2008b, 2008c, 2008d)] extended this idea to develop new methods for estimating parameters in the inverse vibration problems. Liu and Atluri (2008a) have employed the technique of FTIM to solve a large system of nonlinear algebraic equations, and showed that high performance can be achieved by using the FTIM. More recently, Liu has used the FTIM technique to solve the nonlinear complementarity problems, whose numerical results are very well. Then, Liu (2008e) used the FTIM to solve the boundary value problems of elliptic type partial differential equations. Liu and Atluri (2008b) also employed this technique of FTIM to solve mixed-complementarity problems and optimization problems. Then, Liu and Atluri (2008c) using the technique of FTIM solved the inverse Sturm-Liouville problem under specified eigenvalues. In the paper by Ku, Yeih, Liu and Chi (2009) a new time-like function is introduced in the FTIM, which was found being able to speed up the convergence for some problems. For its simple numerical implementation, the FTIM is also used in other places, like as, Liu (2009), Liu and Atluri (2009), Chang and Liu (2009), Chi, Yeih and Liu (2009), and Tsai, Liu and Yeih (2010).

2   The fictitious time integration method

2.1   *Fractional derivatives*

There are several mathematical definitions about the fractional derivative [Samko, Kilbas and Marichev (1987); Podlubny (1999)]. Here, we adopt the two usually used definitions: the Caputo and its reverse operator of the Riemann-Liouville fractional integral. Because the Caputo fractional derivative allows traditional initial condition assumption and boundary conditions, we can compare the initial-boundary values problems of fractional and usual Burgers equations.

The Riemann-Liouville fractional integral is an essential concept to understand the fractional derivatives of Riemann-Liouville and Caputo, and is given by [Samko, Kilbas and Marichev (1987); Chen and Holm (2003)]:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \tag{5}$$

where a and $\alpha > 0$ are constants. The corresponding Riemann-Liouville fractional derivative is written as

$$D_*^\lambda f(x) = \frac{d}{dx}\left[J^{1-\lambda} f(x)\right] = J^{-\lambda} f(x) = \frac{1}{\Gamma(-\lambda)} \int_a^x \frac{f(t)}{(x-t)^{1+\lambda}} dt, \ 0 < \lambda < 1. \tag{6}$$

The Riemann-Liouville fractional derivative, however, has a notable disadvantage in engineering applications of nonzero of the fractional derivative of constant C, e.g., $D_*^\lambda C \neq 0$, which would entail that dissipation does not vanish for a system in equilibrium [Samko, Kilbas and Marichev (1987); Seredynska and Hanyga (2000)], and violates the causality. The Caputo fractional derivative has instead been developed to overcome this drawback [Caputo (1967); Caputo and Mainardi (1971)]:

$$D^\lambda f(x) = J^{1-\lambda}\left[\frac{d}{dx} f(x)\right] = \frac{1}{\Gamma(1-\lambda)} \int_a^x \frac{\dot{f}(t)}{(x-t)^\lambda} dt, \ 0 < \lambda < 1. \tag{7}$$

A simple calculation shows that [Chen and Holm (2003)]

$$D^\lambda f(x) = D_*^\lambda f(x) - \frac{f(a)}{\Gamma(1-\lambda)(x-a)^\lambda}. \tag{8}$$

Comparing Eqs. (6) and (7) it can be seen that when the Riemann-Liouville fractional derivative has a hypersingular improper integral, where the order of singularity is higher than the dimension of integrating variable, it has an advantage that $f(t)$ is used in the integrand, and that when the Caputo fractional derivative has an advantage of a less singular improper integral with the order of singularity being

lower than the dimension of integrating variable, it has a disadvantage that $\dot{f}(t)$ is used in the integrand.

In this paper, we consider Eq. (1) with time- and space-fractional derivative. When $\alpha > 0$, we have

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{\partial^n u(x,s)}{\partial s^n} ds, & n-1 < \alpha < n, \\ \frac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n. \end{cases} \quad (9)$$

The form of the space-fractional derivative is similar to the above and we omit it here.

2.2 Transformation into a new functional PDE

It is known that for the nonlinear PDEs with derivatives of integer order, many methods can be used to find numerical solutions. For example, the numerical solutions of the integer-order Burgers equation with very high Reynold number are reported by Liu (2006a, 2006b). However, for the fractional PDEs, there are only limited approaches, such as the Laplace transform method [Podlubny (1999)], the Fourier transform method, the iteration method [Samko, Kilbas and Marichev (1987)], and the operational method.

By using the above fractional derivatives we can write Eq. (1) as

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_s(x,s)}{(t-s)^\alpha} ds + \varepsilon u u_x = \nu u_{xx} + \frac{\eta}{\Gamma(1-\beta)} \int_a^x \frac{u_s(s,t)}{(x-s)^\beta} ds. \quad (10)$$

We first introduce a fictitious damping coefficient $\nu_0 > 0$ into Eq. (10):

$$\nu_0 \nu u_{xx} - \nu_0 \varepsilon u u_x + \frac{\nu_0 \eta}{\Gamma(1-\beta)} \int_a^x \frac{u_s(s,t)}{(x-s)^\beta} ds - \frac{\nu_0}{\Gamma(1-\alpha)} \int_0^t \frac{u_s(x,s)}{(t-s)^\alpha} ds = 0. \quad (11)$$

Then, we propose the following transformation:

$$v(x,t,\tau) = (1+\tau)^\gamma u(x,t), \ 0 < \gamma \leq 1, \quad (12)$$

such that, by using Eq. (11) we have

$$\frac{\nu_0}{(1+\tau)^\gamma} \left[\nu v_{xx} - \frac{\varepsilon v v_x}{(1+\tau)^\gamma} + \frac{\eta}{\Gamma(1-\beta)} \int_a^x \frac{v_s(s,t,\tau)}{(x-s)^\beta} ds - \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{v_s(x,s,\tau)}{(t-s)^\alpha} ds \right]$$
$$= 0. \quad (13)$$

Recalling that $\partial v / \partial \tau = \gamma (1+\tau)^{\gamma-1} u(x,t)$ by Eq. (12), and adding it on both sides of the above equation we can obtain

$$
\begin{aligned}
\frac{\partial v}{\partial \tau} \;=\;& \frac{v_0}{(1+\tau)^{\gamma}} \left[v v_{xx} - \frac{\varepsilon v v_x}{(1+\tau)^{\gamma}} + \frac{\eta}{\Gamma(1-\beta)} \int_a^x \frac{v_s(s,t,\tau)}{(x-s)^{\beta}} ds \right. \\
& \left. - \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{v_s(x,s,\tau)}{(t-s)^{\alpha}} ds \right] + \gamma(1+\tau)^{\gamma-1} u.
\end{aligned} \tag{14}
$$

Then, by using $u = v/(1+\tau)^{\gamma}$ we can recast Eq. (10) into a new type of functional PDE for v:

$$
\begin{aligned}
\frac{\partial v}{\partial \tau} \;=\;& \frac{v_0}{(1+\tau)^{\gamma}} \left[v v_{xx} - \frac{\varepsilon v v_x}{(1+\tau)^{\gamma}} + \frac{\eta}{\Gamma(1-\beta)} \int_a^x \frac{v_s(s,t,\tau)}{(x-s)^{\beta}} ds \right. \\
& \left. - \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{v_s(x,s,\tau)}{(t-s)^{\alpha}} ds \right] + \frac{\gamma v}{1+\tau}.
\end{aligned} \tag{15}
$$

Upon using

$$
\frac{\partial}{\partial \tau} \left(\frac{v}{(1+\tau)^{\gamma}} \right) = \frac{v_\tau}{(1+\tau)^{\gamma}} - \frac{\gamma v}{(1+\tau)^{1+\gamma}},
$$

after multiplying the integrating factor $1/(1+\tau)^{\gamma}$ on both sides of Eq. (15), we can further reduce it to

$$
\begin{aligned}
\frac{\partial}{\partial \tau} \left(\frac{v}{(1+\tau)^{\gamma}} \right) \;=\;& \frac{v_0}{(1+\tau)^{2\gamma}} \left[v v_{xx} - \frac{\varepsilon v v_x}{(1+\tau)^{\gamma}} \right. \\
& \left. + \frac{\eta}{\Gamma(1-\beta)} \int_a^x \frac{v_s(s,t,\tau)}{(x-s)^{\beta}} ds - \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{v_s(x,s,\tau)}{(t-s)^{\alpha}} ds \right].
\end{aligned} \tag{16}
$$

Now, by using $v/(1+\tau)^{\gamma} = u$ again, we can rearrange Eq. (10) into a new type of functional PDE for u:

$$
\begin{aligned}
u_\tau = \;& \frac{v_0}{(1+\tau)^{\gamma}} \\
& \left[v u_{xx} - \varepsilon u u_x + \frac{\eta}{\Gamma(1-\beta)} \int_a^x \frac{u_s(s,t,\tau)}{(x-s)^{\beta}} ds - \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_s(x,s,\tau)}{(t-s)^{\alpha}} ds \right].
\end{aligned} \tag{17}
$$

The above time τ, corresponding to the real time t, is a fictitious time, which is used to embed Eq. (10) into a new functional PDE in a space of one-dimension higher. Here, we must stress that u is an unknown function with $u = u(x,t,\tau)$ subjecting to the constraints in Eqs. (2) and (3) for all $\tau \geq 0$, and $u(x,t,\tau=0)$ is given initially by a guess.

2.3 Semi-discretization

Let $u_i^j(\tau) := u(x_i, t_j, \tau)$ be a numerical value of u at a grid point (x_i, t_j) and at a fictitious time τ. Applying a semi-discretization to the above Eq. (17) yields

$$
\begin{aligned}
\frac{d}{d\tau} u_i^j(\tau) &= \frac{v_0}{(1+\tau)^\gamma} \left[\frac{v}{(\Delta x)^2} [u_{i+1}^j - 2u_i^j + u_{i-1}^j] - \frac{\varepsilon}{2\Delta x} u_i^j [u_{i+1}^j - u_{i-1}^j] \right. \\
&\quad + \left. \frac{\eta}{\Gamma(1-\beta)} \int_a^{x_i} \frac{u_s(s,t_j,\tau)}{(x_i-s)^\beta} ds - \frac{1}{\Gamma(1-\alpha)} \int_0^{t_j} \frac{u_s(x_i,s,\tau)}{(t_j-s)^\alpha} ds \right],
\end{aligned} \tag{18}
$$

where $\Delta x = (b-a)/(m_1+1)$, $\Delta t = T/(m_2+1)$, $x_i = a+(i-1)\Delta x$ and $t_j = (j-1)\Delta t$. The above two integral terms can be calculated from using a simple rectangular rule and a central difference of $u_s(s,t_j,\tau)$ and a forward Euler difference of $u_s(x_i,s,\tau)$ by

$$
\int_a^{x_i} \frac{u_s(s,t_j,\tau)}{(x_i-s)^\beta} ds = \frac{u(x_2,t_j,\tau) - u(x_1,t_j,\tau)}{(x_i-x_1)^\beta} + \sum_{\ell=2}^{i-1} \frac{u(x_{\ell+1},t_j,\tau) - u(x_{\ell-1},t_j,\tau)}{2(x_i-x_\ell)^\beta},
$$

$$
\int_0^{t_j} \frac{u_s(x_i,s,\tau)}{(t_j-s)^\alpha} ds = \sum_{\ell=1}^{j-1} \frac{u(x_i,t_{\ell+1},\tau) - u(x_i,t_\ell,\tau)}{(t_j-t_\ell)^\alpha}. \tag{19}
$$

For the nonlinear fractional Burgers equation (4), a similar derivation leads to

$$
\begin{aligned}
\frac{d}{d\tau} u_i^j(\tau) &= \frac{v_0}{(1+\tau)^\gamma} \left[\frac{v}{(\Delta x)^2} [u_{i+1}^j - 2u_i^j + u_{i-1}^j] - \frac{\varepsilon}{2\Delta x} u_i^j [u_{i+1}^j - u_{i-1}^j] \right. \\
&\quad - \left. \frac{1}{2\Gamma(-\beta)} \int_a^{x_i} \frac{w(s,t_j,\tau)}{(x_i-s)^{1+\beta}} ds \right],
\end{aligned} \tag{20}
$$

where

$$
w(x,t,\tau) = \left[\frac{1}{\Gamma(\beta-1)} \int_a^x \frac{u(s,t,\tau)}{(x-s)^{2-\beta}} ds \right]^2 \tag{21}
$$

with $w(a,t,\tau) = 0$. The above two integral terms can be calculated from using a simple rectangular rule by

$$
\int_a^{x_i} \frac{u(s,t_j,\tau)}{(x-s)^{2-\beta}} ds = \sum_{\ell=1}^{i-1} \frac{\Delta x u(x_\ell,t_j,\tau)}{(x_i-x_\ell)^{2-\beta}},
$$

$$
\int_a^{x_i} \frac{w(s,t_j,\tau)}{(x_i-s)^{1+\beta}} ds = \sum_{\ell=1}^{i-1} \frac{\Delta x w(x_\ell,t_j,\tau)}{(x_i-x_\ell)^{1+\beta}}. \tag{22}
$$

It can be seen that Eq. (20) is much complex than Eq. (18). It is known that the fractional Burgers equation is hard to be numerically solved; for example, Eqs. (18)

and (20) by letting $du_i^j(\tau)/d\tau = 0$ result in highly-dimensional nonlinear systems of algebraic equations, which are typically solved by some iterative methods. In contrast, we can use the above ODEs to find the numerical solutions rather easily.

2.4 The GPS for differential equations system

Upon letting $\mathbf{u} = (u_1^1, u_1^2, \ldots, u_{m_1}^{m_2})^{\mathrm{T}}$ and $\mathbf{f}$ denoting a vector with the ij-th component being the right-hand side of Eq. (18) we can write it as a vector form:

$$\mathbf{u}' = \mathbf{f}(\mathbf{u}, \tau), \quad \mathbf{u} \in \mathbb{R}^n, \quad \tau \in \mathbb{R}, \tag{23}$$

where $\mathbf{u}'$ denotes the differential of $\mathbf{u}$ with respect to τ, and $n = m_1 m_2$ is the number of total grid points inside the domain $\Omega = (a, b) \times (0, T]$.

Group-preserving scheme (GPS) can preserve the internal symmetry group of the considered ODE system. Although we do not know previously the symmetry group of differential equations system, Liu (2001) has embedded it into an augmented differential system, which concerns with not only the evolution of state variables themselves but also the evolution of the magnitude of the state variables vector. Let us note that

$$\|\mathbf{u}\| = \sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{u}} = \sqrt{\mathbf{u} \cdot \mathbf{u}}, \tag{24}$$

where the superscript T signifies the transpose, and the dot between two n-dimensional vectors denotes their inner product. Taking the derivatives of both the sides of Eq. (24) with respect to τ, we have

$$\frac{d\|\mathbf{u}\|}{d\tau} = \frac{(\mathbf{u}')^{\mathrm{T}}\mathbf{u}}{\sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{u}}}. \tag{25}$$

Then, by using Eqs. (23) and (24) we can derive

$$\frac{d\|\mathbf{u}\|}{d\tau} = \frac{\mathbf{f}^{\mathrm{T}}\mathbf{u}}{\|\mathbf{u}\|}. \tag{26}$$

It is interesting that Eqs. (23) and (26) can be combined together into a simple matrix equation:

$$\frac{d}{d\tau} \begin{bmatrix} \mathbf{u} \\ \|\mathbf{u}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n \times n} & \frac{\mathbf{f}(\mathbf{u}, \tau)}{\|\mathbf{u}\|} \\ \frac{\mathbf{f}^{\mathrm{T}}(\mathbf{u}, \tau)}{\|\mathbf{u}\|} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \|\mathbf{u}\| \end{bmatrix}. \tag{27}$$

It is obvious that the first row in Eq. (27) is the same as the original equation (23), but the inclusion of the second row in Eq. (27) gives us a Minkowskian structure

of the augmented state variables of $\mathbf{X} := (\mathbf{u}^{\mathrm{T}}, \|\mathbf{u}\|)^{\mathrm{T}}$, which satisfies the cone condition:

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = 0, \tag{28}$$

where

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & -1 \end{bmatrix} \tag{29}$$

is a Minkowski metric, and $\mathbf{I}_n$ is the identity matrix of order n. In terms of $(\mathbf{u}, \|\mathbf{u}\|)$, Eq. (28) becomes

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = \mathbf{u} \cdot \mathbf{u} - \|\mathbf{u}\|^2 = \|\mathbf{u}\|^2 - \|\mathbf{u}\|^2 = 0. \tag{30}$$

It follows from the definition given in Eq. (24), and thus Eq. (28) is a natural result. Consequently, we have an $n+1$-dimensional augmented system:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} \tag{31}$$

with a constraint (28), where

$$\mathbf{A} := \begin{bmatrix} \mathbf{0}_{n \times n} & \dfrac{\mathbf{f}(\mathbf{u},\tau)}{\|\mathbf{u}\|} \\ \dfrac{\mathbf{f}^{\mathrm{T}}(\mathbf{u},\tau)}{\|\mathbf{u}\|} & 0 \end{bmatrix}, \tag{32}$$

satisfying

$$\mathbf{A}^{\mathrm{T}}\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}, \tag{33}$$

is a Lie algebra $so(n,1)$ of the proper orthochronous Lorentz group $SO_o(n,1)$. This fact prompts us to devise the group-preserving scheme (GPS), whose discretized mapping $\mathbf{G}$ must exactly preserve the following properties:

$$\mathbf{G}^{\mathrm{T}}\mathbf{g}\mathbf{G} = \mathbf{g}, \tag{34}$$
$$\det \mathbf{G} = 1, \tag{35}$$
$$G_0^0 > 0, \tag{36}$$

where G_0^0 is the 00-th component of $\mathbf{G}$.

Although the dimension of the new system is raised one more, it has been shown that the new system permits a GPS given as follows [Liu (2001)]:

$$\mathbf{X}_{\ell+1} = \mathbf{G}(\ell)\mathbf{X}_\ell, \tag{37}$$

where $\mathbf{X}_\ell$ denotes the numerical value of $\mathbf{X}$ at τ_ℓ, and $\mathbf{G}(\ell) \in SO_o(n,1)$ is the group value of $\mathbf{G}$ at τ_ℓ. If $\mathbf{G}(\ell)$ satisfies the properties in Eqs. (34)-(36), then $\mathbf{X}_\ell$ satisfies the cone condition in Eq. (28).

The Lie group can be generated from $\mathbf{A} \in so(n,1)$ by an exponential mapping,

$$\mathbf{G}(\ell) = \exp[\Delta\tau\mathbf{A}(\ell)] = \left[\begin{array}{cc} \mathbf{I}_n + \frac{a_\ell-1}{\|\mathbf{f}_\ell\|^2}\mathbf{f}_\ell\mathbf{f}_\ell^{\mathrm{T}} & \frac{b_\ell\mathbf{f}_\ell}{\|\mathbf{f}_\ell\|} \\ \frac{b_\ell\mathbf{f}_\ell^{\mathrm{T}}}{\|\mathbf{f}_\ell\|} & a_\ell \end{array} \right], \tag{38}$$

where

$$a_\ell := \cosh\left(\frac{\Delta\tau\|\mathbf{f}_\ell\|}{\|\mathbf{u}_\ell\|}\right), \tag{39}$$

$$b_\ell := \sinh\left(\frac{\Delta\tau\|\mathbf{f}_\ell\|}{\|\mathbf{u}_\ell\|}\right). \tag{40}$$

Substituting Eq. (38) for $\mathbf{G}(\ell)$ into Eq. (37), we obtain

$$\mathbf{u}_{\ell+1} = \mathbf{u}_\ell + \eta_\ell\mathbf{f}_\ell, \tag{41}$$

$$\|\mathbf{u}_{\ell+1}\| = a_\ell\|\mathbf{u}_\ell\| + \frac{b_\ell}{\|\mathbf{f}_\ell\|}\mathbf{f}_\ell \cdot \mathbf{u}_\ell, \tag{42}$$

where

$$\eta_\ell := \frac{b_\ell\|\mathbf{u}_\ell\|\|\mathbf{f}_\ell\| + (a_\ell-1)\mathbf{f}_\ell \cdot \mathbf{u}_\ell}{\|\mathbf{f}_\ell\|^2} \tag{43}$$

is an adaptive factor. From $\mathbf{f}_\ell \cdot \mathbf{u}_\ell \geq -\|\mathbf{f}_\ell\|\|\mathbf{u}_\ell\|$ we can prove that

$$\eta_\ell \geq \left[1 - \exp\left(-\frac{\Delta\tau\|\mathbf{f}_\ell\|}{\|\mathbf{u}_\ell\|}\right)\right]\frac{\|\mathbf{u}_\ell\|}{\|\mathbf{f}_\ell\|} > 0, \quad \forall\Delta\tau > 0. \tag{44}$$

This scheme is group properties preserved for all $\Delta\tau > 0$, and is called the group-preserving scheme (GPS).

2.5 Numerical procedures

Starting from an initial value of $u_i^j(0)$, we can employ the GPS to integrate Eqs. (18) and (20) from $\tau = 0$ to a selected final time τ_f. In the numerical integration process we can check the convergence of u_i^j at the ℓ- and $\ell+1$-steps by

$$\sqrt{\sum_{i,j=1}^{m_1,m_2}[u_i^j(\ell+1) - u_i^j(\ell)]^2} \leq \varepsilon, \tag{45}$$

where ε is a selected convergence criterion. If at a time $\tau_0 \leq \tau_f$ the above criterion is satisfied, then the solution of u is obtained.

In practice, if a suitable τ_f is selected we find that the numerical solution is also approached very well to the true solution, even the above convergence criterion is not satisfied. The viscosity coefficient v_0 introduced in Eqs. (18) and (20) can strengthen the stability of numerical integration, and the parameter γ appeared to have the effect of enhancing the convergence speed. We should emphasize that the present method is a new fictitious time integration method (FTIM). Because it does not need to face the nonlinearity and complexity in the spatial-temporal domain, this new FTIM can calculate the fractional Burgers equation (1) very stably and effectively without needing of any iteration technique. Below we give numerical examples to display some advantages of the present FTIM.

3 Numerical examples

From now on we use some numerical examples to test the performance of our approach by the FTIM to solve the fractional Burgers equations.

3.1 Example 1

For the fractional Burgers equation we consider the following boundary conditions and initial condition:

$$u(0,t) = u(1,t) = 0,$$
$$u(x,0) = \sin \pi x. \tag{46}$$

We apply the FTIM to this example by fixing $\varepsilon = 1$, and $v = 0.05$, and compare the numerical solutions for four cases with (a) $\alpha = 1$, and $\eta = 0$, (b) $\alpha = 1$, $\beta = 0.5$, and $\eta = 1$, (c) $\alpha = 0.5$, and $\eta = 0$, and (d) $\alpha = 0.5$, $\beta = 0.5$, and $\eta = 1$. Case (a) is for the usual Burgers equation, case (b) is for the space-fractional derivative Burgers equation, case (c) is for the time-fractional derivative Burgers equation, and case (d) is for both the space- and time-fractional derivative Burgers equation. In this example we use $m_1 = 19$, $m_2 = 39$, $T = 2$, $v_0 = 2$, $\Delta\tau = 0.01$, $\varepsilon = 10^{-3}$, and $\gamma = 0.3$. Starting from an initial guess of $u_i^j(0) = 0.1$, the numerical solutions are all convergent not more than 160 steps. In Fig. 1 we compare these four numerical solutions over the space of (x,t). As usually, the solution of the original Burgers equation is smoothly damped in time due to the viscosity effect. When the space-fractional derivative is considered, the solution is seriously damped in time more

fast than the first case. When the time-fractional derivative is considered, the solution is fast damped in time before one second, but the effect of the time-fractional derivative is gradually balanced with the viscous dissipation, such that the amplitude of solution is decreased less slowly after one second until the end of time, and the amplitude is almost kept constant. When both the space- and time-fractional derivative are considered, the amplitude of solution is decreased fast than the previous case, but after one second until the end of time the amplitude is also kept constant. In Fig. 2 we use four plots to show the solutions at three different times to demonstrate these phenomena.
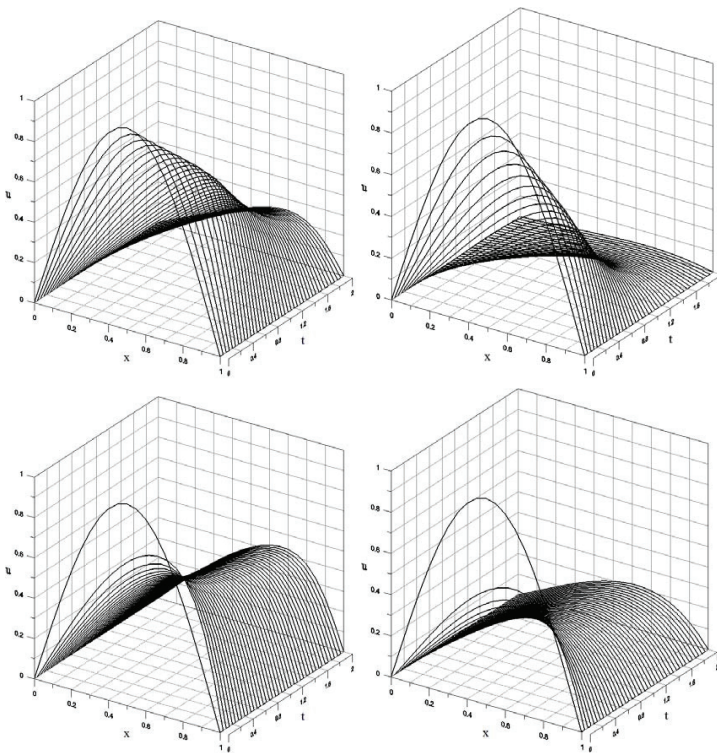

Figure 1: Comparing the numerical solutions by the FTIM for the original Burgers equation (left-top), the space-fractional derivative Burgers equation (right-top), the time-fractional derivative Burgers equation (left-bottom), and both the space- and time-fractional derivative Burgers equation (right-bottom).

Now, we turn our attention to the nonlinear fractional Burgers equation (4). Under the following parameters: $m_1 = 19$, $m_2 = 39$, $T = 2$, $\Delta\tau = 0.001$, $v_0 = 0.1$, $\gamma = 0.5$,
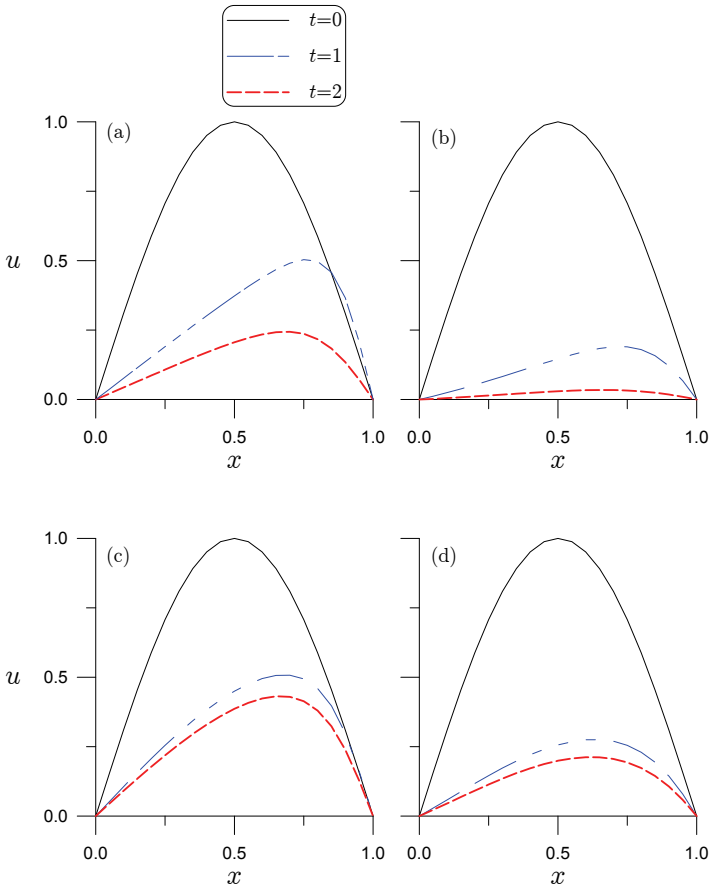
Figure 2: Comparing three different times numerical solutions by the FTIM for (a) the original Burgers equation, (b) the space-fractional derivative Burgers equation, (c) the time-fractional derivative Burgers equation, and (d) both the space- and time-fractional derivative Burgers equation.

and $\varepsilon = 10^{-3}$, we calculate this example for two cases $\beta = 0.5$ (top) and $\beta = 0.75$ (bottom) as shown in Fig. 3. It is interesting that after a few time, the numerical solutions tend to a steady-state solution with a constant profile not varying with time. For the case $\beta = 0.5$, the shape is inclined to the side of $x = 1$ with a unit height as the initial condition is, while for the case $\beta = 0.75$, the shape has a wide plateau with a height about 0.5. For the original Burgers equation the term uu_x is an advective term, but for Eq. (4), the term $D_*^{\beta}(D_*^{1-\beta}u)^2$ becomes a dissipative term, which is balanced with the viscous damping term νu_{xx}, such that $u_t = 0$ after a few
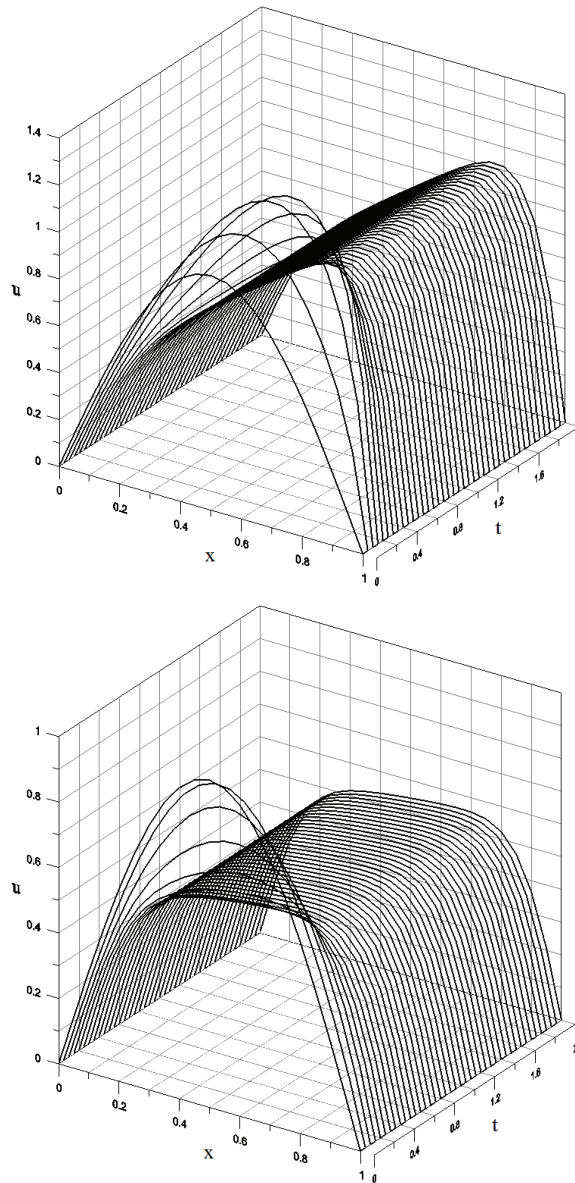
Figure 3: For Example 1 of a nonlinear space-fractional derivative Burgers equation, the numerical results are calculated by the FTIM: $\beta = 0.5$ (top), and $\beta = 0.75$ (bottom).

time. As compared with other fractional Burgers equations as shown in Fig. 1, the numerical solutions in Fig. 3 have a very different behavior.
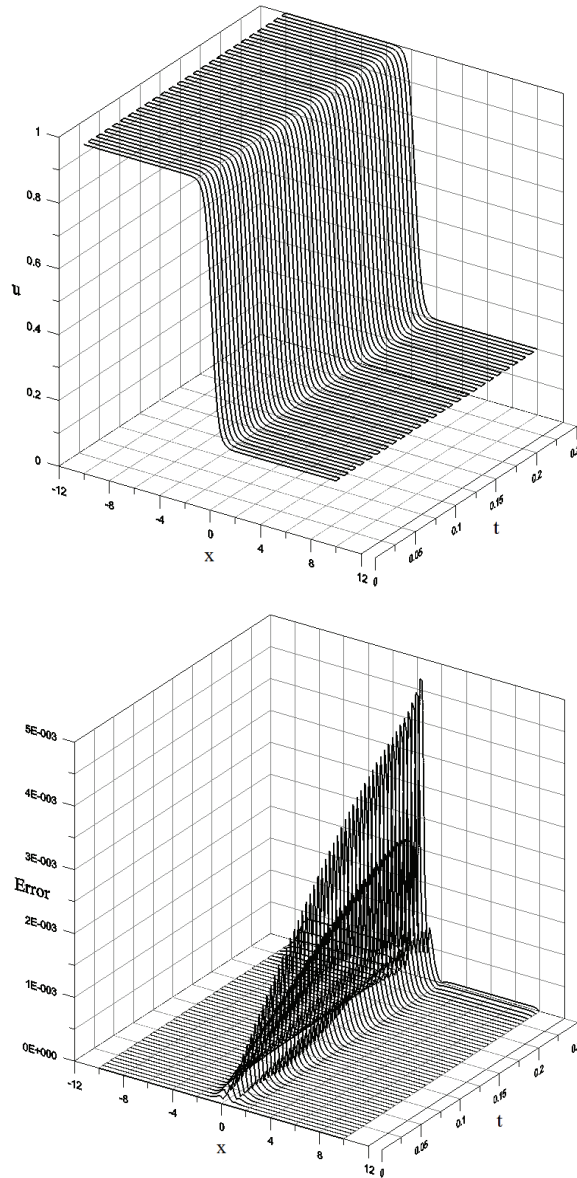

Figure 4: For Example 2 showing the numerical solution (top), and its numerical error (bottom).
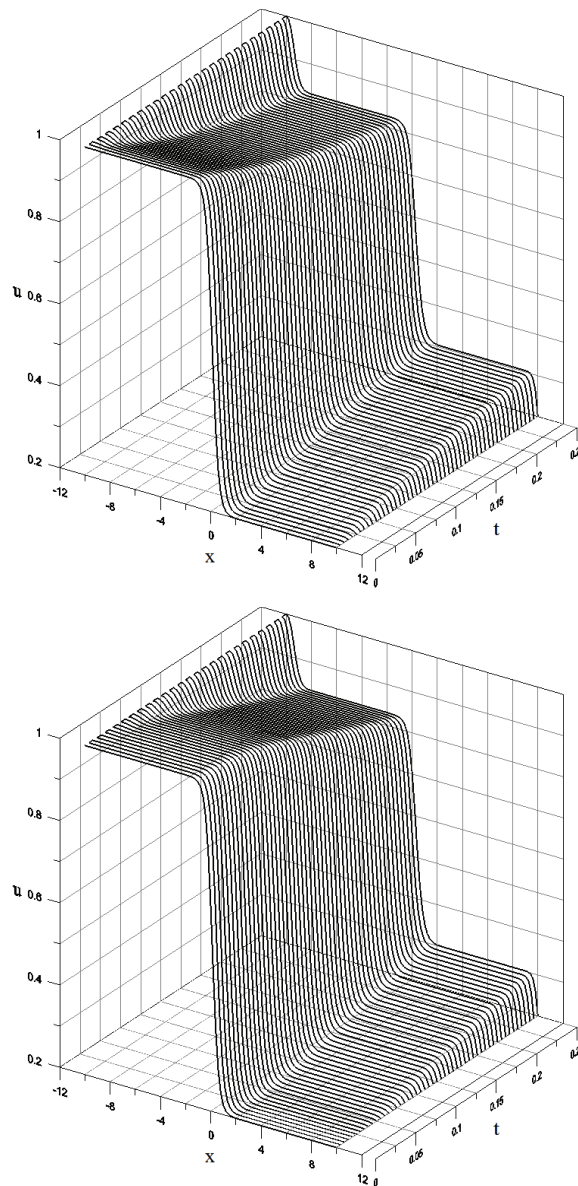
Figure 5: For Example 2 of a time-fractional derivative Burgers equation, the numerical results are calculated by the FTIM, displaying a distortion of the solution with that from the ordinary time-derivative Burgers equation.

3.2 Example 2

Next, we consider a time-fractional derivative Burgers equation:

$$D_t^\alpha u + u u_x - \nu u_{xx} = 0, \quad -10 < x < 10, \ 0 < t < T. \tag{47}$$

When $\alpha = 1$, we have a closed-form solution:

$$u(x,t) = \frac{\mu + \sigma + (\sigma - \mu)\exp F(x,t)}{1 + \exp F(x,t)},$$

$$F(x,t) = \frac{\mu}{\nu}(x - \sigma t - \lambda), \tag{48}$$

where $\nu = 0.1$, $\mu = 0.4$, $\sigma = 0.6$ and $\lambda = 0.125$ are constants been fixed. The initial condition and boundary conditions can be derived from this closed-form solution. When $\alpha < 1$, there exists no such a closed-form solution; however, we also employ the same initial and boundary conditions, in order to focus on the investigation of the effect due to the time-fractional derivative.

By applying the FTIM to this example we first check the accuracy. In the calculations we fix the initial guess of u_i^j by $u_i^j(0) = 0.5$, and the other parameters used are $m_1 = 99$, $m_2 = 49$, $T = 0.25$, $\gamma = 0.5$, $\Delta \tau = 0.01$, $\nu_0 = 0.1$ and $\varepsilon = 10^{-3}$. In Fig. 4 we show the numerical solution of the original Burgers equation, which is very close to the above exact solution in Eq. (48) with the error as shown in the bottom of Fig. 4 being smaller than 0.005. This computational case supported that we may use a FTIM to compute the Burgers equation with a high accuracy. In Fig. 5 we display the numerical solutions with the time-fractional derivatives of $\alpha = 0.5$ (top) and $\alpha = 0.75$ (bottom). It can be seen that the effect of time-fractional derivative obviously distorts the original constant solutions at the both ends of x to a curved surface.

4 Conclusions

In this paper, we have transformed the original fractional Burgers equation into another type of functional PDE in a one-dimension higher space by introducing a fictitious time variable, and adding a fictitious viscous damping coefficient to enhance the stability of numerical integration of the discretized equations by employing the GPS. The constant γ can be suitably chosen, such that the convergence of numerical solutions can be faster. Several numerical examples were worked out, which show that our proposed approach is applicable to the numerical solutions of complicated fractional Burgers equations. When the fractional derivatives are taken into account, the diffusive and dissipative behaviors of the Burgers equation

may be modified. Especially, the nonlinear fractional Burgers equation results in a profoundly changed profile of solutions.

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