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On Solving the Direct/Inverse Cauchy Problems of Laplace Equation in a Multiply Connected Domain, Using the Generalized Multiple-Source-Point Boundary-Collocation Trefftz Method & Characteristic Lengths

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Abstract: In this paper, a multiple-source-point boundary-collocation Trefftz method, with characteristic lengths being introduced in the basis functions, is proposed to solve the direct, as well as inverse Cauchy problems of the Laplace equation for a multiply connected domain. When a multiply connected domain with genus p (p>1) is considered, the conventional Trefftz method (T-Trefftz method) will fail since it allows only one source point, but the representation of solution using only one source point is impossible. We propose to relax this constraint by allowing many source points in the formulation. To set up a complete set of basis functions, we use the addition theorem of Bird and Steele (1992), to discuss how to correctly set up linearly-independent basis functions for each source point. In addition, we clearly explain the reason why using only one source point will fail, from a theoretical point of view, along with a numerical example. Several direct problems and inverse Cauchy problems are solved to check the validity of the proposed method. It is found that the present method can deal with both direct and inverse problems successfully. For inverse problems, the present method does not need to use any regularization technique, or the truncated singular value decomposition at all, since the use of a characteristic length can significantly reduce the ill-posed behavior. Here, the proposed method can be viewed as a general Trefftz method, since the conventional Trefftz method (T-Trefftz method) and the method of fundamental solutions (F-Trefftz method) can be considered as special cases of the presently proposed method.

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1 Introduction

The development of the so-called Trefftz method dates back to 1926 when Trefftz first proposed it in a conference (Trefftz, 1926). Since then, the so-called Trefftz method has been extensively studied and applied to many engineering problems. The basic idea of the Trefftz method is to find a set of the co-called T-complete functions that satisfy the governing equation identically, and then use these functions as the weighting functions of the field quantity (direct Trefftz method) or represent the field quantity by a linear combination of them (indirect method). This idea is very similar to the eigenfunction expansion, so that the Trefftz method is sometimes considered as a generalized eigenfunction expansion method. There exists a voluminous literature about using the Trefftz method. Cheung, Jin and Zienkiewicz (1989) applied the Trefftz method to a harmonic equation, and they also applied it to the plane elasticity problem (Cheung, Jin and Zienkiewicz, 1990), Helmholtz problem (Cheung, Jin and Zienkiewicz, 1991), and the plate bending problem (Cheung, Jin and Zienkiewicz, 1993). Kamiya and Wu (1994) used the Trefftz method to solve the Helmholtz equation. Kita, Kamiya and Ikeda (1995) used the Trefftz formulation to develop a boundary-type sensitivity analysis. Kita, Kamiya and Iio (1999) combined the domain decomposition method and Trefftz method to solve a 2D potential problem. Portela and Charafi (1999) proposed a multi-region Trefftz method to deal with the potential problem in an arbitrarily shaped 2D domain. Kita, Katsuragawa and Kamiya (2004) applied the Trefftz method to simulate the two-dimensional sloshing problem. Chang, Liu, Yeih and Kuo (2002) have used the direct Trefftz method to deal with the free vibration problem of a membrane. Chang, Liu, Kuo and Yeih (2003) developed a symmetric indirect Trefftz method to solve the free vibration problem. Liu, Yeih, Kuo and Chen (2006) have applied the T-Trefftz and F-Trefftz method, an alternative name for the method of fundamental solutions, to solve the Poisson equation. Yeih, Liu, Chang and Kuo (2007) have discussed the ill-posed nature of the Trefftz method. Li, Lu and Hu (2004) adopted a boundary-collocation Trefftz method to solve the biharmonic equation with singularities. Lu, Hu and Li (2004) used the Trefftz method to deal with the Motz problem, and the cracked beam problem. Huang and Li (2006) used the Trefftz method coupled with high order FEM to deal with the singularity problem. Li, Lu, Tsai and Cheng (2006) used the Trefftz method to solve the eigenvalue problem. For a useful survey of literature, one can refer to Kita and Kamiya (1995)

Unlike the conventionally used numerical methods such as the finite difference method (FDM), the finite element method (FEM), the boundary element method (BEM) and so on, the Trefftz method is less popular. In the authors' opinions, there are two reasons that limit the use of the Trefftz method. The first reason is that the system of linear equations which result from the Trefftz method is an ill-posed system, even for a well-posed boundary value problem. As mentioned by Kita and Kamiya (1995), when the number of functions in the Trefftz method increases, the condition number of the resulting dense unsymmetric matrix increases very fast. Yeih, Liu, Chang and Kuo (2007) have discussed the ill-posed nature of the Trefftz method, and they suggested to deal with it by using the Tikhonov's regularization method and L-curve concept. Liu (2007a, 2007b) later proposed a boundary-collocation Trefftz method to deal with the ill-posed behavior, in which a characteristic length is adopted, to scale the basis functions, hereafter referred to as the "modified" Trefftz method. Later, Liu (2008a) extended this method to deal with the potential problem in a 2-D doubly connected domain. Liu (2008b, 2008c) also applied this modified boundary-collocation Trefftz method to solve the inverse Cauchy problem, and found that this method did not require any regularization technique. Liu (2008d) applied the "characteristic-length-scale" or "modified" Trefftz method to solve the biharmonic equation. Since the "modified" collocation Trefftz method has successfully resolved the ill-posed nature of the problem, Liu, Yeih and Atluri (2009) used the resulting matrix of this method to propose a general purpose conditioner which can tackle with the ill-posed behaviors of various systems of linear algebraic equations. The second reason for the lack of popularity of the Trefftz method is that, for the multiply connected domains with genus p (p>1), the conventional Trefftz method (T-Trefftz method) fails. To deal with arbitrary shapes in a 2-D domain, especially for multiply connected domain with genus p (p>1), in the T-Trefftz method, one needs to decompose the problem domain into several simply connected subdomains and use the Trefftz method in each one. On the real boundary, the Trefftz method requires the approximate solution to satisfy the boundary conditions at each collocation point. On the artificial boundary (for dividing the domain into subdomains), the continuity conditions are used to connect the adjacent subdomains. Although this method may successfully resolve the difficulty when facing the multiply connected domain with genus p (p>1), the artificial boundaries introduced in this method, to satisfy the interfacial infinitely conditions, are not unique and depend on the users' preference. In addition, many extra collocation points on the artificial boundary introduce additional unknowns that are generally not good for the numerical method. Due to these two reasons, the Trefftz method is less popular. In this paper, we seek to remedy this situation.

In this paper, we propose a "modified" multiple-source-point boundary-collocation Trefftz method to deal with the Laplace equation. In presently proposed method, we first relax the constraint of the T-Trefftz method, and allow many source points in the system. For each source point, we will discuss how to construct a basis function set. Then, we adopt the "characteristic length" concept to reduce the ill-posed behavior for the Trefftz method. Beside this Introduction section, this article contains the following sections. In section 2, we discuss how to use multiple source points, and how to arrange the basis functions. In section 3, we introduce the concept of a characteristic length. In section 4, we use six examples to show the validity of the proposed method. In the final section, some concluding remarks are given.

2 Trefftz method: source points and basis functions

Consider a two-dimensional domain Ω enclosed by a boundary Γ, and the physical quantity u satisfies the Laplace equation:

$$\nabla^2 u(\mathbf{x}) = 0 \text{ for } \mathbf{x} \in \Omega, \tag{1}$$

$$u = f(\mathbf{x}) \in \Gamma_D \text{ (Dirichlet boundary condition)} \tag{2}$$

$$u_n \equiv \frac{\partial u}{\partial n} = g(\mathbf{x}) \in \Gamma_N \text{(Neumann boundary condition)} \tag{3}$$

$$\alpha u + \beta u_n = h,\ \alpha^2 + \beta^2 \neq 0,\ \mathbf{x} \in \Gamma_R \text{ (Robin boundary condition)} \tag{4}$$

where n denotes the outward normal direction, Γ_D denotes the boundary where the Dirichlet boundary condition is given, Γ_N denotes the boundary where the Neumann boundary condition is given, and Γ_R denotes the boundary where the Robin boundary condition is given.

On each boundary point, if only one type of a boundary condition is given it is referred to, in this paper, as the direct boundary value problem. If on a part of boundary, one is given both the Dirichlet data and Neumann data, but has no information on some other part of the boundary, it is referred to as the inverse Cauchy problem. It is known that the direct boundary value problem is well-posed, while the inverse Cauchy problem is ill-posed.

The conventional Trefftz method (T-Trefftz method) begins with the so-called T-complete functions. For a simply connected domain illustrated in Fig. 1(a), one usually locates the source point inside the domain, and the T-complete basis functions are chosen to be

$$\left\{1, r\cos\theta, r\sin\theta, \cdots, r^k \sin(k\theta), r^k \cos(k\theta), \cdots\right\},$$

where r and θ are the polar coordinate centered at the source-point, as shown in Fig. 1(a).

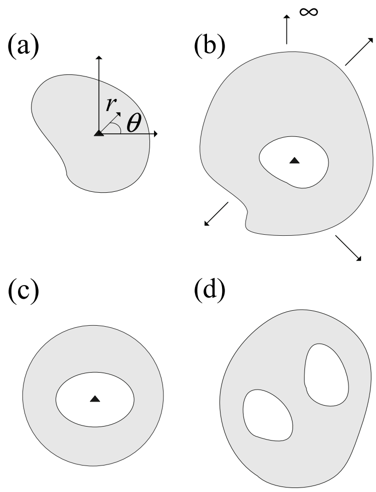

Figure 1: An illustration of (a) simply connected domain; (b) an infinite domain with a cavity; (c) a doubly connected domain, and (d) a multiply connected domain with genus p (p>1). denotes the source point which the conventional T-Trefftz method uses to generate the basis functions.

For an infinite domain with a cavity as illustrated in Fig. 1(b), one usually locates the source point inside the cavity, and the T-complete functions include

$$\left\{\ln r, r^{-1}\cos\theta, r^{-1}\sin\theta, \cdots, r^{-k}\sin(k\theta), r^{-k}\cos(k\theta), \cdots\right\}.$$

For a doubly connected domain (the multiply connected domain with genus 1) as shown in Fig. 1(c), the source point is located inside the cavity, and the T-complete functions include

$$\left\{1, \ln r, r^{\pm 1}\cos\theta, r^{\pm 1}\sin\theta, \cdots, r^{\pm k}\sin(k\theta), r^{\pm k}\cos(k\theta), \cdots\right\}.$$

For a multiply connected domain with genus p (p>1) as shown in Fig. 1(d), one needs to divide the domain into several simply connected domains, and apply the T-Trefftz method in each subdomain. However, as mentioned earlier, the introduction of an artificial boundary increases the number of unknowns and there is no unique way to introduce the imaginary boundaries. It is thus natural for one to ask, if there is simpler way to deal with the problem defined in a multiply connected domain.

Before we answer this question, we introduce another type of Trefftz method, namely the so-called F-Trefftz method, or the so-called method of fundamental solutions (MFS). For the T-Trefftz method, one can see that the number of source points is only one. To represent the field quantity in the T-Trefftz method, one increases the order of the basis functions in r and θ. The F-Trefftz method uses a different concept, in comparison with the T-Trefftz method. Instead of using only one source point and increasing the order of basis functions, the F-Trefftz method allows many source points but uses only one basis function, i.e., the fundamental solution of the differential operator. The concept of the F-Trefftz method is illustrated in Fig. 2. One may ask the questions: can we use many source points, as well as many basis functions at the same time? How does one construct a complete set of basis functions then? In the following, we will propose a generalized multiple-source-point Trefftz method to carry out this idea.

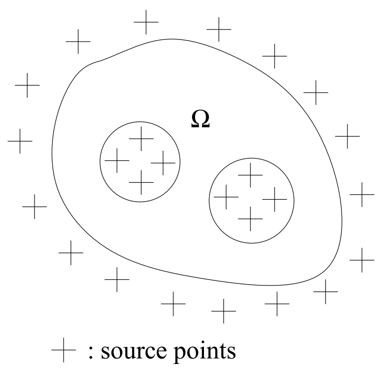

Figure 2: The F-Trefftz method uses many source points outside the domain to generate the fundamental solutions.

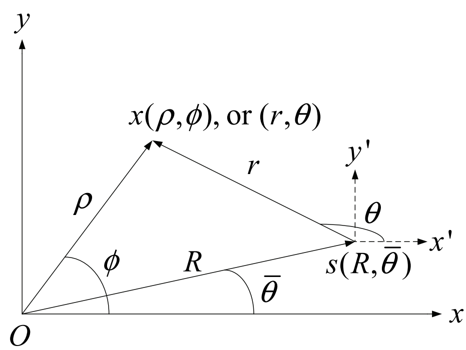

Figure 3: Illustration of the transformation between basis functions using different source points, O and S.

First, we consider a diagram as shown in Fig. 3. We consider two source points say O and S. The observation point is X. Using the polar coordinates centered at O, the coordinates of S are $\left(R, \bar{\theta}\right)$, and those of X are (ρ, ϕ). Using the polar coordinates centered of S, the location of X is (r, θ). Thus X is denoted by either (ρ, ϕ) or (r, θ). Then, we have the following formulae according to the addition theorem (Bird and Steele, 1992):

$$r^n \cos(n\theta) = \sum_{m=0}^{n} C_m^n (-1)^m \rho^{n-m} R^m \cos\left(n\bar{\theta} + m\left(\phi - \bar{\theta}\right)\right), \tag{5}$$

$$r^n \sin(n\theta) = \sum_{m=0}^{n} C_m^n (-1)^m \rho^{n-m} R^m \sin\left(n\bar{\theta} + m\left(\phi - \bar{\theta}\right)\right), \tag{6}$$

where C_m^n is the number of combinations to pick m pieces from n pieces.

From Eqs. (5) and (6), one can see that, for basis functions involving positive powers of the radial distance, the transformation of basis functions is possible. For example, consider the case when n=3, and try to represent $r^3 \cos(3\theta)$ by the basis functions using the source point at O. From Eq. (5), one can easily verify that one can use a combination of positive power basis functions, with power not greater than n=3, on the right-hand-side of Eqs. (5) and (6) to represent $r^3 \cos(3\theta)$. It means that we can use a linear combination of

$$\{1, \rho\cos\phi, \rho\sin\phi, \rho^2\cos 2\phi, \rho^2\sin 2\phi, \rho^3\cos 3\phi, \rho^3\sin 3\phi\}$$

to represent $r^3 \cos(3\theta)$. Therefore, when many source points are used, we may develop a set of positive power (of radial distance) basis functions from any one of these source points. If one uses two sets of positive power (of radial distance) basis functions from two source points, they form a linearly dependent basis, and make the resulting linear equation rank deficient. From this argument, one can easily see that for a simply connected domain as shown in Fig. 1(a), it is not necessary to locate the source point inside the domain, and we can as well locate the source point outside the domain. If we develop two sets of positive power (of radial distance) basis functions of the same order, from Eqs. (5) and (6) one can easily see that they are mathematically equivalent. However, in the existing literature, researchers usually locate the source point inside the domain. The reason may come from the fact that such a choice can avoid possible ill-posed behaviors, and is explained in previous literature (Yeih, Liu, Chang and Kuo, 2007).

If one locates the source point outside the domain then the negative power basis functions and logarithm function can also be employed to expand the solution. The addition theorem also gives the following formulae (Bird and Steele, 1992):

$$\ln r = \left\{ \begin{bmatrix} \ln R - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho}{R}\right)^m \cos\left(m\left(\bar{\theta}-\phi\right)\right),\ R>\rho \\ \ln\rho - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{R}{\rho}\right)^m \cos\left(m\left(\bar{\theta}-\phi\right)\right),\ R<\rho \end{bmatrix} \right. \tag{7}$$

$$r^{-m}\cos(m\theta) = \left\{ \begin{bmatrix} \sum_{k=-1}^{\infty} C_{k+1}^{m+k} \frac{\rho^{k+1}}{R^{k+m+1}} \cos\left[(k+1)\phi - (k+m+1)\bar{\theta}\right],\ R>\rho \\ \sum_{k=-1}^{\infty} (-1)^m C_{k+1}^{m+k} \frac{R^{k+1}}{\rho^{k+m+1}} \cos\left[(k+1)\bar{\theta} - (k+m+1)\phi\right],\ R<\rho \end{bmatrix} \right. \tag{8}$$

$$r^{-m}\sin(m\theta) = \left\{ \begin{bmatrix} \sum_{k=-1}^{\infty} C_{k+1}^{m+k} \frac{\rho^{k+1}}{R^{k+m+1}} \sin\left[(k+1)\phi - (k+m+1)\bar{\theta}\right],\ R>\rho \\ \sum_{k=-1}^{\infty} (-1)^m C_{k+1}^{m+k} \frac{R^{k+1}}{\rho^{k+m+1}} \sin\left[(k+1)\bar{\theta} - (k+m+1)\phi\right],\ R<\rho \end{bmatrix} \right. \tag{9}$$

It can be easily seen that in order to replace the negative power or logarithmic basis function for the source point S, by using the basis functions developed from source point O (see Fig. 3), one may need infinitely many positive power functions and negative power functions depending on the relationship between R and ρ. In other words, such basis functions (negative power and logarithm functions) developed from different source point are not mathematically equivalent. It means that they can be linear independent functions. To explain this in more detail, we consider a multiply connected domain with genus 2 as shown in Fig. 4. The boundary curve for outer circle, Γ_1, is $x^2+y^2=4$. The boundary curve for inner circles enclosing the cavities are: $(x-0.5)^2+y^2=0.4^2$ for Γ_2 and $(x+0.6)^2+y^2=0.3^2$ for Γ_3 respectively. We design the solution as: $\frac{1}{r_A}\sin\theta_A$ where r_A and θ_A are measured from the point A which is a point (0.5,0) in the Cartesian coordinate system. Now we place only one source point at B which is (-0.6,0) inside the cavity and the basis functions are:

$$\left\{1, \ln r_B, r_B^{\pm 1}\cos\theta_B, r_B^{\pm 1}\sin\theta_B, \cdots, r_B^{\pm k}\sin(k\theta_B), r_B^{\pm k}\cos(k\theta_B), \cdots\right\}$$

where r_B and θ_B are measuring from the point B. Now, using Eq. (9) we know that for $R>r_B$ where R is the distance between points A and B, we can use positive power basis functions to represent the solution. However, for $R<r_B$ one then needs negative power basis functions to represent the solution. We recall that, in

the Trefftz method, the undetermined coefficients must be some constants, and cannot be functions of spatial variables. It then can be concluded that for a multiply connected domain with genus p (p>1), we cannot represent all possible solutions if only one source point is used. Following this, we also can have a very important conclusion that inside each cavity at least one source point is required.

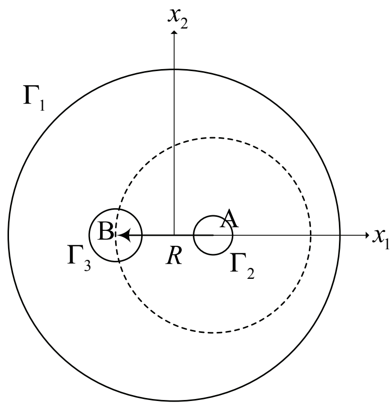

Figure 4: It is impossible to represent the negative power basis functions generated from source point B, by the basis functions from source point A [Linear independence].

Also, when we observe Eqs. (7)-(9), we can find that for different source points the negative power basis functions and logarithm basis functions are linearly independent. If we want to represent one such basis function of source point A, using basis functions developed from source point B, we may require infinitely many basis functions from source point B which is definitely not possible for numerical calculation, because at each source point we can only set up finitely many basis functions.

From the abovementioned reasons, one can conclude that if we allow ourselves to have many source points, and locate at least one source point inside each cavity then we can handle the multiply connected domains with genus p (p>1), without using the domain decomposition as suggested by the previous researches. To summarize the above discussion, we propose the generalized multiple-source-point Trefftz method: for each cavity, at least one source point should be placed, and the negative power and logarithmic basis functions are chosen for each point; and the positive power basis functions can be chosen for only one source point when considering a finite region (one can pick any one source point located in the cavity or just pick one point inside the domain). This generalized multiple-source-point

Trefftz method is the most general one developed so far, because the T-Trefftz and F-Trefftz methods can be treated as the special cases of the presently proposed generalized multiple-source-point Trefftz method. Although now we have resolved the difficulty of treating a multiply connected domain with genus p (p>1) by the concept of multiple-source-point Trefftz method, the ill-posed behavior still needs to be treated. In the following, the concept of characteristic length developed by Liu will be adopted, and the "generalized" multiple-source-point collocation Trefftz method, along with a characteristic length, will be introduced.

3 Generalized multiple-source point collocation Trefftz method, including the concept of a characteristic length

Now we consider a two-dimensional multiply connected domain with genus p. The multiple-source-point Trefftz method uses the following T-complete set:

$$\left\{r_i^{-m}\sin(m\theta_i), r_i^{-m}\cos(m\theta_i), \ln r_i\right\}$$

for the source points inside the cavity or outside the domain of interest, where in the subscript $i$denotes the i-th source points. The positive power basis functions can be arranged at only one source point, say P (P can coincide with one of the previous selected source points inside the cavity). The basis functions are:

$$\left\{1, r_P^m\sin(m\theta_P), r_P^m\cos(m\theta_P)\right\}.$$

Therefore, for the indirect Trefftz formulation one can say the solution is written as the linear combination of these basis functions as:

$$\begin{aligned} u = &\sum_{i=1}^{k}\left\{c_i\ln r_i + \left\{\sum_{m=1}^{h}\left[a_{im}r_i^{-m}\cos(m\theta_i) + b_{im}r_i^{-m}\sin(m\theta_i)\right]\right\}\right\} \\ &+ d + \sum_{m=1}^{h}\left[e_m r_P^m\cos(m\theta_P) + f_m r_P^m\sin(m\theta_P)\right] \end{aligned} \tag{10}$$

when the finite region is considered; and

$$u = \sum_{i=1}^{k}\left\{c_i\ln r_i + \left\{\sum_{m=1}^{h}\left[a_{im}r_i^{-m}\cos(m\theta_i) + b_{im}r_i^{-m}\sin(m\theta_i)\right]\right\}\right\} \tag{11}$$

when an infinite region is considered.

Liu has developed a collocation Trefftz method, involving a characteristic length, in order to make the resulting leading coefficient matrix well-posed. He proposed to use the characteristic lengths to the terms such as $(r)^{\pm m}$, by a characteristic length R, to lead to $\left(\frac{r}{R}\right)^{\pm m}$, such that they will tend to zero for large m. For positive power basis functions, the choice of R should be larger than the maximum distance between any domain point and source point. On the other hand for negative power basis functions, the choice of R should be less than the minimum distance between any domain point and the source point. To adopt this idea, Eqs. (10) and (11) can be rewritten as

$$u=\sum_{i=1}^{k}\left\{\bar{c}_i\ln r_i+\left\{\sum_{m=1}^{h}\left[\bar{a}_{im}\left(\frac{r_i}{R_i}\right)^{-m}\cos(m\theta_i)+\bar{b}_{im}\left(\frac{r_i}{R_i}\right)^{-m}\sin(m\theta_i)\right]\right\}\right\}$$
$$+\bar{d}+\sum_{m=1}^{h}\left[\bar{e}_m\left(\frac{r_P}{R_P}\right)^{m}\cos(m\theta_P)+\bar{f}_m\left(\frac{r_P}{R_P}\right)^{m}\sin(m\theta_P)\right] \tag{12}$$

for the finite region, where R_i and R_P are the characteristic lengths for different source points; and,

$$u=\sum_{i=1}^{k}\left\{\bar{c}_i\ln r_i+\left\{\sum_{m=1}^{h}\left[\bar{a}_{im}\left(\frac{r_i}{R_i}\right)^{-m}\cos(m\theta_i)+\bar{b}_{im}\left(\frac{r_i}{R_i}\right)^{-m}\sin(m\theta_i)\right]\right\}\right\} \tag{13}$$

for an infinite region. The formulations listed in Eqs. (12) and (13) constitute the basis for the generalized multiple-source-point collocation Trefftz method, with characteristic lengths. Using this formulation, we can deal with a multiply connected domain with genus p without introducing any artificial boundaries, and also avoid the ill-posed behavior, such that both the well-posed direct boundary value problem (BVP) as well as ill-posed Cauchy inverse BVP can be equally easily tackled by one unified tool. Before we demonstrate numerical examples, we will make a comment here. The concept of using many source points and the characteristic length has been mentioned in the paper by Bird and Steele (1992). However, they did not explain why one needs to adopt the concept of characteristic length, as Liu has explained (Liu, 2008a, 2008b, 2008c, 2008d). They did not mention that the use of a characteristic length can make the ill-posed system to be a well-posed one. In addition, they did not set up the basis functions for multiple source points correctly. Specifically, they assumed positive power basis functions for all source points, which will make the rank of resulting linear system to be deficient. In their paper, only a multiply connected domain with circular boundaries is considered. Therefore, we believe that the current method provides a more complete description, and is a thorough extension of their work.

In the following, we will use numerical examples to validate the proposed method.

4 Numerical examples

4.1 Example 1

In this example, we solve a well-posed Dirichlet boundary value problem. The domain of interest is a multiply-connected domain with genus 2. The boundaries are described as:

$$\begin{aligned}&\Gamma_1 : \rho_1 = 15,\\ &x_1 = \rho_1 \cos\theta_1,\ y_1 = \rho_1 \sin\theta_1, 0 \le \theta_1 \le 2\pi,\end{aligned} \tag{14}$$

$$\begin{aligned}&\Gamma_2 : \rho_2 = \sqrt{26 - 10\cos 4\theta_2},\\ &x_2 = \rho_2 \cos\theta_2,\ y_2 = \rho_2 \sin\theta_2, 0 \le \theta_2 \le 2\pi,\end{aligned} \tag{15}$$

$$\begin{aligned}&\Gamma_3 : \rho_3 = \sqrt{10 - 6\cos 2\theta_2},\\ &x_3 = \rho_3 \cos\theta_3 + 8,\ y_3 = \rho_3 \sin\theta_3, 0 \le \theta_3 \le 2\pi,\end{aligned} \tag{16}$$

where $(\rho_i,\ \theta_i)$ are the polar coordinates to describe the boundary Γ_ι and $(x_i,\ y_i)$ represent the location of points on the boundary Γ_ι in Cartesian coordinates.

The designed analytical solution is:

$$u(x,y) = \frac{\cos\theta_{s_1}}{r_{s_1}} + \frac{\sin 2\theta_{s_2}}{r_{s_2}^2}, \tag{17}$$

where (r_{s_i}, θ_{s_i}) is the polar coordinates system from point s_i with $(x_{s_1}, y_{s_1}) = (0,0)$, $(x_{s_2}, y_{s_2}) = (7,0)$ representing location of two points s_1 and s_2. It implies that we have:

$$r_{s_1} = \sqrt{(x - x_{s_1})^2 + (y - y_{s_1})^2},\ \theta_{s_1} = \text{atan}(\frac{y - y_{s_1}}{x - x_{s_1}}), \tag{18}$$

$$r_{s_2} = \sqrt{(x - x_{s_2})^2 + (y - y_{s_2})^2},\ \theta_{s_2} = \text{atan}(\frac{y - y_{s_2}}{x - x_{s_2}}). \tag{19}$$

We use the Dirichlet boundary conditions arising out of the solution in (17) as inputs, and use the presently proposed generalized multiple-source-point boundary-collocation Trefftz method with characteristic length scales to compute the solution and compare it with (17).

It can be easily verified that the $s_1$point is located inside the cavity enclosed by the boundary Γ_2, and the s_2 point is located inside the cavity enclosed by the boundary Γ_3. Accordingly, we expect that the conventional Trefftz method will fail.

We now give the locations of source points as:

$$(x_P, y_P) = (1,1),\ (\bar{x}_1, \bar{y}_1) = (1,1),\ (\bar{x}_2, \bar{y}_2) = (8,1). \tag{20}$$

where (x_P, y_P) is the location from which we set up the positive power basis functions, and $(\bar{x}_i, \bar{y}_i)$ represents the i-th source point from where we set up the logarithmic and negative power basis functions. The domain and the source points illustrate in Fig. 5. It can be seen that $(\bar{x}_1, \bar{y}_1)$ is identical with (x_P, y_P), and is located inside the cavity enclosed by the boundary Γ_2 and $(\bar{x}_2, \bar{y}_2)$ locates inside the cavity enclosed by the boundary Γ_3.

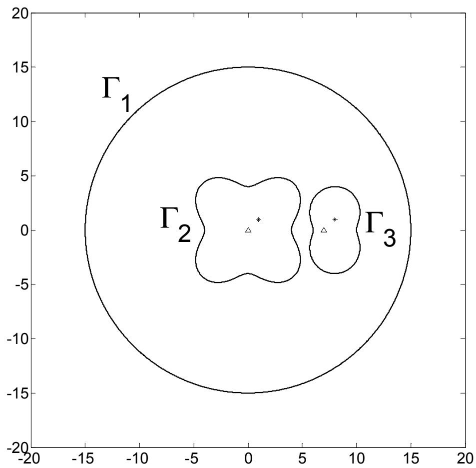

Figure 5: The boundary shapes of Example 1. The symbols Δ denote the origin of the polar coordinate for analytical solution in Eq. (17), and the symbols * denote the source points of the multiple-source-point Trefftz method.

To verify the claim we made previously that for a multiply connected domain with genus p (p>1) it is impossible to solve if one only uses one source point, we first use $(\bar{x}_1, \bar{y}_1)$ as our source point and set up positive power basis functions, negative power basis functions and a logarithmic function. The characteristic lengths are: R_P=15 for the positive power basis functions and R_1=1 for the negative power functions. The maximum order for the power basis functions is m=50. We select a circle with radius equal to 12 and with a center at (0,0), and then plot the field quantity u and its absolute error as the blue dotted lines in Figs. 6(a) and 6(b). It can be seen that using only one source point we cannot obtain an acceptable solution in comparison with the exact one, even though we have already adopted the concept of a characteristic length and 102 basis functions. The reason can be seen from Fig. 6(c). To represent the second term in the right hand side of Eq. (11) using the source point at $(\bar{x}_1, \bar{y}_1)$, we can draw a circle with its radius of $\sqrt{37}$ (the distance between $(\bar{x}_1, \bar{y}_1)$ and (x_{s_2}, y_{s_2})) and center at $(\bar{x}_1, \bar{y}_1)$. It can be seen that for the boundaries Γ_2 and Γ_3, some points are inside the circle, but some points are

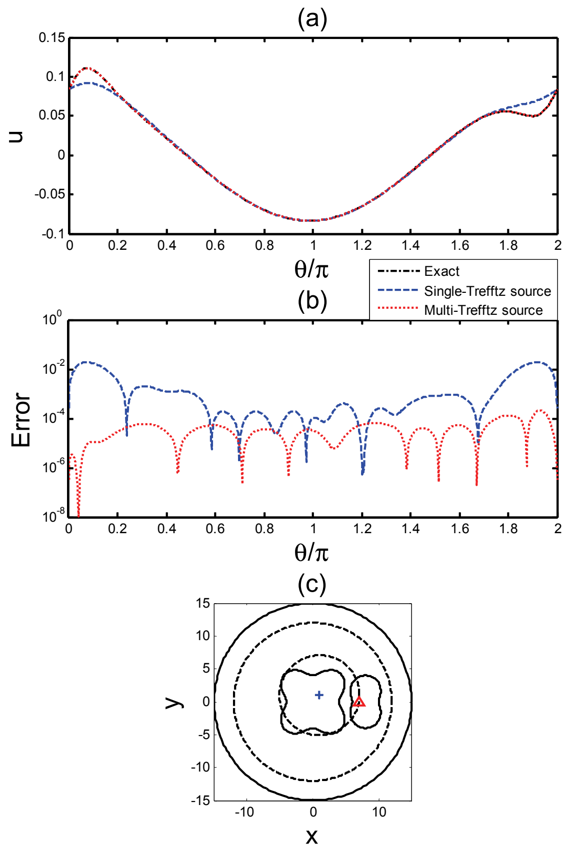

Figure 6: (a) The comparison of the exact solution u on a circle and the numerical solutions obtained by single-Trefftz source and Multi- Trefftz source points; (b) the numerical error of Example 1, and (c) the sketched diagram to explain why using a single source point fails.

not. According to the addition theorem in Eq. (9), we then need two sets of coefficients for the basis functions generated from the source point $(\bar{x}_1, \bar{y}_1)$ to represent the second term of the right hand side in Eq. (11), which is definitely impossible. In numerical reality, the coefficients of the basis functions which we solved by using the conjugate gradient method can only make the boundary collocation error minimum, and it never can be correct. Consequently, when we examine the physical quantities on the circle with radius equal to 12 and with a center at (0,0) it is impossible for us to have accurate results as shown in Fig. 6(a).

Now we adopt the generalized multiple-source point collocation Trefftz method with characteristic lengths, to solve this problem again. The source points are those in Eq. (20). It can be seen that the source point we use to set up the positive power basis functions, and the source point we use to set up the negative power basis functions and logarithm function for the cavity enclosed by the boundary Γ_y are identical. The characteristic lengths used are: R_P=15, R_1=1 and R_2=1. The maximum order for the power basis functions is m=50. It means that now we have 152 basis functions totally. We plot the field quantity u as well by the red dotted lines in Figs. 6(a) and 6(b). It can be seen that now the numerical solution is very close to the exact one when multiple sources are used.

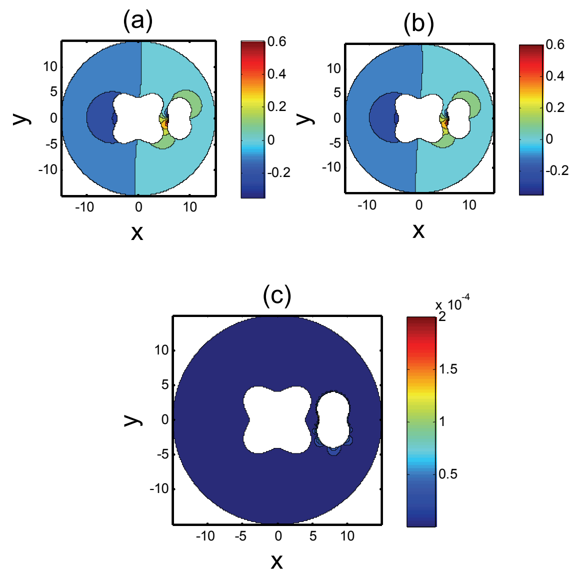

Figure 7: The contour of (a) the exact solution, (b) the numerical solution, and (c) error of Example 1.

In Fig. 7, we also demonstrate the contour plots of the exact solution, numerical solution and absolute error. It can be said that the proposed method successfully solves the difficulty of a multiply connected domain problem with genus p>1 for the Trefftz method.

4.2 Example 2

In this example, we solve the boundary value problem with Dirichlet boundary data on part of the boundary and Neumann boundary data on the remainder. The domain of interest is a multiply connected domain with genus 2, as shown in Fig. 8. The

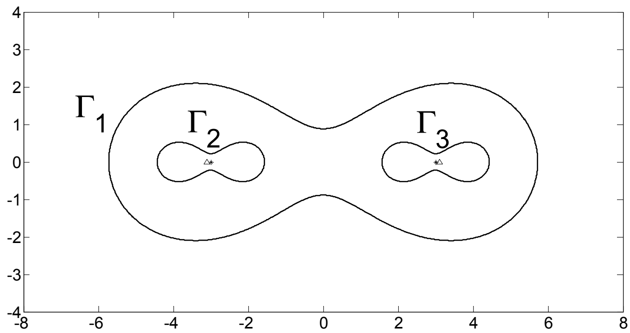

Figure 8: The boundary shapes of Example 2. The symbols Δ denote the origin of the polar coordinate for analytical solution in Eq. (24), and the symbols * denote the source points of the multiple-source-point Trefftz method.

boundaries are described as follows:

$$\Gamma_1:\rho_1=4\sqrt{\cos 2\theta_1+\sqrt{1.1-\sin^2 2\theta_1}},\quad x_1=\rho_1\cos\theta_1,\ y_1=\rho_1\sin\theta_1, 0\le\theta_1\le 2\pi, \tag{21}$$

$$\Gamma_2:\rho_2=\sqrt{\cos 2\theta_2+\sqrt{1.1-\sin^2 2\theta_2}},\quad x_2=\rho_2\cos\theta_2-3,\ y_2=\rho_2\sin\theta_2, 0\le\theta_2\le 2\pi, \tag{22}$$

$$\Gamma_3:\rho_3=\sqrt{\cos 2\theta_3+\sqrt{1.1-\sin^2 2\theta_3}},\quad x_3=\rho_3\cos\theta_3+3,\ y_3=\rho_3\sin\theta_3, 0\le\theta_3\le 2\pi, \tag{23}$$

where $(\rho_i,\ \theta_i)$ is the polar coordinates to describe the boundary Γ_ι and $(x_i,\ y_i)$ represent the location of points on the boundary Γ_ι by the Cartesian coordinates. On boundary Γ_1 the Dirichlet boundary condition is given and on boundaries Γ_2 and Γ_3 the Neumann boundary condition is given.

The designed exact solution is

$$u(x,y)=\exp\left(\frac{x-x_{s_1}}{r_{s_1}^2}\right)\cos\left(\frac{y-y_{s_1}}{r_{s_1}^2}\right)+\exp\left(\frac{y-y_{s_2}}{r_{s_2}^2}\right)\sin\left(\frac{x-x_{s_2}}{r_{s_2}^2}\right) \tag{24}$$

where (r_{s_i},θ_{s_i}) is the polar coordinates system from point s_i with $(x_{s_1},y_{s_1})=(-3.1,0)$, $(x_{s_2},y_{s_2})=(3.1,0)$ representing location of two points s_1 and s_2. It means:

$$r_{s_1}=\sqrt{(x-x_{s_1})^2+(y-y_{s_1})^2}, \tag{25}$$

$$r_{s_2} = \sqrt{(x - x_{s_2})^2 + (y - y_{s_2})^2}. \tag{26}$$

The Dirichlet b.c on Γ_1 and the Neumann b.c on Γ_2 and Γ_3, corresponding to the exact solution (24) are then used as inputs to the present multiple-source-point boundary-collocation Trefftz method, with characteristic lengths, and the computed solution by boundary-collocation is compared to the exact solution (24).

The source points we use for the MMSCT are:

$(x_P, y_P) = (-3, 0)$, $(\bar{x}_1, \bar{y}_1) = (-3, 0)$, $(\bar{x}_2, \bar{y}_2) = (3, 0)$. The domain and the source points illustrate in Fig. 8. Other parameters are: M=30, R_P=9, R_1=0.2 and R_2=0.2.

We select a circle with radius equal to 0.5 and with a center at (0,0) and plot the physical quantity u on it. In Fig. 9(a), the numerical solution is very close to the exact one. In Fig 9(b), both absolute error and relative error are plotted and they all reach the order of 10^{-3}, which shows the proposed method can obtain a very accurate solution.

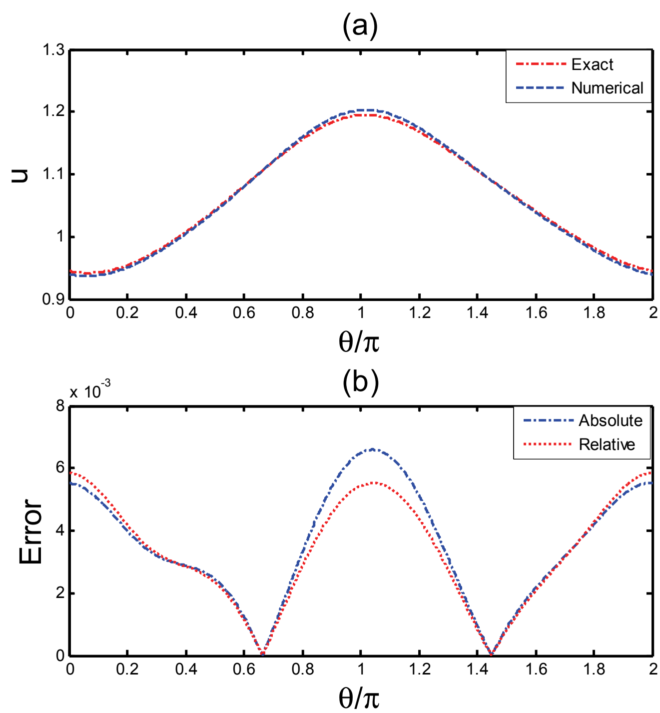

Figure 9: (a) The comparison of the exact solution of u on a circle with radius equal to 0.5 and its center at (0,0) and the numerical solution, and (b) the absolute and the relative error of Example 2.

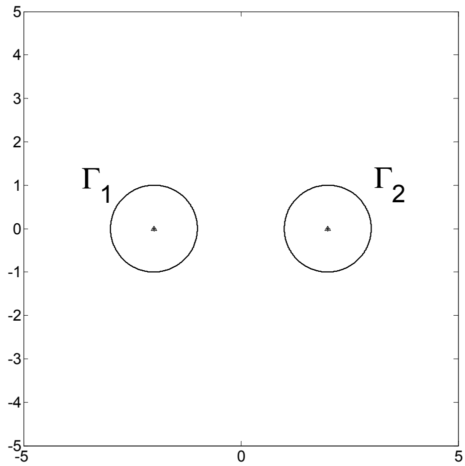

Figure 10: The boundary shapes of Example 3. The symbols Δ denote the origin of the polar coordinate for analytical solution in Eq. (29), and the symbols * denote the source points of the multiple-source-point Trefftz method.

4.3 Example 3

In this example, we consider an infinite domain with two cavities, as shown in Fig. 10. The boundaries of cavities are given as:

$$\Gamma_1 : x_1 = \cos\theta_1 - 2,\ y_1 = \sin\theta_1, 0 \le \theta_1 \le 2\pi, \tag{27}$$

$$\Gamma_2 : x_2 = \cos\theta_2 + 2,\ y_2 = \sin\theta_2, 0 \le \theta_2 \le 2\pi, \tag{28}$$

where θ_1 is the angle between positive x-axis and the line between (-2,0) and the observation point; θ_2 is the angle between positive x-axis and the line between (2,0) and the observation point. The designed exact solution is given as:

$$u(x,y) = \frac{2}{\pi}\text{atan}\left(\frac{2(y-y_{s_1})}{r_{s_1}^2-1}\right) + \frac{2}{\pi}\text{atan}\left(\frac{2(y-y_{s_2})}{r_{s_2}^2-1}\right), \tag{29}$$

where $(x_{s_1}, y_{s_1}) = (-2,0)$, $(x_{s_2}, y_{s_2}) = (2,0)$ and $r_{s_1} = \sqrt{(x-x_{s_1})^2+(y-y_{s_1})^2}$, $r_{s_2} = \sqrt{(x-x_{s_2})^2+(y-y_{s_2})^2}$.

In this example, we can only arrange the negative power basis functions and the logarithm function, since the positive power basis functions tend to infinity when the radius goes to infinity. Therefore two source points we used are:

$(\bar{x}_1, \bar{y}_1) = (-2,0)$ and $(\bar{x}_2, \bar{y}_2) = (2,0)$.

The domain and the source points are illustrated in Fig. 10. Parameters used are: $m = 50$, $R_1 = 1$, $R_2 = 1$. We solve the problem by giving Dirichlet boundary data on boundaries.

We select a circle with radius equal to 0.1 and with a center at (0,0), and then we examine how well the numerical solution is computed, by using the proposed method. The physical quantity u is plotted in Fig. 11(a) and absolute error is plotted in Fig. 11(b). From these figures, one can see that the proposed method can obtain a very good result. It then can be concluded that the proposed method can deal with an infinite domain with many cavities.

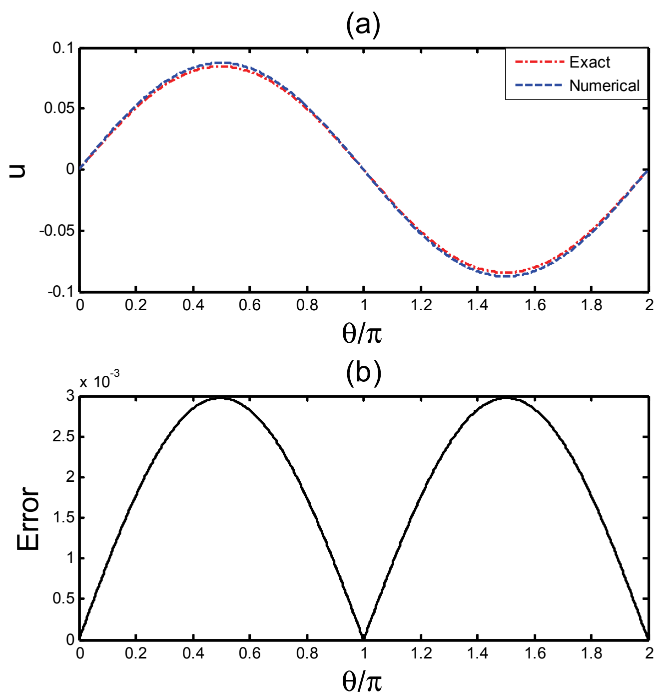

Figure 11: (a) The comparison of the exact solution of u on a circle with radius equal to 0.1 and its center at (0,0) and the numerical solution, and (b) the error of Example 3.

4.4 Example 4

In this example, we consider the inverse Cauchy boundary value problem. The domain is a multiply connected domain with genus 2, as shown in Fig. 12. The

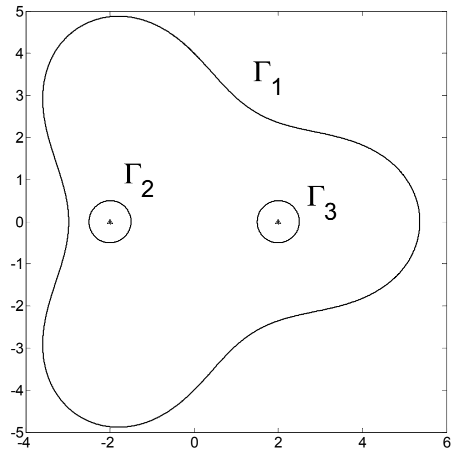

Figure 12: The boundary shapes of Example 4. The symbols Δ denote the origin of the polar coordinate for analytical solution in Eq. (33), and the symbols * denote the source points of the multiple-source-point Trefftz method.

boundaries are given as:

$$\Gamma_1 : \rho_1 = 4(\cos 3\theta_1 + \sqrt{2 - \sin^2 3\theta_1})^{\frac{1}{3}}, \\ x_1 = \rho_1 \cos\theta_1,\ y_1 = \rho_1 \sin\theta_1, 0 \le \theta_1 \le 2\pi, \tag{30}$$

$$\Gamma_2 : x_2 = 0.5\cos\theta_2 - 2,\ y_2 = 0.5\sin\theta_2, 0 \le \theta_2 \le 2\pi, \tag{31}$$

$$\Gamma_3 : x_3 = 0.5\cos\theta_3 + 2,\ y_3 = 0.5\sin\theta_3, 0 \le \theta_3 \le 2\pi, \tag{32}$$

where Cauchy data are given on Γ_1 and no information on Γ_2 and Γ_3.
The designed exact solution is:

$$u(x,y) = \frac{\cos 3\theta_{s_1}}{r_{s_1}^3} + \frac{\sin 3\theta_{s_2}}{r_{s_2}^3}, \tag{33}$$

where $(x_{s_1}, y_{s_1}) = (-2, 0)$, $(x_{s_2}, y_{s_2}) = (2, 0)$ and

$$r_{s_1} = \sqrt{(x - x_{s_1})^2 + (y - y_{s_1})^2},\ \theta_{s_1} = \text{atan}(\frac{y - y_{s_1}}{x - x_{s_1}}), \tag{34}$$

$$r_{s_2} = \sqrt{(x - x_{s_2})^2 + (y - y_{s_2})^2},\ \theta_{s_2} = \text{atan}(\frac{y - y_{s_2}}{x - x_{s_2}}). \tag{35}$$

The Cauchy data on Γ_1, corresponding to the assumed exact solution (33), is assumed as input to the present numerical solution process, and the computed solution and the b.c on Γ_2 and Γ_3 are compared with these from Eq. (33).

The source points are:

$$(x_P, y_P) = (-2,0),\ (\bar{x}_1, \bar{y}_1) = (-2,0),\ (\bar{x}_2, \bar{y}_2) = (2,0). \tag{36}$$

The domain and the source points illustrate in Fig. 12. Parameters used are: $m = 4$, $R_p = 8$, $R_1 = 0.4$, $R_2 = 0.4$.

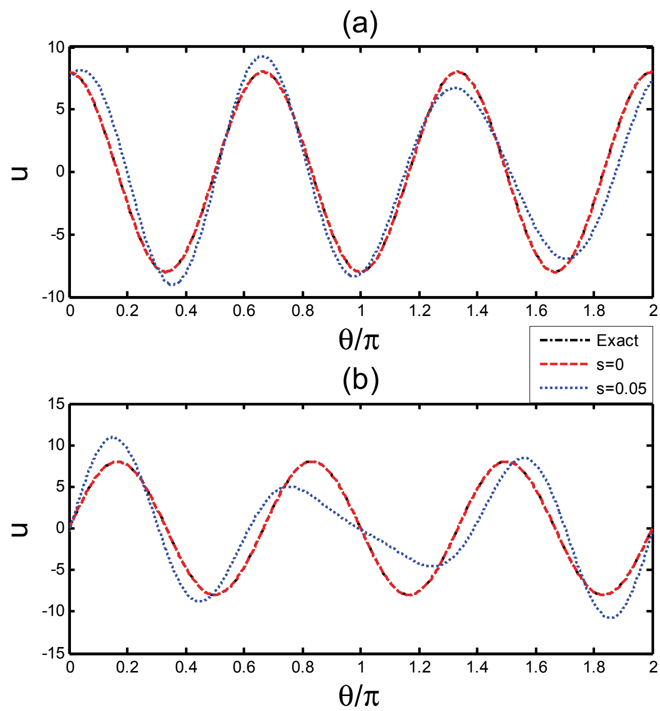

Figure 13: (a) The comparison of the exact solution and the numerical solutions without noise, and under 5% random noise on Γ_2, and (b) on $\Gamma_3$of Example 4.

We study two cases: the boundary data being polluted with a 5% relative random noise, and without noise. The results are illustrated in Figs. 13(a) and 13(b) in which the physical quantity on Γ_2 is plotted in Fig. 13(a) and the physical quantity on Γ_3 is plotted in Fig. 13(b). It can be seen when no noise exists in the boundary data, the recovery of physical quantity on the unknown boundary is very good. However, when the noise appears in data, the recovery of physical quantity for the

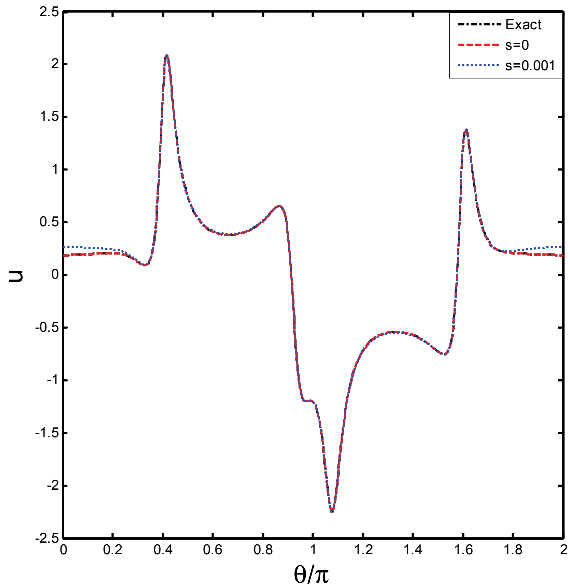

Figure 14: The comparison of the exact solution and the numerical solutions without noise, and under 0.1% random noise on $\Gamma_1$of Example 5.

unknown boundary then becomes not so good, due to the ill-posed nature of the inverse Cauchy problem. Although the numerical solution cannot accurately recover the quantity on unknown boundary under noise, the trend of physical quantity is acceptable. In comparison with conventional methods to deal with the ill-posed problem such as Tikhonov's regularization method (Tikhonov and Arsenin, 1977), or the truncated SVD method, the current method shows many superior merits over them. In Tikhonov's regularization method or truncated SVD method, the determination of regularization parameter or truncation threshold consumes a lot of computational time. In our method, the characteristic length is very easy to determine, such that one can solve the ill-posed problem without spending effort on determining the regularization parameter, or truncation threshold. In this manner, our method can deal with the well-posed problem, as well as the ill-posed problem, by using the same algorithm and this feature makes our method a unified one.

4.5 Example 5

In the previous example, the Cauchy data are given on the outer boundary and no information on the boundaries of cavities is given. In this example, we give the Cauchy data on the boundaries of cavities and no information on the outer boundary is given. The boundaries of interested domain are given as:

$$\Gamma_1 : x_1 = 2\cos\theta_1 + \cos 2\theta_1,\ y_1 = 2\sin\theta_1 - \sin 2\theta_1, 0 \le \theta_1 \le 2\pi, \tag{37}$$

$$\Gamma_2 : x_2 = 0.1(2\cos\theta_2 + \cos 2\theta_2) - 0.5,\ y_2 = 0.1(2\sin\theta_2 - \sin 2\theta_2) - 1, 0 \le \theta_2 \le 2\pi, \tag{38}$$

$$\Gamma_3 : x_3 = 0.1(2\cos\theta_3 + \cos 2\theta_3) - 0.5,\ y_3 = 0.1(2\sin\theta_3 - \sin 2\theta_3) + 1, 0 \le \theta_3 \le 2\pi, \tag{39}$$

where Cauchy data are now given on Γ_2 and Γ_3, and no information on Γ_1.

The designed exact solution is given as:

$$u(x,y) = \frac{\cos\theta_{s_1}}{r_{s_1}} + \frac{\sin\theta_{s_2}}{r_{s_2}}, \tag{40}$$

where $(x_{s_1}, y_{s_1}) = (-0.5, -1)$, $(x_{s_2}, y_{s_2}) = (-0.5, 1)$, and

$$r_{s_1} = \sqrt{(x - x_{s_1})^2 + (y - y_{s_1})^2},\ \theta_{s_1} = \text{atan}(\frac{y - y_{s_1}}{x - x_{s_1}}), \tag{41}$$

$$r_{s_2} = \sqrt{(x - x_{s_2})^2 + (y - y_{s_2})^2},\ \theta_{s_2} = \text{atan}(\frac{y - y_{s_2}}{x - x_{s_2}}). \tag{42}$$

The Cauchy data on the boundaries of cavities, corresponding to Eq. (40), is taken as input to the present numerical solution, and the computed interior solution, as well as the b.c on the outer boundary, are compared with these from Eq. (40).

The source points are: $(x_P, y_P) = (-0.5, -1)$, $(\bar{x}_1, \bar{y}_1) = (-0.5, -1)$, $(\bar{x}_2, \bar{y}_2) = (-0.5, 1)$. The parameters are: $m = 4$, $R_P = 4$, $R_1 = 0.1$, $R_2 = 0.1$.

The cases with 0.1% relative random noise in data and that without noise are studied. The recovery of physical quantity on the unknown boundary is illustrated in Fig. 14. It can be seen that without noise, the proposed method can recover boundary quantity very well. When 0.1% relative noise is added, the numerical result deviates from the exact one a little bit but still very good for the ill-posed inverse Cauchy problem. The contour plots of exact solution, numerical solution without noise and numerical solution with 0.1% relative noise are shown in Figs. 15(a), (b) and (c), respectively.

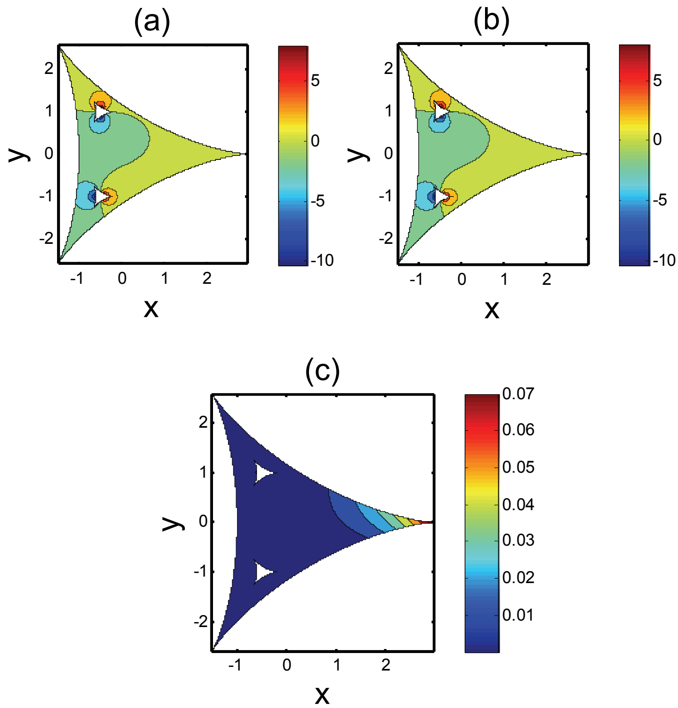

Figure 15: The contours of (a) the exact solution, (b) the numerical solution, and (c) error of Example 5.

4.6 *Example 6*

In this example, we consider to solve the inverse Cauchy boundary value problem of an infinite domain with two cavities. The boundaries of two cavities are given as:

$$\Gamma_1 : x_1 = \cos^3\theta_1 - 2,\ y_1 = \sin^3\theta_1, 0 \le \theta_1 \le 2\pi, \tag{43}$$

$$\Gamma_2 : x_2 = \cos^3\theta_2 - 2,\ y_2 = \sin^3\theta_2, 0 \le \theta_2 \le 2\pi, \tag{44}$$

where Cauchy data are given on Γ_1 and no information is given on Γ_2.
The designed exact solution is:

$$u(x,y) = \frac{\sin 2\theta_{s_1}}{r_{s_1}^2} + \frac{\sin 2\theta_{s_2}}{r_{s_2}^2}, \tag{45}$$

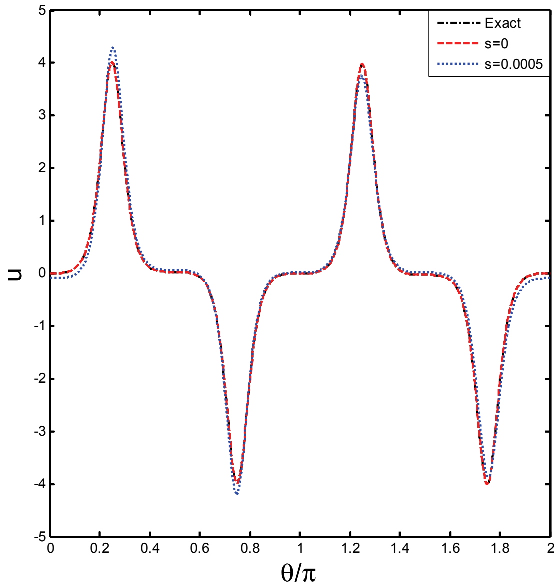

Figure 16: The comparison of the exact solution and the numerical solutions without noise, and under 0.05% random noise on Γ_2, of Example 6.

with$(x_{s_1}, y_{s_1}) = (-2,0)$, $(x_{s_2}, y_{s_2}) = (2,0)$, and

$$r_{s_1} = \sqrt{(x-x_{s_1})^2 + (y-y_{s_1})^2},\ \theta_{s_1} = \text{atan}(\frac{y-y_{s_1}}{x-x_{s_1}}), \tag{46}$$

$$r_{s_2} = \sqrt{(x-x_{s_2})^2 + (y-y_{s_2})^2},\ \theta_{s_2} = \text{atan}(\frac{y-y_{s_2}}{x-x_{s_2}}). \tag{47}$$

The Cauchy data on Γ corresponding to Eq. (45) are used as inputs to the present numerical solution of the Laplace equation and the computed interior solution, as well as the b.c on Γ_2, are compared with these corresponding to Eq. (45).

The source points are:$(\bar{x}_1, \bar{y}_1) = (-2,0)$, $(\bar{x}_2, \bar{y}_2) = (2,0)$. Parameters are: $m = 4$, $R_1 = 0.4$, $R_2 = 0.4$.

We study the cases of introducing a 0.05% relative random noise and that without noise. The results of boundary quantity recovery are shown in Fig 16. From the results, we can find that our method can deal with inverse Cauchy problem of an infinite domain with multiple cavities.

5 Conclusions

In this paper, a generalized multiple-source-point boundary-collocation Trefftz method, with characteristic lengths, is proposed. The reasons why multiple source points are needed is explained, and how to set up linearly independent basis functions is indicated clearly. Due to the use of many source points, the multiply connected domain with genus p (p>1) can be treated without introducing any artificial boundaries. The present method is a unified method, such that the conventional Trefftz method (T-Trefftz method) and the method of fundamental solutions (F-Trefftz method) become special cases of the present method. Owing to the concept of a characteristic length, the ill-posed behaviors can be overcome, such that one can deal well-posed BVP, as well as ill-posed Cauchy BVP, under the present unified method. Numerical examples show that the proposed method can deal with BVPs of a multiply connected domain with genus p (p>1).

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