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An Efficient Reliability-based Optimization Method for Uncertain Structures Based on Non-probability Interval Model

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Abstract: In this paper, an efficient interval optimization method based on a reliability-based possibility degree of interval (RPDI) is suggested for the design of uncertain structures. A general nonlinear interval optimization problem is studied in which the objective function and constraints are both nonlinear and uncertain. Through an interval order relation and a reliability-based possibility degree of interval, the uncertain optimization problem is transformed into a deterministic one. A sequence of approximate optimization problems are constructed based on the linear approximation technique. Each approximate optimization problem can be changed to a traditional linear programming problem, which can be easily solved by the simplex method. An iterative framework is also created, in which the design space is updated adaptively and a fine optimum can be well reached. Two numerical examples are investigated to demonstrate the effectiveness of the present method. Finally, it is employed to perform the optimization design of a practical automobile frame.

Keywords: Structural optimization; Uncertainty; Interval; Reliability; Possibility of degree

1 Introduction

In traditional structural optimization, the evaluations of the objective function and constraints are always based on analytical models with deterministic parameters. However, uncertainties concerned with geometric dimensions, material properties, loads, boundary conditions and etc widely exist in practical engineering problems. For such class of problems, uncertain optimization methods need to be developed for reliable or robust designs.

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One way to deal with uncertainties is to use probabilistic methods Elishakoff (1983); Gurav et al (2005); Liu et al (2003); Kall et al (1982) ; Doltsinis et al (2004,2005) , in which the uncertain parameters are treated as random variables with certain probability distributions. Probabilistic methods require an abundance of experimental data to construct precise probability distributions for the uncertain parameters, however, it is often expensive or even impossible to get sufficient information on the uncertainty. Furthermore, there is research indicating that even a small deviation of the probability distribution is likely to cause a large error of the reliability analysis Ben-Haim et al (1990). Fortunately, in combination with engineering experience, it is always possible to identify the bounds of the uncertain parameters through only a small amount of samples. Consequently, the interval method is developed to deal with this kind of problems without enough uncertainty information.

In the past two decades, the interval method has been attracting more and more attentions as a result of making the uncertainty analysis more convenient and economical. Tanaka et al (1984); Ishibuchi and Tanaka (1990); Rommelfanger (1989) discussed the linear programming problem with interval coefficients in the objective function. Liu and Da (1999) proposed a fuzzy satisfactory degree of interval number to deal with the uncertain constraints. Zhang et al (1999) assumed the interval numbers as random variables with uniform distributions and proposed a possibility degree to solve the multi-criteria decision problem. The described works above are all emphasized on the linear interval number programming. In recent years, many researchers intend to develop some new methods to deal with the nonlinear interval number optimization problems. Levin (1999) seems the first to investigate the nonlinear optimization under interval uncertainty from the mathematical point of view, and an interval Lagrangian function was introduced to solve the interval nonlinear optimization problems. Ma (2002) presented a new approach to solve this kind of problems in which only the uncertain objective function was considered and the uncertain optimization was transformed into a deterministic three-objective optimization. Jiang et al (2008a,2008b) suggested a nonlinear interval number programming (NINP) method based on an interval order relation and a modified possibility degree. This method is suited to solve NINP problems with uncertain coefficients both in the objective function and constraints. Han et al (2009) suggested an efficient NINP method based on the linear sequential programming. Though several NINP models have been developed to transform the uncertain problems to deterministic ones and also some effective techniques have been proposed to improve the optimization efficiency, there still exist severe limitations making NINP method unable to play a bigger role in dealing with the practical problems. Firstly, in the NINP, the possibility degree is generally used to compare intervals and whereby deal with interval constraints. For the current possibility degree, the range of their values is limited within the scope of [0,1], in which 0 and 1 represent that one interval is absolutely larger or smaller than another one. Once the value 0 or 1 is reached, their comparing function will be weakened, and whereby it can't reflect the relative positions of two parameter intervals quantitatively. However, for practical problems different relative positions of two parameter intervals indicate different reliability, whereas the possibility degree can not grasp this important characteristic. As a matter of fact, lacking the ability of reliability analysis limits a wider engineering application of the above NINP methodologies. Secondly, in all the aforementioned NINP methods, the transformed deterministic optimization problems are still nested, which leads to extremely low computational efficiency for practical problems with time-consuming simulation models.

On the basis of the authors' previous work, we propose an efficient interval optimization method using a reliability-based possibility degree of interval, which aims to deal with reliability-based design optimization problems with high optimization efficiency. The following text consists of four major parts. The first part is the formulation of the problem, in which a general interval optimization problem is investigated. In the second part, an NINP method using a reliability-based possibility degree of interval (RPDI) is suggested to transform the uncertain optimization problem into a deterministic optimization problem. In the third part, the linear approximation technique is combined with the proposed NINP method, and hence a sequential nonlinear interval optimization algorithm is developed. A series of approximate optimization problems are constructed. At each iterative step, the approximate interval problem can be changed to a deterministic optimization problem using an order relation and RPDI, which can be easily solved by the simplex method. In the fourth part, the proposed method is applied to two numerical examples to demonstrate its effectiveness, and it is employed to perform the optimization design of a practical automobile frame.

2 Formulation of interval optimization

A general interval optimization problem can be formulated as follows:

$$\min_{\mathbf{X}} f(\mathbf{X},\mathbf{p})$$

subject to

$$g_j(\mathbf{X},\mathbf{p}) \leq b_j^I = \left[b_j^L, b_j^R\right], \quad j = 1,2,...l$$

$$\mathbf{X}_l \leq \mathbf{X} \leq \mathbf{X}_r$$

$$\mathbf{p} \in \mathbf{p}^I = [\mathbf{p}^L, \mathbf{p}^R], p_i \in p_i^I = [p_i^L, p_i^R], \quad i = 1,2,...q \tag{1}$$

where f and g_j are the objective function and the jth constraint, respectively. $\mathbf{X}$ denotes an n-dimensional design vector. In practical engineering, f and g_j are commonly nonlinear functions with respect to $\mathbf{X}$ and $\mathbf{p}$. $\mathbf{p}$ is an q-dimensional uncertain vector, and its uncertainty is modeled by an interval vector $\mathbf{p}^I$. The superscripts I, L, and R denote interval, lower and upper bounds, respectively. The midpoint vector $\mathbf{p}^c$ and radius vector $\mathbf{p}^w$ of $\mathbf{p}^I$ are defined as follows:

$$\mathbf{p}^c = \frac{\mathbf{p}^R + \mathbf{p}^L}{2}, \quad p_i^c = \frac{p_i^R + p_i^L}{2}, \quad i = 1,2,...,q$$

$$\mathbf{p}^w = \frac{\mathbf{p}^R - \mathbf{p}^L}{2}, \quad p_i^w = \frac{p_i^R - p_i^L}{2}, \quad i = 1,2,...,q \tag{2}$$

3 Transformation to a deterministic optimization problem

A nonlinear interval optimization problem in which uncertainties exist in not only the objective function but also constraints is investigated in this paper. A general means is to construct a mathematical model to change it into a deterministic optimization problem. In our formulation given below, an interval order relation and a RPDI will be employed to deal with the uncertain objective function and constraints, respectively.

As a result of the variation of the uncertain parameters, the possible values of the objective function will form an interval for each $\mathbf{X}$. Thus interval comparison is essential for implementing the interval optimization problems. Ishibuchi and Tanaka (1990) proposed an interval order relation $\leq_{cw}$ to compare intervals, and it has the following form for a minimization problem:

$$A^I \leq_{cw} B^I \text{ if } A^c \geq B^c \text{ and } A^w \geq B^w$$

$$A^I <_{cw} B^I \text{ if } A^I \leq_{cw} B^I \text{ and } A^I \neq B^I$$

$$A^c = \frac{A^R + A^L}{2}, \quad B^c = \frac{B^R + B^L}{2}$$

$$A^w = \frac{A^R - A^L}{2}, \quad B^w = \frac{B^R - B^L}{2} \tag{3}$$

where A^I and B^I are two interval numbers. This order relation represents a preference to the midpoint value c and the radius w of the interval number, and based on it, we can transform the interval objective function in Eq. (1) to a deterministic two-objective optimization:

$$\min[f^c(\mathbf{X}), f^w(\mathbf{X})]$$

$$f^c(\mathbf{X}) = \frac{1}{2}(f^L(\mathbf{X}) + f^R(\mathbf{X}))$$

$$f^w(\mathbf{X}) = \frac{1}{2}(f^R(\mathbf{X}) - f^L(\mathbf{X})) \tag{4}$$

where $f^L(\mathbf{X})$ and $f^R(\mathbf{X})$ are lower and upper bounds of $f(\mathbf{X},\mathbf{p})$ at each specific $\mathbf{X}$, respectively. They can be obtained through two optimization processes:

$$f^L(\mathbf{X}) = \min_{\mathbf{p}\in\Gamma} f(\mathbf{X},\mathbf{p}), f^R(\mathbf{X}) = \max_{\mathbf{p}\in\Gamma} f(\mathbf{X},\mathbf{p})$$

$$\mathbf{p} \in \Gamma \in \mathbf{p}^I = [\mathbf{p}^L, \mathbf{p}^R] \tag{5}$$

In the authors' previous work Jiang et al (2010), we suggested a new kind of possibility degree for intervals named RPDI:

$$p_r(A^I \le B^I) = \frac{B^R - A^L}{2(A^w + B^w)}$$

$$A^w = \frac{A^R - A^L}{2}, \quad B^w = \frac{B^R - B^L}{2} \tag{6}$$

where $p_r \in [-\infty, +\infty]$ is used to represent a certain extent that interval A^I is less than B^I. The RPDI can work not only for overlapped intervals but also completely separated intervals, and furthermore its variation trend is in accordance with the variation of the reliability very well. Thus it can exhibit fine properties in the aspects of reliability analysis.

In this study, we adopt the RPDI to deal with the interval constraints in Eq.(1):

$$p_r\left(g_j^I(\mathbf{X}) \le b_j^I\right) = \frac{b_j^R - g_j^L(\mathbf{X})}{2g_j^w(\mathbf{X}) + 2b_j^w} \ge \lambda_j,\ j = 1,2,...,l$$

$$g_j^w(\mathbf{X}) = \frac{g_j^R(\mathbf{X}) - g_j^L(\mathbf{X})}{2}, \quad b_j^w = \frac{b_j^R - b_j^L}{2} \tag{7}$$

where $\lambda_j \in [-\infty, \infty]$ is a predetermined RPDI level for the jth constraint, and it can be adjusted according to the reliability requirement. A large RPDI level indicates a high reliability of the constraint, and whereby a small feasible field for the constraints Eq. (7). $g_j^I(\mathbf{X})$ denotes the interval of the jth constraint at a specific $\mathbf{X}$ caused by the uncertain parameters, and it can be obtained by performing two optimization processes:

$$g_j^L(\mathbf{X}) = \min_{\mathbf{p}\in\Gamma} g_j(\mathbf{X},\mathbf{p}), \quad g_j^R(\mathbf{X}) = \max_{\mathbf{p}\in\Gamma} g_j(\mathbf{X},\mathbf{p})$$

$$\mathbf{p} \in \Gamma \in \mathbf{p}^I = [\mathbf{p}^L, \mathbf{p}^R] \tag{8}$$

Based on the above treatments, a deterministic optimization problem can be finally formulated as follows:

$$\min[f^c(\mathbf{X}), f^w(\mathbf{X})]$$

subject to

$$p_r\left(g_j^I(\mathbf{X}) \leq b_j^I\right) = \frac{b_j^R - g_j^L(\mathbf{X})}{2g_j^w(\mathbf{X}) + 2b_j^w} \geq \lambda_j,\ j = 1, 2, ..., l$$

$$\mathbf{X}_l \leq \mathbf{X} \leq \mathbf{X}_r$$

where

$$f^c(\mathbf{X}) = \frac{1}{2}(f^L(\mathbf{X}) + f^R(\mathbf{X})) = \frac{1}{2}\left(\min_{\mathbf{p}\in\Gamma} f(\mathbf{X}, \mathbf{p}) + \max_{\mathbf{p}\in\Gamma} f(\mathbf{X}, \mathbf{p})\right)$$

$$f^w(\mathbf{X}) = \frac{1}{2}(f^R(\mathbf{X}) - f^L(\mathbf{X})) = \frac{1}{2}\left(\max_{\mathbf{p}\in\Gamma} f(\mathbf{X}, \mathbf{p}) - \min_{\mathbf{p}\in\Gamma} f(\mathbf{X}, \mathbf{p})\right)$$

$$g_j^I(\mathbf{X}) = \left[g_j^L(\mathbf{X}), g_j^R(\mathbf{X})\right] = \left[\min_{\mathbf{p}\in\Gamma} g_j(\mathbf{X}, \mathbf{p}), \max_{\mathbf{p}\in\Gamma} g_j(\mathbf{X}, \mathbf{p})\right] \tag{9}$$

Using the linear combination method to deal with the multi-objective problem, a single-objective optimization problem can be further obtained:

$$\min_{\mathbf{X}} f_d = \beta\left(f^c(\mathbf{X}, \mathbf{p})\right) + (1 - \beta)\left(f^w(\mathbf{X}, \mathbf{p})\right)$$

subject to

$$p_r\left(g_j^I(\mathbf{X}) \leq b_j^I\right) = \frac{b_j^R - g_j^L(\mathbf{X})}{2g_j^w(\mathbf{X}) + 2b_j^w} \geq \lambda_j,\ j = 1, 2, ..., l$$

$$\mathbf{X}_l \leq \mathbf{X} \leq \mathbf{X}_r \tag{10}$$

where $0.0 \leq \beta \leq 1.0$ is a weighting factor.

Generally, Eq. (10) is a nesting optimization problem. The outer optimization layer is used to optimize the design vector, and the inner optimization layer is used to compute the bounds of the objective function and constraints cause by the uncertainties. A direct solve to Eq. (10) will lead to extremely low efficiency inevitably. Thus, in the following section, we will propose an efficient algorithm for this problem.

4 A sequential interval optimization algorithm

In the algorithm, a sequence of approximate sub-optimization problems are generated, and an optimal design vector can be achieved through an iterative process. Based on the first-order Taylor expansion, an approximate optimization problem at the sth iterative step is created as follows:

$$\min_{\mathbf{X}} \tilde{f}(\mathbf{X},\mathbf{p}) \approx f(\mathbf{X}^{(s)},\mathbf{p}^c)+\sum_{i=1}^{n}\frac{\partial f(\mathbf{X}^{(s)},\mathbf{p}^c)}{\partial X_i}(X_i-X_i^{(s)})+\sum_{i=1}^{q}\frac{\partial f(\mathbf{X}^{(s)},\mathbf{p}^c)}{\partial p_i}(p_i-p_i^{(s)})$$

subject to

$$\tilde{g}_j(\mathbf{X},\mathbf{p}) \approx g_j(\mathbf{X}^{(s)},\mathbf{p}^c)+\sum_{i=1}^{n}\frac{\partial g_j(\mathbf{X}^{(s)},\mathbf{p}^c)}{\partial X_i}(X_i-X_i^{(s)})+\sum_{i=1}^{q}\frac{\partial g_j(\mathbf{X}^{(s)},\mathbf{p}^c)}{\partial p_i}(p_i-p_i^{(s)})$$
$$\leq b_j^I = \left[b_j^L, b_j^R\right], j=1,\ldots,l$$

$$\max\left[\mathbf{X}_l, \mathbf{X}^{(s)}-\delta^{(s)}\right] \leq \mathbf{X} \leq \min\left[\mathbf{X}_r, \mathbf{X}^{(s)}+\delta^{(s)}\right] \tag{11}$$

It is obvious that the approximate objective function $\tilde{f}$ and the jth constraint $\tilde{g}_i$ are both linear functions with respect to $\mathbf{X}$ and $\mathbf{p}$. $\delta^{(s)}$ is a move limit vector which forms the current design space with the design vector $\mathbf{X}^{(s)}$ to ensure the approximation accuracy.

4.1 Solution of the approximate optimization problem

Based on the method developed in section 3, Eq. (11) can be transformed to a following deterministic optimization problem:

$$\min_{\mathbf{X}} \tilde{f}_d = \beta\left(\tilde{f}^c(\mathbf{X},\mathbf{p})\right)+(1-\beta)\left(\tilde{f}^w(\mathbf{X},\mathbf{p})\right)$$

$$\tilde{f}^c(\mathbf{X}) = \frac{1}{2}(\tilde{f}^L(\mathbf{X})+\tilde{f}^R(\mathbf{X}))$$

$$\tilde{f}^w(\mathbf{X}) = \frac{1}{2}(\tilde{f}^R(\mathbf{X})-\tilde{f}^L(\mathbf{X}))$$

subject to

$$p_r\left(\tilde{g}_j^I(\mathbf{X}) \leq b_j^I\right) = \frac{b_j^R-\tilde{g}_j^L(\mathbf{X})}{2\tilde{g}_j^w(\mathbf{X})+2b_j^w} \geq \lambda_j,\ j=1,2,\ldots,l$$

$$\tilde{g}_j^w(\mathbf{X}) = \frac{\tilde{g}_j^R(\mathbf{X})-\tilde{g}_j^L(\mathbf{X})}{2}, b_j^w = \frac{b_j^R-b_j^L}{2}$$

$$\max\left[\mathbf{X}_l, \mathbf{X}^{(s)} - \delta^{(s)}\right] \leq \mathbf{X} \leq \min\left[\mathbf{X}_r, \mathbf{X}^{(s)} + \delta^{(s)}\right] \tag{12}$$

where $\tilde{f}^I(\mathbf{X}) = \left[\tilde{f}^L(\mathbf{X}), \tilde{f}^R(\mathbf{X})\right]$ and $\tilde{g}_j^I(\mathbf{X}) = \left[\tilde{g}_j^L(\mathbf{X}), \tilde{g}_j^R(\mathbf{X})\right]$ denote the intervals of the approximate objective function and jth constraint, and here they can be obtained explicitly using the natural interval extension Moore (1979); Qiu (2005) :

$$\tilde{f}^L(\mathbf{X}) = f(\mathbf{X}^{(s)}, \mathbf{p}^c) + \sum_{i=1}^{n} \frac{\partial f(\mathbf{X}^{(s)}, \mathbf{p}^c)}{\partial X_i}\left(X_i - X_i^{(s)}\right) - \sum_{i=1}^{q}\left|\frac{\partial f(\mathbf{X}^{(s)}, \mathbf{p}^c)}{\partial p_i}\right| p_i^w$$

$$\tilde{f}^R(\mathbf{X}) = f(\mathbf{X}^{(s)}, \mathbf{p}^c) + \sum_{i=1}^{n} \frac{\partial f(\mathbf{X}^{(s)}, \mathbf{p}^c)}{\partial X_i}\left(X_i - X_i^{(s)}\right) + \sum_{i=1}^{q}\left|\frac{\partial f(\mathbf{X}^{(s)}, \mathbf{p}^c)}{\partial p_i}\right| p_i^w$$

$$\tilde{g}_j^L(\mathbf{X}) = g_j(\mathbf{X}^{(s)}, \mathbf{p}^c) + \sum_{i=1}^{n} \frac{\partial g_j(\mathbf{X}^{(s)}, \mathbf{p}^c)}{\partial X_i}\left(X_i - X_i^{(s)}\right) - \sum_{i=1}^{q}\left|\frac{\partial g_j(\mathbf{X}^{(s)}, \mathbf{p}^c)}{\partial p_i}\right| p_i^w$$

$$\tilde{g}_j^R(\mathbf{X}) = g_j(\mathbf{X}^{(s)}, \mathbf{p}^c) + \sum_{i=1}^{n} \frac{\partial g_j(\mathbf{X}^{(s)}, \mathbf{p}^c)}{\partial X_i}\left(X_i - X_i^{(s)}\right) + \sum_{i=1}^{q}\left|\frac{\partial g_j(\mathbf{X}^{(s)}, \mathbf{p}^c)}{\partial p_i}\right| p_i^w \tag{13}$$

Substituting Eq. (13) into Eq. (12), we can obtain:

$$\begin{aligned}\min_{\mathbf{X}} \tilde{f}_d = \sum_{i=1}^{n} \frac{\partial f(\mathbf{X}^{(s)}, \mathbf{p}^c)}{\partial X_i}\beta X_i + \beta\left(f(\mathbf{X}^{(s)}, \mathbf{p}^c)\right) - \sum_{i=1}^{n} \frac{\partial f(\mathbf{X}^{(s)}, \mathbf{p}^c)}{\partial X_i} X_i^{(s)} \\ + \sum_{i=1}^{q}\left|\frac{\partial f(\mathbf{X}^{(s)}, \mathbf{p}^c)}{\partial p_i}\right|(1-\beta)\, p_i^w\end{aligned}$$

subject to

$$\begin{aligned}\sum_{i=1}^{n} \frac{\partial g_j(\mathbf{X}^{(s)}, \mathbf{p}^c)}{\partial X_i} X_i \leq \sum_{i=1}^{q}\left|\frac{\partial g_j(\mathbf{X}^{(s)}, \mathbf{p}^c)}{\partial p_i}\right|(1-2\lambda_j)\, p_i^w + \sum_{i=1}^{n} \frac{\partial g_j(\mathbf{X}^{(s)}, \mathbf{p}^c)}{\partial X_i} X_i^{(s)} \\ + (1-\lambda_j)\, b_j^R + \lambda_j b_j^L \max\left[\mathbf{X}_l, \mathbf{X}^{(s)} - \delta^{(s)}\right] \leq \mathbf{X} \leq \min\left[\mathbf{X}_r, \mathbf{X}^{(s)} + \delta^{(s)}\right]\end{aligned} \tag{14}$$

Obviously, Eq. (14) is a traditional linear programming problem, and it can be easily solved by simplex method Nocedal et al (1999).

4.2 Computational procedure

The computational procedure of the proposed algorithm can be described as follows:

1. Select the initial design vector $\mathbf{X}^{(1)}$ and the move limit vector $\delta^{(1)}$. Give the scaling factor $\alpha \in (0,1)$ and the allowable errors $\varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0$. Set the predetermined RPDI level for each constraint, and make the iterative step s=1.

2. Construct the approximate optimization problem Eq. (11) and solve its transformed optimization problem Eq. (14) using the simplex method to obtain an optimal vector $\bar{\mathbf{X}}$.

3. Calculate the bounds of the actual objective function and constraints at the intermediate optimum $\bar{\mathbf{X}}$, based on which the desirability function $f_d\left(\bar{\mathbf{X}}\right)$ and the actual RPDI $p_r\left(\tilde{g}_j^I\left(\bar{\mathbf{X}}\right) \le b_j^I\right),\ j = 1,2,...,l$ can be computed.

4. Judge whether $\bar{\mathbf{X}}$ is a feasible and descending solution through the criterion $\min\left\{\left(p_r\left(g_j^I\left(\bar{\mathbf{X}}\right) \le b_j^I\right) - \lambda_j\right),\ j = 1,2,...,l\right\} > -\varepsilon_1$ and $f_d\left(\bar{\mathbf{X}}\right) < f_d\left(\mathbf{X}^{(s)}\right)$. And

   (a) If it can be satisfied, making $\mathbf{X}^{(s+1)} = \bar{\mathbf{X}}$ and go to step (6);

   (b) Otherwise, making reducing the move limit vector $\delta^{(s)} := \alpha\delta^{(s)}$ by a scaling factor α.

5. Introduce the move limit vector criteria $\min\left\{\delta_i^{(s)},\ i = 1,2,...,n\right\} < \varepsilon_2$.

   (a) If it can be satisfied, $\mathbf{X}^{(s)}$ is obtained as an optimal design vector and the optimization stops;

   (b) Otherwise, go back to step (2).

6. Repeat steps 2 to 5 until the distance between the last two iteration design vectors is smaller than ε_3.

A direct approach to calculate the bounds of the objective function and constraints at the intermediate optimum $\bar{\mathbf{X}}$ in step (3) is to perform several optimization processes. However, it will still influence the optimization efficiency, especially for the complex engineering problems. Here, we can also use the interval analysis technique Qiu (2005) to further improve the optimization efficiency.

Because the intervals of the uncertain parameters are assumed to be relatively small in our study, the objective function f in Eq.(1) can be approximated as a linear function at $\bar{\mathbf{X}}$ within the uncertainty space through the first-order Taylor expansion:

$$f(\bar{\mathbf{X}}) \approx f(\bar{\mathbf{X}}, \mathbf{p}^c) + \sum_{i=1}^{q} \frac{\partial f(\bar{\mathbf{X}}, \mathbf{p}^c)}{\partial p_i}(p_i - p_i^c) \tag{15}$$

where

$$(\mathbf{p}-\mathbf{p}^c)\in[-1,1]\,\mathbf{p}^w,(p_i-p_i^c)\in[-1,1]\,p_i^w,i=1,2,...,q \tag{16}$$

Thus, the bounds of the uncertain objective function at $\bar{\mathbf{X}}$ can be obtained explicitly:

$$f^L(\bar{\mathbf{X}})=f(\bar{\mathbf{X}},\mathbf{p}^c)-\sum_{i=1}^{q}\left|\frac{\partial f(\bar{\mathbf{X}},\mathbf{p}^c)}{\partial p_i}\right|p_i^w$$

$$f^R(\bar{\mathbf{X}})=f(\bar{\mathbf{X}},\mathbf{p}^c)+\sum_{i=1}^{q}\left|\frac{\partial f(\bar{\mathbf{X}},\mathbf{p}^c)}{\partial p_i}\right|p_i^w \tag{17}$$

Similarly, the bounds of the uncertain constraints at $\bar{\mathbf{X}}$ can also be obtained explicitly:

$$g_j^L(\bar{\mathbf{X}})=g_j(\bar{\mathbf{X}},\mathbf{p}^c)-\sum_{i=1}^{q}\left|\frac{\partial g_j(\bar{\mathbf{X}},\mathbf{p}^c)}{\partial p_i}\right|p_i^w$$

$$g_j^R(\bar{\mathbf{X}})=g_j(\bar{\mathbf{X}},\mathbf{p}^c)+\sum_{i=1}^{q}\left|\frac{\partial g_j(\bar{\mathbf{X}},\mathbf{p}^c)}{\partial p_i}\right|p_i^w \tag{18}$$

Obviously, only a small amount of evaluations of the uncertain objective function or each constraint at $\bar{\mathbf{X}}$ need to be calculated.

5 Numerical examples and discussion

5.1 Numerical example 1

Consider the simple plane ten-bar truss structure as shown in Fig. 1, which has been investigated in various optimization contexts Liu et al (1999); Elishakoff et al (1994) . The cross-sectional area A_j of the bars are optimized to obtain a minimum weight design subject to the stress and displacement constraints. The truss is made of aluminum with a weight density of $0.1\text{lb}/\text{in}^3$ and a Young's modulus E of 10^4ksi. The length L of the horizontal and vertical bars is 360in. A constraint of 5in on the vertical displacement of joint 2 is applied. The maximum allowable stress of bar 9 in tension or compression is 75ksi, and the other bars have a same allowable stress in tension or compression which is 25ksi. Joint 4 is subjected to vertical load P_{4y}, and joint 2 is subjected to vertical load P_{2y} and horizontal load P_{2x}. In this numerical example, the load are uncertain, and their nominal values are 100kips,100kips, and 400kips, respectively. The uncertainty level is 10% off from the nominal values of the loads.

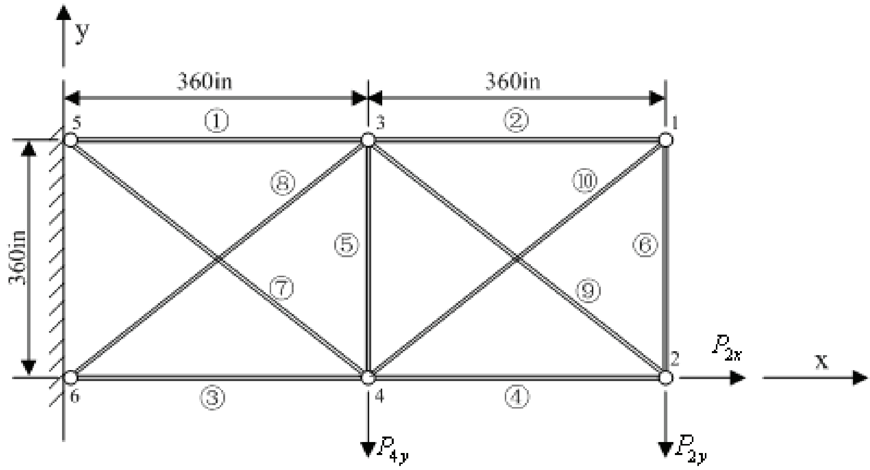

Figure 1: Ten-bar plane truss

The unknown axial force in the bars are denoted by N_i, $i = 1, 2, ..., 10$, and they satisfy the following equilibrium and compatibility equations:

$$N_1 = P_{2y} - \frac{\sqrt{2}}{2} N_8, \quad N_2 = -\frac{\sqrt{2}}{2} N_{10}, \quad N_3 = -P_{4y} - 2P_{2y} + P_{2x} - \frac{\sqrt{2}}{2} N_8 \tag{19}$$

$$N_4 = -2P_{2y} + P_{2x} - \frac{\sqrt{2}}{2} N_{10}, \quad N_5 = -2P_{2y} - \frac{\sqrt{2}}{2} N_8 - \frac{\sqrt{2}}{2} N_{10}, \quad N_6 = \frac{\sqrt{2}}{2} N_{10} \tag{20}$$

$$N_7 = \sqrt{2}(P_{4y} + P_{2y}) + N_8, \quad N_8 = \frac{a_{22}b_1 - a_{21}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$N_9 = \sqrt{2}P_{2y} + N_{10}, \quad N_{10} = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}} \tag{21}$$

$$\begin{aligned} a_{11} &= \left(\frac{1}{A_1} + \frac{1}{A_3} + \frac{1}{A_5} + \frac{2\sqrt{2}}{A_7} + \frac{2\sqrt{2}}{A_8}\right)\frac{L}{2E}, \quad a_{12} = a_{21} = \frac{L}{2A_5E} \\ a_{22} &= \left(\frac{1}{A_2} + \frac{1}{A_4} + \frac{1}{A_6} + \frac{2\sqrt{2}}{A_9} + \frac{2\sqrt{2}}{A_{10}}\right)\frac{L}{2E}, \end{aligned} \tag{22}$$

$$b_1 = \left(\frac{P_{2y}}{A_1} - \frac{P_{4y} + 2P_{2y} - P_{2x}}{A_3} - \frac{P_{2y}}{A_5} - \frac{2\sqrt{2}(P_{4y} + P_{2y})}{A_7}\right)\frac{\sqrt{2}L}{2E} \tag{23}$$

$$b_2 = \left(\frac{\sqrt{2}(P_{2x} - P_{2y})}{A_4} - \frac{\sqrt{2}P_{2y}}{A_5} - \frac{4P_{2y}}{A_7}\right)\frac{L}{2E} \tag{24}$$

The vertical displacement of joint 2 can be obtained from the following expression:

$$\delta_2 = \left[\sum_{i=1}^{6} \frac{N_i^0 N_i}{A_i} + \sqrt{2} \sum_{i=7}^{10} \frac{N_i^0 N_i}{A_i} \right] \frac{L}{E} \tag{25}$$

where N_i^0 can be obtained from Eqs. (19)-(21) with a substitution $P_{4y} = P_{2x} = 0$ and $P_{2y} = 1$.

An optimization problem for minimal weight can be formulated as follows:

$$\min_{\mathbf{A}} W(\mathbf{A}) = \sum_{i=1}^{10} (\rho L_i A_i) = \rho L \left(\sum_{i=1}^{6} A_i + \sqrt{2} \sum_{i=7}^{10} A_i \right)$$

subject to

$$\sigma_i^I(\mathbf{A}) = \frac{|N_i|}{A_i} \le \sigma_{i,allow}, \ i = 1, 2, ..., 10$$

$$\delta_2^I(\mathbf{A}) \le 5\text{in}$$

$$0.1\text{in}^2 \le A_i \le 20\text{in}^2, \quad i = 1, 2, ..., 10 \tag{26}$$

In the optimization process, the weighting factor β is specified as 0.5, and the scaling factor is set to 0.5. ε_1, ε_2, and ε_3 are all taken equal to 0.01. The initial cross-sectional areas for the bar are all given 20in^2, and the initial move limit for each bar is specified 2.0in^2. The same predefined RPDI level is used for each constraint. The optimization results under different RPDI levels are listed in Tables 1-4, it can be found that the predefined RPDI levels are all satisfied at the optima. The minimum weight of the truss decreases with the decreasing of the RPDI level. The relation between the minimum weight and the predefined RPDI level is given in Fig. 2. We can observe that they exhibit an approximate linear relation. For the predefined RPDI level 1.2, the weight of the optimal truss is 2556.14 lb. For the predefined RPDI level 0.8, it reaches a minimum value 2256.24 lb. It can be found that a larger predefined RPDI level is required when a more reliable structural design is needed, however, the manufacturing cost will increase. Thus engineers always face a tradeoff between the design objective and the risk of violating the constraints through adjusting the predefined RPDI levels of the constraints. As a mater of fact, engineers can predefine different RPDI level for each constraint according to the practical problem.

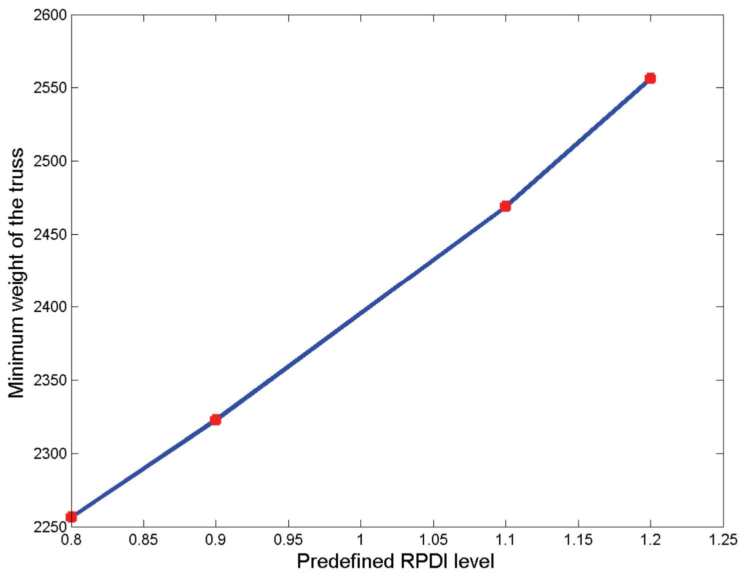

Figure 2: Relation of the predefined RPDI level and the minimum weight of the truss

Table 1: Optimization results under the RPDI level 1.2

Bar's number	Cross-sectional Area(in^2)	Stress Interval(ksi)	RPDI Level
1	17.80	[6.12,8.04]	9.82
2	0.59	[17.51,23.76]	1.20
3	8.57	[7.37,22.04]	1.20
4	15.18	[17.38,23.73]	1.20
5	4.38	[11.90,16.32]	2.96
6	0.59	[17.51,23.76]	1.20
7	11.23	[19.72,24.10]	1.20
8	2.04	[13.13,23.01]	1.20
9	2.80	[40.01,48.90]	3.94
10	0.83	[17.48,23.72]	1.21
Interval of the displacement is [2.42in, 4.57in], RPDI is 1.20. The weight of the optimal truss is 2556.14 lb			

5.2 Numerical example 2

A simple beam design problem as shown in Fig. 3 is investigated, which is modified from a numerical example in the reference Hu (1990). Two cross-sectional

Table 2: Optimization results under the RPDI level 1.1

Bar's number	Cross-sectional Area(in^2)	Stress Interval(ksi)	RPDI Level
1	17.53	[6.13,8.06]	9.75
2	0.35	[17.78,24.35]	1.10
3	8.01	[7.66, 23.39]	1.10
4	14.63	[17.66,24.34]	1.10
5	4.49	[12.95,17.52]	2.64
6	0.35	[17.80,24.35]	1.10
7	11.14	[20.09,24.56]	1.10
8	1.85	[13.28,23.96]	1.10
9	2.91	[40.54,49.55]	3.83
10	0.49	[17.80,24.35]	1.10
Interval of the displacement is [2.46in, 4.76in], RPDI is 1.11. The weight of the optimal truss is 2468.85 lb			

Table 3: Optimization results under the RPDI level 0.9

Bar's number	Cross-sectional Area(in^2)	Stress Interval(ksi)	RPDI Level
1	17.48	[5.91,7.77]	10.27
2	0.10	[18.59,25.71]	0.90
3	6.82	[8.18, 26.87]	0.90
4	13.68	[18.46,25.73]	0.90
5	4.37	[15.43,20.36]	1.94
6	0.10	[18.59,25.71]	0.90
7	11.02	[20.83,25.46]	0.90
8	1.39	[13.55,26.27]	0.90
9	2.98	[41.77,51.05]	3.58
10	0.14	[18.59,25.71]	0.90
Interval of the displacement is [2.46, 5.09], RPDI is 0.90. The weight of the optimal truss is 2322.94 lb			

dimensions X_1 and X_2 are required to be optimized to obtain a minimum vertical deflection of the beam, which is subjected to a cross-sectional area constraint and a stress constraint. The other two cross-sectional dimensions p_1 and p_2 are uncertain parameters, and their midpoints are both 2.0cm, respectively. The Young's Modulus of the beam, blending forces Q_1 and Q_2, length of the beam are $2\times10^4\mathrm{kN/cm}^2$, 600kN, 50kN, 200cm, respectively.

Table 4: Optimization results under the RPDI level 0.8

Bar's number	Cross-sectional Area(in^2)	Stress Interval(ksi)	RPDI Level
1	17.11	[5.68,7.34]	11.62
2	0.10	[19.11,26.47]	0.80
3	6.04	[7.55, 29.34]	0.80
4	13.28	[19.02,26.50]	0.80
5	4.62	[16.40,20.97]	1.88
6	0.10	[19.11,26.47]	0.80
7	11.32	[21.21,25.93]	0.80
8	0.79	[12.85,28.02]	0.80
9	2.90	[42.86,52.39]	3.37
10	0.14	[19.11,26.47]	0.80
Interval of the displacement is [2.36, 5.28], RPDI is 0.90. The weight of the optimal truss is 2256.24 lb			

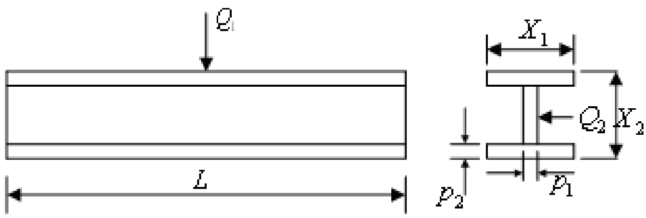

Figure 3: A beam design problem

The uncertain optimization model can be formulated as follows:

$$\min_{x} f(\mathbf{X},\mathrm{p}) = \frac{Q_1 L^3}{48EI} = \frac{5000}{\frac{1}{12}p_1(X_1-2p_2)+\frac{1}{6}X_2p_2^3+2X_2p_2\left(\frac{X_1-p_2}{2}\right)^2}$$

subject to

Cross-sectional area is not more than 300cm^2:

$$g_1\left(\mathbf{X},\mathrm{p}\right) = 2X_1p_2 + p_1\left(X_1-2p_2\right) \leq 300$$

The maximal stress is not more than 10kN/cm^2:

$$g_2(\mathbf{X},\mathrm{p}) = \frac{180000X_1}{p_1(X_1-2p_2)^3+2X_2p_2\left[4p_2+3X_1(X_1-2p_2)\right]} + \frac{15000X_2}{(X_1-2p_2)\,p_1^3+2p_2X_2^3} \le 10$$

$$10.0\text{cm} \le X_1 \le 120.0\text{cm}, 10.0\text{cm} \le X_2 \le 120.0\text{cm} \tag{27}$$

The uncertainty levels are both only $\pm 10\%$ off from the midpoints of the uncertain parameters, namely:

$$p_1 \in [1.8\text{cm}, 2.2\text{cm}], p_2 \in [1.8\text{cm}, 2.2\text{cm}] \tag{28}$$

where the objective function f represents the vertical deflection of the beam.

In the optimization process, the weighting factor β is specified as 0.5, and the scaling factor is set to 0.9. ε_1, ε_2,and ε_3 are all taken equal to 0.01. The initial design vector and move limit vector are set as $(40.00, 40.00)^T$ and $(10.00, 10.00)^T$, respectively. The same predefined RPDI level is used for each constraint. Using the proposed methodology, the optimization results under different predefined RPDI levels are listed in Table 5, it can be found that the RPDI levels of all the constraints are satisfied for the predefined RPDI levels.. For predefined RPDI level 1.1, the obtained interval of the cross sectional area and stress constraints are $[242.64\text{cm}^2, 294.79\text{cm}^2]$ and $[8.11\text{kN/cm}^2, 9.83\text{kN/cm}^2]$ respectively, which are completely less than the corresponding allowable interval. The convergence curve of the proposed algorithm for the case of predefined RPDI level 0.7 is also plotted in Fig. 4. It can be found that the optimization converges at a relatively stationary value only through 10 iterative steps.

5.3 Application

A practical automobile frame model as shown in Fig. 5 is investigated. This frame model is composed by two side beams and eight cross beams. The cross beams are denoted by $b_i, i = 1, 2, ..., 8$. The density of the frame's material is $7.8 \times 10^{-3}\text{Kg/mm}^3$. The frame is a base of the whole automobile, and many parts and unit assemblies are fixed on the frame through the connecting pieces. A static mechanical model of the frame is obtained as shown in Fig. 5. $Q1$, $Q2$, $Q3$, $Q4$ represent the uniform distributed forces acting on the frame. The small triangle denotes the fixed constraints.

The cross beams b_1, b_2, b_3,and b_6 are fixed, and the spans $l_i, i = 1, 2, 3$ of the beams are optimized to obtain a maximum stiffness of the frame in Y direction. The

Table 5: Optimization results under different predefined RPDI levels

Predefined RPDI level	Optimal design vector(cm^2)	Cross sectional Area interval(cm^2)	Stress Interval (kN/ cm^2)	Vertical Displacement interval ($\times 10^{-2}$cm)	RPDI Level
1.1	[78.36,30.00]	[242.64,294.79]	[8.11,9.83]	[1.87,2.24]	1.10,1.10
0.9	[88.25,27.47]	[251.33,305.41]	[8.38,10.19]	[1.49,1.80]	0.90,0.90
0.7	[97.15,25.61]	[260.69,316.85]	[8.68,10.57]	[1.24,1.44]	0.70,0.70

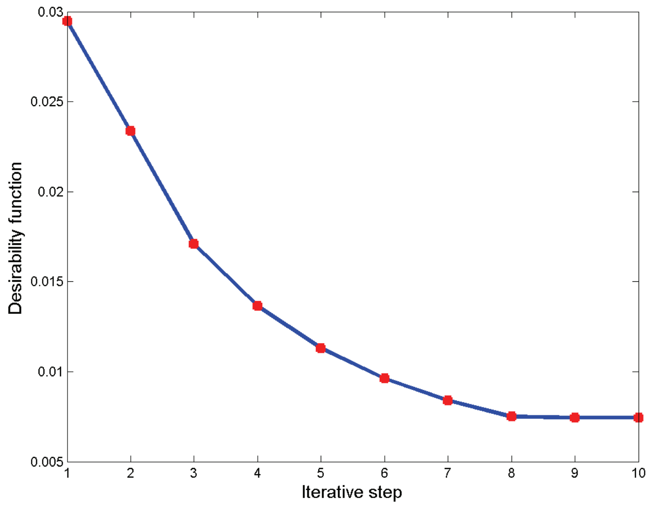

Figure 4: Convergence curve for the predefined RPDI level 0.7

maximum nodal displacement in Y direction can be used to represent the stiffness. As errors are unavoidable during the manufacturing and measurement processes, Young's Modulus E and Poisson's ratio ν are uncertain. The midpoints of E and ν are 2.0×10^5 MPa and 0.3, respectively.

A following uncertain optimization problem can be formulated:

$$\min_{\mathbf{l}} d_{\max}(\mathbf{l}, E, \nu)$$

Subject to

$$\sigma_{\max}(\mathbf{l}, E, \nu) \leq 90\text{MPa}$$

$$500\text{mm} \leq l_i \leq 1200\text{mm}, \; i = 1, 2, 3$$

$$E \in \left[1.8 \times 10^5 \text{ MPa}, 2.2 \times 10^5 \text{ MPa}\right], \; \nu \in [0.27, 0.33] \tag{29}$$

Where $\mathbf{l}$ represent a three-dimensional design vector. The objective function $d_{\max}$ and constraint $\sigma_{\max}$ denote the maximum displacement in Y direction and the maximum equivalent stress in the frame, respectively.

Finite element method is employed to calculate $d_{\max}$ and $\sigma_{\max}$. In the optimization process, the weighting factor β is specified as 0.5, and the RPDI level of the constraint is predefined as 3.8. The scaling factor is set to 0.5. ε_1, ε_2, and ε_3 are all taken equal to 0.01. The initial design vector and move limit vector are set to

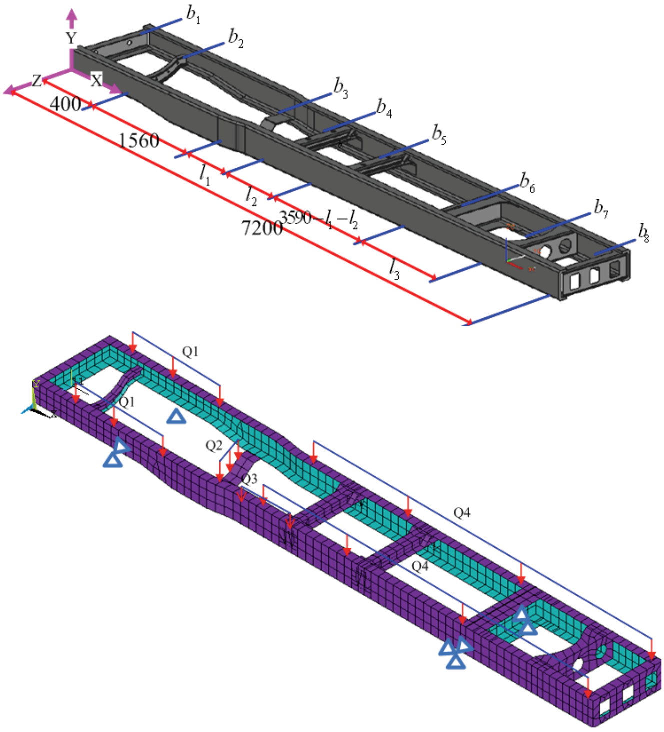

Figure 5: An automobile frame

$[800.00, 800.00, 800.00]^T$ and $[50.00, 50.00, 50.00]^T$, respectively. The optimization results are listed in Table 6, it can be found that the optimal design vector is [777.29,775.83,825.83], at which the interval of the maximum displacement in Y direction is [1.34mm,1.64mm] and the RPDI of the stress constraint is satisfied for the predefined RPDI level. The convergence curve is shown in Fig. 6. It can be found that only 8 iterative steps are needed to obtain the maximum stiffness of the frame. In the optimization process, a total of 72 FEM evaluations are needed.

Table 6: Optimization results of the automobile frame

Optimal design vector(mm)	Displacement interval in Y direction(mm)	Stress interval(Mpa)	RPDI Level
[777.29,775.83,825.83]	[1.34,1.64]	87.17	3.83

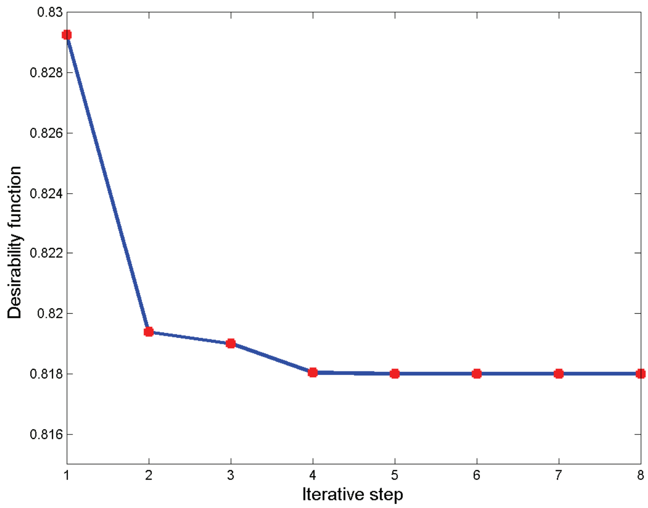

Figure 6: Convergence curve for the design of the automobile frame

6 Conclusion

In this paper, we propose an efficient interval optimization method based on a RPDI, which is more suited for reliability analysis and reliability-based optimization. The simplex method is employed to solve the approximate optimization problem at each iterative step. Additionally, a computational framework is constructed to guarantee the algorithm to converge efficiently. To further improve the efficiency, an approximate method is also suggested to compute the bounds of the objective function and constraints at each intermediate optimum. In the first two numerical examples, different predefined RPDI levels are investigated. The optimization results indicate that the proposed method behaves a relatively good performance. it is also applied to a practical engineering problem, and fine results are obtained with a relatively small number of FEM evaluations.

It is also should be noticed that the intervals of the objective and constraints are calculated based on the Taylor expansion at each iterative step, and thus the uncertainty level of the problem is relatively small. Fortunately, this is always satisfied in most practical engineering problems.

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