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Viscous Equations of Fluid Film Dynamics

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Abstract: We model viscosity in the framework of the exact nonlinear equations of fluid film dynamics. The proposed approach yields monotonic dissipation of energy and guarantees that viscous forces are not engaged when the film undergoes rigid motion. With the addition of viscosity, the governing system has all the essential elements – inertia, surface tension, interaction with the ambient medium, influence of external fields and, now, viscosity – for accurate prediction and interpretation of experimental observations.

The fluid film is modelled as a two-dimensional manifold. The film's thickness is represented by a surface density function. The resulting system is the fluid film equivalent of the classical Navier-Stokes equations. The domain of definition of the fluid film equations is a deforming manifold which makes computer simulation the mostly likely course for obtaining solutions.

1 Introduction

The exact system of fluid film dynamics arose from the Least Action Principle with a natural Lagrangian. That is, it arose in a manner that was prized during the era when analytical methods reigned supreme. Yet only the most fundamental properties of this system can be established by analytical methods. The system is defined on a deforming manifold, which makes pursuit of closed form solutions rather challenging. However, when allied with modern computational methods, this system may hold significant potential for accurate prediction and interpretation of physical effects.

Thin fluid films are as fascinating as they are important. A soap film is one of the simplest systems of this kind and has for centuries attracted interest among mathematicians, physicists, chemists and even artists. In a monograph devoted to the mathematical aspects of capillary effects Finn (1986), Robert Finn follows the history of fluid film investigations from the year 1712. His captivating review contains a plethora of information about the contributions of such giants of mathematical

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physics as Laplace, Young, Poincare, and Rayleigh. The book by Cyril Isenberg Isenberg (1992) contains an overview more narrowly focused on soap films.

Fluid films continue to play an important role in the calculus of variations, and in the study of minimal surfaces. New interesting problems, some rooted in physical experiments Jacobsen (2007), continue to emerge. Setting aside the fundamental nature of fluid films differential geometry and topology, the modeling of fluid films is of paramount importance in numerous applications. In respiratory physiology, the lung tissue if frequently modeled as a fluid film. In particular, the pulmonary alveoli, where the gas exchange takes place in the lung in some way act like soap bubbles. Modeling of fluid film dynamics is essential in foams. Surface tension flows play an important role in manufacturing, such as fabrication of glass tubing Griffiths and Howell (2009).

Fluid films display an astonishing array of physical effects: static and dynamic, macroscopic and nanoscale – at times, simultaneously in a single film. Static effects go well beyond the study of minimal surfaces, as films display variations in thickness, surfactant density, and interaction with external fields Dean and Horgan (2002). Predictably, the dynamics is even richer. Fluid films display turbulence Rivera et al (1998), Martin et al (1998), tremendous variations in thickness Greffier et al (2002), Rivera et al (1998), Nierop et al (2008), the Marangoni effect Tran et al (2009), draining and reverse draining Moulton and Pelesko (2010), ejection of droplets Drenckhan et al (2008), rupture Debregeas et al (1995), self-adaptation Boudaoud et al (1999), and chaotic behavior Gilet and Bush (2009). In this paper, we provide a governing system of equations that has all the essential elements required to explain the experimental findings mentioned in the preceding paragraph. The exact nonlinear system of inviscid equations was presented in Grinfeld (2009), Grinfeld (2010a), and Grinfeld (2010b). Interaction with the ambient gas was added in Grinfeld and Grinfeld (2010). Heretofore, the last missing crucial element has been viscosity. It is the goal of this paper to present a model of viscosity within the framework of the existing system. The proposed approach results in equations that are a fluid film analogue of the classical Navier-Stokes equations. A most complete system is given in equations (49a)-(49c) for a rather general material model of the fluid film, and in equations (52a)-(52c) for the most common material model. We believe that a numerical simulation based on this system can faithfully simulate the remarkable experiment being performed by a remarkable experimentalist in Figure 1.

Figure 1: A girl blowing bubbles: a remarkable experiment by a remarkable experimentalist. In order to simulate this complicated physical system, the governing equations must combine intertial effects, surface tension, interaction with the ambient medium and *viscosity*. Source:Wikimedia

2 Exact nonlinear inviscid dynamic equations

2.1 Assumptions and notation

In recent publications Grinfeld (2009), Grinfeld (2010a) and Grinfeld (2010b), the author proposed an exact system for the dynamics of fluid films modeled as two dimensional manifolds. The thickness of the fluid film is captured by two dimensional density τ, the total mass M of the fluid film represented by the manifold S being

$$M = \int_S \tau dS \tag{1}$$

We decompose the fluid film velocity field $\mathbf{V}$ into the normal component C and the two tangential component V^α, $\alpha = 1,2$. The relevant notation is illustrated in Figure 2.

3 Material model

The energy density function e represents the material model of the fluid film. Many of the material properties of fluid films can be captured by energy density function

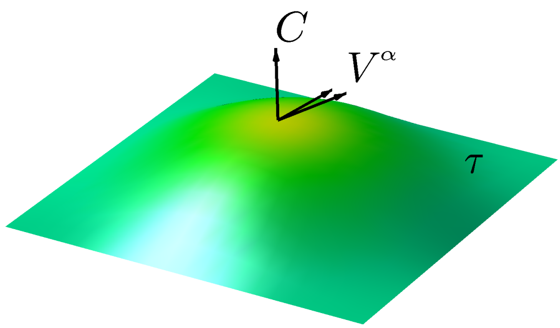

Figure 2: Illustration of the notation. C is the normal component of the velocity field, V^1 and V^2 are the two tangential components, and τ is the surface density that in actuality captures the variable thickness of the film.

that depends only on τ

$$e \equiv e(\tau) \tag{2}$$

In some important applications, including biological membranes and crystalline films, more detailed models are required and the argument list of e may need to be amended with other geometric parameters, such as curvature, or external parameters, such as a director vector. We note that e is density per unit mass. Thus, the total internal energy I is

$$I = \int_S \tau e(\tau)\, dS. \tag{3}$$

The standard static model of surface tension, in which the total potential energy stored in a fluid film is directly proportional to its total area, corresponds to

$$e(\tau) = \frac{\sigma}{\tau}, \tag{4}$$

where σ is the surface tension density. We refer to this model is the *Laplace model*, since Laplace is believed to be first to analyze fluid films from the variational point of view and to associate fluid films with minimal surfaces.

Let the derivative of ε be denoted by e_τ

$$e_\tau = \frac{de(\tau)}{d\tau}. \tag{5}$$

In classical barotropic models, the quantity $\tau^2 e_\tau$ is associated with pressure. We therefore introduce the fluid film pressure $P(\tau)$ according to

$$P(\tau) = \tau^2 e_\tau. \tag{6}$$

For the classical choice (4), the resulting pressure is a negative constant

$$P(\tau) = -\sigma. \tag{7}$$

In the following discussion we let P_τ be the derivative of $P(\tau)$:

$$P_\tau(\tau) = \frac{dP(\tau)}{d\tau}. \tag{8}$$

4 The inviscid equations

We use the full framework of tensor calculus in our presentation of the fluid film equations. The ambient space is referred to arbitrary curvilinear coordinates. The two dimensional manifold S that represents the fluid film is referred to its own arbitrary coordinate system which arbitrarily evolves in time in differentiable fashion. The symbol ∇_α denotes the covariant surface derivative, B^α_β is the curvature tensor and B^α_α, that is mean curvature, is its trace. The indices are lowered and raised with the help of the co- and contravariant metric tensor $S_{\alpha\beta}$ and $S^{\alpha\beta}$. Finally, the $\delta/\delta t$-derivative as a generalization of the partial time derivative $\partial/\partial t$ that is at the heart of the calculus of moving surfaces. It produces tensors out of tensors and, among a number of other remarkable properties, satisfies the product rule and commutes with contraction.

The exact nonlinear system for the dynamics of fluid films consists of two scalar and one vector equations:

$$\frac{\delta\tau}{\delta t} + \nabla_\alpha(\tau V^\alpha) = \tau C B^\alpha_\alpha \tag{9a}$$

$$\tau\left(\frac{\delta C}{\delta t} + 2V^\alpha\nabla_\alpha C + B_{\alpha\beta}V^\alpha V^\beta\right) = -PB^\alpha_\alpha \tag{9b}$$

$$\tau\left(\frac{\delta V^\alpha}{\delta t} + V^\beta\nabla_\beta V^\alpha - C\nabla^\alpha C - 2CV^\beta B^\alpha_\beta\right) = -\nabla^\alpha P \tag{9c}$$

The first equation (9a) is analogous to the continuity equation and is responsible for conservation of mass. The second equation (9b) is Newton's second law for the normal component of the velocity. The last equation (9c) has two components and represents Newton's second law for the tangential components.

4.1 Fundamental properties

The fundamental properties of the exact system (9a)-(9c) follow from its Hamiltonian nature. The system can be derived by the Least Action Principle from the natural Lagrangian

$$L = \frac{1}{2}\int_S \tau\left(C^2 + V^2\right) dS - \int_S \tau e(\tau)\, dS. \tag{10}$$

As a result, the equations conserve the total energy

$$E = \frac{1}{2}\int_S \tau\left(C^2 + V^2\right) dS - \int_S \tau e(\tau)\, dS, \tag{11}$$

circulation Γ around a closed material loop γ

$$\Gamma = \oint_\gamma \mathbf{V} \cdot \mathbf{d}\gamma \tag{12}$$

and the scaled vorticity ω/τ at each material point, where vorticity ω is defined as

$$\omega = \varepsilon^{\alpha\beta}\nabla_\alpha V_\beta \tag{13}$$

Proofs of these properties can be found in Grinfeld (2009).

4.2 Normal coordinates

The exact equations (9a)-(9c) rely on the $\delta/\delta t$-derivative from the calculus of moving surfaces which is essential in achieving invariance under a change of surface coordinates. However, if we restrict ourselves to a particular family of coordinate systems, known as normal coordinates, then we can replace the $\delta/\delta t$-derivative with the partial time derivative $\partial/\partial t$.

Normal coordinates, illustrated in Figure 3, are constructed as follows. At the initial moment, the coordinate system is chosen arbitrarily. Thereafter, the system is evolved in such a way that lines of constant coordinates are orthogonal to the surface. Thus, the evolution of this coordinate system depends on the entire history of membrane configurations and must be constructed continuously as the evolution takes place. The advantages offered by normal coordinates come from the fact that, in a sense, normal coordinates are orthogonal in time. In normal coordinates, provided that the ambient space is referred to affine coordinates, the governing

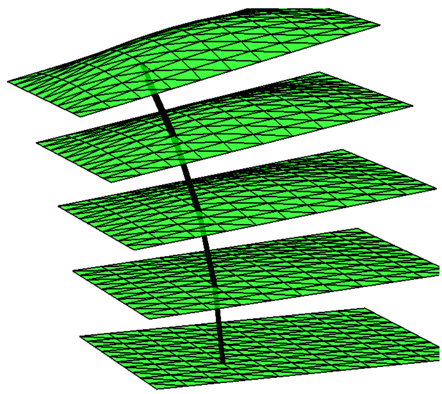

Figure 3: A normal coordinate system. At the initial moment, the coordinate system is chosen arbitrarily. Thereafter, the system is evolved in such a way that lines of constant coordinates are orthogonal to the surface.

equations read

$$\frac{\partial \tau}{\partial t} + \nabla_\alpha (\tau V^\alpha) = \tau C B_\alpha^\alpha \tag{14a}$$

$$\tau \left(\frac{\partial C}{\partial t} + 2V^\alpha \nabla_\alpha C + B_{\alpha\beta} V^\alpha V^\beta \right) = -P B_\alpha^\alpha \tag{14b}$$

$$\tau \left(\frac{\partial V^\alpha}{\partial t} + V^\beta \nabla_\beta V^\alpha - C \nabla^\alpha C - 2C V^\beta B_\beta^\alpha \right) = -\nabla^\alpha P. \tag{14c}$$

Normal coordinates are a natural choice for collocation numerical methods: when a surface marker is advanced in the normal direction with a rate proportional to C and is allowed to retain its surface coordinates, the result is a discrete analogue of normal coordinates.

4.3 Equivalence to the Euler equations for a planar flow

Consider the special motion of a fluid film that originates and remains flat. In this case, we have $B_{\alpha\beta} = 0$, $C = 0$ and the governing system reduces to one scalar and one vector equation

$$\frac{\delta \tau}{\delta t} + \nabla_\alpha \left(\tau V^\alpha\right) = 0 \tag{15a}$$

$$\tau \left(\frac{\delta V^\alpha}{\delta t} + V^\beta \nabla_\beta V^\alpha \right) = -\nabla^\alpha P. \tag{15b}$$

This system is completely equivalent to Euler's classical hydrodynamic equations. When we introduce viscosity and consider a planar flow, we expect a system equivalent to the Navier-Stokes equations.

5 Introduction of Viscosity

In constructing our model of viscosity, we aim to satisfy two key properties: 1. that viscous forces are not engaged when the film undergoes rigid motion and 2. that the total energy monotonically diminishes with time. In order to satisfy the first desired property, we postulate that the viscous force is proportional to the *rate of strain tensor* $E_{\alpha\beta}$, defined as

$$E_{\alpha\beta} = \frac{1}{2} \left(\nabla_\alpha V_\beta + \nabla_\beta V_\alpha - 2CB_{\alpha\beta} \right). \tag{16}$$

The tensor $E_{\alpha\beta}$ vanishes when the film undergoes rigid motion thus assuring that the viscous forces vanish as well. Let the viscous force T^i be

$$T^i = \nabla_\beta \left(Z^i_\alpha M^{\alpha\beta\gamma\delta} E_{\gamma\delta} \right), \tag{17}$$

where the tensor $M^{\alpha\beta\gamma\delta}$ is symmetric in the first two and the last two indices:

$$M^{\alpha\beta\gamma\delta} = M^{\beta\alpha\gamma\delta} \tag{18}$$

$$M^{\alpha\beta\gamma\delta} = M^{\alpha\beta\delta\gamma}, \tag{19}$$

and is positive definite in the sense that

$$M^{\alpha\beta\gamma\delta} x_{\alpha\beta} x_{\gamma\delta} > 0 \tag{20}$$

for any nontrivial $x_{\alpha\beta}$. A viable possibility for $M^{\alpha\beta\gamma\delta}$ is the *Newtonian* form

$$M^{\alpha\beta\gamma\delta} = \lambda S^{\alpha\beta} S^{\gamma\delta} + \mu \left(S^{\alpha\gamma} S^{\beta\delta} + S^{\alpha\delta} S^{\beta\gamma} \right). \tag{21}$$

We would like to decompose T^i in the normal and tangential components $T^{(N)}$ and T^α. To this end, apply the product rule to the definition (17) of T^i:

$$T^i = N^i M^{\alpha\beta\gamma\delta} B_{\alpha\beta} E_{\gamma\delta} + Z^i_\alpha \nabla_\beta \left(M^{\alpha\beta\gamma\delta} E_{\gamma\delta} \right). \tag{22}$$

Therefore, the normal and the tangential components of T^i are

$$T^{(N)} = N^i M^{\alpha\beta\gamma\delta} B_{\alpha\beta} E_{\gamma\delta} \tag{23}$$

$$T^{\alpha} = \nabla_\beta \left(M^{\alpha\beta\gamma\delta} E_{\gamma\delta} \right) \tag{24}$$

The Newtonian $M^{\alpha\beta\gamma\delta}$ (21) vanishes under the covariant surface derivative, therefore

$$T^{(N)} = N^i M^{\alpha\beta\gamma\delta} B_{\alpha\beta} E_{\gamma\delta} \tag{25}$$

$$T^{\alpha} = M^{\alpha\beta\gamma\delta} \nabla_\beta E_{\gamma\delta}. \tag{26}$$

We will now present the full viscous system for fluid film dynamics.

5.1 The full system

The governing system for fluid film dynamics that incorporates the influences of viscous forces reads

$$\frac{\delta\tau}{\delta t} + \nabla_\alpha \left(\tau V^\alpha \right) = \tau C B^\alpha_\alpha \tag{27a}$$

$$\tau \left(\frac{\delta C}{\delta t} + 2V^\alpha \nabla_\alpha C + B_{\alpha\beta} V^\alpha V^\beta \right) = -P B^\alpha_\alpha + T^{(N)} \tag{27b}$$

$$\tau \left(\frac{\delta V^\alpha}{\delta t} + V^\beta \nabla_\beta V^\alpha - C \nabla^\alpha C - 2C V^\beta B^\alpha_\beta \right) = -\nabla^\alpha P + T^\alpha. \tag{27c}$$

In the following sections, we will show that this system reduces to the Navier-Stokes equations for planar flows and leads to a monotonic decay of the total energy.

5.2 Equivalence to the Navier-Stokes equations for a planar flow

Let us once again consider the special case of a planar flow characterized by $B_{\alpha\beta} = 0$ and $C = 0$. For a general $M^{\alpha\beta\gamma\delta}$, we have

$$\frac{\delta\tau}{\delta t} + \nabla_\alpha \left(\tau V^\alpha \right) = 0 \tag{28a}$$

$$\tau \left(\frac{\delta V^\alpha}{\delta t} + V^\beta \nabla_\beta V^\alpha \right) = -\nabla^\alpha P + \nabla_\beta \left(M^{\alpha\beta\gamma\delta} E_{\gamma\delta} \right). \tag{28b}$$

Furthermore, for the proposed form (21) of $M^{\alpha\beta\gamma\delta}$, the system reads

$$\frac{\delta\tau}{\delta t} + \nabla_\alpha (\tau V^\alpha) = 0 \tag{29a}$$

$$\tau\left(\frac{\delta V^\alpha}{\delta t} + V^\beta \nabla_\beta V^\alpha\right) = -\nabla^\alpha P + \lambda \nabla^\alpha \nabla_\beta V^\beta + \mu\left(\nabla^\alpha \nabla_\beta V^\beta + \nabla_\beta \nabla^\beta V^\alpha\right). \tag{29b}$$

This system is equivalent to compressible Navier-Stokes equations. Upon the further assumption of incompressibility

$$\nabla_\beta V^\beta = 0 \tag{30}$$

this system becomes

$$\frac{\delta\tau}{\delta t} + \nabla_\alpha (\tau V^\alpha) = 0 \tag{31a}$$

$$\tau\left(\frac{\delta V^\alpha}{\delta t} + V^\beta \nabla_\beta V^\alpha\right) = -\nabla^\alpha P + \mu \nabla_\beta \nabla^\beta V^\alpha. \tag{31b}$$

which is equivalent to the classical incompressible Navier-Stokes equations. We note that P in equation (31b) is determined by the system (31a)-(31b) in combination with boundary conditions, rather than by equation (6).

5.3 Rate of dissipation

We calculate the rate of dissipation of the total energy by following the approach presented in Grinfeld (2009), where energy conservation was demonstrated in the inviscid case. The key to deriving the rate of change of the total energy E is the formula that governs time evolution of surface integrals

$$\frac{d}{dt}\int_S F\,dS = \int_S \frac{\delta F}{\delta t} dS - \int_S C B^\alpha_\alpha F\,dS. \tag{32}$$

Let q be the absolute value of total velocity:

$$q^2 = C^2 + V_\alpha V^\alpha. \tag{33}$$

In terms of q, the total energy E is given by

$$E = \int_S \tau\left(\frac{1}{2}q^2 + e\right). \tag{34}$$

The evolution of E is obtained by analyzing this integral according to equation (32):

$$\frac{dE}{dt} = \int_S \left(\frac{\delta\tau}{\delta t} - \tau C B^\alpha_\alpha\right)\left(\frac{1}{2}q^2 + e\right) + \tau\left(q\frac{\delta q}{\delta t} + e_\tau \frac{\delta\tau}{\delta t}\right). \tag{35}$$

The quantity $\delta\tau/\delta t$ is available from the mass conservation equation (9a). The analysis of $\delta q/\delta t$ is based on the application of the dynamic equations. First, apply the $\delta/\delta t$-derivative to the definition of q, equation (33). By the product rule, we have

$$q\frac{\delta q}{\delta t} = C\frac{\delta C}{\delta t} + \frac{1}{2}\left(\frac{\delta V_\alpha}{\delta t}V^\alpha + \frac{\delta V^\alpha}{\delta t}V_\alpha\right), \tag{36}$$

and we observe that quantities that need to be determined are $\delta C/\delta t$, $\delta V_\alpha/\delta t$ and $\delta V^\alpha/\delta t$. Equation (9b) gives us $\delta C/\delta t$

$$\frac{\delta C}{\delta t} = -\frac{1}{\tau}PB^\alpha_\alpha + \frac{1}{\tau}N_iT^i - 2V^\alpha\nabla_\alpha C - B_{\alpha\beta}V^\alpha V^\beta. \tag{37}$$

The quantity $\partial V^\alpha/\delta t$ comes from equation (9c):

$$\frac{\delta V^\alpha}{\delta t} = -\frac{1}{\tau}\nabla^\alpha P + \frac{1}{\tau}T^\alpha - V^\beta\nabla_\beta V^\alpha + C\nabla^\alpha C + 2CV^\beta B^\alpha_\beta. \tag{38}$$

By lowering the index α, it follows that

$$\frac{\delta V_\alpha}{\delta t} = -\frac{1}{\tau}\nabla_\alpha P + \frac{1}{\tau}T_\alpha - V^\beta\nabla_\beta V_\alpha + C\nabla_\alpha C. \tag{39}$$

Combining equations (37)-(39), we obtain the quantity $q\delta q/\delta t$:

$$q\frac{\delta q}{\delta t} = -\frac{1}{\tau}V_\alpha\nabla^\alpha P - \frac{1}{\tau}PCB^\alpha_\alpha - \frac{1}{2}V^\beta\nabla_\beta q + \frac{1}{\tau}T^\alpha V_\alpha. \tag{40}$$

We are now able to express dE/dt in terms of the primary elements of the dynamic system:

$$\frac{dE}{dt} = -\int_S \left(\begin{array}{c} \nabla_\alpha\left(\frac{1}{2}q^2\tau V^\alpha\right) + \nabla_\alpha\left(\tau V^\alpha\right) + V_\alpha\nabla^\alpha P \\ +\frac{1}{\tau}P\nabla_\alpha\left(\tau V^\alpha\right) - T^\alpha V_\alpha - CN_iT^i \end{array}\right) dS \tag{41}$$

The rest of the analysis proceeds by a repeated application of Gauss's theorem. The first term in the integrand yields

$$\int_S \nabla_\alpha\left(\frac{1}{2}q^2\tau V^\alpha\right) dS = \int_\gamma \frac{1}{2}q^2\tau V^\alpha n_\alpha d\gamma, \tag{42}$$

where γ is the stationary contour of the fluid film and n_α is the normal (that lies in the tangent plane to S) to the contour γ. Since V^α at the boundary must point along the contour γ, is orthogonal to the normal n_α. Thus, the integrand is identically zero and this term vanishes. An application of Gauss's theorem to the second term leads to

$$\int_{\alpha} \nabla_{\alpha} (\tau V^{\alpha}) e \, dS = -\int_S \tau V^{\alpha} \nabla_{\alpha} e \, dS = -\int_S \tau V^{\alpha} e_{\tau} \nabla_{\alpha} \tau \, dS. \tag{43}$$

Regarding the remaining terms, elementary calculus shows that they form a single divergence expression

$$-\tau V^{\alpha} e_{\tau} \nabla_{\alpha} \tau + V_{\alpha} \nabla^{\alpha} \left(\tau^2 e_{\tau}\right) + \tau e_{\tau} \nabla_{\alpha} (\tau V^{\alpha}) = \nabla_{\alpha} \left(\tau^2 e_{\tau} V^{\alpha}\right).$$

The integral of this term vanishes by Guass's theorem. We have therefore shown that

$$\frac{dE}{dt} = \int_S \left(T^{\alpha} V_{\alpha} + C T^{(N)}\right) dS, \tag{44}$$

or

$$\frac{dE}{dt} = \int_S \left(\nabla_{\beta} \left(M^{\alpha\beta\gamma\delta} E_{\gamma\delta}\right) V_{\alpha} + C M^{\alpha\beta\gamma\delta} B_{\alpha\beta} E_{\gamma\delta}\right) dS. \tag{45}$$

Assuming that the gradient of $\nabla_{\beta} M^{\alpha\beta\gamma\delta}$ vanishes (as it does for $M^{\alpha\beta\gamma\delta}$ given by equation (21)), we have

$$\frac{dE}{dt} = -\int_S M^{\alpha\beta\gamma\delta} E_{\gamma\delta} \left(\nabla_{\beta} V_{\alpha} - C B_{\alpha\beta}\right) dS \tag{46}$$

Recalling the definition (16) of the rate of strain tensor $E_{\alpha\beta}$ and the postulated symmetries of $M^{\alpha\beta\gamma\delta}$, we have

$$\frac{dE}{dt} = -2 \int_S M^{\alpha\beta\gamma\delta} E_{\alpha\beta} E_{\gamma\delta} dS, \tag{47}$$

which is nonpositive because $M^{\alpha\beta\gamma\delta}$ is assumed positive definite. We have therefore proven that the total energy E monotonically diminishes in the course of evolution of the fluid film.

5.4 Boundary conditions

The viscous system (24a)-(24c) is a second order system of partial differential equations. We must therefore specify an additional boundary condition. By analogy with three-dimensional hydrodynamics, we add the condition of traction. The full set of three boundary conditions reads

$$C = 0 \tag{48a}$$

$$V^1 = 0 \tag{48b}$$

$$V^2 = 0. \tag{48c}$$

5.5 A most complete system

In this section, we combine all the available elements to present a system that has a most complete list of necessary elements for explaining and predicting experimental data. The included physical influences are

1. Surface tension and related surface influences captured by the internal energy density function $e(\tau)$

2. Influences of external conservative forces represented by the potential function U

3. Viscosity effects introduced in this paper

4. Interaction with the ambient medium. We suppose that the ambient medium is a liquid or a gas governed by the classical equation of hydrodynamics. Depending on the physical problem, the ambient medium can be assumed compressible on incompressible, viscous or inviscid. Let p denote the pressure of the ambient medium and $[p]$ be the discontinuity jump in p across the surface of the fluid film.

The system that includes these elements reads

$$\frac{\delta\tau}{\delta t}+\nabla_\alpha\left(\tau V^\alpha\right)=\tau CB_\alpha^\alpha \tag{49a}$$

$$\tau\left(\frac{\delta C}{\delta t}+2V^\alpha\nabla_\alpha C+B_{\alpha\beta}V^\alpha V^\beta\right)=-PB_\alpha^\alpha+T^{(N)}-\frac{\partial U}{\partial N}+[p] \tag{49b}$$

$$\tau\begin{pmatrix}\frac{\delta V^\alpha}{\delta t}+V^\beta\nabla_\beta V^\alpha\\ -C\nabla^\alpha C-2CV^\beta B_\beta^\alpha\end{pmatrix}=-\nabla^\alpha P+T^\alpha-\nabla^\alpha U. \tag{49c}$$

Finally, we present the system for the Laplace choice of the internal energy

$$e\left(\tau\right)=\frac{\sigma}{\tau} \tag{50}$$

and for the Newtonian model of viscosity (21). We note that the Laplace model of surface tension is the simples material model of fluid films. It leads to internal energy I that is directly proportional to the total surface area S

$$I=\sigma S. \tag{51}$$

This form of energy captures a broad range of dynamical effects, but also leads to a defect (discussed in Grinfeld (2010a) and Grinfeld (2009)) with regard to the equilibrium density distribution.

For the Laplace material model of surface tension (50) and Newtonian viscosity (21), the full governing system reads

$$\frac{\delta \tau}{\delta t} + \nabla_\alpha \left(\tau V^\alpha\right) = \tau C B^\alpha_\alpha \tag{52a}$$

$$\tau \left(\frac{\delta C}{\delta t} + 2V^\alpha \nabla_\alpha C + B_{\alpha\beta} V^\alpha V^\beta \right) = \left(\begin{array}{l} \sigma B^\alpha_\alpha - \frac{\partial U}{\partial N} + [p] \\ \quad + \lambda B^\alpha_\alpha \left(\nabla_\beta V^\beta - C B^\beta_\beta \right) \\ \quad\quad + 2\mu B^\alpha_\beta \left(\nabla_\alpha V^\beta - C B^\beta_\alpha \right) \end{array} \right) \tag{52b}$$

$$\tau \left(\begin{array}{l} \frac{\delta V^\alpha}{\delta t} + V^\beta \nabla_\beta V^\alpha \\ \quad - C \nabla^\alpha C - 2 C V^\beta B^\alpha_\beta \end{array} \right) = \left(\begin{array}{l} -\nabla^\alpha U + \lambda \nabla^\alpha \nabla_\beta V^\beta \\ \quad + \mu \left(\nabla^\alpha \nabla_\beta V^\beta + \nabla_\beta \nabla^\beta V^\alpha \right) \end{array} \right). \tag{52c}$$

We believe that this system is capable of explaining the existing experimental data as well as predicting real physical phenomena.

6 Conclusion

In Section 5, we presented a way to introduce viscosity into the exact equations of fluid film dynamics. The resulting system, that is the fluid film analogue of the classical Navier-Stokes equations is given in equations (27a)-(27c). The boundary conditions that describe a viscous fluid film that spans a stationary contour are given in equations (48a)-(48c). The addition of viscosity, in a sense, completes the essential puzzle of realism and opens the door to comparison with experiment. The full system (49a)-(49c) combines inertial effects material properties that can be captured by an internal energy density function $e(\tau)$ that is a function only of the two dimensional density τ, with viscosity, interaction with the ambient medium and the influence of external conservative forces.

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