# Numerical Formulations for the Prediction of Deformation, Strain and Stress of Un-patterned ETFE Cushions 

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#### Abstract

ETFE cushions are increasingly being used to form high-profile facades and structural forms. This investigation aims to extend an analytical theory of large deformation in order to predict the shape and stress distributions of an unpatterned square ETFE cushion without the need to resort to discretised numerical methods. In order to assess the validity of the theoretical procedure a prototype cushion has been analysed using a finite element simulation. The theoretical procedure is also compared with alternative approximate equations proposed for the design of ETFE cushions.


Keywords: ETFE, pneumatic, cushion, form finding, analytical solution.

## 1 Introduction

ETFE foil is a relatively new construction material; its first structural use was in the form of a pneumatic cladding system for Burger's Zoo in the Netherlands in 1982 [4]. The primary properties of ETFE foil which have popularised it as a construction material include an ability to transmit up to $97 \%$ of visible light, a low density and the ability to elongate up to $400 \%$ prior to breaking [1]. Due to its relatively novel application in the construction sector there is not yet a published standard design guide for the structural use of ETFE foil. Development of design procedures has generally been conducted on a project by project basis and published with the inclusion of only small amounts of technical data. For example, the cushions for the Eden project, Figure 1a, were the largest built by Vector Foiltech at the time and the contract therefore included full scale prototyping and physical testing [4]. However, ETFE is no longer a material reserved for prestigious structures and is increasingly being used for smaller scale projects such as leisure facilities, bus shelters and shopping centres. One such example, a bus shelter clad

[^0] 7RU
by ETFE cushion units, was installed by Architen Landrell in 2008 at North West Bus Interchange in Westfield London (Figure 1b)


Figure 1: (a) Eden Project and (b) North West Bus Interchange, Westfield London ${ }^{2}$

As ETFE cushion cladding systems become more and more common cost effective solutions for their design and manufacture will be in high demand. The development of a closed form numerical methodology for the prediction of shape and stress distributions would allow designers to develop efficient designs for un-patterned ETFE cushion systems without using sophisticated computational tools such as the finite element method. The manufacturing process is also simplified, minimising the need for welding, and maximising material usage with the avoidance of patterning. Un-patterned cushions offer the simplest approach to cushion manufacture for reasonable coverage per panel.
The shape of an inflated ETFE cushion is a function of the internal applied pressure and the elastic properties of the ETFE. The design of ETFE cushions is therefore complex as the shape of the cushion and the stress in the foil are interdependent. This problem leads to the development of solutions which rely on trial and error methods of successive approximation, common in the design of pre-stressed tensile structures. Borgart presents an approximate calculation method for air inflated cushion which aims to provide a simpler method of analysis for the initial stages of design without disregarding the geometric non-linearity of ETFE cushions [3]. However, the analytical method presented is reliant on the identification of a design height and is therefore difficult to apply to the initial design of un-patterned cushions. The availability of further published studies is extremely limited and comprises a small number of Chinese and German language papers ([8]-[10]). Most promisingly, a classical analytical approach has been proposed by Trostel [6].
The solutions to a formulation based on the theories of large and small deformations allow the prediction of the shape and stress distribution of an inflated membrane.

The formulation uses shape functions and the elastic properties of the membrane to predict the three dimensional Cartesian surface co-ordinates of the deflected cushion surface. The translations may then be converted to strains which in turn are transformed into stresses in accordance with a predetermined elastic model. In this paper the formulation and it's suitability for the prediction of the deformation, stress and strain of un-patterned ETFE cushions is examined. The key hypothesis is that the further development and refinement of this theory will provide the answers to the problem presented by the interdependence of stress and shape in a closed form solution, without the need to undertake a detailed finite element analysis.


Figure 2: Surface described by position vector $\mathfrak{r}=\mathfrak{r}(\alpha, \beta)$, [6]

## 2 Development of the theoretical formulation

The development of the generalised variational problem in line with the theory of large deformation is summarised following the work of Trostel [6]. This problem is then adapted for the specific case of a square membrane through the definition of displacement and load vectors which reflect the boundary conditions of the cushion.

### 2.1 Geometric fundamentals

A position vector $\mathfrak{r}=\mathfrak{r}(\alpha, \beta)$, which depends on two scalar parameters $\alpha$ and $\beta$, may be used to describe a surface. Two groups of parametric curves, $\alpha$-curves where $\beta$ is constant and $\beta$-curves where $\alpha$ is constant, describe a non-orthogonal
network upon this surface. Unit tangent vectors, $\mathfrak{e}_{\alpha}$ and $\mathfrak{e}_{\beta}$, are constructed to the parametric curves in order to specify the orientation of the tangent plane at a specific point, see Figure 2.
The tangent plane may also be specified by a single unit normal vector $\mathfrak{e}_{\gamma}$ (Figure 2). The unit tangent and unit normal vectors can be defined as follows,
$\mathfrak{e}_{\alpha}=\frac{\partial \mathfrak{r}}{\partial s_{\alpha}}$
$\mathfrak{e}_{\beta}=\frac{\partial \mathfrak{r}}{\partial s_{\beta}}$
$\mathfrak{e}_{\gamma}=\frac{\mathfrak{e}_{\alpha} \times \mathfrak{e}_{\beta}}{\sin \sigma_{\alpha \beta}}$

### 2.2 The equilibrium and deformation conditions

The membrane forces in the system $\left(\mathfrak{e}_{\alpha}, \mathfrak{e}_{\beta}, \mathfrak{e}_{\gamma}\right)$ may be defined as,
$\mathfrak{n}_{\alpha}=n_{\alpha} \mathfrak{e}_{\alpha}+n_{\alpha \beta}{ }^{\mathfrak{e}}{ }_{\beta}$
$\mathfrak{n}_{\beta}=n_{\beta \alpha} \mathfrak{e}_{\beta}+n_{\beta} \mathfrak{e}_{\beta}$
Every surface element ( $d F=d s_{\alpha} d s_{\beta} \sin \sigma_{\alpha \beta}$ ) of the membrane is bounded by pairs of parametric curves as shown in Figure 3. If equilibrium conditions are imposed on each element, the sectional loads acting on the membrane must form an equilibrium system with the surface loads. These surface loads are defined by a load vector $\mathfrak{p}$ where $p$ denotes pressure.
$\mathfrak{p}=p_{\alpha} \mathfrak{e}_{\alpha}+p_{\beta} \mathfrak{e}_{\beta}+p_{\gamma} \mathfrak{e}_{\gamma}$
It can be deduced (Figure 3) that the equilibrium condition is as follows,

$$
\begin{equation*}
\frac{\partial}{\partial s_{\alpha}}\left(\mathfrak{n}_{\beta} d s_{\alpha}\right) d s_{\beta}+\frac{\partial}{\partial s_{\beta}}\left(\mathfrak{n}_{\alpha} d s_{\beta}\right) d s_{\alpha}=-\mathfrak{p} d s_{\alpha} d s_{\beta} \sin \sigma_{\alpha \beta} \tag{4}
\end{equation*}
$$

The displacement of a membrane surface may be represented by a displacement vector $\mathfrak{b}(\alpha, \beta)$ which may be constrained for specific boundary conditions using unit tangent vectors combined with scalar components. If a membrane surface is described by a position vector $\mathfrak{r}=\mathfrak{r}(\alpha, \beta)$ then the deformed membrane surface is defined as follows,
$\mathfrak{r}^{\prime}(\alpha, \beta)=\mathfrak{r}(\alpha, \beta)+\mathfrak{b}(\alpha, \beta)$


Figure 3: Surface element dF undergoing equilibrium conditions, [5]

The deformations of a membrane element may be characterised by the three dimensionless components of the symmetrical strain tensor $\mathfrak{D}$.
$\mathfrak{D}=\left[\begin{array}{ll}d_{\alpha \alpha} & d_{\alpha \beta} \\ d_{\beta \alpha} & d_{\beta \beta}\end{array}\right]$
where the shear component $d_{\alpha \beta}=d_{\beta \alpha}$.
The components of $\mathfrak{D}$ are found to be dependent on the derivatives of the displacement vector $\mathfrak{b}(\alpha, \beta)$.
$d_{\alpha \alpha}=\mathfrak{e}_{\alpha} \frac{\partial \mathfrak{b}}{\partial s_{\alpha}}+\frac{1}{2}\left(\frac{\partial \mathfrak{b}}{\partial s_{\alpha}}\right)^{2}$
$d_{\beta \beta}=\mathfrak{e}_{\beta} \frac{\partial \mathfrak{b}}{\partial s_{\beta}}+\frac{1}{2}\left(\frac{\partial \mathfrak{b}}{\partial s_{\beta}}\right)^{2}$
$d_{\alpha \beta}=\frac{1}{2}\left[\mathfrak{e}_{\alpha} \frac{\partial \mathfrak{b}}{\partial s_{\beta}}+\mathfrak{e}_{\beta} \frac{\partial \mathfrak{b}}{\partial s_{\alpha}}+\frac{\partial \mathfrak{b}}{\partial s_{\alpha}} \frac{\partial \mathfrak{b}}{\partial s_{\beta}}\right]$
If large displacements are assumed, then to find the sectional loads, the exact characteristics corresponding to the deformed surface $\mathfrak{r}^{\prime}(\alpha, \beta)$ must be used. It should be noted that characteristics corresponding to the deformed surface element are denoted by an apostrophe. Therefore, the membrane forces in the system ( $\left.\mathfrak{e}^{\prime}{ }_{\alpha}, \mathfrak{e}^{\prime}{ }_{\beta}, \mathfrak{e}^{\prime}{ }_{\gamma}\right)$ are as follows,

$$
\begin{align*}
& \mathfrak{n}_{\alpha}^{\prime}=n^{\prime}{ }_{\alpha} \mathfrak{e}^{\prime}{ }_{\alpha}+n^{\prime}{ }_{\alpha \beta} \mathfrak{e}^{\prime}{ }_{\beta} \\
& \mathfrak{n}_{\beta}^{\prime}=n^{\prime}{ }_{\beta \alpha} \mathfrak{e}^{\prime}{ }_{\beta}+n^{\prime}{ }_{\beta} \mathfrak{e}^{\prime}{ }_{\beta} \tag{8}
\end{align*}
$$



Figure 4: Deformed surface element $\mathrm{dF}^{\prime}$ undergoing equilibrium conditions, [5]

The stress state of a membrane may be defined by a symmetrical load tensor $\mathfrak{N}$ in a similar fashion to the deformations.
$\mathfrak{N}=\left[\begin{array}{cc}n_{\alpha \alpha} & n_{\alpha \beta} \\ n_{\beta \alpha} & n_{\beta \beta}\end{array}\right]=\left[\begin{array}{cc}n^{\prime}{ }_{\alpha} \frac{1+\varepsilon_{\beta}}{1+\varepsilon_{\alpha}} & n^{\prime}{ }_{\alpha \beta} \\ n^{\prime}{ }_{\beta \alpha} & n^{\prime}{ }_{\beta} \frac{1+\varepsilon_{\alpha}}{1+\varepsilon_{\beta}}\end{array}\right]$
where the shear component $n_{\alpha \beta}=n_{\beta \alpha}$.

### 2.3 Load strain relationships

The generalised Kappus Law is used in this formulation to describe the load-strain relationship within this theory of large deformations. Hooke's Law, the simplest and most commonly used law of elasticity is written as,
$n_{\alpha}=\frac{D}{1-v^{2}} \varepsilon_{\alpha}$,
in which $D$ is the elastic constant and $v$ is Poisson's ratio of the material in question. Kappus's Law (11) is a non linear extension of Hooke's Law and is expected to be applicable to rubber polymers which undergo considerable strain hardening, and is Hooke's Law (10) with the addition of a non linear term, as in,

$$
\begin{align*}
n_{\alpha} & =\frac{D}{1-v^{2}} \varepsilon_{\alpha}\left(1+\varepsilon_{\alpha}\right)\left(1+\frac{\varepsilon_{\alpha}}{2}\right) \\
& =\frac{D}{1-v^{2}}\left(\varepsilon_{\alpha}+\varepsilon_{\alpha}{ }^{2}+\frac{\varepsilon_{\alpha}{ }^{2}}{2}+\frac{\varepsilon_{\alpha}{ }^{3}}{2}\right)  \tag{11}\\
& =\frac{D}{1-v^{2}} \varepsilon_{\alpha}+\frac{D}{1-v^{2}}\left(\frac{3 \varepsilon_{\alpha}^{2}}{2}+\frac{\varepsilon_{\alpha}^{3}}{2}\right)
\end{align*}
$$

The generalised Kappus Law leads to the following specific relationships between the components of the load and strain tensors $\mathfrak{D}$ and $\mathfrak{N}$.

$$
\begin{align*}
& n_{\alpha \alpha}=\frac{D\left(d_{\alpha \alpha}+v d_{\beta \beta}\right)}{1-v^{2}} \\
& n_{\beta \beta}=\frac{D\left(d_{\beta \beta}+v d_{\alpha \alpha}\right)}{1-v^{2}} \\
& n_{\alpha \beta}=\frac{D d_{\alpha \beta}}{2(1+v)} \tag{12}
\end{align*}
$$

### 2.4 Development of the generalised variational problem

The generalised variational problem is based on the theory of virtual work and is developed by multiplying the equilibrium equation of the surface by an arbitrary displacement variation $\delta_{\mathfrak{b}}$ and integrating over the entire un-deformed membrane to give,

$$
\begin{array}{r}
\iint_{(F)}\left\{n_{\alpha \alpha} \delta_{\mathfrak{b}} d_{\alpha \alpha}+n_{\beta \beta} \delta_{\mathfrak{b}} d_{\beta \beta}+2 n_{\alpha \beta} \delta_{\mathfrak{b}} d_{\alpha \beta}-\mathfrak{p}^{\prime}\left(1+\varepsilon_{\alpha}\right)\left(1+\varepsilon_{\beta}\right) \sin \sigma_{\alpha \beta}^{\prime} \delta_{\mathfrak{b}} \mathfrak{b}\right\} \\
d F=0 \tag{13}
\end{array}
$$

The problem is developed for Kappus's Law by substituting the load tensor components of $\mathfrak{N}$ with the displacement tensor components of $\mathfrak{D}$ in accordance with the specific load strain relationships of (12) to give,

$$
\begin{align*}
& \delta_{\mathfrak{b}} \iint_{(F)} \frac{D}{2\left(1-v^{2}\right)}\left\{d_{\alpha \alpha^{2}}+d_{\beta \beta^{2}}+\right. 2 v d_{\alpha \alpha} d_{\beta \beta}+(1-v) d_{\alpha \beta}^{2} \\
&\left.-\mathfrak{p}^{\prime}\left(1+\varepsilon_{\alpha}\right)\left(1+\varepsilon_{\beta}\right) \sin \sigma^{\prime}{ }_{\alpha \beta} \mathfrak{b}\right\} d F=0 \tag{14}
\end{align*}
$$

The components of $\mathfrak{D}$ are then substituted for their derivative identities (4) to give the generalised variational problem in terms of the displacement vector $\mathfrak{b}$ and the unit tangent vectors which describe the orientation of the surface of the membrane.

$$
\begin{align*}
& \delta_{\mathfrak{b}} \iint_{(F)}\left\{\frac { D } { 2 ( 1 - v ^ { 2 } ) } \left[\left(\mathfrak{e}_{\alpha} \frac{\partial \mathfrak{b}}{\partial s_{\alpha}}+\frac{1}{2}\left(\frac{\partial \mathfrak{b}}{\partial s_{\alpha}}\right)^{2}\right)^{2}+\left(\mathfrak{e}_{\beta} \frac{\partial \mathfrak{b}}{\partial s_{\beta}}+\frac{1}{2}\left(\frac{\partial \mathfrak{b}}{\partial s_{\beta}}\right)^{2}\right)^{2}\right.\right. \\
& +2 v\left(\mathfrak{e}_{\alpha} \frac{\partial \mathfrak{b}}{\partial s_{\alpha}}+\frac{1}{2}\left(\frac{\partial \mathfrak{b}}{\partial s_{\alpha}}\right)^{2}\right)\left(\mathfrak{e}_{\beta} \frac{\partial \mathfrak{b}}{\partial s_{\beta}}+\frac{1}{2}\left(\frac{\partial \mathfrak{b}}{\partial s_{\beta}}\right)^{2}\right) \\
& \left.\left.+\frac{1-v}{2}\left(\mathfrak{e}_{\alpha} \frac{\partial \mathfrak{b}}{\partial s_{\beta}}+\mathfrak{e}_{\beta} \frac{\partial \mathfrak{b}}{\partial s_{\alpha}}+\frac{\partial \mathfrak{b}}{\partial s_{\alpha}} \frac{\partial \mathfrak{b}}{\partial s_{\beta}}\right)^{2}\right]-\mathfrak{p}^{\prime}\left(1+\varepsilon_{\alpha}\right)\left(1+\varepsilon_{\beta}\right) \sin \sigma^{\prime}{ }_{\alpha \beta} \mathfrak{b}\right\} d F=0 \tag{15}
\end{align*}
$$

### 2.5 Development of the specific variational problem for a square membrane

To solve the generalised variational problem for a specific case, in this case a square membrane fixed along each side, a definition for the displacement vector is required. The vector is defined in terms of a scalar component and unit tangent vectors. The scalar component takes the form of displacements $u, v$ and $w$ which are defined by basic displacement functions $u_{1}, v_{1}$ and $w_{1}$ and constants $C_{1}$ and $C_{w}$. The basic displacement functions will be developed later according to the boundary conditions of the specific problem and constants $C_{1}$ and $C_{w}$ are derived from the resulting specific variational problem.
When considering the specific problem the scalar parameters $\alpha$ and $\beta$ become $x$ and $y$ respectively. For an initially plane rectangular membrane the displacement vector is defined as follows,
$\mathfrak{b}(x, y)=u(x, y) \mathfrak{e}_{x}+v(x, y) \mathfrak{e}_{y}+w(x, y) \mathfrak{e}_{z}$
$u(x, y)=C_{u} u_{1}$
$v(x, y)=C_{v} v_{1}$
$w(x, y)=C_{w} w_{1}$
Differentiation of $\mathfrak{b}$ along the x and y parametric curves leads to the following displacement derivatives,

$$
\begin{align*}
\frac{\partial \mathfrak{b}}{\partial x} & =\frac{d u}{d x} \mathfrak{e}_{x}+\frac{d v}{d x} \mathfrak{e}_{y}+\frac{d w}{d x} \mathfrak{e}_{z} \\
\frac{\partial \mathfrak{b}}{\partial y} & =\frac{d u}{d y} \mathfrak{e}_{x}+\frac{d v}{d y} \mathfrak{e}_{y}+\frac{d w}{d y} \mathfrak{e}_{z} \tag{18}
\end{align*}
$$

According to the rules of unit vector cross multiplication,
$\mathfrak{e}_{x} \times \mathfrak{e}_{y}=\mathfrak{e}_{z}, \mathfrak{e}_{y} \times \mathfrak{e}_{z}=\mathfrak{e}_{x}, \quad \mathfrak{e}_{z} \times \mathfrak{e}_{x}=\mathfrak{e}_{y}$
$\mathfrak{e}_{y} \times \mathfrak{e}_{x}=-\mathfrak{e}_{z}, \mathfrak{e}_{z} \times \mathfrak{e}_{y}=-\mathfrak{e}_{x}, \mathfrak{e}_{x} \times \mathfrak{e}_{z}=-\mathfrak{e}_{y}$
$\mathfrak{e}_{x} \times \mathfrak{e}_{x}=\mathfrak{e}_{y} \times \mathfrak{e}_{y}=\mathfrak{e}_{z} \times \mathfrak{e}_{z}=0$
The following identities are found
$\mathfrak{e}_{x} \frac{\partial \mathfrak{b}}{\partial x}=\frac{d u}{d x}, \mathfrak{e}_{y} \frac{\partial \mathfrak{b}}{\partial y}=\frac{d v}{d y}, \mathfrak{e}_{x} \frac{\partial \mathfrak{b}}{\partial y}=\frac{d u}{d y}, \mathfrak{e}_{y} \frac{\partial \mathfrak{b}}{\partial x}=\frac{d v}{d x}$

$$
\begin{equation*}
\left(\frac{\partial \mathfrak{b}}{\partial x}\right)^{2}=\left(\frac{d u}{d x}\right)^{2}+\left(\frac{d v}{d x}\right)^{2}+\left(\frac{d w}{d x}\right)^{2},\left(\frac{\partial \mathfrak{b}}{\partial y}\right)^{2}=\left(\frac{d u}{d y}\right)^{2}+\left(\frac{d v}{d y}\right)^{2}+\left(\frac{d w}{d y}\right)^{2} \tag{20}
\end{equation*}
$$

$\frac{\partial \mathfrak{b}}{\partial x} \frac{\partial \mathfrak{b}}{\partial y}=\frac{d u}{d x} \frac{d u}{d x}+\frac{d v}{d x} \frac{d v}{d y}+\frac{d w}{d x} \frac{d w}{d y}$
As for the displacement vector $\mathfrak{b}$ the load vector $\mathfrak{p}$ must be defined for the specific problem. Only deformations due to the internal pressure of the cushion are to be considered. Therefore, the surface load vector can be simplified to contain only the unit normal vector to the deformed surface $\mathfrak{e}^{\prime}{ }_{z}$.
$\mathfrak{p}^{\prime}=p \times \mathfrak{e}_{z}^{\prime}$
As stated previously the un-deformed surface $\mathfrak{r}(x, y)$ is defined by unit tangent vectors $\mathfrak{e}_{x}, \mathfrak{e}_{y,} \mathfrak{e}_{z}$ and the deformed surface $\mathfrak{r}^{\prime}(x, y)$ is defined by unit tangent vectors $\mathfrak{e}_{x}^{\prime}, \mathfrak{e}^{\prime}{ }_{y,} \mathfrak{e}_{z}^{\prime}$. The deformed tangent vectors may be defined via the position and displacement vector derivatives according to basic geometric principles and the identities for $\mathfrak{e}_{x}, \mathfrak{e}_{y}, \mathfrak{e}_{z}$ shown in (1).
$\mathfrak{e}^{\prime}{ }_{x}=\frac{1}{1+\varepsilon_{x}}\left(\frac{\partial \mathfrak{r}}{\partial x}+\frac{\partial \mathfrak{b}}{\partial x}\right)=\frac{1}{1+\varepsilon_{x}}\left(\mathfrak{e}_{x}+\frac{\partial \mathfrak{b}}{\partial x}\right)$
$\mathfrak{e}^{\prime}{ }_{y}=\frac{1}{1+\varepsilon_{y}}\left(\mathfrak{e}_{y}+\frac{\partial \mathfrak{b}}{\partial y}\right)$
$\mathfrak{e}_{z}^{\prime}=\frac{\mathfrak{e}^{\prime}{ }_{x} \times \mathfrak{e}^{\prime}{ }_{y}}{\sin \sigma_{x y}^{\prime}}=\frac{\left(\mathfrak{e}_{x}+\frac{\partial \mathfrak{b}}{\partial x}\right)\left(\mathfrak{e}_{y}+\frac{\partial \mathfrak{b}}{\partial y}\right)}{\left(1+\varepsilon_{x}\right)\left(1+\varepsilon_{y}\right) \sin \sigma_{x y}^{\prime}{ }_{x y}}$
Taking this and the identities of (20) into account the load term within the generalised variational problem (15) becomes,

$$
\begin{align*}
& \mathfrak{p}^{\prime}\left(1+\varepsilon_{x}\right)\left(1+\varepsilon_{y}\right) \sin \sigma_{x y}^{\prime}=p\left[\frac{\left(\mathfrak{e}_{x}+\frac{\partial \mathfrak{b}}{\partial x}\right)\left(\mathfrak{e}_{y}+\frac{\partial \mathfrak{b}}{\partial y}\right)}{\left(1+\varepsilon_{x}\right)\left(1+\varepsilon_{y}\right) \sin \sigma_{x y}^{\prime}}\right]\left(1+\varepsilon_{x}\right)\left(1+\varepsilon_{y}\right) \sin \sigma_{x y}^{\prime} \\
& =p\left(\mathfrak{e}_{x}+\frac{\partial \mathfrak{b}}{\partial x}\right) \times\left(\mathfrak{e}_{y}+\frac{\partial \mathfrak{b}}{\partial y}\right) \\
& =p\left[\left(\frac{\partial v}{\partial x} \frac{\partial w}{\partial y}-\left(1+\frac{\partial v}{\partial y}\right) \frac{\partial w}{\partial x}\right) \mathfrak{e}_{x}+\left(\frac{\partial u}{\partial y} \frac{\partial w}{\partial x}-\left(1+\frac{\partial u}{\partial x}\right) \frac{\partial w}{\partial y}\right) \mathfrak{e}_{y}\right. \\
& \left.\quad+\left(\left(1+\frac{\partial u}{\partial x}\right)\left(1+\frac{\partial v}{\partial y}\right)-\frac{\partial u}{\partial y} \frac{\partial v}{\partial x}\right) \mathfrak{e}_{z}\right] \tag{24}
\end{align*}
$$

Through the introduction of the displacement vector $\mathfrak{b}$ to the surface load term it becomes,

$$
\begin{align*}
& p\left[\left(\frac{\partial v}{\partial x} \frac{\partial w}{\partial y}-\left(1+\frac{\partial v}{\partial y}\right) \frac{\partial w}{\partial x}\right) \mathfrak{e}_{x}+\left(\frac{\partial u}{\partial y} \frac{\partial w}{\partial x}-\left(1+\frac{\partial u}{\partial x}\right) \frac{\partial w}{\partial y}\right) \mathfrak{e}_{y}\right. \\
& \left.+\left(\left(1+\frac{\partial u}{\partial x}\right)\left(1+\frac{\partial v}{\partial y}\right)-\frac{\partial u}{\partial y} \frac{\partial v}{\partial x}\right) \mathfrak{e}_{z}\right] \mathfrak{b} \\
& =p\left[\left(\frac{\partial v}{\partial x} \frac{\partial w}{\partial y}-\left(1+\frac{\partial v}{\partial y}\right) \frac{\partial w}{\partial x}\right) \mathfrak{e}_{x} \mathfrak{b}+\left(\frac{\partial u}{\partial y} \frac{\partial w}{\partial x}-\left(1+\frac{\partial u}{\partial x}\right) \frac{\partial w}{\partial y}\right) \mathfrak{e}_{y} \mathfrak{b}\right. \\
& \left.+\left(\left(1+\frac{\partial u}{\partial x}\right)\left(1+\frac{\partial v}{\partial y}\right)-\frac{\partial u}{\partial y} \frac{\partial v}{\partial x}\right) \mathfrak{e}_{z} \mathfrak{b}\right] \\
& =p\left[\left(\frac{\partial v}{\partial x} \frac{\partial w}{\partial y}-\left(1+\frac{\partial v}{\partial y}\right) \frac{\partial w}{\partial x}\right) u+\left(\frac{\partial u}{\partial y} \frac{\partial w}{\partial x}-\left(1+\frac{\partial u}{\partial x}\right) \frac{\partial w}{\partial y}\right) v\right. \\
& \left.+\left(\left(1+\frac{\partial u}{\partial x}\right)\left(1+\frac{\partial v}{\partial y}\right)-\frac{\partial u}{\partial y} \frac{\partial v}{\partial x}\right) w\right] \tag{25}
\end{align*}
$$

The final specific variational; problem is therefore,
$\delta_{\mathrm{u}, \mathrm{v}, \mathrm{w}} \Pi=0$
where,

$$
\begin{align*}
& \Pi=\iint_{(F)}\left\{\frac { D } { 2 ( 1 - v ^ { 2 } ) } \left[\left(\frac{\partial u}{\partial x}+\frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^{2}+\left(\frac{\partial v}{\partial x}\right)^{2}+\left(\frac{\partial w}{\partial x}\right)^{2}\right]\right)^{2}\right.\right. \\
& +\left(\frac{\partial v}{\partial y}+\frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^{2}+\left(\frac{\partial v}{\partial y}\right)^{2}+\left(\frac{\partial w}{\partial y}\right)^{2}\right]\right)^{2} \\
& +2 v\left(\frac{\partial u}{\partial x}+\frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^{2}+\left(\frac{\partial v}{\partial x}\right)^{2}+\left(\frac{\partial w}{\partial x}\right)^{2}\right]\right) \\
& \begin{array}{l}
\left(\frac{\partial v}{\partial y}+\frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^{2}+\left(\frac{\partial v}{\partial y}\right)^{2}+\left(\frac{\partial w}{\partial y}\right)^{2}\right]\right) \\
\left.+\frac{1-v}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x} \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \frac{\partial v}{\partial y}+\frac{\partial w}{\partial x} \frac{\partial w}{\partial y}\right)^{2}\right] \\
-p\left[\left(\frac{\partial v}{\partial x} \frac{\partial w}{\partial y}-\left(1+\frac{\partial v}{\partial y}\right) \frac{\partial w}{\partial x}\right) u+\left(\frac{\partial u}{\partial y} \frac{\partial w}{\partial x}-\left(1+\frac{\partial u}{\partial x}\right) \frac{\partial w}{\partial y}\right) v\right. \\
\left.\left.+\left(\left(1+\frac{\partial u}{\partial x}\right)\left(1+\frac{\partial v}{\partial y}\right)-\frac{\partial u}{\partial y} \frac{\partial v}{\partial x}\right) w\right]\right\} d x d y
\end{array}
\end{align*}
$$

Partial differentiation of $\Pi$ (26) with respect to $C_{u}, C_{v}$ and $C_{w}$ leads to the system of equations,

$$
\begin{equation*}
\frac{\partial \Pi}{\partial C_{u}}=0 ; ; \frac{\partial \Pi}{\partial C_{v}}=0 ; ; \frac{\partial \Pi}{\partial C_{w}}=0 \tag{27}
\end{equation*}
$$

For a square cushion with a side length $a, C_{u}=C_{v}=C_{1}$. The problem is therefore simplified and the system of three cubic equations becomes a system of two. Due to the symmetry of the resulting state of deformation the displacements must be given by $u=f(x, y)$ and $v=f(y, x)$. It can therefore be stated that $u_{1}(\xi, \eta)=v_{1}(\eta, \xi)$ and $v_{1}(\xi, \eta)=u_{1}(\eta, \xi)$ where $\xi=\frac{x}{a}$ and $\eta=\frac{y}{a}$. The displacements are therefore approximated by the equations below.
$u(x, y)=C_{1} u_{1}(\xi, \eta)$
$v(x, y)=C_{1} v_{1}(\xi, \eta)$
$w(x, y)=C_{w} w_{1}(\xi, \eta)$
$\Pi$ now depends solely on $C_{w}$ and $C_{1}$. The extremum conditions $\frac{\partial \Pi}{\partial\left(C_{1} / a\right)}=0$ and $\frac{\partial \Pi}{\partial\left(C_{w} / a\right)}=0$ lead to the following system of cubic equations.

$$
\begin{align*}
& \left(\frac{C_{w}}{a}\right)^{2}\left[\lambda_{1} \frac{C_{1}}{a}+\lambda_{2}\right]-2 \varkappa \frac{C_{w}}{a}\left[\lambda_{3} \frac{C_{1}}{a}-\lambda_{4}\right]+\lambda_{5}\left(\frac{C_{1}}{a}\right)^{3}+3 \lambda_{6}\left(\frac{C_{1}}{a}\right)^{2} \\
& +2 \lambda_{7}\left(\frac{C_{1}}{a}\right)=0 \\
& \left(\frac{C_{1}}{a}\right)^{2}\left[\lambda_{1}-2 \lambda_{8} \frac{\varkappa a}{C_{w}}\right]+2 \frac{C_{1}}{a}\left[\lambda_{2}-\lambda_{9} \frac{\varkappa a}{C_{w}}\right]+\lambda_{10}\left(\frac{C_{w}}{a}\right)^{2}-2 \lambda_{11} \frac{\varkappa a}{C_{w}}=0 \tag{29}
\end{align*}
$$

Where;
$\varkappa=\frac{p a\left(1-v^{2}\right)}{D}$

$$
\begin{array}{r}
\lambda_{1}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[\left(w_{1 \xi^{2}}^{2}+v w_{1 \eta^{2}}^{2}\right)\left(u_{1 \xi}^{2}+v_{1 \xi}^{2}\right)+\left(w_{1 \eta^{2}}^{2}+v w_{1 \xi}{ }^{2}\right)\left(u_{1 \eta}^{2}+v_{1 \eta}^{2}\right)\right. \\
\\
\left.+2(1-v) w_{1 \xi} w_{1 \eta}\left(u_{1 \xi} u_{1 \eta}+v_{1 \xi} v_{1 \eta}\right)\right] d \xi d \eta
\end{array}
$$

$$
\begin{aligned}
\lambda_{2}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[w_{1 \xi}^{2}\left(u_{1 \xi}+v v_{1 \eta}\right)+w_{1 \eta}^{2}\right. & \left(v_{1 \eta}+v u_{1 \xi}\right) \\
& \left.+(1-v) w_{1 \xi} w_{1 \eta}\left(u_{1 \eta}+v_{1 \xi}\right)\right] d \xi d \eta
\end{aligned}
$$

$\lambda_{3}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[u_{1}\left(v_{1 \xi} w_{1 \eta}-v_{1 \eta} w_{1 \xi}\right)+v_{1}\left(u_{1 \eta} w_{1 \xi}-u_{1 \xi} w_{1 \eta}\right)\right] d \xi d \eta$
$\lambda_{4}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[u_{1} w_{1 \xi}+v_{1} w_{1 \eta}\right] d \xi d \eta ;$
$\lambda_{5}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[\left(u_{1} \xi^{2}+v_{1} \xi^{2}\right)^{2}+\left(u_{1 \eta}{ }^{2}+v_{1 \eta^{2}}^{2}\right)^{2}+2 v\left(u_{1} \xi^{2}+v_{1} \xi^{2}\right)\left(u_{1 \eta}^{2}+v_{1 \eta^{2}}^{2}\right)\right.$ $\left.+2(1-v)\left(u_{1} \xi u_{1 \eta}+v_{1 \xi} v_{1 \eta}\right)^{2}\right] d \xi d \eta ;$
$\lambda_{6}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[u_{1 \xi}\left(u_{1} \xi^{2}+v_{1} \xi^{2}\right)+v_{1 \eta}\left(u_{1} \eta^{2}+v_{1 \eta^{2}}^{2}\right)+v u_{1 \xi}\left(u_{1} \xi^{2}+v_{1} \xi^{2}\right)\right.$ $\left.+v v_{1 \eta}\left(u_{1 \eta}^{2}+v_{1 \eta^{2}}\right)+(1-v)\left(u_{1 \eta}+v_{1 \xi}\right)\left(u_{1 \xi} u_{1 \eta}+v_{1 \xi} v_{1 \eta}\right)^{2}\right] d \xi d \eta$
$\lambda_{7}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[u_{1} \xi^{2}+v_{1 \eta}^{2}+2 v u_{1 \xi} v_{1 \eta}+\frac{(1-v)}{2}\left(u_{1 \eta}+v_{1 \xi}\right)^{2}\right] d \xi d \eta$
$\lambda_{8}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[u_{1} \xi v_{1 \eta}-u_{1 \eta} v_{1 \xi}\right] w_{1} d \xi d \eta$
$\lambda_{9}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[u_{1 \xi}+v_{1 \eta}\right] w_{1} d \xi d \eta=-\lambda_{4} ;$
$\lambda_{10}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[w_{1} \xi^{2}+w_{1 \eta^{2}}^{2}\right]^{2} d \xi d \eta ;$
$\lambda_{11}=\int_{\xi=0}^{1} \int_{\eta=0}^{1} w_{1} d \xi d \eta$
The subscripts $\xi$ and $\eta$ denote partial differentiation with respect to $\xi$ and $\eta$.

### 2.6 Calculation of principle strains, sectional loads and direction.

The principle strains and therefore the principle sectional loads in the surface of the cushion along with their directions can be derived from the displacements $(u, v, w)$. The equations required are developed from the fundamentals of the membrane theory of small deformations. The values of the dimensionless components of the
strain tensor $\mathfrak{D}$ shown in (6) may be found using the identities defined in (20) and the partial derivatives of (28) to form the set of equations shown below.

$$
\begin{align*}
& d_{x x}=\mathfrak{e}_{x} \frac{\partial \mathfrak{b}}{\partial s_{x}}+\frac{1}{2}\left(\frac{\partial \mathfrak{b}}{\partial s_{x}}\right)^{2}=\frac{\partial u}{\partial x}+\frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^{2}+\left(\frac{\partial v}{\partial x}\right)^{2}+\left(\frac{\partial w}{\partial x}\right)^{2}\right] \\
&=\frac{C_{1}}{a} u_{1 \xi}+\frac{1}{2}\left[\left(\frac{C_{1}}{a}\right)^{2}\left(u_{1 \xi^{2}}+v_{1 \xi}{ }^{2}\right)+\left(\frac{C_{w}}{a}\right)^{2} w_{1 \xi^{2}}^{2}\right] \\
& \begin{aligned}
& d_{y y}=\mathfrak{e}_{y} \frac{\partial \mathfrak{b}}{\partial s_{y}}+\frac{1}{2}\left(\frac{\partial \mathfrak{b}}{\partial s_{y}}\right)^{2}=\frac{\partial v}{\partial y}+\frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^{2}+\left(\frac{\partial v}{\partial y}\right)^{2}+\left(\frac{\partial w}{\partial y}\right)^{2}\right] \\
&=\frac{C_{1}}{a} v_{1 \eta}+\frac{1}{2}\left[\left(\frac{C_{1}}{a}\right)^{2}\left(u_{1 \eta}^{2}+v_{1 \eta}^{2}\right)+\left(\frac{C_{w}}{a}\right)^{2} w_{1 \eta^{2}}^{2}\right] \\
& 2 d_{x y}=\mathfrak{e}_{x} \frac{\partial \mathfrak{b}}{\partial s_{y}}+\mathfrak{e}_{y} \frac{\partial \mathfrak{b}}{\partial s_{x}}+\frac{\partial \mathfrak{b}}{\partial s_{x}} \frac{\partial \mathfrak{b}}{\partial s_{y}} \\
&=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\left[\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)+\left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right)+\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\right] \\
&=\frac{C_{1}}{a}\left(u_{1 \eta}+v_{1 \xi}\right)+\left(\frac{C_{1}}{a}\right)^{2}\left(u_{1 \eta} u_{1 \xi}+v_{1 \eta} v_{1 \xi}\right)+\left(\frac{C_{w}}{a}\right)^{2} w_{1 \eta} w_{1 \xi}
\end{aligned}
\end{align*}
$$

The maximum and minimum extensions of an infinitely small element may be determined from (32) through (33).


Figure 5: Triangular surface element before and after deformation, [5]

$$
\begin{equation*}
d_{1,2}=\frac{1}{2}\left[d_{x x}+d_{y y} \pm \sqrt{\left(d_{x x}-d_{y y}\right)^{2}+4 d_{x y}^{2}}\right] \tag{33}
\end{equation*}
$$

$1+\varepsilon_{1,2}=\sqrt{1+2 d_{1,2}}$
The principle sectional loads may then be calculated according to Kappus's Law using the identities for $n_{\alpha \alpha}, n_{\beta \beta}$ and $n_{\alpha \beta}$ defined in (9) and (12).
$n^{\prime}{ }_{1,2}=\frac{D}{1-v^{2}} \frac{1+\varepsilon_{1,2}}{1+\varepsilon_{2,1}}\left(d_{1,2}+v d_{2,1}\right)=\frac{D}{1-v^{2}} \frac{\sqrt{1+2 d_{1,2}}}{\sqrt{1+2 d_{2,1}}}\left(d_{1,2}+v d_{2,1}\right)$
The functions for the direction of the principle stresses shown below are derived from an arbitrary initially right-angled surface element, see Figure 5. $\varphi$ denotes the initial principle direction and is referred to the x-direction. $\varphi^{\prime}$ denotes the principle direction after deformation and is referred to the deformed $x$-direction. The second principle direction is perpendicular to $\varphi$.
$\tan \varphi^{\prime}=\frac{1+\varepsilon_{2}}{1+\varepsilon_{1}} \tan \varphi=\frac{\sqrt{1+2 d_{2}}}{\sqrt{1+2 d_{1}}} \tan \varphi$
$\tan 2 \varphi=\frac{2 d_{x y}}{d_{x x}-d_{y y}}$
Equivalent expressions may be derived, in principle, for other (non-square) cushion geometries. An example of a rectangular cushion is demonstrated in Appendix A.

## 3 Development of displacement functions.

Displacement functions may be developed by considering the central region of an initially plane rectangular membrane. At the centre of the membrane the pattern of deformation is as shown in Figure 6. When considering a square cushion the pattern of deformation is the same in both the X and Y directions, this leads to the following system of equations,

$$
\begin{align*}
& w_{1(\xi, \eta)}=w_{1(\xi)} \times w_{1(\eta)} \\
& u_{1(\xi, \eta)}=u_{1(\xi)} \times u_{1(\eta)} \\
& v_{1(\xi, \eta)}=v_{1(\xi)} \times v_{1(\eta)} \tag{38}
\end{align*}
$$

### 3.1 Trigonometric displacement functions

Trigonometric functions satisfying the conditions of immovable edges and the symmetry of the problem are [6],
$u_{1}=-\sin (2 \pi \xi) \sin (\pi \eta)$


Figure 6: Graphical representation of the potential cushion deformation

$$
\begin{align*}
& v_{1}=-\sin (\pi \xi) \sin (2 \pi \eta) \\
& w_{1}=\sin (\pi \xi) \sin (\pi \eta) \tag{39}
\end{align*}
$$

In order to complete the complex integrations in (31) to define the values of $\lambda$ the partial differentiation of $u_{1}, v_{1}$ and $w_{1}$ with respect to $\xi$ and $\eta$ is first completed.
$u_{1 \xi}=-2 \pi \cos (2 \pi \xi) \sin (\pi \eta) ; ; u_{1 \eta}=-\sin (2 \pi \xi) \pi \cos (\pi \eta)$
$v_{1 \xi}=-\pi \cos (\pi \xi) \sin (2 \pi \eta) ; ; v_{1 \eta}=-\sin (\pi \xi) 2 \pi \cos (2 \pi \eta)$
$w_{1 \xi}=\pi \cos (\pi \xi) \sin (\pi \eta) ; ; w_{1 \eta}=\sin (\pi \xi) \pi \cos (\pi \eta)$
These displacement functions and partial derivatives yield the following values for $\lambda$,
$\lambda_{1}=\frac{15}{16} \pi^{4}+\frac{15}{16} \pi^{4} v ; ; \lambda_{2}=\pi^{2} v-\frac{5}{3} \pi^{2} ; ; \lambda_{3}=\frac{3}{8} \pi^{2} ; ;$
$\lambda_{4}=-\frac{4}{3} ; ; \lambda_{5}=\frac{193}{32} \pi^{4}+\frac{17}{8} \pi^{4} v ; ;$
$\lambda_{6}=-\frac{4}{3} \pi^{2}+4 \pi^{2} v ; ; \lambda_{7}=\frac{16}{9} v+\frac{16}{9}-\frac{1}{4} \pi^{2} v+\frac{9}{4} \pi^{2} ; ; \lambda_{8}=\frac{3}{16} \pi^{2} ; ; \lambda_{9}=\frac{4}{3} ; ;$
$\lambda_{10}=\frac{5}{16} \pi^{4} ; ; \lambda_{11}=\frac{4}{\pi^{2}}$
By substituting the above values of $\lambda$ into (29) the following pair of equations is produced where the only unknowns are the values of $\mathrm{C}_{w}$ and $\mathrm{C}_{1}$.

$$
\begin{align*}
& \left(\frac{C_{w}}{a}\right)^{2}\left(\frac{C_{1}}{a}\left(\frac{15}{16} \pi^{4}+\frac{15}{16} \pi^{4} v\right)+\pi^{2} v-\frac{5}{3} \pi^{2}\right)-2 \varkappa \frac{C_{w}}{a}\left(\frac{3}{8} \pi^{2} \frac{C_{1}}{a}+\frac{4}{3}\right) \\
& +\left(\frac{C_{1}}{a}\right)^{3}\left(\frac{193}{32} \pi^{4}+\frac{17}{8} \pi^{4} v\right)+\frac{C_{1}^{2}}{a^{2}} 3\left(-\frac{4}{3} \pi^{2}+4 \pi^{2} v\right) \\
& \\
& +\frac{C_{1}}{a} 2\left(\frac{16}{9} v+\frac{16}{9}-\frac{1}{4} \pi^{2} v\right)=0 \\
& \left(\frac{C_{1}}{a}\right)^{2}\left(\frac{15}{16} \pi^{4}+\frac{15}{16} \pi^{4} v-\frac{3}{8} \pi^{2} \frac{\varkappa a}{C_{w}}\right)+\frac{C_{1}}{a} 2\left(\pi^{2} v-\frac{5}{3} \pi^{2}-\frac{4}{3} \frac{\varkappa a}{C_{w}}\right)  \tag{41}\\
& \\
& +\left(\frac{C_{w}}{a}\right)^{2} \frac{5}{16} \pi^{4}-\frac{\varkappa a}{C_{w}}\left(\frac{8}{\pi^{2}}\right)=0
\end{align*}
$$

The values of $\mathrm{C}_{w}$ and $\mathrm{C}_{1}$ at varying pressure may now be found by solving (41) simultaneously. The deformation of the membrane due to internal pressure can then be calculated according to (28) and (39) from which the three dimensional Cartesian co-ordinates of the cushion's deformed surface may be derived.

### 3.2 Polynomial displacement functions

To allow for increased manipulation of the displacement functions beyond that available from trigonometric equations, $9^{t h}$ order polynomial equations may be developed. Candidate characteristic polynomials for $u_{1(\xi)}$ and $v_{1(\eta)}$ are:

$$
\begin{array}{r}
u_{1(\xi)}=A_{0}+A_{1}(\xi)+A_{2}(\xi)^{2}+A_{3}(\xi)^{3}+A_{4}(\xi)^{4}+A_{5}(\xi)^{5}+A_{6}(\xi)^{6}+A_{7}(\xi)^{7} \\
+A_{8841)} 88(\xi)^{8}+A_{9}(\xi)^{9} \\
\begin{array}{r}
v_{1(\eta)}=
\end{array} A_{0}+A_{1}(\eta)+A_{2}(\eta)^{2}+A_{3}(\eta)^{3}+A_{4}(\eta)^{4}+A_{5}(\eta)^{5}+A_{6}(\eta)^{6}+A_{7}(\eta)^{7} \\
+A_{8}(\eta)^{8}+A_{9}(\eta)^{9} \tag{42}
\end{array}
$$

with,

$$
\begin{array}{r}
\frac{d\left(u_{1}\right)}{d(\xi)}=A_{1}+2 A_{2}(\xi)+3 A_{3}(\xi)^{2}+4 A_{4}(\xi)^{3}+5 A_{5}(\xi)^{4}+6 A_{6}(\xi)^{5}+7 A_{7}(\xi)^{6} \\
+8 A_{8}(\xi)^{7}+9 A_{9}(\xi)^{8}
\end{array}
$$

$$
\begin{array}{r}
\frac{d\left(v_{1}\right)}{d(\eta)}=A_{1}+2 A_{2}(\eta)+3 A_{3}(\eta)^{2}+4 A_{4}(\eta)^{3}+5 A_{5}(\eta)^{4}+6 A_{6}(\eta)^{5}+7 A_{7}(\eta)^{6} \\
+8 A_{8}(\eta)^{7}+9 A_{9}(\eta)^{8} \tag{43}
\end{array}
$$

Essential boundary conditions are defined in Table 1. Initially the boundary conditions are set to exactly reproduce those prescribed by the trigonometric equations. However, these boundary conditions may be redefined on the basis of experimental data to produce alternative representations of the membrane deformation.
The coefficients $A_{0}-A_{9}$ are obtained by solving the expressions described in Table 1 , in this case, using Gauss elimination. Substituting the results for $A_{0}-A_{9}$ into (42), the functions for $u_{1(\xi)}$ and $v_{1(\eta)}$, are therefore defined as,

$$
\begin{aligned}
u_{1}(\xi)=-6.280 \xi+0.120 \xi^{2}+39.375 \xi^{3} & +10.507 \xi^{4}-105.351 \xi^{5}+11.947 \xi^{6} \\
& +124.018 \xi^{7}-95.573 \xi^{8}+21.239 \xi^{9}
\end{aligned}
$$

$$
\begin{array}{r}
v_{1}(\eta)=-6.280 \eta+0.120 \eta^{2}+39.375 \eta^{3}+10.507 \eta^{4}-105.351 \eta^{5}+11.947 \eta^{6} \\
+124.018 \eta^{7}-95.573 \eta^{8}+21.239 \eta^{9} \tag{44}
\end{array}
$$

The functions for $u_{1(\eta)}, v_{1(\xi)}, w_{1(\xi)}$ and $w_{1(\eta)}$ are developed in a similar fashion to $u_{1(\xi)}$ and $v_{1(\eta)}$, based on the functions given in (45) and (46) and the boundary conditions in Table 2. The resulting expressions are given in (47).

$$
\begin{aligned}
u_{1(\eta)}=w_{1(\eta)}=B_{0}+B_{1}(\eta)+B_{2}(\eta)^{2}+B_{3}(\eta)^{3}+ & B_{4}(\eta)^{4}+B_{5}(\eta)^{5}+B_{6}(\eta)^{6} \\
& +B_{7}(\eta)^{7}+B_{8}(\eta)^{8}+B_{9}(\eta)^{9}
\end{aligned}
$$

$$
v_{1(\xi)}=w_{1(\xi)}=B_{0}+B_{1}(\xi)+B_{2}(\xi)^{2}+B_{3}(\xi)^{3}+B_{4}(\xi)^{4}+B_{5}(\xi)^{5}+B_{6}(\xi)^{6}
$$

$$
\begin{equation*}
+B_{7}(\xi)^{7}+B_{8}(\xi)^{8}+B_{9}(\xi)^{9} \tag{45}
\end{equation*}
$$

$$
\begin{aligned}
\frac{d u_{1(\eta)}}{d \eta}=\frac{d w_{1(\eta)}}{d \eta}=B_{1}+2 B_{2}(\eta)+3 B_{3}(\eta)^{2}+ & 4 B_{4}(\eta)^{3}+5 B_{5}(\eta)^{4}+6 B_{6}(\eta)^{5} \\
& +7 B_{7}(\eta)^{6}+8 B_{8}(\eta)^{7}+9 B_{9}(\eta)^{8}
\end{aligned}
$$

$$
\frac{d v_{1(\xi)}}{d \xi}=\frac{d w_{1(\xi)}}{d \xi}=B_{1}+2 B_{2}(\xi)+3 B_{3}(\xi)^{2}+4 B_{4}(\xi)^{3}+5 B_{5}(\xi)^{4}+6 B_{6}(\xi)^{5}
$$

$$
\begin{equation*}
+7 B_{7}(\xi)^{6}+8 B_{8}(\xi)^{7}+9 B_{9}(\xi)^{8} \tag{46}
\end{equation*}
$$

Table 1: Initial boundary conditions for the derivation of $\mathbf{u}_{1(x / a)}$ and $\mathbf{v}_{1(y / a)}$

| $\xi, \eta$ | $u_{1(\xi)}, v_{1(\eta)}$ |  |
| :---: | :---: | :---: |
| 0 | 0 | $\begin{aligned} & =A_{0}+A_{1}(0)+A_{2}(0)^{2}+A_{3}(0)^{3}+A_{4}(0)^{4}+A_{5}(0)^{5}+ \\ & A_{6}(0)^{6}+A_{7}(0)^{7}+A_{8}(0)^{8}+A_{9}(0)^{9} \end{aligned}$ |
| 0.25 | -1 | $\begin{aligned} & =A_{0}+A_{1}(0.25)+A_{2}(0.25)^{2}+A_{3}(0.25)^{3}+ \\ & A_{4}(0.25)^{4}+A_{5}(0.25)^{5}+A_{6}(0.25)^{6}+A_{7}(0.25)^{7}+ \\ & A_{8}(0.25)^{8}+A_{9}(0.25)^{9} \end{aligned}$ |
| 0.5 | 0 | $\begin{aligned} & =A_{0}+A_{1}(0.5)+A_{2}(0.5)^{2}+A_{3}(0.5)^{3}+A_{4}(0.5)^{4}+ \\ & A_{5}(0.5)^{5}+A_{6}(0.5)^{6}+A_{7}(0.5)^{7}+A_{8}(0.5)^{8}+ \\ & A_{9}(0.5)^{9} \end{aligned}$ |
| 0.75 | 1 | $\begin{aligned} & =A_{0}+A_{1}(0.75)+A_{2}(0.75)^{2}+A_{3}(0.75)^{3}+ \\ & A_{4}(0.75)^{4}+A_{5}(0.75)^{5}+A_{6}(0.75)^{6}+A_{7}(0.75)^{7}+ \\ & A_{8}(0.75)^{8}+A_{9}(0.75)^{9} \end{aligned}$ |
| 1 | 0 | $\begin{aligned} & =A_{0}+A_{1}(1)+A_{2}(1)^{2}+A_{3}(1)^{3}+A_{4}(1)^{4}+A_{5}(1)^{5}+ \\ & A_{6}(1)^{6}+A_{7}(1)^{7}+A_{8}(1)^{8}+A_{9}(1)^{9} \end{aligned}$ |
|  | $\frac{d\left(u_{1}(\xi)\right)}{d(\xi)}, \frac{d\left(v_{1}(\eta)\right)}{d(\eta)}$ |  |
| 0 | $6.28$ | $\begin{aligned} & =A_{1}+2 A_{2}(0)+3 A_{3}(0)^{2}+4 A_{4}(0)^{3}+5 A_{5}(0)^{4}+ \\ & 6 A_{6}(0)^{5}+7 A_{7}(0)^{6}+8 A_{8}(0)^{7}+9 A_{9}(0)^{8} \end{aligned}$ |
| 0.25 | 0 | $\begin{array}{ll} =A_{1}+2 A_{2}(0.25)+3 A_{3}(0.25)^{2}+4 A_{4}(0.25)^{3} & + \\ 5 A_{5}(0.25)^{4} & + \\ 7 A_{7}(0.25)^{6}+8 A_{8}(0.25)^{7}+9 A_{9}(0.25)^{8} & \\ \hline \end{array}$ |
| 0.5 | 6.28 | $\begin{aligned} & =A_{1}+2 A_{2}(0.5)+3 A_{3}(0.5)^{2}+4 A_{4}(0.5)^{3}+ \\ & 5 A_{5}(0.5)^{4}+6 A_{6}(0.5)^{5}+7 A_{7}(0.5)^{6}+8 A_{8}(0.5)^{7}+ \\ & 9 A_{9}(0.5)^{8} \end{aligned}$ |
| 0.75 | 0 | $\begin{array}{lll} =A_{1}+2 A_{2}(0.75)+3 A_{3}(0.75)^{2}+4 A_{4}(0.75)^{3} & + \\ 5 A_{5}(0.75)^{4} & + & 6 A_{6}(0.75)^{5} \\ 7 A_{7}(0.75)^{6}+8 A_{8}(0.75)^{7}+9 A_{9}(0.75)^{8} & + \\ \hline \end{array}$ |
| 1 | $6.28$ | $\begin{aligned} & =A_{1}+2 A_{2}(1)+3 A_{3}(1)^{2}+4 A_{4}(1)^{3}+5 A_{5}(1)^{4}+ \\ & 6 A_{6}(1)^{5}+7 A_{7}(1)^{6}+8 A_{8}(1)^{7}+9 A_{9}(1)^{8} \end{aligned}$ |

Table 2: Initial boundary conditions for the derivation of $u_{1(\eta)}, w_{1(\eta)}, v_{1(\xi)}$ and $w_{1(\xi)}$

| $\xi, \eta$ | $u_{1(\eta)}, w_{1(\eta)}, v_{1(\xi)}, w_{1(\xi)}$ |  |
| :---: | :---: | :---: |
| 0 | 0 | $\begin{aligned} & =B_{0}+B_{1}(0)+B_{2}(0)^{2}+B_{3}(0)^{3}+B_{4}(0)^{4}+B_{5}(0)^{5}+ \\ & B_{6}(0)^{6}+B_{7}(0)^{7}+B_{8}(0)^{8}+B_{9}(0)^{9} \end{aligned}$ |
| 0.25 | 0.707 | $\begin{aligned} & =B_{0}+B_{1}(0.25)+B_{2}(0.25)^{2}+B_{3}(0.25)^{3}+ \\ & B_{4}(0.25)^{4}+B_{5}(0.25)^{5}+B_{6}(0.25)^{6}+B_{7}(0.25)^{7}+ \\ & B_{8}(0.25)^{8}+B_{9}(0.25)^{9} \end{aligned}$ |
| 0.5 | 1 | $\begin{aligned} & =B_{0}+B_{1}(0.5)+B_{2}(0.5)^{2}+B_{3}(0.5)^{3}+B_{4}(0.5)^{4}+ \\ & B_{5}(0.5)^{5}+B_{6}(0.5)^{6}+B_{7}(0.5)^{7}+B_{8}(0.5)^{8}+ \\ & B_{9}(0.5)^{9} \end{aligned}$ |
| 0.75 | 0.707 | $\begin{aligned} & =B_{0}+B_{1}(0.75)+B_{2}(0.75)^{2}+B_{3}(0.75)^{3}+ \\ & B_{4}(0.75)^{4}+B_{5}(0.75)^{5}+B_{6}(0.75)^{6}+B_{7}(0.75)^{7}+ \\ & B_{8}(0.75)^{8}+B_{9}(0.75)^{9} \end{aligned}$ |
| 1 | 0 | $\begin{aligned} & =B_{0}+B_{1}(1)+B_{2}(1)^{2}+B_{3}(1)^{3}+B_{4}(1)^{4}+B_{5}(1)^{5}+ \\ & B_{6}(1)^{6}+B_{7}(1)^{7}+B_{8}(1)^{8}+B_{9}(1)^{9} \end{aligned}$ |
|  | $\frac{d u_{1(\eta)}}{d \eta}, \frac{d w_{1(\eta)}}{d \eta}, \frac{d v_{1(\xi)}}{d \xi}, \frac{d w_{1(\xi)}}{d \xi}$ |  |
| 0 | 3.14 | $\begin{aligned} & =B_{1}+2 B_{2}(0)+3 B_{3}(0)^{2}+4 B_{4}(0)^{3}+5 B_{5}(0)^{4}+ \\ & 6 B_{6}(0)^{5}+7 B_{76} \\ & 9 B_{9}(0)^{8} \end{aligned}$ |
| 0.25 | 2.22 | $\begin{array}{lcc} =B_{1}+2 B_{2}(0.25)+3 B_{3}(0.25)^{2}+4 B_{4}(0.25)^{3} & + \\ 5 B_{5}(0.25)^{4} & + & 6 B_{6}(0.25)^{5} \\ 7 B_{76} & + \\ 9 B_{9}(0.25)^{8} & (0.25)^{6}+8 B_{8}(0.25)^{7} & + \\ \hline \end{array}$ |
| 0.5 | 0 | $\begin{array}{lccc} = & B_{1} & + & 2 B_{2}(0.5)+3 B_{3}(0.5)^{2}+4 B_{4}(0.5)^{3} \\ 5 B_{5}(0.5)^{4} & + & 6 B_{6}(0.5)^{5} & + \\ 7 B_{76} & + & (0.5)^{6}+8 B_{8}(0.5)^{7} & + \\ 9 B_{9}(0.5)^{8} & & & \\ \hline \end{array}$ |
| 0.75 | $2.22$ | $\begin{array}{lcc} =B_{1}+2 B_{2}(0.75)+3 B_{3}(0.75)^{2}+4 B_{4}(0.75)^{3} & + \\ 5 B_{5}(0.75)^{4} & + & 6 B_{6}(0.75)^{5} \\ 7 B_{76} & + \\ 9 B_{9}(0.75)^{8} & (0.75)^{6}+8 B_{8}(0.75)^{7} & + \\ \hline \end{array}$ |
| 1 | $3.14$ | $\begin{aligned} & =B_{1}+2 B_{2}(1)+3 B_{3}(1)^{2}+4 B_{4}(1)^{3}+5 B_{5}(1)^{4}+ \\ & 6 B_{6}(1)^{5}+7 B_{76} \\ & 9 B_{9}(1)^{8} \end{aligned}$ |

$$
\begin{align*}
u_{1(\eta)}=w_{1(\eta)}=3.142 \eta-5.171 \eta^{3}+0.024 \eta^{4}+2.458 \eta^{5} & +0.206 \eta^{6} \\
& -0.879 \eta^{7}+0.220 \eta^{8} \\
v_{1(\xi)}=w_{1(\xi)}=3.142 \xi-5.171 \xi^{3}+0.024 \xi^{4}+2.458 \xi^{5}+ & 0.206 \xi^{6} \\
- & 0.879 \xi^{7}+0.220 \xi^{8} \tag{47}
\end{align*}
$$

The functions for $u_{1(\xi)}$ and $u_{1(\eta)}$ are plotted in Figure 7 alongside their trigonometric counterparts. The polynomial functions are shown to successfully reproduce the results of the trigonometric function having the same boundary conditions, and therefore validating the use of $9^{t h}$ order polynomials. However, the polynomial is considerably more versatile in the present context.


Figure 7: Graphical representation of polynomial and trigonometric functions for $u_{1(\xi)}$ and $u_{1(\eta)}$

Substituting (44) and (47) into (38) yields the final form of the polynomial displacement functions $u_{1}, v_{1}$ and $w_{1}$.

$$
\begin{aligned}
& u_{1}=\left(-6.280 \xi+0.120 \xi^{2}+39.375 \xi^{3}+10.507 \xi^{4}-105.351 \xi^{5}\right. \\
& \left.+11.947 \xi \xi^{6}+124.018 \xi^{7}-95.573 \xi^{8}+21.239 \xi^{9}\right) \\
& \times\left(3.142 \eta-5.171 \eta^{3}+0.024 \eta^{4}+2.458 \eta^{5}+0.206 \eta^{6}-0.879 \eta^{7}+0.220 \eta^{8}\right) \\
& \begin{array}{r}
v_{1}=\left(3.142 \xi-5.171 \xi^{3}+0.024 \xi^{4}+2.458 \xi^{5}+0.206 \xi^{6}-0.879 \xi^{7}+0.220 \xi^{8}\right) \\
\times\left(-6.280 \eta+0.120 \eta^{2}+39.375 \eta^{3}+10.507 \eta^{4}-105.351 \eta^{5}+11.947 \eta^{6}\right. \\
\\
\left.+124.018 \eta^{7}-95.573 \eta^{8}+21.239 \eta^{9}\right)
\end{array}
\end{aligned}
$$

$$
\begin{align*}
& w_{1}=\left(3.142 \xi-5.171 \xi^{3}+0.024 \xi^{4}+2.458 \xi^{5}+0.206 \xi^{6}-0.879 \xi^{7}+0.220 \xi^{8}\right) \\
& \times\left(3.142 \eta-5.171 \eta^{3}+0.024 \eta^{4}+2.458 \eta^{5}+0.206 \eta^{6}-0.879 \eta^{7}+0.220 \eta^{8}\right) \tag{48}
\end{align*}
$$

The partial differentiation of $u_{1}, v_{1}$ and $w_{1}$ yields,

$$
\begin{aligned}
& u_{1 \xi}=\left(-6.280+0.240 \xi+118.125 \xi^{2}+42.028 \xi^{3}-526.755 \xi^{4}+71.682 \xi^{5}\right. \\
& \left.+868.126 \xi^{6}-764.548 \xi^{7}+191.151 \xi^{8}\right) \\
& \times\left(3.142 \eta-5.171 \eta^{3}+0.024 \eta^{4}+2.458 \eta^{5}+0.206 \eta^{6}-0.879 \eta^{7}+0.220 \eta^{8}\right) \\
& u_{1 \eta}=\left(-6.280 \xi+0.120 \xi^{2}+39.375 \xi^{3}+10.507 \xi^{4}-105.351 \xi^{5}+11.947 \xi^{6}\right. \\
& \left.+124.018 \xi^{7}-95.573 \xi^{8}+21.239 \xi^{9}\right) \\
& \times\left(3.142-15.513 \eta^{2}+0.096 \eta^{3}+2.458 \eta^{4}+0.206 \eta^{5}-0.879 \eta^{6}+0.220 \eta^{7}\right)
\end{aligned}
$$

$$
v_{1 \xi}=\left(3.142-15.513 \xi^{2}+0.096 \xi^{3}+2.458 \xi^{4}+0.206 \xi^{5}-0.879 \xi^{6}+0.220 \xi^{7}\right)
$$

$$
\times\left(-6.280 \eta+0.120 \eta^{2}+39.375 \eta^{3}+10.507 \eta^{4}-105.351 \eta^{5}+11.947 \eta^{6}\right.
$$

$$
\left.+124.018 \eta^{7}-95.573 \eta^{8}+21.239 \eta^{9}\right)
$$

$$
v_{1 \eta}=\left(3.142 \xi-5.171 \xi^{3}+0.024 \xi^{4}+2.458 \xi^{5}+0.206 \xi^{6}-0.879 \xi^{7}+0.220 \xi^{8}\right)
$$

$$
\times\left(-6.280+0.240 \eta+118.125 \eta^{2}+42.028 \eta^{3}-526.755 \eta^{4}+71.682 \eta^{5}\right.
$$

$$
\left.+868.126 \eta^{6}-764.548 \eta^{7}+191.151 \eta^{8}\right)
$$

$$
\begin{gathered}
w_{1 \xi}=\left(3.142-15.513 \xi^{2}+0.096 \xi^{3}+2.458 \xi^{4}+0.206 \xi^{5}-0.879 \xi^{6}+0.220 \xi^{7}\right) \\
\times\left(3.142 \eta-5.171 \eta^{3}+0.024 \eta^{4}+2.458 \eta^{5}+0.206 \eta^{6}-0.879 \eta^{7}+0.220 \eta^{8}\right)
\end{gathered}
$$

$$
w_{1 \eta}=\left(3.142 \xi-5.171 \xi^{3}+0.024 \xi^{4}+2.458 \xi^{5}+0.206 \xi^{6}-0.879 \xi^{7}+0.220 \xi^{8}\right)
$$

$$
\begin{equation*}
\times\left(3.142-15.513 \eta^{2}+0.096 \eta^{3}+2.458 \eta^{4}+0.206 \eta^{5}-0.879 \eta^{6}+0.220 \eta^{7}\right) \tag{49}
\end{equation*}
$$

Substituting these expressions into (31), $\lambda_{i}, i=1 \rightarrow 11$, are defined as,
$\lambda_{1}=30.431 v+91.278 ; ; \lambda_{2}=9.868 v-16.446 ; ; \lambda_{3}=3.700 ; ; \lambda_{4}=-1.333 ; ;$
$\lambda_{5}=206.896 v+584.913 ; ; \lambda_{6}=39.465 v-13.124 ; ; \lambda_{7}=-0.690 v+23.975 ; ;$
$\lambda_{8}=1.850 ; ; \lambda_{9}=1.333 ; ; \lambda_{10}=30.440 ; ; \lambda_{11}=0.405$
From (29) with (50), the following simultaneous equations are obtained in terms of $\frac{C_{w}}{a}$ and $\frac{C_{1}}{a}$.

$$
\begin{aligned}
& \left(\frac{C_{w}}{a}\right)^{2}\left(\frac{C_{1}}{a}(30.431 v+91.278)+(9.868 v-16.446)\right) \\
& -2 \varkappa \frac{C_{w}}{a}\left(-\frac{C_{1}}{a} 3.700+1.333\right)+\left(\frac{C_{1}}{a}\right)^{3}(206.896 v+584.913) \\
& \quad+3\left(\frac{C_{1}}{a}\right)^{2}(39.465 v-13.124)+2 \frac{C_{1}}{a}(-0.690 v+23.975)=0
\end{aligned}
$$

$$
\left(\frac{C_{1}}{a}\right)^{2}\left(30.431 v+91.278-\frac{\varkappa a}{C_{w}} 3.700\right)+2 \frac{C_{1}}{a}\left(9.868 v-16.446-\frac{\varkappa a}{C_{w}} 1.333\right)
$$

$$
\begin{equation*}
+\left(\frac{C_{w}}{a}\right)^{2} 30.440-\frac{\varkappa a}{C_{w}} 0.811=0 \tag{51}
\end{equation*}
$$

## 4 Calculation of the theoretical surface

The elastic modulus of ETFE $(D)$ is determined through testing to be approximately $190.6 \mathrm{~kg} / \mathrm{cm}$. Non SI units are used throughout the formulation in order to maintain continuity. However, all final answers are converted to SI units retrospectively. It should also be noted that up to first yield Kappus's Law exhibits a good fit with the stress-strain relationship of ETFE foil [2]. It is anticipated that the membrane will not yield prior to internal pressure beyond 800 Pa and therefore Kappus's Law will remain a valid load strain relationship for the purposes of this formulation. $D$, along with a Poisson's ratio $(v)$ of approximately 0.333 , a cushion side length (a) of 100 cm and internal pressures ( $p$ ) Pressure units are converted from Pa to atmospheres for the calculation at increments of 100 Pa are used to derive the values of $\varkappa$ from (30), summarised in Table 3. For the values of $\varkappa$ the pairs of cubic equations defined for each increment of pressure may then be derived and solved simultaneously in order to find the corresponding values of $C_{1}$ and $C_{w}$ (Table 3).

The trigonometric values calculated using (41) shown alongside the equivalent polynomial values calculated using (51), are clearly equivalent, reflecting identical boundary conditions and the ability of the polynomial to match the trigonometric function (see Figure 7) for different levels of geometric non-linearity. These values of $C_{1}$ and $C_{w}$ may now be used to calculate the surface coordinates of the cushion
for pressures in the range $0-800 \mathrm{~Pa}$ at 100 Pa intervals. (Equivalent values for two example rectangular cushions are provided in Appendix A).
Assuming symmetry of the cushion, 3 dimensional Cartesian co-ordinates for the entire inflated cushion may be found from one quarter. A grid of points 10 cm by 10 cm divides a quarter of a 100 cm square pillow. The displaced surface coordinates of each point have been derived for the inflated cushion at each pressure increment shown in Table 3. A graphical representation of the resulting surfaces is provided in Figure 8.

Table 3: Final Values for $\mathrm{C}_{1}$ and $\mathrm{C}_{w}$ using trigonometric and polynomial displacement functions

|  | $p$ <br> $(\mathrm{~Pa})$ | 100 | 200 | 300 | 400 | 500 | 600 | 700 | 800 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | $\varkappa$ | 0.0005 | 0.0009 | 0.0014 | 0.0018 | 0.0023 | 0.0028 | 0.0032 | 0.0037 |
| Trigonometric | $C_{w}$ | 2.527 | 3.184 | 3.646 | 4.013 | 4.324 | 4.595 | 4.838 | 5.059 |
|  | $C_{1}$ | 0.018 | 0.028 | 0.037 | 0.045 | 0.052 | 0.059 | 0.065 | 0.072 |
| Polynomial | $C_{w}$ | 2.527 | 3.184 | 3.646 | 4.013 | 4.324 | 4.595 | 4.838 | 5.059 |
|  | $C_{1 w}$ | 0.018 | 0.028 | 0.037 | 0.045 | 0.052 | 0.059 | 0.065 | 0.072 |



Figure 8: Graphical representation of theoretical inflated cushion surfaces

The principle section loads and their directions for one quarter of a cushion with an internal pressure of 800 Pa obtained from (35) and (36) are illustrated in Figure 9. It may be noted that the values are identical for both the trigonometric and polynomial functions as the boundary conditions of the polynomial functions have been selected such that the displacement functions match.

Areas of negative (compressive) stress in the corners of the cushion can be identified (Figure 9). This may indicate that the approximation is invalid for these zones [6]. However, it has been proposed that in the case of un-patterned square cushions with insufficient internal pressure, anticlastic surfaces naturally form in the corners leading to compressive stresses and therefore wrinkles in these areas [3].


Figure 9: Principle sectional loads and directions of a cushion at a pressure of 800 Pa

## 5 Comparison of theoretical surface with a finite element model

### 5.1 FE model produced using GSA

To provide a comparison for the theoretical surfaces and stress distribution presented in the preceding sections an equivalent un-patterned 1 metre square cushion was analysed using a geometrically non-linear finite element model produced in Oasys-GSA [7]. The model was discretised using 0.02 m square 2D Quad4 elements with properties equivalent to those used in the formulation, (Table 4).

Table 4: GSA Fabric Properties

| Elastic modulus | $186.3 \mathrm{kN} / \mathrm{m}$ |
| :--- | :--- |
| Poisson's ratio | 0.333 |
| Shear modulus | $69.86 \mathrm{kN} / \mathrm{m}$ |

### 5.2 GSA model translations

Simulation of the cushion inflated to 800 Pa is selected for detailed comparison. Points on the original grid used for calculation of the displacements are displayed to the right of each graphic produced in GSA (left).


Figure 10: 2D element translation returned by GSA model with selected points for comparison.

The Z-translations (w), along the centre line of the cushion, predicted by the finite element simulation are compared with the translations computed using the trigonometric function (in polynomial form as described in Table 3 and denoted here "trigonometric equivalent") (Figure 11). Given a converged mesh and the geometrically non-linear kinematics of the model used in GSA, it is assumed here that the finite element model (GSA) accurately predicts the equilibrated geometry and stresses in the cushion, and as such that the GSA solution represents target values to be matched by the closed-form predictions. Based on this principle it is clear that the prediction of the z-co-ordinate translation by the trigonometric equivalent polynomial requires improvement.


Figure 11: Centre line comparison of theoretical and GSA surfaces along centre lines of cushion

### 5.3 Development of polynomial basic displacement function

The magnitude of displacement predicted by the theoretical results is controlled by the values of $C_{1}$ and $C_{w}$. The magnitude of $C_{1}$ and $C_{w}$ is dependent upon the partial derivatives of the basic displacement functions. The basic displacement functions also control the predicted shape of the inflated cushion. It is therefore the basic displacement functions which may be adjusted in order to refine the formulation.
The GSA results may be used to define a new set of boundary conditions for the development of improved displacement functions. The GSA Z-translations normal to the centre line of the cushion in the x -direction at a pressure of 800 Pa are normalised. Using these values the boundary conditions detailed in Table 5 are established and used to produce the polynomials shown below Table 5. It should be noted that the first derivative boundary conditions used to reproduce the original trigonometric equivalent polynomial displacement functions have been omitted in order to avoid issues caused by over constraining the equation. The displacement function therefore becomes a $4^{\text {th }}$ order polynomial.

Table 5: Boundary conditions based on finite element simulation data for the development of $u_{1(\eta)}, w_{1(\eta)}, v_{1(\xi)}$ and $w_{1(\xi)}$

| $\xi, \eta$ | $u_{1(\eta)}, w_{1(\eta)}, v_{1(\xi)}, w_{1(\xi)}$ |  |
| :--- | :--- | :--- |
| 0 | 0 | $=B_{0}+B_{1}(0)+B_{2}(0)^{2}+B_{3}(0)^{3}+B_{4}(0)^{4}$ |
| 0.25 | 0.76 | $=B_{0}+B_{1}(0.25)+B_{2}(0.25)^{2}+B_{3}(0.25)^{3}+B_{4}(0.25)^{4}$ |
| 0.5 | 1 | $=B_{0}+B_{1}(0.5)+B_{2}(0.5)^{2}+B_{3}(0.5)^{3}+B_{4}(0.5)^{4}$ |
| 0.75 | 0.76 | $=B_{0}+B_{1}(0.75)+B_{2}(0.75)^{2}+B_{3}(0.75)^{3}+B_{4}(0.75)^{4}$ |
| 1 | 0 | $=B_{0}+B_{1}(1)+B_{2}(1)^{2}+B_{3}(1)^{3}+B_{4}(1)^{4}$ |

$$
\begin{align*}
& u_{1(\eta)}=w_{1(\eta)}=4.213 \eta-5.067 \eta^{2}+1.707 \eta^{3}-0.853 \eta^{4} \\
& v_{1(\xi)}=w_{1(\xi)}=4.213 \xi-5.067 \xi^{2}+1.707 \xi^{3}-0.853 \xi^{4} \tag{52}
\end{align*}
$$

By combining (57) with the original polynomial functions for $u_{1(\xi)}$ and $v_{1(\eta)}$ in (44) the new functions for $u_{1}, v_{1}$ and $w_{1}$ are found,

$$
\begin{aligned}
& u_{1}=\left(-6.280 \xi+0.120 \xi^{2}+39.375 \xi^{3}+10.507 \xi^{4}-105.351 \xi^{5}+11.947 \xi^{6}\right. \\
& \left.+124.018 \xi^{7}-95.573 \xi^{8}+21.239 \xi^{9}\right) \\
& \times\left(4.213 \eta-5.067 \eta^{2}+1.707 \eta^{3}-0.853 \eta^{4}\right)
\end{aligned}
$$

$$
v_{1}=\left(4.213 \xi-5.067 \xi^{2}+1.707 \xi^{3}-0.853 \xi^{4}\right)
$$

$$
\times\left(-6.280 \eta+0.120 \eta^{2}+39.375 \eta^{3}+10.507 \eta^{4}-105.351 \eta^{5}\right.
$$

$$
\left.+11.947 \eta^{6}+124.018 \eta^{7}-95.573 \eta^{8}+21.239 \eta^{9}\right)
$$

$$
w_{1}=\left(4.213 \xi-5.067 \xi^{2}+1.707 \xi^{3}-0.853 \xi^{4}\right)
$$

$$
\begin{equation*}
\times\left(4.213 \eta-5.067 \eta^{2}+1.707 \eta^{3}-0.853 \eta^{4}\right) \tag{53}
\end{equation*}
$$

The new basic displacement functions (denoted here "improved polynomial") are then used in the usual manner to calculate a new theoretical surface. The functions
of (59) yield the following pair of quadratic equations,

$$
\begin{align*}
& \left(\frac{C_{w}}{a}\right)^{2}\left(\frac{1}{a}(28.681 v+139.201) C_{1}+9.936 v-243.451\right) \\
& -2 \varkappa \frac{C_{w}}{a}\left(\frac{C_{1}}{a} 3.554+1.360\right)+\left(\frac{C_{1}}{a}\right)^{3}(220.738 v+693.702) \\
& \quad+\left(\frac{C_{1}}{a}\right)^{2} 3(40.908 v-19.020)+\frac{C_{1}}{a} 2(-1.142 v+25.668)=0 \\
& \left(\frac{C_{1}}{a}\right)^{2}\left(28.681 v+139.201-\frac{\varkappa a}{C_{w}} 3.554\right) \\
& \quad+\frac{C_{1}}{a} 2\left(9.936 v-23.451-\frac{\varkappa a}{C_{w}} 1.360\right)+\left(\frac{C_{w}}{a}\right)^{2} 49.560-\frac{\varkappa a}{C_{w}} 0.908=0 \tag{54}
\end{align*}
$$

Table 6: Values for $\mathrm{C}_{1}$ and $\mathrm{C}_{w}$ using trigonometric equivalent and improved polynomial displacement functions

|  | $p(\mathrm{~Pa})$ | 100 | 200 | 300 | 400 | 500 | 600 | 700 | 800 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | $\varkappa$ | 0.0005 | 0.0009 | 0.0014 | 0.0018 | 0.0023 | 0.0028 | 0.0032 | 0.0037 |
| Trigonometric <br> equivalent | $C_{w}$ | 2.527 | 3.184 | 3.646 | 4.013 | 4.324 | 4.595 | 4.838 | 5.059 |
|  | $C_{1}$ | 0.018 | 0.028 | 0.037 | 0.045 | 0.052 | 0.059 | 0.065 | 0.072 |
| Improved <br> polynomial | $C_{w}$ | 2.319 | 2.923 | 3.346 | 3.683 | 3.968 | 4.217 | 4.440 | 4.643 |
|  | $C_{1}$ | 0.021 | 0.034 | 0.045 | 0.054 | 0.063 | 0.071 | 0.079 | 0.086 |

### 5.4 Comparison of translation results

The improved polynomial Z-translations are compared to the trigonometric equivalent and GSA results and found to be an improved fit along the centre line of the cushion, (Figure 12).

The improved polynomial produces a better fit with the X translations, along the $\mathrm{x}=\mathrm{y}$ line, (Figure 13a). However along the $\mathrm{y}=500$ centre line the improvement is less pronounced, (Figure 13b). This is due to the fact that the shape of deformation along the centre line is controlled by the polynomial functions for $u_{1(\xi)}$ and $v_{1(\eta)}$ which have not been altered. It is therefore only the increased value of $C_{1}$ that improves the fit with the GSA results. The same pattern of error is mirrored by the $y$-translations due to the symmetry of the cushion.


Figure 12: Centre line comparison of 'improved' theoretical and GSA surfaces along centre lines of cushion


Figure 13: Comparison of X-translation along key lines (a) $X=Y$, (b) $Y=500$

To further investigate the improved polynomial formulation the absolute and percentage errors compared to selected GSA results for both the trigonometric equivalent and improved polynomial formulations are presented, (Figure 14a,b). The error is greatly decreased in the central zone of the cushion. However the absolute error and to a greater degree the percentage error increases towards the edges of the cushion, particularly in the corners, (Figure 14b). The maximum absolute error in the z-translation produced by the improved polynomial formulation is -3.62 mm
(0r 36\%) at (100,100). Increased errors are also exhibited by the lateral X and Y-translations towards the edges.


Figure 14: Comparison of absolute and percentage error in Z-translation (a) Trigonometric equivalent, (b) Improved polynomial

### 5.5 GSA model principle stress results

The GSA 2D element derived principle forces (Nmax and Nmin) at 800Pa are presented in a similar manner to the translations with corresponding symmetric stress values provided by the trigonometric equivalent and improved polynomial models (Figure 16a-c).

When considering the stress plots, the trigonometric equivalent polynomial produces considerably smaller errors in the area of maximum stress, $(0,500),+9.2 \%$ compared to $+85.0 \%$ given by the improved polynomial. However, the trigonometric equivalent produces some larger negative errors in the area of minimum stress towards the corner of the cushion, with $-88.9 \%$ compared to $-60.0 \%$ at $(0,100)$ and$58.63 \%$ compared to $0.39 \%$ at $(0,200)$, (Figure 16a). This pattern is not reproduced along the diagonal, (Figure 16b). At $(200,200)$ the trigonometric equivalent is in


| $x / y$ | 0 | 100 | 200 | 300 | 400 | 500 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0.007 | 0.617 | 1.054 | 1.358 | 1.530 | 1.582 |
| 100 | 0.662 | 0.786 | 1.056 | 1.323 | 1.487 | 1.541 |
| 200 | 1.087 | 1.087 | 1.134 | 1.286 | 1.427 | 1.478 |
| 300 | 1.379 | 1.348 | 1.307 | 1.297 | 1.373 | 1.417 |
| 400 | 1.541 | 1.501 | 1.441 | 1.385 | 1.354 | 1.376 |
| 500 | 1.582 | 1.537 | 1.473 | 1.414 | 1.375 | 1.364 |
| $x / y$ | 0 | 100 | 200 | 300 | 400 | 500 |
| 0 | -0.002 | 0.194 | 0.364 | 0.468 | 0.524 | 0.531 |
| 100 | 0.172 | 0.366 | 0.558 | 0.645 | 0.686 | 0.675 |
| 200 | 0.347 | 0.527 | 0.791 | 0.913 | 0.940 | 0.919 |
| 300 | 0.455 | 0.616 | 0.886 | 1.111 | 1.175 | 1.156 |
| 400 | 0.515 | 0.661 | 0.913 | 1.154 | 1.306 | 1.314 |
| 500 | 0.539 | 0.697 | 0.945 | 1.177 | 1.324 | 1.363 |
|  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |

Figure 15: 2D element derived principle forces returned by GSA model with selected points for comparison
error by $-35.42 \%$ compared to $-55.76 \%$. It can be seen that in general, disregarding certain coincident points, the pattern of stress is more closely reproduced by the trigonometric equivalent function. This is especially true along the centre line of the cushion, (Figure 16c).
In the case of the minimum principle stress (Nmin) the trigonometric equivalent polynomial again produces significantly smaller errors in the area of maximum stress, $(0,500)$, with $+6.6 \%$ compared to $+78.7 \%$. Larger negative errors are also produced in the area of minimum stress towards the corner of the cushion, $-161.2 \%$ compared to $-134.3 \%$ at $(0,100)$ and $-86.47 \%$ compared to $-33.61 \%$ at $(0,200)$, (Figure 16a). It is noteworthy that both polynomial formulations give much larger areas of negative minimum stress in the corners than that given by the GSA model. Both extend over 100 mm from the cushion edges whereas the GSA model only gives a negative area extending just 20 mm from the cushion edges. This leads to very large indicated percentage errors.


Figure 16: Comparison of absolute and percentage error $\operatorname{Nmax}\left(\mathrm{n}_{1}\right)$ and $\operatorname{Nmin}\left(\mathrm{n}_{2}\right)$ (a) Edge stresses $(\mathrm{X}=0)$, (b) Stresses across the diagonal ( $\mathrm{X}=\mathrm{Y}$ ), (c) Centreline stresses ( $\mathrm{Y}=500$ )

A comparison of predicted z-translation and maximum and minimum principle stresses indicate that predicting the z-translations accurately with an analytical model does not necessarily result in accurate predictions of the stress in the cushion foil. For example, the "improved polynomial" performed less well the then trigonometric equivalent when comparing principle stresses with the finite element model.

## 6 Comparison of theoretical surface with two approximations reported in [3]

The approximated equation below has been proposed to describe the surface of an inflated membrane on a square base with a distributed membrane force $n$ and internal pressure $p$ [3], see Figure 17.
$z(x, y)=\frac{5 p}{8 n} \times \frac{1}{r_{x}^{2}+r_{y}^{2}}\left(r_{x}^{2}-x^{2}\right)\left(r_{y}^{2}-y^{2}\right)$


Figure 17: Square cushion layout for approximated equation (52) [3]

The equation for the distributed membrane force $n$ is derived from the second derivative of (55) and relies on the assumption that maximum stress occurs at the
summit of the cushion where, $r_{x}=r_{y}, \kappa_{x}=\kappa_{y}, R_{x}=R_{y}$ and $x=y=0$.
$\frac{\partial^{2} z}{\partial x^{2}}=\frac{5 p}{8 n} \times \frac{1}{2 r_{x}^{2}}\left(-2 r_{x}^{2}+2 y^{2}\right)=-\frac{5 P}{8 n}=\kappa=\frac{1}{R}$
Therefore at the summit,
$n=\frac{5}{8} \times p \times R$
This equation for the maximum distributed membrane force requires the input of a design height $H$ in order to calculate the curvature $\kappa$ and hence radius of curvature $R$ of the inflated membrane. The radius of curvature for a membrane with a square base in terms of the design height $H$ and side length $r$ is given as,
$R=\frac{H^{2}+r^{2}}{2 H}$
The approximated equation (52) therefore becomes the following,
$z(x, y)=\frac{2 H}{H^{2}+r^{2}} \times \frac{1}{r_{x}^{2}+r_{y}^{2}}\left(r_{x}^{2}-x^{2}\right)\left(r_{y}^{2}-y^{2}\right)$
Equation (56) relies solely on the selected design height and side length of the cushion and does not account for the mechanical properties of the membrane or the internal pressure.
An alternative to (56) is a more straightforward parabolic approximation which again relies solely on the selected design height [3].
$z(x, y)=H\left(1-\left(\frac{x}{r_{x}}\right)^{2}\right)\left(1-\left(\frac{y}{r_{y}}\right)^{2}\right)$
The geometric formulations described in (59) and (60) are compared with the Ztranslation predicted by the equilibrium formulation using the trigonometric equivalent polynomial and a pressure of 800 Pa in Figure $18 \mathrm{a} \& \mathrm{~b}$.
The trigonometric equivalent formulation and approximated surfaces are in reasonable agreement. Once again the main area of difference is found to be in the corners of the cushion with both approximated solutions giving a greater rise than the trigonometric equivalent formulation. However, as both approximated equations (59) and (60) rely solely on the selection of an accurate design height, which in a normal design situation would be unknown, they are less effective for the design of un-patterned cushions.


Figure 18: Comparison of trigonometric equivalent formulation with approximated surfaces (a) along the centreline $(\mathrm{Y}=500)$, (b) across the diagonal $(\mathrm{X}=\mathrm{Y})$

It should be noted that the derivation of (59) relies on the assumption that the maximum distributed membrane force, $n$, occurs at the summit of the cushion. However both the GSA model and the equilibrium formulations indicate that maximum stress occurs at the cushion's edges and not in the centre. According to (57) the maximum distributed membrane force is $1.26 \mathrm{~N} / \mathrm{mm}$ which is lower than the GSA models central stress of $1.36 \mathrm{~N} / \mathrm{mm}$ and lower still than its maximum stress of $1.58 \mathrm{~N} / \mathrm{mm}$. It may therefore, be concluded that the assumption of maximum stress at the summit of the cushion is incorrect.

## 7 Conclusions

A theoretical procedure for the design of un-patterned ETFE cushions has been developed. The procedure utilises polynomial functions with essential boundary conditions. The validity of the theoretical procedure has been investigated using a finite element model produced using GSA and analysed via GsRelax non-linear analysis. The original trigonometric equivalent polynomial created a theoretical surface which showed relatively poor agreement with the GSA model. However, through the use of boundary conditions developed from the model Z-translation output along the centre line an improved polynomial could be produced. The improved polynomial formulation showed a much greater level of agreement with the GSA model X, Y and Z-translations.
The force distributions, however, displayed mixed improvement. The trigonometric equivalent polynomial formulation showed the best agreement in areas of high stress, whereas the improved polynomial formulation showed slightly better agreement in some areas of lower stress. Both polynomial formulations predicted much larger areas of negative stress in the corners than the GSA model. Therefore, a better prediction of displaced geometry does not necessarily imply an accurate stress distribution.

The closed-form analysis has also been compared with two existing design procedures. The results were found to be similar despite the procedure being a complex analytical formulation based on continuum mechanics. However, the procedure developed in this paper has the advantage of not requiring the selection of a design height making it more appropriate for the design of un-patterned cushions. It has also been shown that the assumption of maximum of stress at the summit of the inflated cushion used in the derivation of one of the approximations is incorrect.

In conclusion, the theoretical procedure presented in this paper offers an effective approximate solution for the design of un-patterned ETFE cushions and removes the need for complex and computationally expensive finite element analysis at the early design stage.

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## Appendix A: Rectangular formulation

To provide a more general formulation the equations required for the analysis of a rectangular cushion are presented here. Displacements are approximated by;

$$
u(x, y)=C_{u} u_{1}(\xi, \eta)
$$

$$
v(x, y)=C_{v} v_{1}(\xi, \eta)
$$

$$
\begin{equation*}
w(x, y)=C_{w} w_{1}(\xi, \eta) \tag{61}
\end{equation*}
$$

As stated above, in order to produce the system of equations required for the derivation of $C_{u}, C_{v}$ and $C_{w}$ the partial differentiation of $\pi$ (26) with respect to $C_{u}, C_{v}$ and
$C_{w}$ must be undertaken.This yields the following system of equations;

$$
\begin{aligned}
& \frac{\partial \pi}{\partial C_{u}}=0 \\
& \begin{aligned}
=\frac{D}{b^{2}\left(2-2 v^{2}\right)} & \left(C_{u} \lambda_{1}+\frac{3 C_{u}{ }^{2}}{a} \lambda_{2}+\frac{C_{u}{ }^{3}}{a^{2}} \lambda_{3}+\frac{C_{v} b}{a} \lambda_{4}+\frac{C_{v}{ }^{2}}{a} \lambda_{5}+\frac{C_{w}{ }^{2}}{a} \lambda_{6}\right. \\
& \left.+\frac{2 C_{v} C_{u} b}{a^{2}} \lambda_{7}+\frac{C_{v}{ }^{2} C_{u}}{a^{2}} \lambda_{8}+\frac{C_{w}^{2} C_{u}}{a^{2}} \lambda_{9}\right)-\frac{P}{a}\left(\frac{C_{v} C_{w}}{b} \lambda_{10}+C_{w} \lambda_{11}\right)
\end{aligned}
\end{aligned}
$$

$$
\frac{\partial \pi}{\partial C_{v}}=0
$$

$$
=\frac{D}{a^{2}\left(2-2 v^{2}\right)}\left(C_{v} \lambda_{12}+\frac{3 C_{v}^{2}}{b} \lambda_{13}+\frac{C_{v}^{3}}{b^{2}} \lambda_{14}+\frac{C_{u} a}{b} \lambda_{4}+\frac{C_{u}^{2}}{b} \lambda_{15}+\frac{C_{w}^{2}}{b} \lambda_{16}\right.
$$

$$
\left.+\frac{2 C_{v} C_{u} a}{b^{2}} \lambda_{17}+\frac{C_{u}^{2} C_{v}}{b^{2}} \lambda_{8}+\frac{C_{w}^{2} C_{v}}{b^{2}} \lambda_{18}\right)-\frac{P}{b}\left(\frac{C_{v} C_{w}}{a} \lambda_{10}+C_{w} \lambda_{19}\right)
$$

$\frac{\partial \pi}{\partial C_{w}}=0$

$$
\begin{array}{r}
=\frac{D}{a^{2} b^{2}\left(2-2 v^{2}\right)}\left(C_{w}{ }^{3} \lambda_{\mathbf{2}}+2 C_{u} C_{w} a \lambda_{6}+C_{u}{ }^{2} C_{w} \lambda_{9}+2 C_{v} C_{w} b \lambda_{16}+C_{v}{ }^{2} C_{w} \lambda_{\mathbf{1} 8}\right) \\
-\frac{P}{\mathbf{a} b}\left(C_{u} b \lambda_{\mathbf{1}}+C_{v} a \lambda_{\mathbf{1 9}}+C_{u} C_{v} \lambda_{\mathbf{1 0}}+\mathbf{a} b \lambda_{\mathbf{2 0}}\right) \tag{62}
\end{array}
$$

Where;
$\lambda_{1}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[\frac{2 u_{1 \xi}^{2} b^{2}}{a^{2}}+u_{1 \eta^{2}}{ }^{2}(1-v)\right] d \xi d \eta ;$
$\lambda_{2}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[\frac{u_{1 \xi}{ }^{3} b^{2}}{a^{2}}+u_{1 \xi} u_{1 \eta}^{2}\right] d \xi d \eta ;$
$\lambda_{3}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[\frac{u_{1 \xi}{ }^{4} b^{2}}{a^{2}}+\frac{u_{1 \eta}{ }^{4} a^{2}}{b^{2}}+2 u_{1 \xi}{ }^{2} u_{1 \eta}{ }^{2}\right] d \xi d \eta ;$
$\lambda_{4}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[2 u_{1 \xi} v_{1 \eta} v+v_{1 \xi} u_{1 \eta}(1-v)\right] d \xi d \eta ;$
$\lambda_{5}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[\frac{v_{1 \xi}{ }^{2} u_{1 \xi} b^{2}}{a^{2}}+u_{1 \xi} v_{1 \eta}{ }^{2} v+v_{1 \xi} v_{1 \eta} u_{1 \eta}(1-v)\right] d \xi d \eta ;$

$$
\begin{aligned}
& \lambda_{6}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[\frac{w_{1 \xi}{ }^{2} u_{1 \xi} b^{2}}{a^{2}}+u_{1 \xi} w_{1 \eta}^{2} v+w_{1 \xi} w_{1 \eta} u_{1 \eta}(1-v)\right] d \xi d \eta \\
& \lambda_{7}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[\frac{u_{1 \eta}{ }^{2} v_{1 \eta} a^{2}}{b^{2}}+u_{1 \xi}{ }^{2} v_{1 \eta} v+u_{1 \xi} u_{1 \eta} v_{1 \xi}(1-v)\right] d \xi d \eta \\
& \begin{array}{r}
\lambda_{8}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[\frac{v_{1 \xi}{ }^{2} u_{1 \xi} \xi^{2} b^{2}}{a^{2}}+\frac{v_{1 \eta}{ }^{2} u_{1 \eta}^{2} a^{2}}{b^{2}}+u_{1 \xi}{ }^{2} v_{1 \eta}{ }^{2} v+v_{1 \xi}{ }^{2} u_{1 \eta}^{2} v\right. \\
\\
\left.+2 u_{1 \xi} u_{1 \eta} v_{1 \xi} v_{1 \eta}(1-v)\right] d \xi d \eta \\
\lambda_{9}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[\frac{w_{1 \xi}{ }^{2} u_{1 \xi}{ }^{2} b^{2}}{a^{2}}+\frac{w_{1 \eta}{ }^{2} u_{1 \eta}^{2} a^{2}}{b^{2}}+u_{1 \xi}{ }^{2} w_{1 \eta}{ }^{2} v+w_{1 \xi}{ }^{2} u_{1 \eta}{ }^{2} v\right. \\
\\
\left.\lambda_{1 \xi} u_{1 \eta} w_{1 \xi} w_{1 \eta}(1-v)\right] d \xi d \eta
\end{array}
\end{aligned}
$$

$$
\lambda_{10}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[\left(v_{1 \xi} w_{1 \eta}-w_{1 \xi} v_{1 \eta}\right) u_{1}+\left(w_{1 \xi} u_{1 \eta}-u_{1 \xi} w_{1 \eta}\right) v_{1}\right.
$$

$$
\left.+\left(u_{1 \xi} v_{1 \eta}-v_{1 \xi} u_{1 \eta}\right) w_{1}\right] d \xi d \eta
$$

$\lambda_{11}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[u_{1 \xi} w_{1}-w_{1 \xi} u_{1}\right] d \xi d \eta ;$
$\lambda_{12}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[\frac{2 v_{1 \eta}{ }^{2} a^{2}}{b^{2}}+v_{1} \xi^{2}(1-v)\right] d \xi d \eta ;$
$\lambda_{13}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[\frac{v_{1 \eta}^{3} a^{2}}{b^{2}}+v_{1 \xi}{ }^{2} v_{1 \eta}\right] d \xi d \eta ;$
$\lambda_{14}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[\frac{v_{1 \xi}{ }^{4} b^{2}}{a^{2}}+\frac{v_{1 \eta}{ }^{4} a^{2}}{b^{2}}+2 v_{1 \xi}{ }^{2} v_{1 \eta^{2}}{ }^{2}\right] d \xi d \eta ;$
$\lambda_{15}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[\frac{u_{1 \eta}{ }^{2} v_{1 \eta} a^{2}}{b^{2}}+u_{1 \xi}{ }^{2} v_{1 \eta} v+u_{1 \xi} u_{1 \eta} v_{1 \xi}(1-v)\right] d \xi d \eta ;$
$\lambda_{16}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[\frac{w_{1 \eta}{ }^{2} v_{1 \eta} b^{2}}{a^{2}}+w_{1 \xi}{ }^{2} v_{1 \eta} v+w_{1 \xi} w_{1 \eta} v_{1 \xi}(1-v)\right] d \xi d \eta ;$
$\lambda_{17}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[\frac{v_{1 \xi}{ }^{2} u_{1 \xi} b^{2}}{a^{2}}+u_{1 \xi} v_{1 \eta}^{2} v+v_{1 \xi} v_{1 \eta} u_{1 \eta}(1-v)\right] d \xi d \eta ;$

$$
\begin{aligned}
\lambda_{18}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[\frac{w_{1 \xi} \xi^{2} v_{1 \xi^{2}} b^{2}}{a^{2}}+\frac{w_{1 \eta}^{2} v_{1 \eta}{ }^{2} a^{2}}{b^{2}}\right. & +v_{1 \xi}{ }^{2} w_{1 \eta}{ }^{2} v+w_{1 \xi}{ }^{2} v_{1 \eta}^{2} v \\
& \left.+2 v_{1 \xi} v_{1 \eta} w_{1 \xi} w_{1 \eta}(1-v)\right] d \xi d \eta
\end{aligned}
$$

$\lambda_{19}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[v_{1 \eta} w_{1}-w_{1 \eta} v_{1}\right] d \xi d \eta ;$
$\lambda_{20}=\int_{\xi=0}^{1} \int_{\eta=0}^{1}\left[w_{1}\right] d \xi d \eta ;$

Equation (33) remains the same as do the definitions for the principle section loads and principle directions. However, the definitions for the maximum and minimum extensions of an infinitely small element (32) become;

$$
\begin{aligned}
& d_{x x}=\mathfrak{e}_{x} \frac{\partial \mathfrak{b}}{\partial s_{x}}+\frac{1}{2}\left(\frac{\partial \mathfrak{b}}{\partial s_{x}}\right)^{2} \\
& =\frac{\partial u}{\partial x}+\frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^{2}+\left(\frac{\partial v}{\partial x}\right)^{2}+\left(\frac{\partial w}{\partial x}\right)^{2}\right] \\
& =\frac{C_{u}}{a} u_{1 \xi}+\frac{1}{2}\left[\left(\frac{C_{u}}{a}\right)^{2} u_{1 \xi^{2}}^{2}+\left(\frac{C_{v}}{a}\right)^{2} v_{1 \xi^{2}}^{2}+\left(\frac{C_{w}}{a}\right)^{2} w_{1 \xi^{2}}^{2}\right] \\
& \begin{array}{r}
d_{y y}=\mathfrak{e}_{y} \frac{\partial \mathfrak{b}}{\partial s_{y}}+\frac{1}{2}\left(\frac{\partial \mathfrak{b}}{\partial s_{y}}\right)^{2} \\
=\frac{\partial v}{\partial y}+\frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^{2}+\left(\frac{\partial v}{\partial y}\right)^{2}+\left(\frac{\partial w}{\partial y}\right)^{2}\right] \\
\quad=\frac{C_{v}}{b} v_{1 \eta}+\frac{1}{2}\left[\left(\frac{C_{u}}{b}\right)^{2} u_{1 \eta}^{2}+\left(\frac{C_{v}}{b}\right)^{2} u_{1 \eta}^{2}+\left(\frac{C_{w}}{b}\right)^{2} w_{1 \eta}^{2}\right]
\end{array}
\end{aligned}
$$

$$
\begin{align*}
& 2 d_{x y}=\mathfrak{e}_{x} \frac{\partial \mathfrak{b}}{\partial s_{y}}+\mathfrak{e}_{y} \frac{\partial \mathfrak{b}}{\partial s_{x}}+\frac{\partial \mathfrak{b}}{\partial s_{x}} \frac{\partial \mathfrak{b}}{\partial s_{y}} \\
& =\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\left[\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)+\left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right)+\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\right] \\
& \quad=\frac{C_{u}}{b} u_{1 \eta}+\frac{C_{v}}{a} v_{1 \xi}+\frac{C_{u}}{a b} u_{1 \eta} u_{1 \xi}+\frac{C_{v}{ }^{2}}{a b} v_{1 \eta} v_{1 \xi}+\frac{C_{w}^{2}}{a b} w_{1 \eta} w_{1 \xi} \tag{64}
\end{align*}
$$

Investigation of the most effective displacement functions for a rectangular cushion is beyond the scope of this paper. However, to demonstrate the effectiveness of this formulation, the trigonometric displacement functions (39) will be used in order to analyse two cushion configurations the first measuring 2 m by 3 m and the second 1 m by 6 m .
The first cushion ( $a=200, b=300$ ) yields the following values of $\lambda$,
$\lambda_{1}=\frac{19}{4} \pi^{2}-\frac{1}{4} \pi^{2} v ; ; \lambda_{2}=0 ; ; \lambda_{3}=\frac{21}{4} \pi^{4} ; ; \lambda_{4}=\frac{16}{9}+\frac{16}{9} v ; ; \lambda_{5}=-\frac{4}{3} \pi^{2}+2 \pi^{2} v ; ;$
$\lambda_{6}=-\frac{5}{3} \pi^{2}+\frac{1}{2} \pi^{2} v ; ; \lambda_{7}=-\frac{10}{27} \pi^{2}+2 \pi^{2} v ; ; \lambda_{8}=\frac{97}{144} \pi^{4}+\frac{17}{16} \pi^{4} v ; ;$
$\lambda_{9}=\frac{85}{96} \pi^{4}+\frac{5}{32} \pi^{4} v ; ; \lambda_{10}=\frac{9}{16} \pi^{2} ; ; \lambda_{11}=\frac{4}{3} ; ; \lambda_{12}=\frac{41}{36} \pi^{2}-\frac{1}{4} \pi^{2} v ; ; \lambda_{13}=0 ; ;$
$\lambda_{14}=\frac{369}{256} \pi^{4} ; ; \lambda_{15}=-\frac{10}{27} \pi^{2}+2 \pi^{2} v ; ; \lambda_{16}=-\frac{25}{54} \pi^{2}+\frac{1}{2} \pi^{2} v ; ; \lambda_{17}=-\frac{4}{3} \pi^{2}+2 \pi^{2} v$
$\lambda_{18}=\frac{145}{384} \pi^{4}+\frac{5}{32} \pi^{4} v ; ; \lambda_{19}=\frac{4}{3} ; ; \lambda_{20}=\frac{4}{\pi^{2}} ; ; \lambda_{21}=\frac{105}{256} \pi^{4}$
By substituting the above values of $\lambda$ into (62) the following system of equations is produced where the only unknowns are the values of $C_{u}, C_{v}$ and $C_{w}$.

$$
\begin{aligned}
& \frac{\partial \pi}{\partial C_{u}}=0 \\
& =\frac{D}{b^{2}\left(2-2 v^{2}\right)}\left(C_{u}\left(\frac{19}{4} \pi^{2}+\frac{1}{4} \pi^{2} v\right)+\frac{C_{u}{ }^{3}}{a^{2}}\left(\frac{21}{4} \pi^{4}\right)\right. \\
& +\frac{C_{v} b}{a}\left(\frac{16}{9}+\frac{16}{9} v\right)+\frac{C_{v}}{a}\left(2 \pi^{2} v-\frac{4}{3} \pi^{2}\right)+\frac{C_{w}^{2}}{a}\left(\frac{1}{2} \pi^{2} v-\frac{5}{3} \pi^{2}\right) \\
& +\frac{2 C_{v} C_{u} b}{a^{2}}\left(2 \pi^{2} v-\frac{10}{27} \pi^{2}\right)+\frac{C_{v}{ }^{2} C_{u}}{a^{2}}\left(\frac{97}{144} \pi^{4}+\frac{17}{16} \pi^{4} v\right) \\
& \left.\quad+\frac{C_{w}{ }^{2} C_{u}}{a^{2}}\left(\frac{85}{96} \pi^{4}+\frac{5}{3} \pi^{4} v\right)\right)-\frac{P}{a}\left(\frac{C_{v} C_{w}}{b}\left(\frac{9}{16} \pi^{2}\right)+C_{w}\left(\frac{4}{3}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial \pi}{\partial C_{v}}=0 \\
& =\frac{D}{a^{2}\left(2-2 v^{2}\right)}\left(C_{v}\left(\frac{41}{36} \pi^{2}-\frac{1}{4} \pi^{2} v\right)+\frac{C_{v}{ }^{3}}{b^{2}}\left(\frac{369}{256} \pi^{4}\right)\right. \\
& +\frac{C_{u} a}{b}\left(\frac{16}{9}+\frac{16}{9} v\right)+\frac{C_{u}^{2}}{b}\left(2 \pi^{2} v-\frac{10}{27} \pi^{2}\right)+\frac{C_{w}^{2}}{b}\left(\frac{1}{2} \pi^{2} v-\frac{25}{54} \pi^{2}\right) \\
& +\frac{2 C_{v} C_{u} a}{b^{2}}\left(2 \pi^{2} v-\frac{4}{3} \pi^{2}\right)+\frac{C_{u}{ }^{2} C_{v}}{b^{2}}\left(\frac{97}{144} \pi^{4}+\frac{17}{16} \pi^{4} v\right) \\
& \left.\quad+\frac{C_{w}{ }^{2} C_{v}}{b^{2}}\left(\frac{145}{384} \pi^{4}+\frac{5}{32} \pi^{4} v\right)\right)-\frac{P}{b}\left(\frac{C_{u} C_{w}}{a}\left(\frac{9}{16} \pi^{2}\right)+C_{w}\left(\frac{4}{3}\right)\right)
\end{aligned}
$$

$\frac{\partial \pi}{\partial C_{w}}=0$

$$
\begin{align*}
& =\frac{D}{a^{2} b^{2}\left(2-2 v^{2}\right)}\left(C_{w}^{3}\left(\frac{105}{256} \pi^{4}\right)+2 C_{u} C_{w} a\left(\frac{1}{2} \pi^{2} v-\frac{5}{3} \pi^{2}\right)\right. \\
& +C_{u}^{2} C_{w}\left(\frac{85}{96} \pi^{4}+\frac{5}{3} \pi^{4} v\right)+2 C_{v} C_{w} b\left(\frac{1}{2} \pi^{2} v-\frac{25}{54} \pi^{2}\right) \\
& \left.+C_{v}^{2} C_{w}\left(\frac{145}{384} \pi^{4}+\frac{5}{32} \pi^{4} v\right)\right)-\frac{P}{a b}\left(C_{u} b\left(\frac{4}{3}\right)+C_{v} a\left(\frac{4}{3}\right)\right. \\
& \left.+C_{u} C_{v}\left(\frac{9}{16} \pi^{2}\right)+a b\left(\frac{4}{\pi^{2}}\right)\right) \tag{65}
\end{align*}
$$

The second cushion ( $a=100, b=600$ ) yields the following values for $\lambda$,
$\lambda_{1}=\frac{289}{4} \pi^{2}-\frac{1}{4} \pi^{2} v ; ; \lambda_{2}=0 ; ; \lambda_{3}=\frac{20769}{256} \pi^{4} ; ; \lambda_{4}=\frac{16}{9}+\frac{16}{9} v ; ;$
$\lambda_{5}=2 \pi^{2} v-\frac{58}{3} \pi^{2} ; ;$
$\lambda_{6}=\frac{1}{2} \pi^{2} v-\frac{145}{6} \pi^{2} ; ; \lambda_{7}=2 \pi^{2} v-\frac{4}{27} \pi^{2} ; ; \lambda_{8}=\frac{1297}{144} \pi^{4}+\frac{17}{16} \pi^{4} v ; ;$
$\lambda_{9}=\frac{5185}{386} \pi^{4}+\frac{5}{32} \pi^{4} v ; ; \lambda_{10}=\frac{9}{16} \pi^{2} ; ; \lambda_{11}=\frac{4}{3} ; ; \lambda_{12}=\frac{11}{36} \pi^{2}-\frac{1}{4} \pi^{2} v ; ; \lambda_{13}=0 ; ;$
$\lambda_{14}=\frac{21}{4} \pi^{4} ; ; \lambda_{15}=2 \pi^{2} v-\frac{4}{27} \pi^{2} ; ; \lambda_{16}=\frac{1}{2} \pi^{2} v-\frac{5}{27} \pi^{2} ; ; \lambda_{17}=2 \pi^{2} v-\frac{58}{3} \pi^{2}$
$\lambda_{18}=\frac{325}{96} \pi^{4}+\frac{5}{32} \pi^{4} v ; ; \lambda_{19}=\frac{4}{3} ; ; \lambda_{20}=\frac{4}{\pi^{2}} ; ; \lambda_{21}=\frac{1305}{256} \pi^{4}$

By substituting the above values of $\lambda$ into (62) the following system of equations is produced where the only unknowns are the values of $C_{u}, C_{v}$ and $C_{w}$.

$$
\begin{aligned}
& \frac{\partial \pi}{\partial C_{u}}=0 \\
& =\frac{D}{b^{2}\left(2-2 v^{2}\right)}\left(C_{u}\left(\frac{289}{4} \pi^{2}-\frac{1}{4} \pi^{2} v\right)+\frac{C_{u}{ }^{3}}{a^{2}}\left(\frac{20769}{256} \pi^{4}\right)+\frac{C_{v} b}{a}\left(\frac{16}{9}+\frac{16}{9} v\right)\right. \\
& +\frac{C_{v}{ }^{2}}{a}\left(2 \pi^{2} v-\frac{58}{3} \pi^{2}\right)+\frac{C_{w}{ }^{2}}{a}\left(\frac{1}{2} \pi^{2} v-\frac{145}{6} \pi^{2}\right)+\frac{2 C_{v} C_{u} b}{a^{2}}\left(2 \pi^{2} v-\frac{4}{27} \pi^{2}\right) \\
& \left.+\frac{C_{v}{ }^{2} C_{u}}{a^{2}}\left(\frac{1297}{144} \pi^{4}+\frac{17}{16} \pi^{4} v\right)+\frac{C_{w}{ }^{2} C_{u}}{a^{2}}\left(\frac{5185}{386} \pi^{4}+\frac{5}{32} \pi^{4} v\right)\right) \\
& \quad-\frac{P}{a}\left(\frac{C_{v} C_{w}}{b}\left(\frac{9}{16} \pi^{2}\right)+C_{w}\left(\frac{4}{3}\right)\right)
\end{aligned}
$$

$$
\frac{\partial \pi}{\partial C_{v}}=0
$$

$$
=\frac{D}{a^{2}\left(2-2 v^{2}\right)}\left(C_{v}\left(\frac{11}{36} \pi^{2}-\frac{1}{4} \pi^{2} v\right)+\frac{C_{v}{ }^{3}}{b^{2}}\left(\frac{21}{4} \pi^{4}\right)+\frac{C_{u} a}{b}\left(\frac{16}{9}+\frac{16}{9} v\right)\right.
$$

$$
+\frac{C_{u}^{2}}{b}\left(2 \pi^{2} v-\frac{4}{27} \pi^{2}\right)+\frac{C_{w}^{2}}{b}\left(\frac{1}{2} \pi^{2} v-\frac{5}{27} \pi^{2}\right)+\frac{2 C_{v} C_{u} a}{b^{2}}\left(2 \pi^{2} v-\frac{58}{3} \pi^{2}\right)
$$

$$
\left.+\frac{C_{u}^{2} C_{v}}{b^{2}}\left(\frac{1297}{144} \pi^{4}+\frac{17}{16} \pi^{4} v\right)+\frac{C_{w}^{2} C_{v}}{b^{2}}\left(\frac{325}{96} \pi^{4}+\frac{5}{32} \pi^{4} v\right)\right)
$$

$$
-\frac{P}{b}\left(\frac{C_{u} C_{w}}{a}\left(\frac{9}{16} \pi^{2}\right)+C_{w}\left(\frac{4}{3}\right)\right)
$$

$$
\frac{\partial \pi}{\partial C_{w}}=0
$$

$$
=\frac{D}{a^{2} b^{2}\left(2-2 v^{2}\right)}\left(C_{w}{ }^{3}\left(\frac{1305}{256} \pi^{4}\right)+2 C_{u} C_{w} a\left(\frac{1}{2} \pi^{2} v-\frac{145}{6} \pi^{2}\right)\right.
$$

$$
+C_{u}^{2} C_{w}\left(\frac{5185}{386} \pi^{4}+\frac{5}{32} \pi^{4} v\right)+2 C_{v} C_{w} b\left(\frac{1}{2} \pi^{2} v-\frac{5}{27} \pi^{2}\right)
$$

$$
\left.+C_{v}^{2} C_{w}\left(\frac{325}{96} \pi^{4}+\frac{5}{32} \pi^{4} v\right)\right)
$$

$$
\begin{equation*}
-\frac{P}{a b}\left(C_{u} b\left(\frac{4}{3}\right)+C_{v} a\left(\frac{4}{3}\right)+C_{u} C_{v}\left(\frac{9}{16} \pi^{2}\right)+a b\left(\frac{4}{\pi^{2}}\right)\right) \tag{66}
\end{equation*}
$$

Setting the elastic modulus of ETFE (D) to $190.6 \mathrm{~kg} / \mathrm{cm}$ and Poisson's ratio $(v)$ to approximately 0.333 values of $C_{u}, C_{v}$ and $C_{w}$ are derived for a range of pressures for each cushion as shown in Table 7.

Table 7: Values for $C_{u}, C_{v}$ and $C_{w}$

|  | $p(\mathrm{~Pa})$ | 100 | 200 | 300 | 400 | 500 | 600 | 700 | 800 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{a}=200$ <br> $\mathrm{~b}=300$ | $C_{u}$ | 0.093 | 0.148 | 0.194 | 0.237 | 0.276 | 0.312 | 0.347 | 0.381 |
|  | $C_{v}$ | 0.043 | 0.069 | 0.091 | 0.112 | 0.131 | 0.149 | 0.167 | 0.184 |
|  | $C_{w}$ | 7.706 | 9.716 | 11.100 | 12.260 | 13.210 | 14.041 | 14.790 | 15.465 |
| $\mathrm{a}=100$, <br> $\mathrm{b}=600$ | $C_{u}$ | 0.039 | 0.062 | 0.081 | 0.099 | 0.115 | 0.130 | 0.144 | 0.158 |
|  | $C_{v}$ | -0.005 | -0.008 | -0.010 | -0.012 | -0.013 | -0.014 | -0.016 | -0.017 |
|  | $C_{w}$ | 3.410 | 4.297 | 4.926 | 5.420 | 5.844 | 6.212 | 6.542 | 6.841 |


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