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A Nonlinear Optimization Algorithm for Lower Bound Limit and Shakedown Analysis

G. Gang[1] **and Y.H. Liu**[2]

Abstract: Limit and shakedown analysis theorems are the theories of classical plasticity for the direct computation of the load-carrying capacity under proportional and varying loads. Based on Melan's theorem, a solution procedure for lower bound limit and shakedown analysis of three-dimensional (3D) structures is established making use of the finite element method (FEM). The self-equilibrium stress fields are expressed by linear combination of several basic self-equilibrium stress fields with parameters to be determined. These basic self-equilibrium stress fields are elastic responses of the body to imposed permanent strains obtained through elastic-plastic incremental analysis by the three-dimensional finite element method (3D-FEM). The Complex method is used to solve the resulting nonlinear programming directly and determine the maximal load amplifier. The numerical results show that it is efficient and accurate to solve three-dimensional limit and shakedown analysis problems by using the 3D-FEM and the Complex method. The limit analysis is treated here as a special case of shakedown analysis in which only proportional loading is considered.

Keywords: Limit and shakedown analysis, 3D-FEM, self-equilibrium stress, nonlinear programming, the Complex method

1 Introduction

The computational problems associated with structural design and safety evaluation remain a challenge to many engineering problems. In the course of structural design and safety evaluation, traditional linear elastic analysis always gives conservative results of engineering problems, and hence the load-carrying capacities of structures can't be brought into play effectively. The plastic limit and shakedown loads, which can determine the load-carrying capacities of structures, are important parameters in performing structural integrity assessment. Therefore, the methods

[1] China Special Equipment Inspection and Research Center, Beijing 100013, China
[2] Department of Engineering Mechanics, Tsinghua University, Beijing 100084, China

of determining the limit and shakedown loads efficiently and accurately have attracted the attention of many researchers [Ciria H. (2008), da Silva M.V. (2007), Vu D.C. (2007), Vu D.K. (2004), Liu, Y. H. (1995,2000)].

The elastic-plastic incremental analysis is clearly more general and yields more information (sometimes unnecessarily abundant) often at higher computational effort. Its solution has been investigated for many years, primarily in the context of the finite element method [Zienkiewicz, O. C. (1977), Wunderlich, W. (1981)], and in the context of the boundary element method [Brebbia, C. A. (1980), Banerjee, P. K. (1981)]. However, to many practical engineering problems, only limit and shakedown loads and collapse mode are needed, and the stress and strain field histories are unnecessary to be known. Intended to avoid elastic –plastic incremental computation which is usually time-consuming, the limit and shakedown analysis method is considered to be applied to this kind of problems. Limit and shakedown analysis theorems are the theories of classical plasticity for the direct computation of the load-carrying capacity under proportional and varying loads.

As a simplified method, shakedown and limit analysis has higher computational efficiency and is more practical than incremental analysis. Theoretically this method can avoid the elastic-plastic incremental computation which is usually time-consuming, but on the other hand, it faces great difficulty in numerical computation. With solution procedure, it is mostly centered on mathematical programming [Cohn, M. Z.(1979), Maier, G. (1982)]. Because this mathematical programming has excessive independent variables and constraint conditions, and in general is a non-linear programming, the scale of solving is quite large. Therefore, how to establish an effective and reliable solution procedure to overcome this difficulty is very crucial for the application of limit and shakedown analysis method in engineering practice.

In this paper, a solution procedure of shakedown analysis is established by the finite element method based on Melan's theorem. The self-equilibrium stress fields are constructed by linear combination of several basic self-equilibrium stress fields with parameters to be determined. These basic self-equilibrium stress fields are expressed as elastic responses of the body to imposed permanent strains obtained through elastic-plastic incremental analysis by the 3D-FEM. The resulting nonlinear programming is solved effectively by the Complex method. The limit analysis is treated as a special case of shakedown analysis in which only proportional loading is considered.

2 Static theorem of shakedown analysis

The static shakedown theorem [Martin, J. B. (1975)] can be stated as follows: for a structure to shake down to the prescribed loading range (the body force is neglected here for brevity) if, and only if, there exists a time-independent self-equilibrium stress ρ_{ij} which, superimposed on the fictitious elastic stress σ_{ij}^{E}, yields the stress σ_{ij} not violating the yield condition at any point of the structure and for all possible load combinations, namely:

$$\varphi[\sigma_{ij}(x,t)] = \varphi[\sigma_{ij}^{\mathrm{E}}(x,t) + \rho_{ij}(x)] \leq 0 \quad \forall\, x \in \boldsymbol{\Omega} \tag{1}$$

here, φ is the yield function, $\sigma_{ij}(x,t)$ is the actual stress in a solid subjected to surface tractions $\boldsymbol{p}(x,t)$, $\sigma_{ij}^{\mathrm{E}}(x,t)$ denotes the fictitious stress that would appear had the structure responded to the applied loads in a purely elastic manner, and $\rho_{ij}(x)$ represents a self-equilibrium stress field that must satisfy equilibrium requirements within the body and vanish on the part of the surface where tractions are prescribed:

$$\rho_{ij,j} = 0 \quad \text{in} \quad \boldsymbol{\Omega} \tag{2}$$

$$\rho_{ij}n_j = 0 \quad \text{on} \quad \boldsymbol{\Gamma}_p \tag{3}$$

Strains appearing in the solid are sufficiently small so that geometric changes in configuration of the structure can be disregarded. Elastic strains $\varepsilon_{ij}^{\mathrm{E}}$ and plastic strains σ_{ij}^{P} are thus additive and the total strains are linear in terms of the displacement gradient, $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$. Plastic strain rates are governed by the associated flow law.

So, based on the static shakedown theorem, for a structure made up of elastic-perfectly plastic material, the maximum enlarging of the load variation domain allowing still for shakedown, characterized by a factor β, can be obtained by solving the following optimization problem:

$$\max : \beta(\beta \rightarrow \beta_s) \tag{4}$$

$$\text{s.t.} \quad \varphi[\sigma_{ij}(x,t)] = \varphi[\beta\sigma_{ij}^{e}(x,t) + \rho_{ij}(x)] \leq 0 \quad \forall\, x \in \Omega \tag{5}$$

$$\rho_{ij,j}(x) = 0 \quad \forall x \in \boldsymbol{\Omega} \tag{6}$$

$$\rho_{ij}(x)n_j = 0 \quad \forall x \in \boldsymbol{\Gamma}_p \tag{7}$$

In Eq(5), $\sigma_{ij}^{e}(x)$ denotes the fictitious elastic stress of structure under the basic variable loads. The above static formulation is of bounding character. It means that if we can find a self-equilibrium stress field $\rho_{ij}(x)$ and a corresponding safe factor β such that the yield condition (5) is satisfied for all $x \in \boldsymbol{\Omega}$ and for all $t > 0$, then β provides a lower bound to the actual shakedown factor β_s.

3 Mathematical programming of discretized structure

In the displacement finite element method, the geometry of the problem domain is first discretized. The load-dependent elastic stress field $\boldsymbol{\sigma}_i^{\mathrm{E}} = \boldsymbol{\sigma}^{\mathrm{E}}(\boldsymbol{x}_i)$ at the Gaussian points $\boldsymbol{x}_i$ can be calculated by means of the FEM, where the index i denotes the ith Gaussian point of the discretized structure. The equilibrium conditions (6) and (7) for the self-equilibrium stress can be transformed into the equivalent weak form given as

$$\int_{\Omega} \delta \boldsymbol{\varepsilon}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\rho}(\boldsymbol{x})\mathrm{d}\Omega = 0 \tag{8}$$

where $\delta\boldsymbol{\varepsilon}$ is an arbitrary virtual kinematically admissible strain.

According to the interpolation approximation, the virtual displacement $\delta\boldsymbol{u}(\boldsymbol{x})$ can be obtained as

$$\delta\boldsymbol{u}(\boldsymbol{x}) = \sum_{i=1}^{n} \phi_i(\boldsymbol{x})\delta\hat{\boldsymbol{u}}_i \tag{9}$$

over each element, and n is the number of nodes for each element. The corresponding virtual strain

$$\delta\boldsymbol{\varepsilon}(\boldsymbol{x}) = \boldsymbol{B}(\boldsymbol{x})\delta\hat{\boldsymbol{u}} \tag{10}$$

can be derived according to $\delta\varepsilon_{ij} = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i})$, where $\boldsymbol{B}(\boldsymbol{x})$ is the strain matrix which expresses the relationship between strain vector and nodal displacement vector, while $\hat{\boldsymbol{u}}$ is the nodal displacement vector for each element assembling all components of the fictitious nodal displacement $\hat{\boldsymbol{u}}_i$.

Using this relation and introducing the unknown self-equilibrium stress vector $\boldsymbol{\rho}(\boldsymbol{x}_i) = \boldsymbol{\rho}_i$ at each Gaussian point $\boldsymbol{x}_i$, the equilibrium condition (8) is integrated numerically with the Gauss technique. Denoting the weighting factor for numerical integration for the ith Gaussian point by w_i, the numerical integration over the structure yields

$$\int_{\Omega} (\delta\varepsilon(\boldsymbol{x}))^{\mathrm{T}}\boldsymbol{\rho}(\boldsymbol{x})\mathrm{d}\Omega \int_{\Omega} (\boldsymbol{B}(\boldsymbol{x})\delta\hat{\boldsymbol{u}})^{\mathrm{T}}\boldsymbol{\rho}(\boldsymbol{x})\mathrm{d}\Omega = (\delta\hat{\boldsymbol{u}})^{\mathrm{T}} \sum_{i=1}^{NG} \boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\rho}_i w_i = 0 \tag{11}$$

Because $\delta\hat{\boldsymbol{u}}$ is arbitrary, the above equation can be satisfied only if

$$\int_{\Omega} (\boldsymbol{B}(\boldsymbol{x}))^{\mathrm{T}}\boldsymbol{\rho}(\boldsymbol{x})\mathrm{d}\Omega = \sum_{i=1}^{NG} \boldsymbol{C}_i\boldsymbol{\rho}_i = \boldsymbol{C}\boldsymbol{\rho} = 0 \tag{12}$$

as the linear, discretized equilibrium condition for the self-equilibrium stress.

In Eq(12), $\boldsymbol{C}$ is a constant matrix that is uniquely defined by the discretized structure, $\boldsymbol{\rho}$ is the global self-equilibrium stress vector and NG is the total number of the Gaussian points of the discretized structure. The relations between $\boldsymbol{C}$, $\boldsymbol{C}_i$, $\boldsymbol{\rho}$ and $\boldsymbol{\rho}_i$ are respectively given by:

$$\boldsymbol{C} = (\boldsymbol{C}_1, \ldots, \boldsymbol{C}_i, \ldots, \boldsymbol{C}_{NG}) \tag{13}$$

$$\boldsymbol{\rho}^{\mathrm{T}} = (\boldsymbol{\rho}_1^{\mathrm{T}}, \ldots, \boldsymbol{\rho}_i^{\mathrm{T}}, \ldots, \boldsymbol{\rho}_{NG}^{\mathrm{T}}) \tag{14}$$

Actually, the yield condition (5) should be satisfied for all points of the discretized structure. However, according to the general approach in numerical analysis we only require that this condition is fulfilled at all Gaussian points of the structure. Now we are able to give the following discretized static formulation for the determination of the limit load factor β^s

$$\beta^s = \max \beta \tag{15}$$

$$\text{s.t.} \quad \sum_{i=1}^{NG} \boldsymbol{C}_i \boldsymbol{\rho}_i = \boldsymbol{C}\boldsymbol{\rho} = 0 \tag{16}$$

$$\varphi[\beta \boldsymbol{\sigma}_i^{\mathrm{E}} + \boldsymbol{\rho}_i] \leq 0 \quad i = 1 \sim NG \tag{17}$$

After the discretization of the structure, the load-dependent elastic stress field $\sigma_i^e(t) = \sigma^e(x_i, t)$ can be calculated by the FEM. Here x_i means the point where the yielding condition is checked. Then $\sigma_i^e(t)$ denotes the fictitious elastic stress at the ith point under the basic load domain. Let ρ_i denote the self-equilibrium stress at point i, the constraint condition (5) of optimization problem can be reformulated as:

$$\varphi[\sigma_i(t)] = \varphi[\beta \sigma_i^e(t) + \rho_i] \leq 0, \quad i = 1 \sim NG \tag{18}$$

where the NG is the total number of points where the yielding condition be checked. The discretized formulation (18) contains still the time t as variable. It means that the number of yield conditions is infinite because there are infinite possible fictitious elastic stress fields caused by the variable load in the basic load domain. The problem can essentially be simplified if we restrict ourselves to the basic load domain with a form of a convex polygon with M corners ($M = 1$ when limit analysis is considered). With respect to the convexity of the yield condition $\varphi(\cdot)$, it can easily be shown that if

$$\varphi[\beta \sigma_i^e(j) + \rho_i] \leq 0, \quad i = 1 \sim NG; \quad j = 1 \sim M \tag{19}$$

is satisfied for all $j = 1 \sim M$, then

$$\varphi[\beta \sigma_i^e(t) + \rho_i] \leq 0, \quad i = 1 \sim NG \tag{20}$$

is fulfilled at any time t.

By doing so, the constraint condition (18) can be replaced by the following one:

$$\varphi[\beta \sigma_i^e(j) + \rho_i] \leq 0, \quad i = 1 \sim NG; \quad j = 1 \sim M \tag{21}$$

where $\sigma_i^e(j)$ denotes the fictitious elastic stress vector at the ith point associated with the jth corner of the load domain.

After the discretization of space and time domain, the dimension of the mathematical programming is still very high so that the solving of the problem is very difficult or even impossible. To overcome this difficulty, a reduced-basis technique is used to simulate the self-equilibrium stress field [Stein, E. (1993)].

In the discretized sense, the unknown self-equilibrium stress vector associated with the best load factor β_s in the formulation (21) can be expressed by the linear combination of all the independent self-equilibrium stress vectors. The purpose of the reduced-basis technique is to look for several self-equilibrium stress basis vectors whose linear combination can lead to the appropriate ρ through the optimization, namely

$$\rho_i = C_1\rho_i^1 + C_2\rho_i^2 + \cdots + C_R\rho_i^R \quad i = 1 \sim NG \tag{22}$$

here, R is the number of basis vectors, $\rho_i^1, \rho_i^2, \ldots \rho_i^k, \ldots, \rho_i^R$ are the selected self-equilibrium stress basis vectors, and $C_1 \sim C_R$ are the parameters to be determined.

By doing so, the resulting mathematical programming of discretized body is as follows:

$$\max : \beta(\beta \to \beta_s) \tag{23}$$

$$\text{s.t.}\varphi[\beta\sigma_i^e(j) + C_1\rho_i^1 + C_2\rho_i^2 + \cdots + C_R\rho_i^R] \le 0 \quad i = 1 \sim NG, \quad j = 1 \sim M \tag{24}$$

Because $\rho_i^1, \rho_i^2, \ldots, \rho_i^R$, are the selected self-equilibrium stress basis vectors, the constraint conditions for ρ (Eq.(6), (7)) have been satisfied automatically. By the reduced basis technique the nonlinear programming problem can be solved in a sequence of reduced residual stress spaces with very low dimensions. In this way, large scale finite element systems of different structures can be treated successfully.

It should be noticed that after the discretization of the structure, the above static formulation (23) and (24) does not keep its bounding character in the strict meaning. This is because the fictitious elastic stress field is not calculated precisely, the equilibrium conditions for the self-equilibrium stress field are satisfied only in a weak form, and instead of all $x \in \mathbf{\Omega}$ we require only that the yield condition be fulfilled at the all stress points. In any case, in shakedown problems the elastic solution is crucial and even minor approximations on local values make the bounding nature of results questionable. But, if the discretization is sufficiently fine, one can hope that the computational result β provides a reliable value to the actual shakedown factor β_s.

4 The construction of equilibrium stress field

As shown in the inequality (21), the total equilibrium stress field is made up of two parts: the first part is the fictitious elastic stress field $\sigma_i^e(j)$ in the jth corner of basic load domain with load factor β, and the second is the self-equilibrium stress field ρ_i. The fictitious elastic stress field $\sigma_i^e(j)$ can be computed by the FEM directly.

The self-equilibrium stress field can be expressed as linear combination of a group of "basis vectors" with parameters to be determined. Every "basis vector" is a self-equilibrium stress field, which should satisfy the constraint relations (6) and (7). So, the total equilibrium stress field can be written as follows:

$$\sigma_i(j) = \beta \sigma_i^e(j) + C_1\rho_i^1 + C_2\rho_i^2 + \cdots + C_R\rho_i^R; \quad i = 1 \sim NG, \quad j = 1, M \tag{25}$$

every "basis vector" is expressed by the difference between the stress fields of different iteration steps at same loading incremental step of elastic-plastic incremental computation. It is essentially the elastic stress response to plastic strain obtained through incremental elastic-plastic BEM procedure and undoubtedly satisfies the constraint relations (6) and (7).

The whole process of solving this problem can be divided into some sub-problems. The iteration index, indicating each sub-problem in a corresponding reduced self-equilibrium stress space, is denoted by $k(k = 1, 2, \ldots)$. The solution algorithm is as follows:

At the beginning of the kth sub-problem we have a known state represented by a load factor $\beta^{(k-1)}$ and a self-equilibrium stress distribution $\rho_i^{(k-1)}$ that satisfies:

$$\varphi[\beta^{(k-1)}\sigma_i^{\mathrm{EB}}(j) + \rho_i^{(k-1)}] \le 0 \quad i = 1 \sim NG, \quad j = 1, M \tag{26}$$

This inequality (28) indicates that $[\beta^{(k-1)}, \rho_i^{(k-1)}]$ is a feasible point of the mathematical programming (26). Therefore $\beta^{(k-1)}$ is a lower bound to the shakedown factor β_s of the discretized system (but not necessarily a lower bound of the original problem).

In the kth sub-problem, on the basis of the $(k-1)$th total stress $\sigma_i^{(k-1)}$, adding the kth load increment $\Delta\beta^k(j)$ to the body, we can get a group of new "basis vectors" (i. e. $\rho_i^{1(k)}, \rho_i^{2(k)}, \ldots$) for every load corner of basic loading domain by the elastic-plastic iteration computation. Then take the self-equilibrium stress field $\rho_i^{(k-1)}$ of last sub-problem as one of the "basis vectors" $\rho_i^{R(k)}$, so we get R "basis vectors" in all in this sub-problem. Using the Complex method (which will be introduced latter) to solve the nonlinear programming of this sub-problem, we can get the kth approximate solution $\beta^{(k)}$, the corresponding self-equilibrium stress field $\rho_i^{(k)}$ and total stress field $\sigma_i^{(k)} = \beta^{(k)}\sigma_i^e + \rho_i^{(k)}$. For $k = 1$, through computing the Mises' equivalent stress at every stress point associated with every corner of basic loading domain, we can get the elastic limit load amplifier $\beta^e(j)$ respectively. Set $\beta^{(0)} =$

$\min[\beta^e(1) \sim \beta^e(M)]$ and $\rho_i^0 = \mathbf{0}$, so we got $\beta^{(0)}$ and $\rho_i^{(0)}$ as the initial values of the whole solving process.

The above computational process will be repeated until the following convergence criterion is fulfilled:

$$\frac{\beta^{(k)} - \beta^{(k-1)}}{\beta^{(k-1)}} \leq \varepsilon^0 \quad k \geq 2 \tag{27}$$

where ε^0 is a given error tolerance. Our numerical experiences show that, in general, when k is equal to or even less than five, $\beta^{(k)}$ is already a good approximate solution to the actual shakedown factor and the whole solution procedure has stable convergence. According to the numerical experiment, in general, the number of "basis vectors" can be choose as $R = (3 \sim 5) \times M$ in the first sub-problem and $R = (3 \sim 5) \times M + 1$ in the following sub-problems.

5 Solving of nonlinear programming

Here the von Mises' yield condition is adopted. Taking advantage of the above relationships in Section 5, the unified version of all sub-problems can be written as:

$$\max : \beta(\beta \to \beta_s) \tag{28}$$

$$\text{s.t.}\varphi[\beta\sigma_i^e(j) + C_1\rho_i^1 + C_2\rho_i^2 + \cdots + C_R\rho_i^R] \leq 0 \quad i = 1 \sim NG; \quad j = 1 \sim M \tag{29}$$

The value of fictitious elastic stress and every "basis vector" at each stress point can be computed by the FEM before the programming problem (28) and (29) is solved. There are $(NG \times M)$ constraint inequalities in the above sub-problem. The optimal variables include the objective function β and R parameters to be determined.

This nonlinear programming has these features:

1. The number of optimal variables is relatively small ($R = (3 \sim 5) \times M + (1 \sim 2)$);
2. The number of constraint conditions is quite large ($NG \times M$);
3. All the constraint conditions are quadratic inequalities.

Taking account of these characteristics, we use the Complex method to solve this programming problem [Xi, S. L. (1983)]. The solving process can be divided into two steps.

Step 1: for given numerical values of $C_1', C_2', \ldots, C_R'$, get the corresponding load factor β''.

Because all the constraint conditions are quadratic inequalities, they can be treated as quadratic functions with independent variable β:

$$\begin{aligned} Q_k(\beta) &= \varphi[\beta\sigma_i^{\mathrm{EB}}(j) + C_1'\rho_i^1 + C_2'\rho_i^2 + \cdots + +C_R'\rho_i^R] \\ &= a_k\beta^2 + b_k\beta + c_k \\ &\le 0 \qquad\qquad k = 1,2,\ldots,NG\times M \end{aligned} \tag{30}$$

In the above expression, the parameter a_k is made up of stress deviators (known) of fictitious elastic stress field at ith stress point under the load of jth corner, and b_k, c_k are made up of stress deviators (known) of both fictitious elastic stress field and every "basis vector". It can be easily proved that a_k must be a non-negative number (i. e. $a_k \ge 0$). So, the value of β_k' which satisfy the kth inequality (30) must be between the two roots of the corresponding equation:

$$Q_k(\beta_k') = 0, \quad k = 1,2,\ldots, \quad NG\times M \tag{31}$$

Let these two roots of kth equation be marked by $\beta_{1(k)}'$ and $\beta_{2(k)}'$. Without losing generality, we assume these two roots satisfy $\beta_{1(k)}' \le \beta_{2(k)}'$. The solving of Eq(31) can be divided into two cases:

I. $\forall a_k > 0$, if:

$$\Delta_k = b_k^2 - 4a_kc_k \ge 0 \tag{32}$$

the Eq(31) has two roots:

$$\beta_{1(k)}' = \frac{-b_k - \sqrt{b_k^2 - 4a_kc_k}}{2a_k}, \quad \beta_{2(k)}' = \frac{-b_k + \sqrt{b_k^2 - 4a_kc_k}}{2a_k}, \tag{33}$$

II. $\forall\, a_k = 0$, this means that the corresponding fictitious elastic stress field is equal to zero. So the value of b_k must be equal to zero. The inequality (30) can be simplified as:

$$c_k \le 0 \tag{34}$$

If inequality (34) is satisfied, then inequality (30) can be satisfied with any value of β_k'. So we can prescribe the roots of the Eq(31):

$$\beta_{1(k)}' = -\infty, \quad \beta_{2(k)}' = +\infty, \tag{35}$$

Because β must satisfy all the inequalities (30), so the following expression must be satisfied for any i and j:

$$\max_i \beta_{1(i)}' \le \min_j \beta_{2(j)}' \quad \forall\, i,j = 1,2,\ldots,NG\times M \tag{36}$$

If the above conditions (32) and (36) $\forall\, a_k > 0$ or (34) and (36) $\forall\, a_k = 0$ can be satisfied, the possible value range of the β' should be:

$$\max\{\beta'_{1(1)},\ldots,\beta'_{1(k)},\ldots,\beta'_{1(NG\times M)}\} \le \beta' \le \min\{\beta'_{2(1)},\ldots,\beta'_{2(k)},\ldots,\beta'_{2(NG\times M)}\} \tag{37}$$

So the maximum likelihood of β' is:

$$\beta'' = \max\{\beta'\} = \min\{\beta'_{2(1)},\beta'_{2(2)},\ldots,\beta'_{2(k)},\ldots,\beta'_{2(NG\times M)}\} \tag{38}$$

So, for the arbitrarily given numerical values of $C'_1, C'_2, \ldots, C'_R$, (feasible to that problem, i. e. satisfy (32) and (36) $\forall\, a_k > 0$ or (35) and (36) $\forall\, a_k = 0$), we can get a corresponding numerical value of β''. This kind of relationship can be expressed as the following function form:

$$\beta'' = \psi(C'_1, C'_2, \ldots, C'_R) \tag{39}$$

Step 2: Look for optimal values of $C'^*_1, C'^*_2, \ldots, C'^*_R$, let the corresponding load factor $\beta''^* \to \beta^s$.

For transforming this problem into the standard formulation of the Complex method, the objective function (39) can be substituted by:

$$\beta'' = \psi(C'_1, C'_2, \ldots, C'_R) = -F(C'_1, C'_2, \ldots, C'_R) \tag{40}$$

Then the nonlinear programming (28) and (29) can be represented by the following new form:

$$\beta''^* = \max \beta'' = -\min F \tag{41}$$

$$\text{s.t.} \quad F = F(C'_1, C'_2, \ldots, C'_R) = -\psi(C'_1, C'_2, \ldots, C'_R) \tag{42}$$

$$\forall\, a_k > 0, \Delta_k = b_k^2 - 4a_k c_k \ge 0 \quad \text{(No index summation)} \tag{43}$$

$$\forall\, a_k = 0, \quad c_k \le 0 \quad k = 1,2,\ldots,NG\times M \tag{44}$$

$$\beta'_{1(i)} \le \beta'_{2(j)} \quad \forall\, i,j = 1,2,\ldots,NG\times M \tag{45}$$

This is a standard non-linear programming formulation which can be solved without any difficulties by the Complex method. The solution process of the Complex method [Xi, S. L. (1983)] is as follows:

(1) Form the initial Complex configurations, namely, find out $(2R+1)$ initial points in a R-dimensional space. The coordinates of every point are denoted by a group of numbers $C'_1, C'_2, \ldots, C'_R$, which must satisfy the constraint conditions (43) and (44) in advance.

(2) After the formation of initial Complex configurations, the following iteration will proceed:

(a) Find the best point $x^{(b)}$ (it means that the objective function has minimal value at this point) and the worst point $x^{(w)}$, then compute the coordinates (marked by $x^{\wedge}$) of central point of all the points except $x^{(w)}$:

$$x^{\wedge} = \frac{1}{2R}\left(\sum_{i=1}^{2R+1} x^{(i)} - x^{(w)}\right) \tag{46}$$

It can be easily proved that the central point $x^{\wedge}$ satisfies the constraint conditions (43) and (44).

(b) Seek for the reflecting point of $x^{(w)}$ with respect $x^{\wedge}$ to and mark this point by x^{Δ}:

$$x^{\Delta} = (1+\lambda)x^{\wedge} - \lambda x^{(w)} \tag{47}$$

Here $\lambda > 0$ is the reflecting factor (generally we let $\lambda = 1,3$). If the point x^{Δ} does not satisfy the constraint conditions (43) and (44)., then move the point x^{Δ} to the central point $x^{\wedge}$ by half distance, namely:

$$x^{\Delta}(\text{new}) = 0.5(x^{\Delta}(\text{old}) + x^{\wedge}) \tag{48}$$

If the new x^{Δ} still does not satisfy the constraint conditions, then use the formula (48)repeatedly until this point satisfies the constraint conditions.

(c) Compute the value of $F(x^{\Delta})$. If

$$F(x^{\Delta}) < \max_{i=1\sim 2R+1, i\neq w} (F(x^{i})) \tag{49}$$

then let the point x^{Δ} substitute the point x^{w} and go to (d), otherwise let the point x^{Δ} move half distance towards central the point $x^{\wedge}$ (use the formula (48) again) until formula (49) is satisfied.

(d) For a prescribed error tolerance ε^{1}, if

$$\|x^{b} - x^{w}\| < \varepsilon^{1} \tag{50}$$

then take x^{b} as the appropriate solution of this sub-problem, otherwise go back to (a).

The numerical computations of present solution procedure show that both initial Complex configuration and initial solution have little influence on the computational results.

6 Numerical examples

Some problems which are analyzed using the proposed solution procedure are presented in this section. These examples are employed to verify the performance of the numerical technique. In the following examples, 8-node isoparametric elements are used for the discretization of structures, see Fig.1. All the bodies are made up of von Mises' material. The material parameters are as follows: elastic modulus $E = 2.1 \times 10^5$MPa, Poisson's ratio $v = 0.3$, and yielding stress $\sigma_0 = 200$MPa.

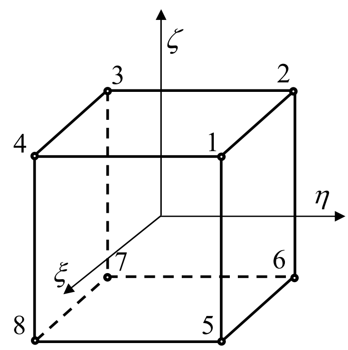

Figure 1: Sketch map of finite element.

(1) Limit and shakedown analysis of the square plate with a central circular hole

A square plate with a central circular hole is considered (see Fig.2). This is a classical problem in numerical limit and shakedown analysis. The length of the plate is L and the ratio between the diameter of the hole and the length of the plate is 0.2. The system is subjected to the biaxial uniform loads P_1 and P_2. Both can vary independently of each other between zero $P_1^{\max}$ and $P_2^{\max}$ certain maximum magnitudes and $(0 \leq P_1 \leq P_1^{\max}, 0 \leq P_2 \leq P_2^{\max})$.

The problem has also been investigated by [Belytschko, T. (1972), Corradi, L. (1974), Nguyen, D. H. (1979), Genna, F. (1988), Stein, E. (1992), Gross-Weege, J. (1997), Carvelli, V. (1999)].

Fig.3 represents finite element distribution of the adopted discretization. The mesh consists of 96 elements.

The computational results by the present method and reasonable agreement are observed in Fig.4 with the counterpart limit analysis results by [Gross-Weege, J. (1997)], the shakedown analysis results by [Gross-Weege, J. (1997)] and [Carvelli, V. (1999)], achieved respectively by a static approach and a kinematic approach.

The numerically detailed comparisons with available earlier works are summarized

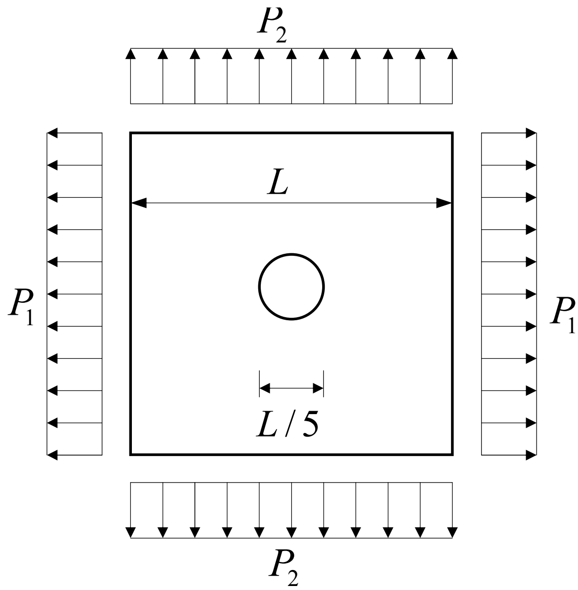

Figure 2: Sketch map of the plate

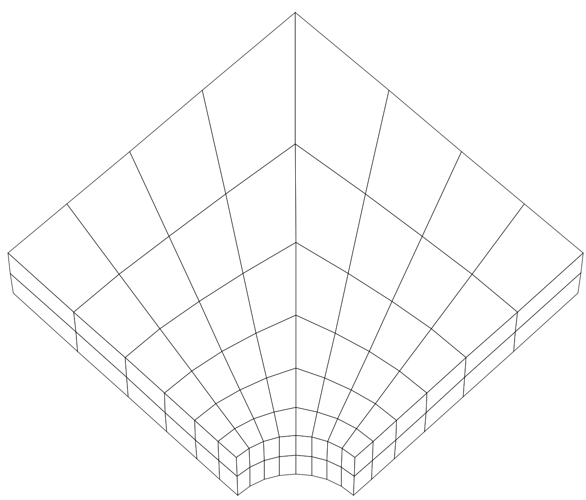

Figure 3: Finite element mesh of the plate

in Table 1 for three special load combinations of P_1 and P_2. It should be noticed that the results are based on different approaches concerning both the discretization of the problem and the numerical solution technique. Tab. 1 shows that our results in general are close to the available numerical results. The comparisons illustrate the validity of the present solution procedure.

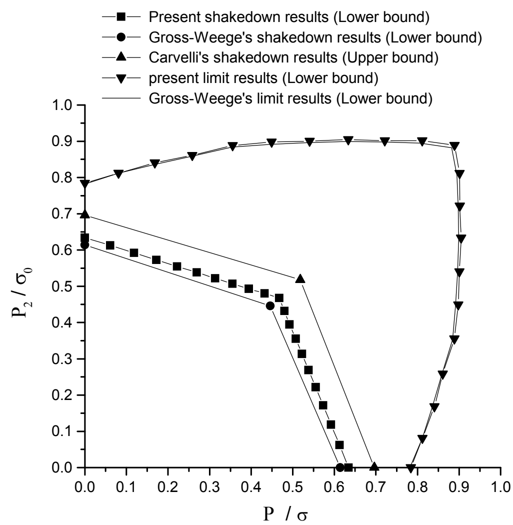

Figure 4: The shakedown and limit load domains for the plate

Table 1: Comparison of different numerical solutions for limit and shakedown analysis (P_1/σ_0)

Problems	Authors & Methods	Loading cases		
		$P_2 = P_1$	$P_2 = P_1/2$	$P_2 = 0$
Limit analysis	Nguyen & Palgen (1979), Lower bound	0.704	—	0.564
	Corradi & Zavelani (1974), Upper bound	0.767	—	0.691
	Gross-Weege (1997), Lower bound	0.882	0.891	0.782
	Present solution, Lower bound	0.889	0.898	0.784
Shakedown Analysis	Belytschko (1972), Lower bound	0.431	0.501	0.571
	Corradi & Zavelani (1974), Upper bound	0.504	0.579	0.654
	Nguyen & Palgen (1979), Lower bound	0.431	0.532	0.557
	Genna (1988), Lower bound	0.478	0.566	0.653
	Stein & Zhang (1992), Lower bound	0.453	0.539	0.624
	Gross-Weege (1997), Lower bound	0.446	0.524	0.614
	Carvelli et al. (1999), Upper bound	0.518	0.607	0.696
	Present solution, Lower bound	0.467	0.538	0.634

(2) Shakedown analysis of a thick-walled cylinder subjected to internal pressure.

Shakedown analysis is performed for a thick-walled cylinder subjected to a fluctuating uniform internal pressure P (Fig.5). This is a benchmark problem with analytical solution. For this problem, we calculate the shakedown loads of thick-walled cylinders with different ratios of a/b (internal radius / external radius). The cylinder is discretized by 48 elements when $a/b = 1/2$, as shown in Fig.6. The numerical results compared with the analytical solutions are shown in Fig.7. The analytical solution of shakedown problem of thick-walled cylinder subjected to fluctuating uniform internal pressure is as follows:

$$P_{SD} = \min\{P_L, 2 \times P_e\}. \tag{51}$$

Where

$$P_L = \frac{2}{\sqrt{3}}\sigma_0 \ln\frac{b}{a},\ P_e = \tfrac{2}{\sqrt{3}}\sigma_0[1-(a^2/b^2)]$$

When $b/a \leq 2.22$, non-shakedown is plastic collapse; when $b/a > 2.22$, non-shakedown is alternating plasticity.

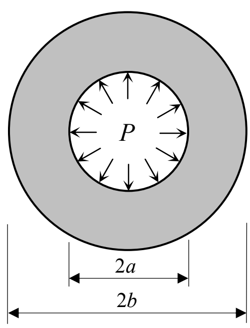

Figure 5: Thick-walled cylinder subjected to internal pressure

We can see that the numerical results are in good agreement with the analytical solutions.

(3) Limit analysis of a defective pipeline subjected to internal pressure, axial tension and bending moment.

The limit loads of a 3-D defective pipeline are computed here using the proposed method. The geometry of the pipeline with a small slot subjected to the combined action of internal pressure P, axial tension N (caused by internal pressure P, i.e, $N = P\pi R_i^2$, where R_i is the inner radius of pipeline) and bending moment M at

Table 2: Some numerical results compared with analytical solutions (MPa).

b/a	Plastic limit	Elastic limit ($\times 2$)	Shakedown limit	Present solution	error
1.5	93.64	128.30	93.64	91.26	2.54%
2	160.08	173.21	160.08	157.04	1.90%
3	253.71	205.28	205.28	201.39	1.89%
4	320.15	216.50	216.50	212.26	1.96%

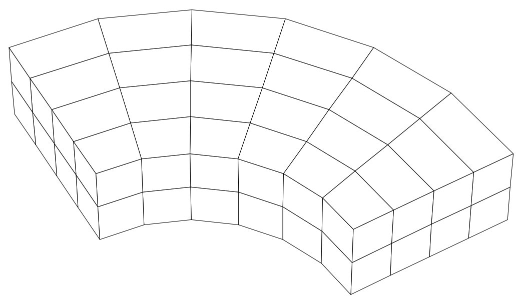

Figure 6: Finite element mesh of the cylinder

both ends is shown in Fig.8. The geometric parameters of the pipeline adopted here are as follows: $R_o = 70$mm; $R_i = 50$mm; $A = 30$mm; $B = 22$mm; $C = 10$mm; $T = 20$mm and $\theta = 18^\circ$.

Considering the symmetry, we take a quadrant of the 3-D defective pipeline to discretize the structure and the corresponding displacement constraints are imposed on the symmetric boundaries. The element mesh of the pipeline adopted are shown in Fig.9.

We define the following non-dimensional parameters:

$$m = M/M_0 \tag{52}$$

$$p = P/P_0 \tag{53}$$

where $M_0 = \frac{4}{3}\sigma_0(R_o^3 - R_i^3)$ is the theoretical limit bending moment of pipeline without defect, and $P_0 = \frac{2}{\sqrt{3}}\sigma_0 \ln \frac{R_o}{R_i}$ is the theoretical limit internal pressure of pipeline without defect.

The calculated lower bound limit load domain of the defective pipeline is plotted in Fig.10, for combined internal pressure and bending moment. For the purpose of comparison, we also employ the commercial software ANSYS to compute the limit loads of pipeline by 3-D elastic-plastic incremental analysis. In this example, when the defective pipeline is acted by only internal pressure or only

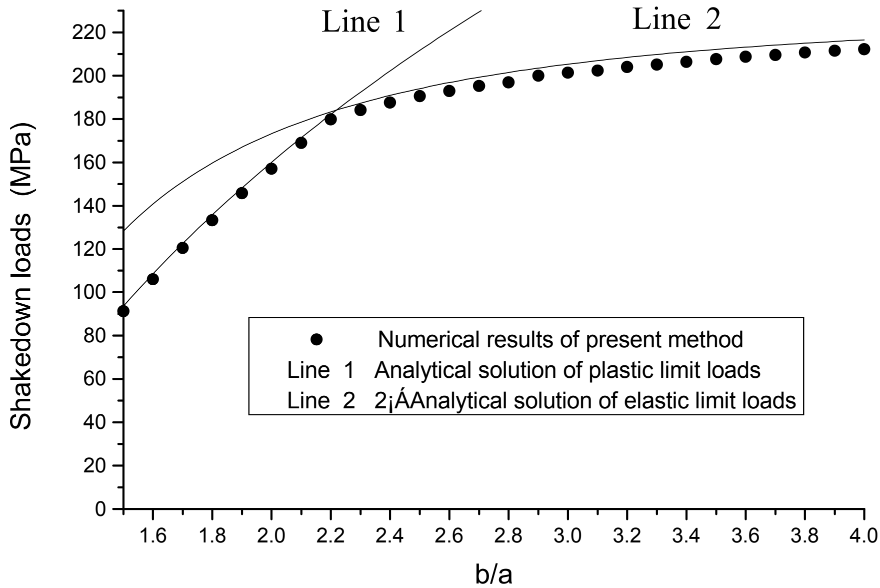

Figure 7: Comparison of present solutions of shakedown load with analytical solutions

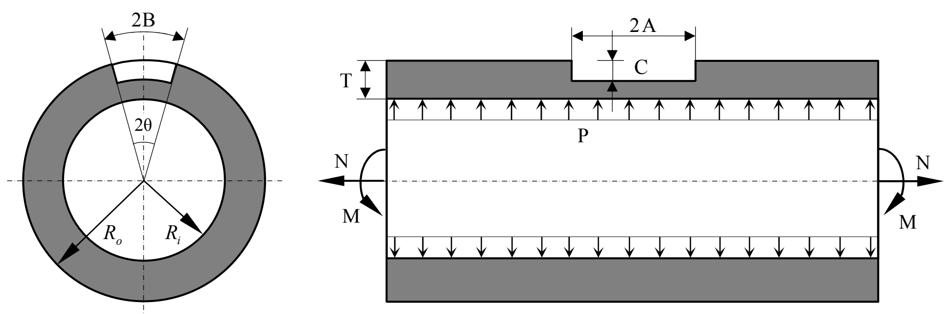

Figure 8: The geometry of pipeline with a slot subjected to internal pressure, axial

bending moment, we get the limit internal pressure $P = 63.42$MPa (the solution of ANSYS $P = 64.05$MPa) and $M = 5.496 \times 107$KN·m (the solution of ANSYS $M = 5.527 \times 107$KN·m).

Generally, the computational time by the present method is about $1/3 \sim 1/6$ that by the incremental analysis of ANSYS.

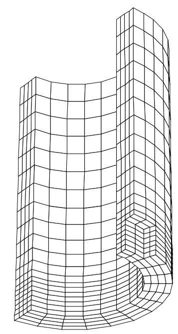

Figure 9: Finite element mesh of the pipeline

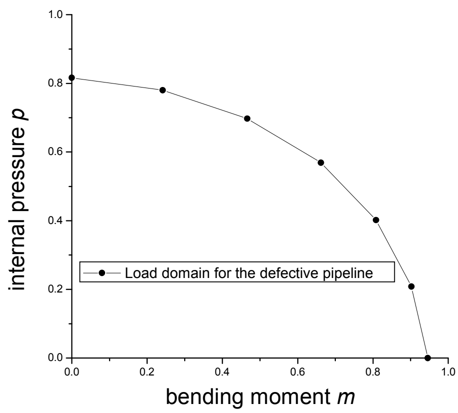

Figure 10: The limit load domain of the defective pipeline subjected to internal pressure and bending moment

7 Conclusions

A numerical solution procedure is proposed for lower bound limit and shakedown analysis by the 3D-FEM. A reduced-basis technique and the Complex method are adopted herein. By doing these, the proposed numerical method yields good results and reduces greatly the computational cost. Numerical examples are given to demonstrate the efficiency and accuracy of the present method. Through the above study and analysis, we can draw the following conclusions:

(1) The FEM limit and shakedown analysis by the static approach can be effectively performed with the reduced-basis technique and the Complex method. The whole solution procedure turns out to be significantly cost-effective with respect

to other approaches, particularly with respect to evolutive step-by-step analysis by the commercial finite element codes. The numerical results show that the FEM can provide an effective procedure for limit and shakedown analysis.

(2) The Complex method represents a cost-effective, numerically stable and reliable tool for the mathematical programming problem of lower bound limit and shakedown analysis. The numerical results of the solution procedure adopted herein appear to be satisfactory and rather insensitive to the choice of the initial complex configurations and load increments used to create basis self-equilibrium stress vectors.

(3) The discretization of structure by the FEM turns out to be an efficient way to get the fictitious elastic stress field and to compute the basis self-equilibrium stress vectors associated with the incremental elastic-plastic analysis.

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